Strategy-Proofness in Markets with Indivisibilities

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  - Hybrid Models
- Applications: Entry-level labor markets, (on-campus) housing, school choice, kidney exchange, etc..

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- Other solution concepts

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- Here: (Strong) Core $\Leftrightarrow$ IR + Pairwise Stability (no blocking pair).

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- Note that T4 $\Rightarrow$ T3 & T2 $\Rightarrow$ T1.

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- Farsighted stable sets: Mauleon, Vannetelbosch and Vergote (2011 TE).

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- **Anonymity (AN):** Names do not matter.

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T9 (Ehlers, 2012 CIREQ-WP Appendix):
SP+IR+Weak EFF+CONS $\varphi \Leftrightarrow \varphi = Core$.

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- There are R with $Core(R) = \emptyset$.
- $Comp(R)$ is always non-empty and is obtained by breaking ties in R in any manner $ST(R)$ and applying TTC, i.e.

$$Comp(R) = \cup_{R' \in ST(R)} TTC(R') = UnionCore(R)$$

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- TTC with fixed tie-breaking $\succ = (\succ_i)_{i \in N}$: for all weak R, $f^{\succ}(R) = TTC(\succ(R))$.

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T18 (Ehlers, Klaus and Papai, 2002 JME, Ehlers and Klaus, 2004 IJGT, 2006 GEB, 2007 ET): *For* $\mathcal{P}^N$ *or* $\mathcal{P}_0^N$ *SP+EFF+Solidarity/Consistency* $\varphi \Leftrightarrow \varphi$ *is a mixed dictator-pairwise exchange rule.*

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Serial Dictatorship (SD): For each R, using this order let each agent choose his most preferred house from the remaining ones. Below we assume that $|H| = |N|$ (and implicitly that the null object is not available)

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T21 (Bogomolnaia, Deb and Ehlers, 2005 JET): *For $\mathcal{W}_0^N$ SP+(weak) NB+EFF $\varphi \Leftrightarrow \varphi$ is a bi-polar serial dictatorship.*

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Acyclicity of $\succ$: There are no $a, b \in H$ and $i, j, k \in N$ such $i \succ_a j \succ_a k \succ_b i$.

T23 (Ergin, 2002 ECON): *For strict* $\succ$ the following are equivalent

(i) Agent-proposing DA is EFF
(ii) Agent-proposing DA is GSP
(iii) Agent-proposing DA is CONS
(iv) $\succ$ is acyclic.

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House Allocation with Existing Tenants (Abdulkadiroglu and Sönmez, 1999 JET): ...

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- When are stable+CEFF+SP compatible? (=solvability of $\succeq$)

Weak Priorities IV: Solvability

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If $\exists$ stable+CEFF+SP φ, then $\succeq$ is "acyclic", $\succeq$ has no strong priority reversals, and $\succeq$ has no inconsistent weak priority reversal.

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- **T29 (Ehlers and Westkamp, 2011 CIREQ-WP):**
  Let $|N| \geq 4$ and for all $i, j \in N$ there exist $a, b \in H$ such that $i \succ_a j$ and $j \succ_b i$.
  If $\exists$ stable+CEFF+SP φ, then $\succeq$ is strict or $\succeq$ is "house allocation with existing tenants".

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- New applications of matching: military, assignment of landing slots.