Trends in Computational Social Choice

16

Cite as: László Á. Kóczy, Péter Biró, and Balázs Sziklai. US vs. European Apportionment Practices: The Conflict between Monotonicity and Proportionality. In Ulle Endriss (editor), *Trends in Computational Social Choice*, chapter 16, pages 309–325. AI Access, 2017.

CHAPTER 16

US vs. European Apportionment Practices: The Conflict between Monotonicity and Proportionality

László Á. Kóczy, Péter Biró, and Balázs Sziklai

16.1 Introduction

In a representative democracy citizens exert their influence via elected representatives. Representation will be fair if the citizens have more or less the same (indirect) influence, that is, if each representative stands for the same number of citizens. This idea was explicitly declared in the 14th Amendment of the US Constitution, but dates back even earlier to the times of the Roman Republic.

> "Representatives shall be apportioned among the several States according to their respective numbers, counting the whole number of persons in each State, excluding Indians not taxed. (14th Amendment, Section 2)

Establishing electoral districts with equal numbers of voters becomes nontrivial, when they must fit into the existing administrative structure of a country. For instance the distribution of three seats between two equally populated regions will necessarily lead to inequalities. This example may seem artificial, but under more realistic circumstances with many regions and a high number of seats to be allocated the problem remains hard. The general problem of allocating seats between regions in a fair way is known as the apportionment problem.

Proportional apportionment is one, but not the only ingredient of fair representation. Other, monotonicity-related issues — studying changes in the allocation subject to changes in the input parameters — emerged in the past 150 years. The most notable one is the so-called Alabama paradox. During the 1880 US census the Chief Clerk of the Census Office considered an enlargement of the House of Representatives and noted that moving from 299 to 300 seats would result in a loss of a seat for the State Alabama. This anomaly together with the later discovered population and new state paradoxes pressed the legislators to revise the apportionment rules again and again. The currently used seat distribution method is free from such anomalies. However, it does not satisfy the so called Hare-quota, a basic guarantee of proportionality (Balinski and Young, 1975).

While virtually every Western-type democracy adopted the principle laid down in the US Constitution, their approaches differ on how they deal with the arising paradoxes and anomalies.

The European Commission for Democracy through Law, better known as the Venice Commission, a recent entrant to this debate, published a comprehensive guidebook on good electoral laws in 2002. The Code of Good Practice in Electoral Matters (Venice Commission, 2002) — consequently used in reviewing Albania's and Estonia's electoral law in 2011 (OSCE/ODIHR, 2011; Venice Commission and OSCE/ODIHR, 2011) and forming an apparent model to the modifications Hungary introduced to its electoral law in 2012 —, contains original recommendations for a good practice of apportionment.

> "**Equality in voting power**, where the elections are not being held in one single constituency, requires constituency boundaries to be drawn in such a way that seats in the lower chambers representing the people are distributed equally among the constituencies, in accordance with a specific apportionment criterion, e.g., the number of residents in the constituency, the number of resident nationals (including minors), the number of registered electors, or possibly the number of people actually voting ... Constituency boundaries may also be determined on the basis of geographical criteria and the administrative or indeed historic boundary lines, which often depend on geography ... The maximum admissible departure from the distribution criterion adopted depends on the individual situation, although it should seldom exceed 10% and never 15%, except in really exceptional circumstances (a demographically weak administrative unit of the same importance as others with at least one lower-chamber representative, or concentration of a specific national minority)." (Venice Commission, 2002, §§13–15 in Section 2.2)

The recommendation leaves some details open. Does the maximum admissible departure refer to the difference of population between any two constituencies or the difference of the population of any constituency from the average constituency size? The latter approach is more permissive and more common around the world (see Table 16.1). Indeed, the final version of the 2012 electoral law of Hungary replaced the former with 10-15% departure limits with the latter with 15-20% departure limits. Without this significant relaxation the rule was mathematically impossible to satisfy (Biró et al., 2012).

Similar thresholds exist in many other countries (Table 16.1), but the values differ greatly from country to country. The strictest limits are set in the United States that permits no inequalities by its Constitution. Zero-tolerance, however, remains a theoretical objective. Real life is widely different: the constituencies of Montana are almost twice as large as the ones in Rhode Island. Assuming that the voters' influence is proportional to the size of the constituencies, the voters of Rhode Island have 88% more influence than the voters of Montana. A shocking gap, but dwarfed by the differences in Georgia where the electoral law of 1999 did not set rules about the sizes of constituencies. The number of voters per (single-seat) constituencies ranged from 3,600 in the Lent'ekhi or 4,200 in the

Country	Thresholds	Country	Thresholds
Albania	5%	New Zealand	5%
Armenia	15%	Papua New Guinea	20%
Australia	10%	Singapore	30%
Canada	25%	Ukraine	10%
Czech Republic	15%	UK	5%
France	20%	USA	0%
Germany	15%	Yemen	5%
Hungary	15% (20%)	Zimbabwe	20%
Italy	10%		

Table 16.1: Thresholds (thresholds under “extraordinary circumstances”) for the maximum difference from the average constituency size (Handley, 2007).

Kazbegi districts to over 138,000 in Kutaisi City, hugely favouring voters in the former regions.

Setting a limit on the maximum departure from the average size is a very natural condition, but already such a mild requirement conflicts with well-established apportionment standards: for certain apportionment problems all allocations that respect the given limits violate properties such as Hare-quota and monotonicity (Biró et al., 2015). Furthermore, the recommendation of the Venice Committee does not generally specify a unique solution, so it still leaves possibilities of manipulation. This second problem may be overcome by a new apportionment rule, constructed in the spirit of the recommendation. The Leximin Method efficiently computes a solution where the differences from the average size are lexicographically minimized (Biró et al., 2015).

In this chapter we survey the apportionment methods and the impact of the latest policy recommendation by the Venice Commission. First, in Section 16.2 we give an overview on the classical apportionment methods and the Leximin Method, and discuss their properties. Then we illustrate the usage of the Leximin Method compared to the solutions by the current legislations from a wide range of countries. These examples are based on our own calculations that in turn are made using information on voting systems and population data gathered from a wide range of sources. The details together with a systematic study of voting systems will be published elsewhere.

16.2 Overview of Apportionment Methods

In this section we introduce the apportionment problem; we introduce and characterise methods to solve it.

16.2.1 The Apportionment Problem

In a representative democracy higher level decisions are made by a group of elected representatives. In most countries each representative speaks for citizens living in a certain geographical area and is elected in one of several voting districts or constituencies. Generally a constituency elects a single candidate, although in some countries, like Ireland or Singapore a constituency may elect multiple representatives. Other countries, like the Netherlands or Israel, has no non-trivial constituencies, but all representatives are elected at the national level with no geographical attachment — we regard this as a trivial case with a single constituency. Yet others have combinations of these (Csató, 2015, 2016) — we will focus on the voting districts. The basis of geographical representation is that people living in certain regions, such as New Yorkers or Scotsmen are not just arbitrary voters, but people sharing certain cultural or geographical interests. Constituencies are consequently organised into geographical, political or administrative regions.

We look for a fair an proportional representation. However natural this approach seems, it is not universal. The Cambridge Compromise, an academic-driven proposal for a mathematical method to allocate the seats of the European Parliament among the member states, for instance, takes proportionality as only one of the aspects to be taken into account (Grimmett, 2012). In weighted voting the weights are also not proportional. During the negotiations of the Lisbon Treaty that, among others, reformed voting in the Council of the European Union the Jagellonian Compromise proposed to use the Penrose square-root law, where the allocated weights are proportional to the square root of populations (Penrose, 1946; Słomczyński and Życzkowski, 2006; Kóczy, 2012). While these are examples where proportionality is knowingly violated, but for the purposes of fairness, there are many voting systems (Canada and Denmark are examples) where certain *territories*, such as rural regions, or less populate states, are overrepresented by law.

Our interest thus lies in the allocation of representatives among these regions in a fair way. Allocating seats among parties in party-list proportional representation, the biproportional apportionment problem (see Chapter 3 in this book) or voting with multi-winner approval rules (Brill et al., 2017) is analogous and the general problem of apportionment can go well beyond the districting problem and can deal with the allocation of any finite, indivisible good among heterogenous claimants in a fair, proportional way. While the methodology can be used, for instance for discrete clearing in the bankruptcy literature (Csóka and Herings, 2016), in the following we keep the voting terminology and also take such applications and examples. We assume that the task is to allocate the *seats* of a legislature or *House* among several, n *states* — and elegantly skip the problem of districting (Tasnádi, 2011; Puppe and Tasnádi, 2015), the laying out of the actual districts, that can introduce additional inefficiencies. Before going any further, we formally define the problem and introduce some of the best known methods to solve the apportionment problem.

An *apportionment problem* $(\boldsymbol{p}, H)$ is a pair consisting a vector

$$\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$$

of state populations, where $P = \sum_{i=1}^{n} p_i$ is the population of the country and $H \in \mathbb{N}_+$ denotes the number of seats in the House (where $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$).

Our task is to determine the non-negative integers $a_1, a_2, \ldots, a_n$ with $\sum_{i=1}^{n} a_i = H$ representing the number of constituencies in states $1, 2, \ldots, n$.

Let $\boldsymbol{p} \in \mathbb{N}_+^n$ and $\boldsymbol{a} \in \mathbb{N}^n$ be the n-dimensional vectors that contain the population sizes and the allotted number of seats, respectively. An *apportionment method* or *rule* is a function M that assigns an allotment for each apportionment problem $(\boldsymbol{p}, H)$. An apportionment method specifies exactly how many House seats each of the states gets. The resulting apportionment is not necessarily unique although for a good method the multiplicity only emerges in artificial examples. Let $A = \frac{P}{H}$ denote the average size of a constituency. The fraction $\frac{p_i}{P}H = \frac{p_i}{A}$ is called the *respective share* of state i. Let δ_i be the difference in percentage, displayed by the constituencies of state i and let d_i be the *departure*, its absolute value. Formally,

$$\delta_i = \frac{\frac{p_i}{a_i} - A}{A} \quad \text{and} \quad d_i = |\delta_i| \tag{16.1}$$

Throughout the paper we will employ the following notation: let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we say that $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_i \geq y_i$ for $i = 1, 2, \ldots, n$.

16.2.2 Apportionment Methods

The fundamental idea of apportionment methods is that a representative should speak for the same number of voters irrespective of the state or region she represents. Ideally a state i should get a proportional part $\frac{p_i}{P}H$ of the seats. This number is the *standard quota*. If not all standard quotas are integers and most of the time they are not, we must diverge from the ideal numbers. Rounding the numbers down does not immediately solve the problem as the total number of seats to be distributed is fixed, so if the standard quota is rounded down for some, it must be rounded up for others, immediately creating inequalities. Many of the best known methods only differ in rounding up or down the standard quotas differently. See also Chapter 3 where some remarkably different methods coming from a different stream of literature are presented.

Largest Remainder Methods

The largest remainder methods all rely on the logic of calculating the “price” of a seat in terms of the number of voters, allocating the fully “paid” seats. The remaining seats are allocated to the states with the *largest remainders*, that is, the states with the largest fractional seat. Several methods exist using different ways to calculate the price, the Hamilton method is the simplest and best known.

The Hamilton method (also known as Hare-Niemeyer or Vinton method) sets the price as the standard or Hare divisor $D_S = \frac{P}{H}$, which is the same as the average constituency size A. By dividing the population of a state by the standard divisor D_S we calculate the ideal number of constituencies in the given state. From this we can calculate how many seats does the state's population suffice for: each state is guaranteed to get the integer part of the quota, the *lower quota*. The remaining seats are distributed in the same way as for other largest remainder methods.

We are not aware of a specification of a tiebreaking rule when the remainders are identical, although with real life data this is a non-issue. The Hamilton method was the first proposal to allocate the seats of the United States Congress between states, but this was vetoed by president Washington.

Other largest remainder methods differ in the way their quotas are calculated. The Hagenbach-Bischoff quota (Hagenbach-Bischoff, 1888) is calculated with the divisor $D_{\text{H-B}} = \frac{P}{H+1}$, while the Droop and Imperiali (named after Belgian Senator Pierre Imperiali) quotas with the only very slightly different $D_{\text{D}} = \lfloor \frac{P}{H+1} + 1 \rfloor$ (Droop, 1881) and $D_{\text{I}} = \frac{P}{H+2}$. The Droop quota is typically used in single transferable vote systems, where voters rank candidates and if their top choice has sufficient votes to get elected, the vote goes to the second choice and so on. The Droop divisor is the lowest number satisfying that the number of claimable resources, such as seats does not exceed the House. In this sense the Hagenbach-Bischoff and especially the Imperiali method may allocate seats that must later be taken back.

Divisor Methods

Divisor methods (sometimes called *highest average* or *highest quotient methods*) follow a slightly different logic by adjusting the quotient itself. When the (lower) quotas are calculated there will be some left-over seats. By lowering the divisor — effectively the price of a seat — states will be able to afford more. Divisor methods are mathematically equivalent to procedural apportionment methods such as e.g. the D'Hondt method, which distribute seats one at a time to the state with the highest claim, then update the claims after each iteration until all the seats are allocated.

The Jefferson or D'Hondt method, introduced by Thomas Jefferson in 1791 and by Victor D'Hondt in 1878 in two mathematically very different, though equivalent forms is the simplest of all divisor methods. Under the Jefferson method the standard divisor $D_S = \frac{P}{H}$ is calculated. The lower quotas generally do not add up to the size of the House, so in this method the standard divisor is gradually lowered by "trial and error" until they do. While this is not a precise mathematical algorithm, note that the modified divisor will generally satisfy this for a whole range of values, so an appropriate value is easy to find.

The D'Hondt method uses the following claim function

$$\text{D'Hondt method} \qquad q_i^{\text{H}}(s) = \frac{p_i}{s+1}$$

showing how many voters would a representative, on average, represent if an additional seat were given to the state i already having s seats.

Some voting systems use variants of the D'Hondt method that bias the results in favour or against larger claimants, such as states with larger voting population or parties with many votes in a party-list voting system. These include the following

Adams method	$q_i^{\mathrm{A}}(s) = \dfrac{p_i}{s}$
Danish method	$q_i^{\mathrm{D}}(s) = \dfrac{p_i}{s+1/3}$
Huntington-Hill method/EP	$q_i^{\mathrm{HH}}(s) = \dfrac{p_i}{\sqrt{s(s+1)}}$
Sainte-Laguë/Webster method	$q_i^{\mathrm{SL}}(s) = \dfrac{p_i}{s+1/2}$
Imperiali method	$q_i^{\mathrm{I}}(s) = \dfrac{p_i}{s+2}$
Macau method	$q_i^{\mathrm{M}}(s) = \dfrac{p_i}{2^s}$

displaying an increasing bias against large states with the Adams, Danish Huntington-Hill and Sainte-Laguë methods favouring large states more than the D'Hondt, Imperiali or especially the Macau method (Marshall et al., 2002; Bittó, 2017). The Huntington-Hill method, also known as the Method of Equal Proportions (EP) is the method currently used in the United States House of Representatives.

The Leximin Method

The Leximin Method (Biró et al., 2015) is fundamentally different from the methods discussed so far. While these were based on finding the standard quota and then trying to find a good way to round these numbers, the Leximin Method looks at relative differences. It *minimizes the absolute value of the largest relative difference from the average constituency size* — the *maximum departure* — and does this in a recursive fashion.

To have a more precise definition, we need to introduce some terminology. Lexicographic is like alphabetic ordering where words are compared letter-by-letter and the ordering is based on the first difference. When it comes to real vectors the ordering is based on the first coordinates where these vectors differ. Formally vector $x \in \mathbb{R}^m$ is *lexicographically smaller* than $y \in \mathbb{R}^m$ (denoted by $x \prec y$) if $x \neq y$ and there exists a number $1 \leq j \leq m$ such that $x_i = y_i$ if $i < j$ and $x_j < y_j$.

Returning to our model, given an apportionment problem $(\boldsymbol{p}, H)$ and an allotment $\boldsymbol{a}$, let $\boldsymbol{\Delta}(\boldsymbol{a})$ denote a nonnegative n-dimensional vector, where the differences $d_i(\boldsymbol{a})$ are contained in a *non-increasing order*. A solution $\boldsymbol{a}$ is said to be *lexicographically minimal*, or simply *leximin*, if there is no other allotment $\boldsymbol{a}'$ where $\boldsymbol{\Delta}(\boldsymbol{a}')$ is lexicographically smaller than $\boldsymbol{\Delta}(\boldsymbol{a})$. The *Leximin Method* chooses an allocation of seats, such that the non-increasingly ordered vector of differences is lexicographically minimal. This method is somewhat more complex than the earlier ones, but while other methods make sure that states do not get too many seats, the Leximin Method takes both under- and overrepresentation into account. Perhaps it is not so obvious here, but the method is well-defined and Biró et al. (2015) gave an efficient algorithm to calculate it.

16.2.3 Properties and Paradoxes

There are several apportionment methods and while in most cases they all produce nearly identical results, we would like to understand the reasons for the small differences that may be observed. The way to argue in favour or against these methods is by looking at their properties. In the following we list some properties that apportionment methods satisfy.

Quota

Exact proportional representation is seldom possible as the respective shares of the states are hardly ever integer numbers. However if such a case occurs, that is, the fractions $a_i = \frac{p_i}{P}H$ are integers for all $i \in \{1, \ldots, n\}$ then the allotment $\boldsymbol{a}$ is said to have the *exact quota* property.

In any other case taking one of the nearest integers to the exactly proportional share is a natural choice or at least some methods explicitly try to allocate seats accordingly. An allotment $\boldsymbol{a}$ satisfies *lower (upper) quotas*, if no state receives less (more) constituencies than the lower (upper) integer part of its respective share, that is $a_i \geq \left\lfloor \frac{p_i}{P}H \right\rfloor$ for all $i \in \{1, \ldots, n\}$ and $a_i \leq \left\lceil \frac{p_i}{P}H \right\rceil$ for all $i \in \{1, \ldots, n\}$, respectively. An allotment satisfies the *Hare-quota* or simply the *quota* property if it satisfies both upper and lower quota.

Similarly, we say that an apportionment method $M(\boldsymbol{p}, H)$ satisfies lower (upper) quota if for any apportionment problem (p, H), $M(\boldsymbol{p}, H)_i \geq \lfloor \frac{p_i}{P}H \rfloor$ or $M(\boldsymbol{p}, H)_i \leq \lceil \frac{p_i}{P}H \rceil$ respectively for all $i \in \{1, \ldots, n\}$ and satisfies *Hare-quota* if it satisfies both of them.

Monotonicity

Monotonicity properties describe how changes in the number of available seats or the (relative) claims made by the states should affect the number of allocated seats.

House-monotonicity states that the individual states should not lose seats when more seats are available in the House.

Definition 16.1. *An apportionment method M is* house-monotonic *if $M(\boldsymbol{p}, H') \geq M(\boldsymbol{p}, H)$ for any apportionment problem $(\boldsymbol{p}, H)$ and House sizes $H' > H$.*

A scenario where increasing the House size would decrease the number of seats allotted to a state is often considered undesirable, perhaps even paradoxical. An apportionment rule where this is possible is said to exhibit the *Alabama paradox* referring to a historical occurrence of the phenomenon for state Alabama. House-monotonic apportionment methods are free from this paradox.

There is a related monotonicity requirement and an associated paradox when populations are considered. The *population paradox* arises when the population of two states increases at different rates. Then it is possible that the state with more rapid growth actually loses seats to the state with slower growth. Biró et al. (2015) present an example where the population paradox emerges; Tasnádi (2008) surveys the emergence of this paradox historically in the apportionment among parties in Hungary.

Definition 16.2. *An apportionment rule* M *is population-monotonic if* $M(\boldsymbol{p}', H)_i \geq M(\boldsymbol{p}, H)_i$ *for any House size* H *and population sizes* $\boldsymbol{p}, \boldsymbol{p}'$ *such that* $p_i' > p_i$, $p_j' > p_j$ *and* $\frac{p_i'}{p_i} \geq \frac{p_j'}{p_j}$ *while* $p_k' = p_k$ *for* $k \in \{1, 2, \ldots, n\}, k \neq i, j$.

Note that there are several alternative definitions of this property. The one presented here is slightly weaker than some others used in the literature (Lauwers and Van Puyenbroeck, 2008; Balinski and Young, 1982). However, as we will see even this weaker property is violated by some rules.

Departure from the Exact Quota

If it is not possible to distribute the seats according to the exact quota there will be necessarily some inequality. *Departure* is the relative difference between the average number of represented voters per representative in a given state and nationwide.

Several countries specify an explicit limit on the permitted departure from the average in their electoral law in accordance with the recommendation of the Venice Commission (2002). An apportionment satisfies the q-*permitted departure property* if all departures are smaller than the given limit q. Then an apportionment method satisfies the *admissible departure property* if for each apportionment problem, for which there exists an apportionment satisfying the permitted departure property, it produces such an apportionment. Formally

An apportionment satisfies the *Venice* or *Smallest maximum admissible departure property* if for apportionment problem it produces an apportionment where the largest departure is the smallest. For a given apportionment problem $(\boldsymbol{p}, H)$ let $\alpha_{(\boldsymbol{p},H)}$ be the smallest maximum admissible departure that can be achieved with an allotment, i.e.,

$$\alpha_{(\boldsymbol{p},H)} = \min_{\boldsymbol{a} \in A(n,H)} \max_{i \in \{1,\ldots,n\}} \{d_i\} \tag{16.2}$$

where $A(n, H)$ denotes the set of n-dimensional non-negative vectors for which the sum of the coordinates is H.

Definition 16.3. *An apportionment rule* M *satisfies the* smallest maximum admissible departure property *if* $\left| \frac{\frac{p_i}{M(\boldsymbol{p},H)_i} - A}{A} \right| \leq \alpha_{(\boldsymbol{p},H)}$ *for any apportionment problem* (p, H) *and for each* $i \in \{1, \ldots, n\}$.

16.3 Choosing Methods

The reason for looking at the various properties has been to be able to evaluate the different methods. In Table 16.2 we present some of the known comparison results about these methods. Apportionment has a long history in the United States and the method has already been altered several times. Over the years many new states joined, populations increased dramatically and correspondingly, the House was expanded, too, and we have seen properties violated several

Table 16.2: A comparison of apportionment methods.

	quota	House monotonicity	population monotonicity	Venice
Hamilton	both	no	no	no
Jefferson/D'Hondt	lower	yes	yes	no
Webster/Sainte-Laguë	mostly	yes	yes	no
Huntingdon-Hill/EP	no	yes	yes	no
Leximin	no	no	no	yes

times. While apart from the initial use of the Jefferson method, Hamilton and Webster were used together, Hamilton was found to exhibit both the Alabama paradox, when house-monotonicity is violated, the population monotonicity and also the new state paradox that we did not discuss here. As a result the method has been replaced by the Huntingdon-Hill, or Equal Proportions method that is still used today.

Even if we treat the Venice property separately, notice that there is no method that would satisfy all other requirements. Balinski and Young (1975) introduced the so-called *Quota method* that is house-monotonic and fulfills the quota property as well, but proved that no method that is free from both the Alabama and the population paradoxes satisfies quota (Balinski and Young, 1982). On the other hand Biró et al. (2015) have shown that the Venice property is not compatible with any of the remaining properties. Notice that the result is also true if we look at admissible departures only. For a low enough admissible departure the same counterexamples can be presented. This means that the recommendation of Venice Commission (2002) inherently violates quota and the monotonicity properties.

When we say that a method violates a property we mean that there exists an apportionment problem where the given property is violated. These counterexamples are sometimes artificial. They may for instance rely on symmetries that are extremely unlikely in real life. In the following we look at real apportionment problems gathered from countries all over the world. In the next couple of sections we test the properties on this real data set.

16.3.1 Bounds on the Maximum Departure

Let us fix an apportionment problem $(\boldsymbol{p}, H)$. Obviously d_i is the smallest if state i receives either its lower or upper quota, although it matters which one. Note that the closest integer to the respective share does not always yield the smallest difference from the average. Let us elaborate on this relationship a bit further.

Let $l_i = \left\lfloor \frac{p_i}{P} H \right\rfloor$ and $u_i = \left\lceil \frac{p_i}{P} H \right\rceil$, respectively, denote the lower and upper quotas of state i and let β_i and ω_i denote the minimum and maximum difference achievable for state i when it gets the lower or upper integer part of its respective share. The maximum of the β_i values, denoted by β (for *best* case), is a natural lower bound on the maximum departure for any apportionment, which satisfies the Hare-quota property. Similarly the maximum of the ω_i values, denoted by

ω (for *worst* case), is an upper bound for any apportionment which satisfies the Hare-quota. Formally:

$$\beta_i = \min\left(\left|\frac{\frac{p_i}{l_i} - A}{A}\right|, \left|\frac{\frac{p_i}{u_i} - A}{A}\right|\right), \qquad \beta = \max_{i \in N} \beta_i. \tag{16.3}$$

$$\omega_i = \max\left(\left|\frac{\frac{p_i}{l_i} - A}{A}\right|, \left|\frac{\frac{p_i}{u_i} - A}{A}\right|\right), \qquad \omega = \max_{i \in N} \omega_i. \tag{16.4}$$

Suppose we would like to minimize the differences from the average constituency size. We calculate the standard quota for every state and start rounding it up or down depending on which one yields a smaller difference. Unfortunately the resulting allotment is infeasible if we have distributed too few or too many seats. The *best case scenario* is when the allotted number of seats add up to the House size. In such cases we can guarantee that the departure is not bigger than β. Even if some states are rounded in the wrong direction, β is achievable if we rounded the critical states well. The *worst case scenario* is when the critical states are rounded in the wrong direction, in such cases the difference will be ω. Note that it is always possible to allocate the seats in such way that the apportionment satisfies the quota property, hence if the goal is to minimize the differences from the average then ω is achievable even in the worst case.

In contrast the maximum difference α can be implemented by the Leximin Method, By design, $\beta \leq \alpha \leq \omega$, thus the Leximin Method always yields an apportionment that falls within these bounds. Somewhat surprisingly, empirical data shows that divisor methods, which are known to violate the quota property never exceed these bounds either (see Figures 16.1 and 16.2).

16.3.2 Monotonicity vs. Quota vs. Maximum Departure

The Leximin Method fails to be monotonic because it focuses solely on reducing the maximum departure from the average constituency size. In effect this means that the Leximin Method will reallocate seats from big states to small ones if the resulting apportionment has smaller departure. Large states with many seats serve as puffers where excess seats can be allocated or seats can be acquired if there are needed elsewhere as these changes do not affect the average size of constituencies dramatically. For the exact same reasons the Leximin Method violates quota as well.

Divisor methods are all immune from the Alabama paradox. The reason is clear: by enlarging the House, the price of a seat decreases, thus each state can afford more. Similarly, divisor methods are immune from both the population- and new state paradoxes. In fact if a method avoids the population paradox it must be a divisor method (Balinski and Young, 1982). As a consequence divisor methods sometime fail to produce quota apportionments. Interestingly, quota failures just as for leximin affect only large states (see Tables 16.3 and 16.4).

Quota failures are more common for problems with substantially different state/county sizes. In case of Hungary the capital Budapest has eight times more voters than the smallest county, Nógrád. In comparison the Irish administrative

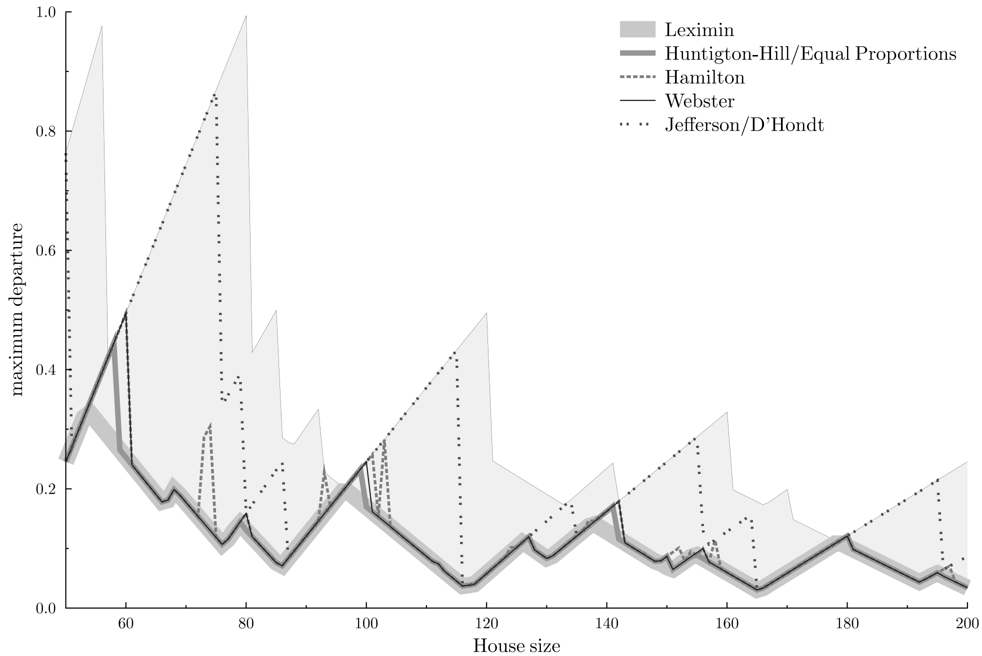

Figure 16.1: Apportionment over Belgian regions. Leximin coincides with β; EP, Webster are near. Ironically, D'Hondt performs poorly, reaching ω several times.

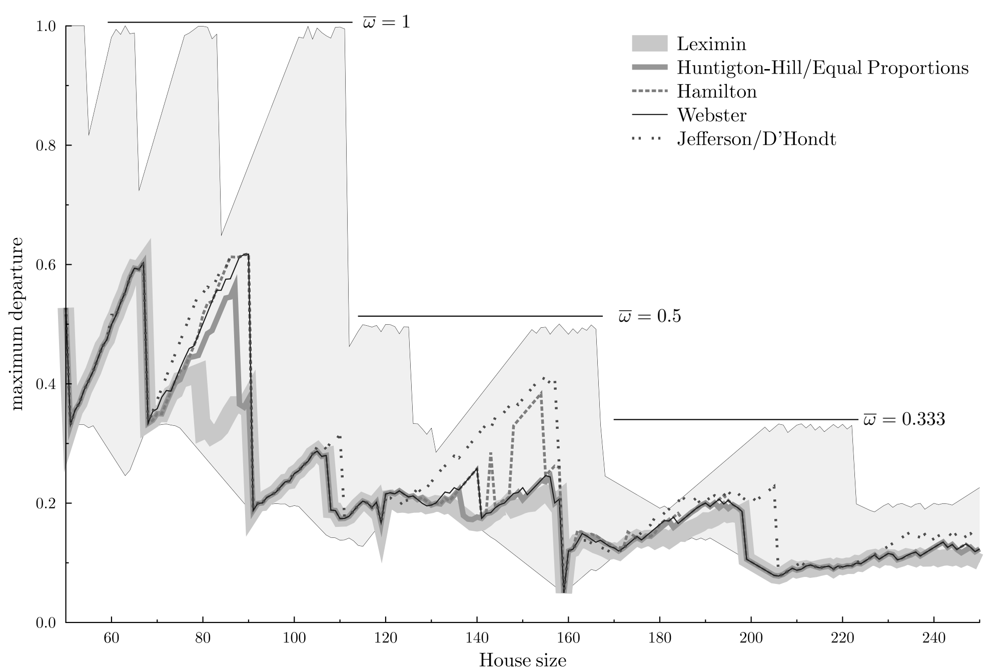

Figure 16.2: Apportionment over Irish counties. Leximin performs best, then EP, Webster, but all struggle to evenly distribute seats due to regular county sizes.

	Leximin	EP	Jeff/D'Hondt	Adams	Webster
Largest county (Budapest)	27	1	74	67	0
2nd largest county (Pest)	4	0	17	21	0
Elsewhere	0	0	0	0	0

Table 16.3: Number of quota failures based on Hungarian constituency data when House size varies between 100 and 200.

	Leximin	EP	Jeff/D'Hondt	Adams	Webster
Largest state (California)	112	2	201	201	0
2nd largest state (Texas)	30	0	198	192	0
3rd largest state (New York)	12	0	120	67	0
4th largest state (Florida)	6	0	105	21	0
Elsewhere	0	0	19	24	0

Table 16.4: Number of quota failures based on US constituency data when House size varies between 335 and 535 (that is current House size ± 100).

regions do not vary that much. The population ratio of the largest (Donegal) and the smallest (South-West Cork) county is only 1.83. Even on a broader range of House sizes (50-250) the Adams, EP and Webster methods do not violate the quota property and the leximin and the Jefferson/D'Hondt methods only violate it 3 times each (again at the two largest counties).

The leximin and EP methods, although conceptually very different, in practice tend to produce similar apportionments. They coincide for the apportionment problems in Austria, Denmark, Finland, Ireland, Luxemburg and Portugal, differ for the US House of Representatives and in England by 1 and 2 seats respectively. This small difference, however, accounts for the worse (better) departure statistic and for the (lack of) monotonicity.

The β and ω bounds indicate that proportional representation rests on whether we can round the critical states in a good direction. Enforcing quota ensures that the departure will not exceed ω but the additional constraint also makes it difficult to stay close to β, since it does not allow us to use states as buffers to lend/borrow problematic or desperately needed seats for critical states without creating too much inequality. What are the critical states? Critical states are small states which are only a few times as big as the average constituency size. It is easy to prove the following upper bounds

$$\beta \leq \overline{\beta} \stackrel{def}{=} \frac{1}{2l_{sm}+1} \tag{16.5}$$

$$\omega \leq \overline{\omega} \stackrel{def}{=} \begin{cases} \frac{1}{l_{sm}} & \text{if } l_{sm} > 0, \\ \infty & \text{if } l_{sm} = 0. \end{cases} \tag{16.6}$$

where l_{sm} denotes the lower integer part of the smallest state's respective share.

$l_i - u_i$	p_i^*	$\overline{\beta}$	$\hat{p}_i$	$\overline{\omega}$
$0-1$	0	1	$< A$	∞
$1-2$	$\frac{4}{3}A$	1/3	A or $2A$	1
$2-3$	$\frac{12}{5}A$	1/5	$2A$ or $3A$	1/2
$3-4$	$\frac{24}{7}A$	1/7	$3A$ or $4A$	1/3
$4-5$	$\frac{40}{9}A$	1/9	$4A$ or $5A$	1/4
$5-6$	$\frac{60}{11}A$	1/11	$5A$ or $6A$	1/5

Table 16.5: Critical state populations. The first column shows the lower and upper quotas. If state i's population is close to p_i^* then β_i will be close to $\overline{\beta}$. If state i's population is close to $\hat{p}_i$ then ω_i will be close to $\overline{\omega}$.

Figure 16.2 demonstrates the meaning of Table 16.5. As the House size increases from 111 to 112 the average constituency size becomes so small that even the smallest county is at least twice as big as A. As a result ω drops significantly and never anymore exceeds 50%.

The reason why we are interested in $\overline{\beta}$ rather than in $\overline{\omega}$ is that some methods like the EP and Webster can reach β and the Leximin Method often coincides with it even for a wide range of House sizes. Since β is achievable it is a valid question where β takes its maximum and how can we lower it. Equation 16.5 highlights the relationship of β and lower quota of the smallest state. For example, if the average constituency size is sufficiently small, less than half of the smallest state, then the maximum departure will be less than 20% (assuming we achieve β).

The Leximin Method will coincide with β if the House size is not too small and there are puffer states that enable seat reconfiguration. That means there are at least one or two large states.

16.4 Conclusion

Several alternative methods exist for the allocation of seats among states or regions and while all these methods have the same goal, fair representation, each approaches fairness from a different angle. Fairness can be captured by several incompatible properties and our interest lies in uncovering the principles that lead to one or another choice. In particular, we want to understand the incompatibility of the quota and maximum difference properties. The latter is a mathematical formulation of a good practice recommended by the Venice Commission (2002) to ensure near-equal representation. The Quota Property on the other hand puts the states first and guarantees that the states or regions get very close to their fair share. The conflict between the two views is far from obvious, but we soon learned that fairness at the state level contributes to larger inequalities among voters elsewhere.

The actual apportionments in certain European countries fall quite far from both the recommendation of the Venice Commission and the method used in the US. While the differences can, surely be attribute to the lack of a scientific approach, certain countries introduce systematic biases, often to counter the overrepresentation of the urban areas. Corrections are not needed for a country with homogeneous constituencies, but if some share common interest, voting blocks may emerge and proportionality is no longer fair.

For instance the Spanish Congress of Deputies consists of 350 members, but only 248 are apportioned according to the population data. Each of the fifty provinces is entitled to an initial minimum of two seats, while the cities of Ceuta and Melilla get one each. As a result the constituencies of Teruel are roughly 65% smaller, Madrid's are 30% larger than the average; the vote of a Teruelian citizen is worth nearly four times more than that of a Madrilenian. The Danish apportionment, on the other hand, uses the classical D'Hondt method, but based on the sum of the (1) population, (2) voting population, and (3) 20 times the area in square kilometres (as a rural bonus) for each region. Other countries have special clauses specifying the seat allocated to certain states explicitly, outside the apportionment procedure. While this is generally to ensure the fair treatment of a peripheral or underpopulated region, favourable developments of the population often turns such measures unnecessary or even harmful for the region. Such anomalies are very interesting from both a theoretical and practical point of view, but elaborating on them further would be beyond the limits of this paper and we present them in a companion paper with a systematic study of apportionment methods and practices.

Acknowledgments

Biró was supported by the Hungarian Academy of Sciences under its Momentum Programme (LP2016-3/2016) and by OTKA grant no. K108673. Kóczy and Sziklai were supported by OTKA grant no. K109354. Sziklai was supported by the ÚNKP-16-4-I. New National Excellence Program of the Ministry of Human Capacities.

Bibliography

M. Balinski and H. P. Young. The quota method of apportionment. *American Mathematical Monthly*, 82(7):701–730, 1975.

M. Balinski and H. P. Young. *Fair Representation: Meeting the Ideal of One Man, One Vote*. Yale University Press, New Haven, 1982.

P. Biró, L. Á. Kóczy, and B. Sziklai. Választókörzetek igazságosan? *Közgazdasági Szemle*, 59:1165–1186, 2012.

P. Biró, L. Á. Kóczy, and B. Sziklai. Fair apportionment in the view of the Venice Commission's recommendation. *Mathematical Social Sciences*, 77:32–41, 2015.

V. Bittó. Az Imperiali és Macau politikai választókörzet-kiosztási módszerek empirikus vizsgálata. MPRA Paper 79554, Munich Personal RePEc Archive, 2017.

M. Brill, J.-F. Laslier, and P. Skowron. Multiwinner Approval Rules as Apportionment Methods. In *Proceedings of the 31st AAAI Conference on Artficial Intelligence (AAAI)*, pages 414–420, 2017.

L. Csató. Between plurality and proportionality: an analysis of vote transfer systems. Technical report, arXiv, 2015. URL `http://arxiv.org/abs/1507.01477`.

L. Csató. A mathematical evaluation of vote transfer systems. Technical report, arXiv, 2016. URL `http://arxiv.org/abs/1607.02879`.

P. Csóka and P. J.-J. Herings. Decentralized clearing in financial networks. Technical Report 14/2016, Corvinus University Budapest, Budapest, 2016.

H. R. Droop. On Methods of Electing Representatives. *Journal of the Statistical Society of London*, 44(2):141–202, 1881.

G. R. Grimmett. European apportionment via the Cambridge Compromise. *Mathematical Social Sciences*, 63:68–73, 2012.

E. Hagenbach-Bischoff. *Die Frage der Einführung einer Proportionalvertretung statt des absoluten Mehres.* H. Georg, Basel, 1888.

L. Handley. Boundary Delimitation. In *Challenging the Norms and Standards of Election Administration*, pages 59–74. International Foundation for Electoral Systems, 2007.

L. Á. Kóczy. Beyond Lisbon: Demographic trends and voting power in the European Union Council of Ministers. *Mathematical Social Sciences*, 63(2):158–152, 2012.

L. Lauwers and T. Van Puyenbroeck. Minimally Disproportional Representation: Generalized Entropy and Stolarsky Mean-Divisor Methods of Apportionment. Working Papers 2008/24, Hogeschool-Universiteit Brussel, Faculteit Economie en Management, 2008.

A. W. Marshall, I. Olkin, and F. Pukelsheim. A majorization comparison of apportionment methods in proportional representation. *Social Choice and Welfare*, 19(4), 2002.

OSCE/ODIHR. Estonia Parliamentary Elections, 6 March 2011. Election assessment mission report, Organization for Security and Co-operation in Europe, Office for Democratic Institutions and Human Rights, Warsaw, May 2011.

L. S. Penrose. The elementary statistics of majority voting. *Journal of the Royal Statistical Society*, 109(1):53–57, 1946.

C. Puppe and A. Tasnádi. Axiomatic districting. *Social Choice and Welfare*, 44: 31–50, 2015.

W. Słomczyński and K. Życzkowski. Penrose voting system and optimal quota. *Acta Physica Polonica B*, 37:3133, 2006.

A. Tasnádi. The extent of the population paradox in the Hungarian electoral system. *Public Choice*, 134(3-4):293–305, 2008.

A. Tasnádi. The political districting problem: A survey. *Society and Economy*, 33 (3):543–554, 2011.

Venice Commission. Code of Good Practice in Electoral Matters. *Conseil de l'Europe-AD*, 23(190):1–33, 2002.

Venice Commission and OSCE/ODIHR. On the Electoral Law and the Electoral Practice of Albania. Joint opinion, Venice Commission and Organization for Security and Co-operation in Europe, Office for Democratic Institutions and Human Rights, 2011.