Fairness vs. efficiency

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Allocation problems

- Allocating
  - cakes, divisible/indivisible items (goods or chores)
- How?
  - The input is given to the algorithm
  - The algorithm makes queries
- Fairness notions
  - Proportionality, envy-freeness
- More allocation restrictions
  - E.g., for cakes: contiguous or non-contiguous

Allocation problem instances

- Indivisible items setting
  - a set M of **m items** to be allocated to
  - **n agents** from a set N
  - agent i has **utility** $V_i(j)$ for item j
  - **additive utilities**: when allocated a set of items S, agent i has utility $V_i(S)$ equal to the sum of her utility for the items in the set

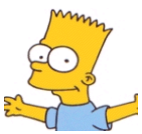
$$V_i(S) = \sum_{j \in S} V_i(j)$$

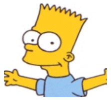
- Notation:
  - **allocation A = (A_1, A_2, …, A_n)**: disjoint partition of items into n sets where A_i is the set of items agent i gets

An example

indivisible items (goods)

agents

				
	3	0	5	12
	0	2	2	1

utility of agent  for item 

An example

indivisible items (goods)

agents

	3	0	5	12
	0	2	2	1

utility of agent for item

allocation A = ({ }, { })

Envy-freeness

- Definition: an allocation $A = (A_1, A_2, \ldots, A_n)$ is called **envy-free** if for every pair of agents i, j, it holds $V_i(A_i) \geq V_i(A_j)$
- Informally: nobody envies the bundle of items allocated to another agent

Proportionality

- Definition: an allocation $A = (A_1, A_2, \ldots, A_n)$ is called **proportional** if $V_i(A_i) \geq V_i(M)/n$ for every agent i

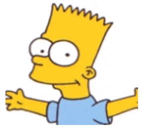
- Informally: every agent believes she gets a fair share

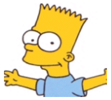
- For 2 agents: proportionality = envy-freeness

Envy-free allocations: examples

items

agents

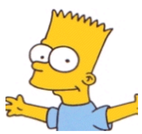
				
	3	0	5	12
	0	2	2	1

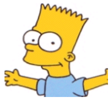
 

allocation ({  }, {    } is EF

Envy-free allocations: examples

items / **agents**

				
	3	0	5	12
	0	2	2	1

allocation ({  }, {    } is EF

allocation ({   }, {   } is EF

Efficiency

- Economic efficiency
  - Pareto-optimality
  - Social welfare maximization
- Computational efficiency
  - Polynomial-time computation
  - Low query complexity

Efficiency

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  - Pareto-optimality
  - Social welfare maximizati
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a property of allocations

a property of allocation algorithms

Warming up: Pareto-optimality vs fairness

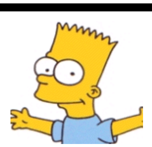
- Definition: an allocation $A = (A_1, A_2, \ldots, A_n)$ is called **Pareto-optimal** if there is no allocation $B = (B_1, B_2, \ldots, B_n)$ such that $V_i(B_i) \geq V_i(A_i)$ for every agent i and $V_{i'}(B_{i'}) > V_{i'}(A_{i'})$ for some agent i'

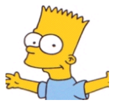
- Informally: there is no allocation in which all agents are at least as happy and some agent is strictly happier

Envy-freeness vs. Pareto-optimality

items

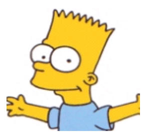
agents

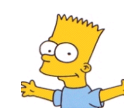
				
	3	0	5	12
	0	2	2	1

- Observation: In a Pareto-optimal allocation, agent  does not get  and agent  does not get 

Envy-freeness vs. Pareto-optimality

items

agents				
	3	0	5	12
	0	2	2	1

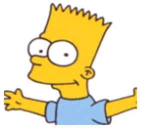
- Observation: In a Pareto-optimal allocation, agent  does not get  and agent  does not get 

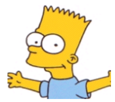
An envy-free allocation that is not Pareto-optimal

Envy-freeness vs. Pareto-optimality

items

agents

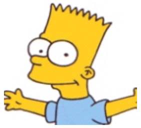
				
	3	0	5	12
	0	2	2	1

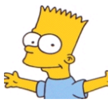
		PO	EF
		?	?

Envy-freeness vs. Pareto-optimality

items

agents

				
	3	0	5	12
	0	2	2	1

		PO	EF
		YES	NO

Envy-freeness vs. Pareto-optimality

items

agents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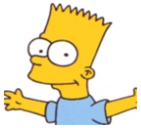
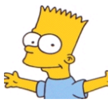
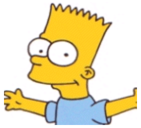
	0	2	2	1

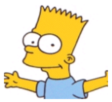
		PO	EF
		YES	NO
		?	?

Envy-freeness vs. Pareto-optimality

items

agents

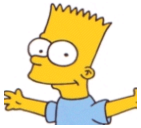
				
	3	0	5	12
	0	2	2	1

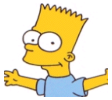
		PO	EF
		YES	NO
		NO	NO

Envy-freeness vs. Pareto-optimality

items

agents

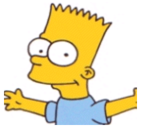
				
	3	0	5	12
	0	2	2	1

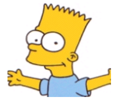
		PO	EF
		YES	NO
		NO	NO
		?	?

Envy-freeness vs. Pareto-optimality

items

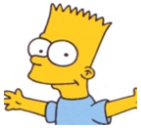
agents

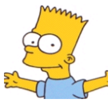
				
	3	0	5	12
	0	2	2	1

		PO	EF
		YES	NO
		NO	NO
		YES	YES

Envy-freeness vs. Pareto-optimality

items

agents				
	3	0	5	12
	0	2	2	1

		PO	EF
		YES	NO
		NO	NO
		YES	YES
		?	?

Envy-freeness vs. Pareto-optimality

items

agents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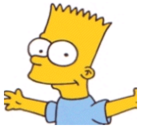
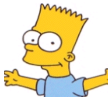
	0	2	2	1

		PO	EF
		YES	NO
		NO	NO
		YES	YES
		YES	NO

Envy-freeness vs. Pareto-optimality

- Theorem: Consider an allocation instance with 2 agents that has at least one EF allocation. Then, **there is an EF allocation that is simultaneously PO**.

Envy-freeness vs. Pareto-optimality

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- Proof. Sort the EF allocations in lexicographic order of agents' utilities. The first allocation in this order is clearly PO.

Envy-freeness vs. Pareto-optimality

- Theorem: Consider an allocation instance with 2 agents that has at least one EF allocation. Then, **there is an EF allocation that is simultaneously PO**.
- Proof. Sort the EF allocations in lexicographic order of agents’ utilities. The first allocation in this order is clearly PO.
- **Question:** What about 3-agent instances?
- **Question:** What about Proportionality vs PO?

Social welfare

- **Social welfare** is a measure of global value of an allocation
- **Utilitarian social welfare** of an allocation A:
  - the total utility of the agents for the items allocated to them in A

$$uSW(A) = \sum_{i \in N} V_i(A_i)$$

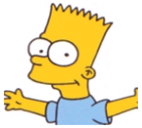
- **Egalitarian social welfare**: $eSW(A) = \min_{i \in N} V_i(A_i)$
- **Nash social welfare**: $nSW(A) = \prod_{i \in N} V_i(A_i)$

An example

- SW-maximizing allocations?

items

agents

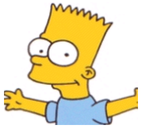
				
	15	0	40	45
	0	30	30	40

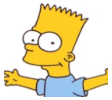
An example

- SW-maximizing allocations?

items

agents

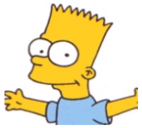
				
	15	0	40	45
	0	30	30	40

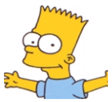
		
uSW	?	?
eSW	?	?
nSW	?	?

An example

- SW-maximizing allocations?

items / **agents**

				
	15	0	40	45
	0	30	30	40

		
uSW		
eSW	?	?
nSW	?	?

Give each item to the agent who values it the most
uSW=130

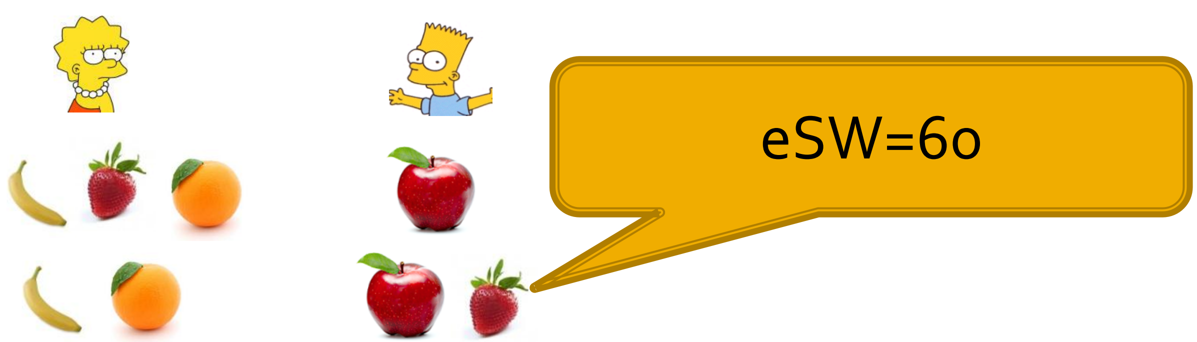
An example

- SW-maximizing allocations?

items

agents

	banana	apple	strawberry	orange
Lisa	15	0	40	45
Bart	0	30	30	40



uSW

eSW

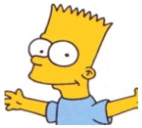
nSW ? ?

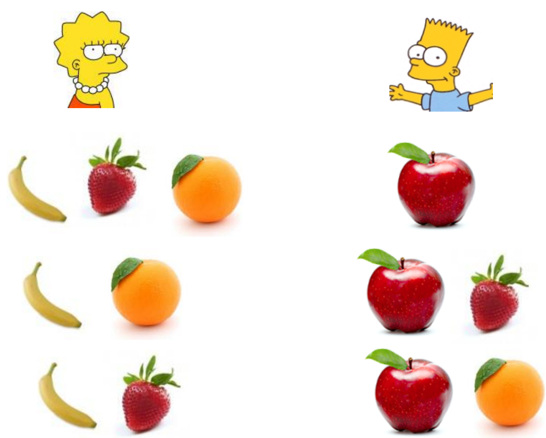
An example

- SW-maximizing allocations?

items

agents

				
	15	0	40	45
	0	30	30	40



uSW

eSW

nSW

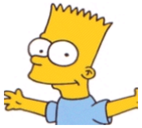
nSW=3850

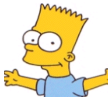
An example

- SW-maximizing allocations?

items

agents

				
	15	0	40	45
	0	30	30	40

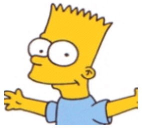
			EF
uSW			?
eSW			?
nSW			?

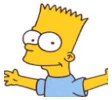
An example

- SW-maximizing allocations?

items

agents

				
	15	0	40	45
	0	30	30	40

			**EF**
uSW			**NO**
eSW			**YES**
nSW			**YES**

Price of fairness

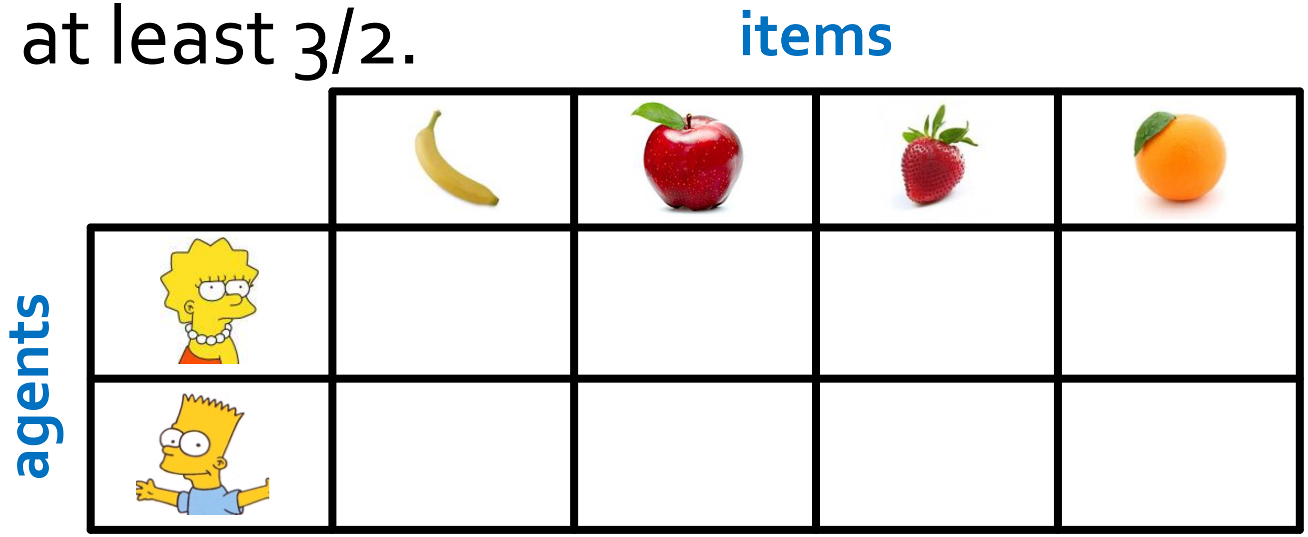
- **Price of fairness** (in general)
  - how far from its maximum value can the social welfare of the best fair allocation be?
- More specifically:
  - Which definition of social welfare to use?
  - Which fairness notion to use?
- Answer:
  - **Any combination of them**

PoP & uSW for 2 agents

- The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is 3/2.

PoP & uSW for 2 agents

- The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is at least 3/2.

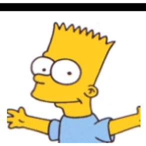


PoP & uSW for 2 agents

- The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is at least 3/2.

items

agents

				
	0.5-ε	0.5-ε	ε	ε
	0.25+ε	0.25+ε	0.25-ε	0.25-ε

PoP & uSW for 2 agents

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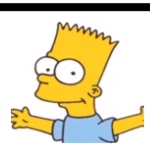
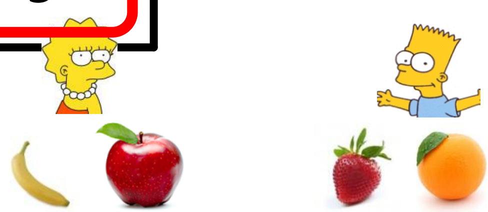
items

agents

				
	0.5-ε	0.5-ε	ε	ε
	0.25+ε	0.25+ε	0.25-ε	0.25-ε

- Optimal allocation (uSW ≈ 1.5)



PoP & uSW for 2 agents

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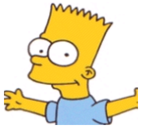
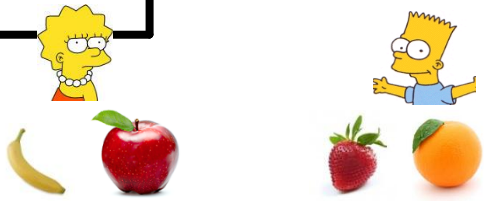
items

agents

				
	0.5-ε	0.5-ε	ε	ε
	0.25+ε	0.25+ε	0.25-ε	0.25-ε



- Optimal allocation (uSW ≈ 1.5)
- Best proportional allocation

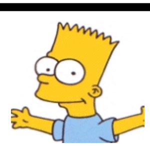
? ?

PoP & uSW for 2 agents

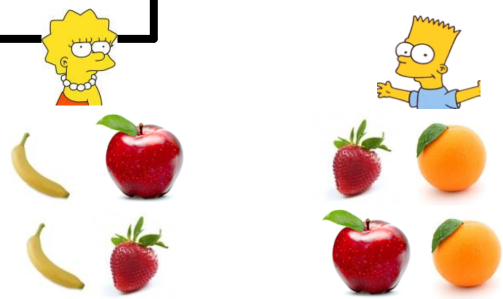
- The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is at least 3/2.

items

agents

				
	0.5-ε	0.5-ε	ε	ε
	0.25+ε	0.25+ε	0.25-ε	0.25-ε

- Optimal allocation (uSW ≈ 1.5)
- Any prop. allocation has uSW ≈ 1



PoP & uSW for 2 agents

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PoP & uSW for 2 agents

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- Proof: If the uSW-maximizing allocation is proportional, then PoP=1. So, assume otherwise. Then, some agent has utility less than 1/2 for a total of at most 3/2. In any proportional allocation, uSW=1.

PoP & uSW for 2 agents

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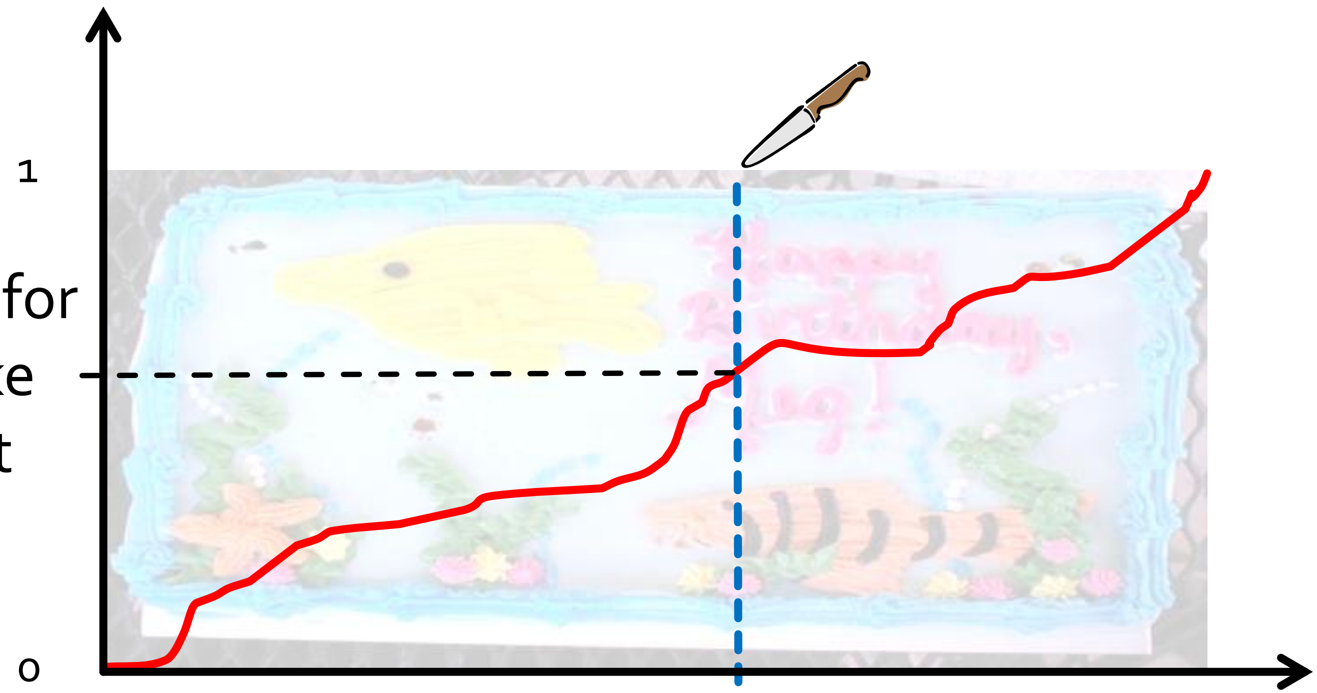
- Proof: If the uSW-maximizing allocation is proportional, then PoP=1. So, assume otherwise. Then, some agent has utility less than 1/2 for a total of at most 3/2. In any proportional allocation, uSW=1.

- **Question:** PoP/PoEF wrt uSW for many agents?

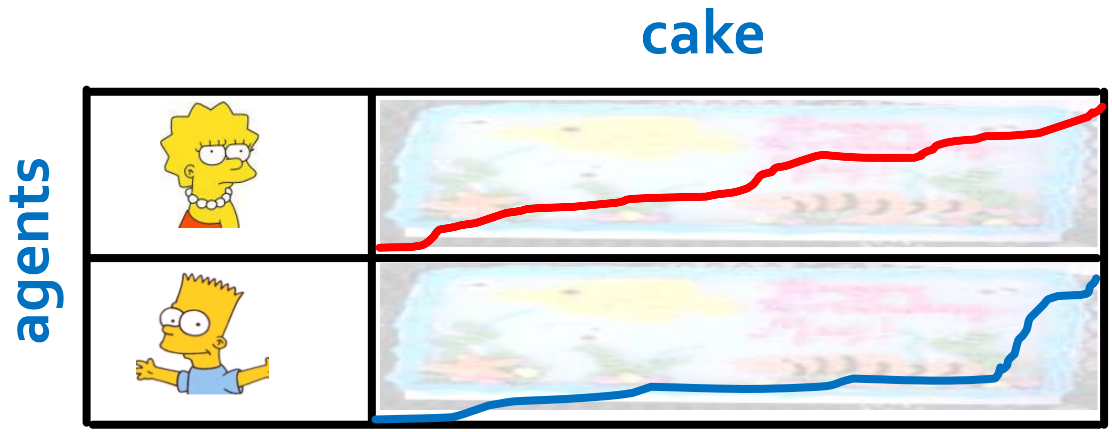
Fairness vs efficiency in cake-cutting

Fairness vs efficiency in cake-cutting

Utility of the agent for the piece of the cake at the left of the cut

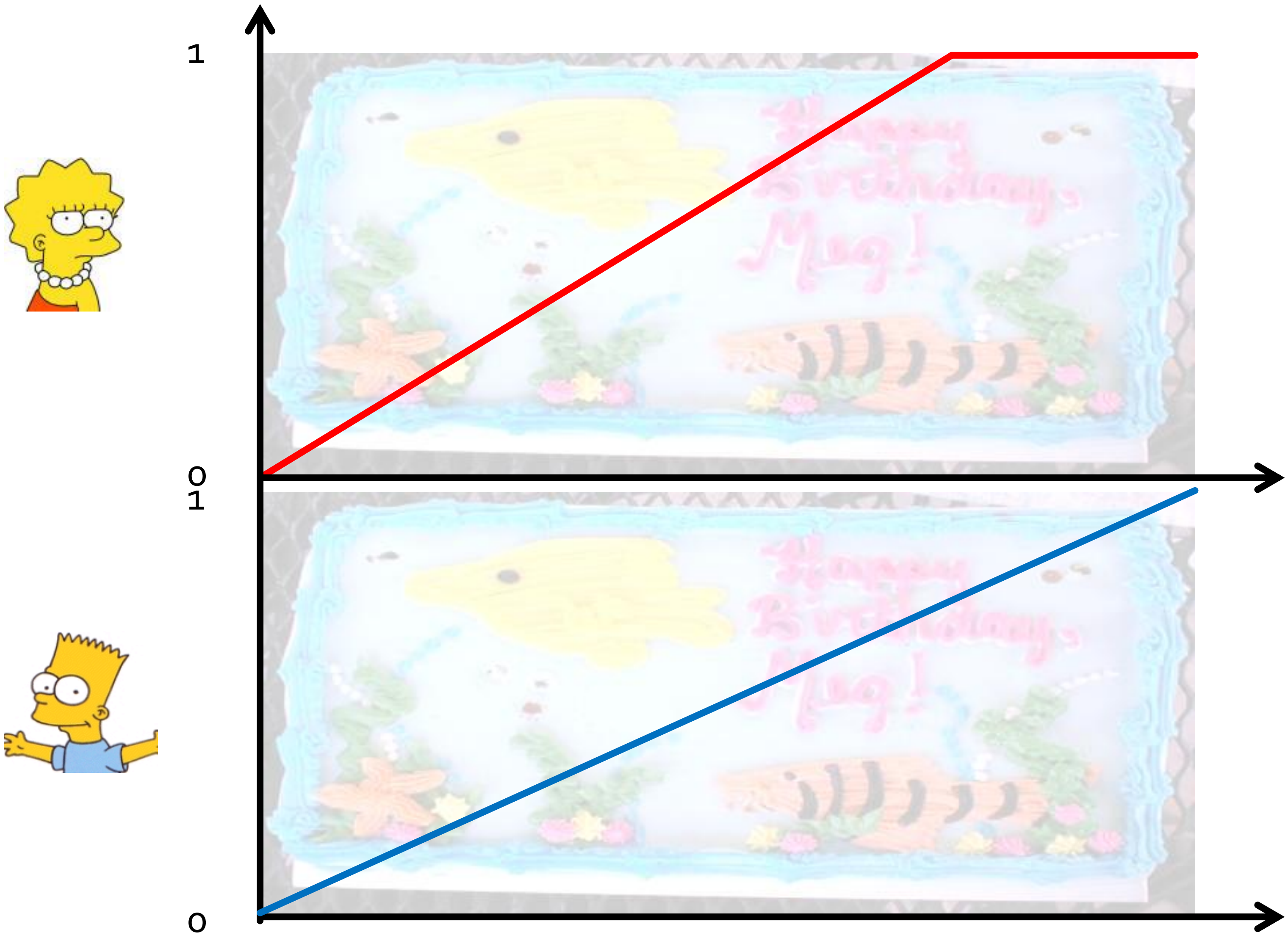


The cake-cutting setting

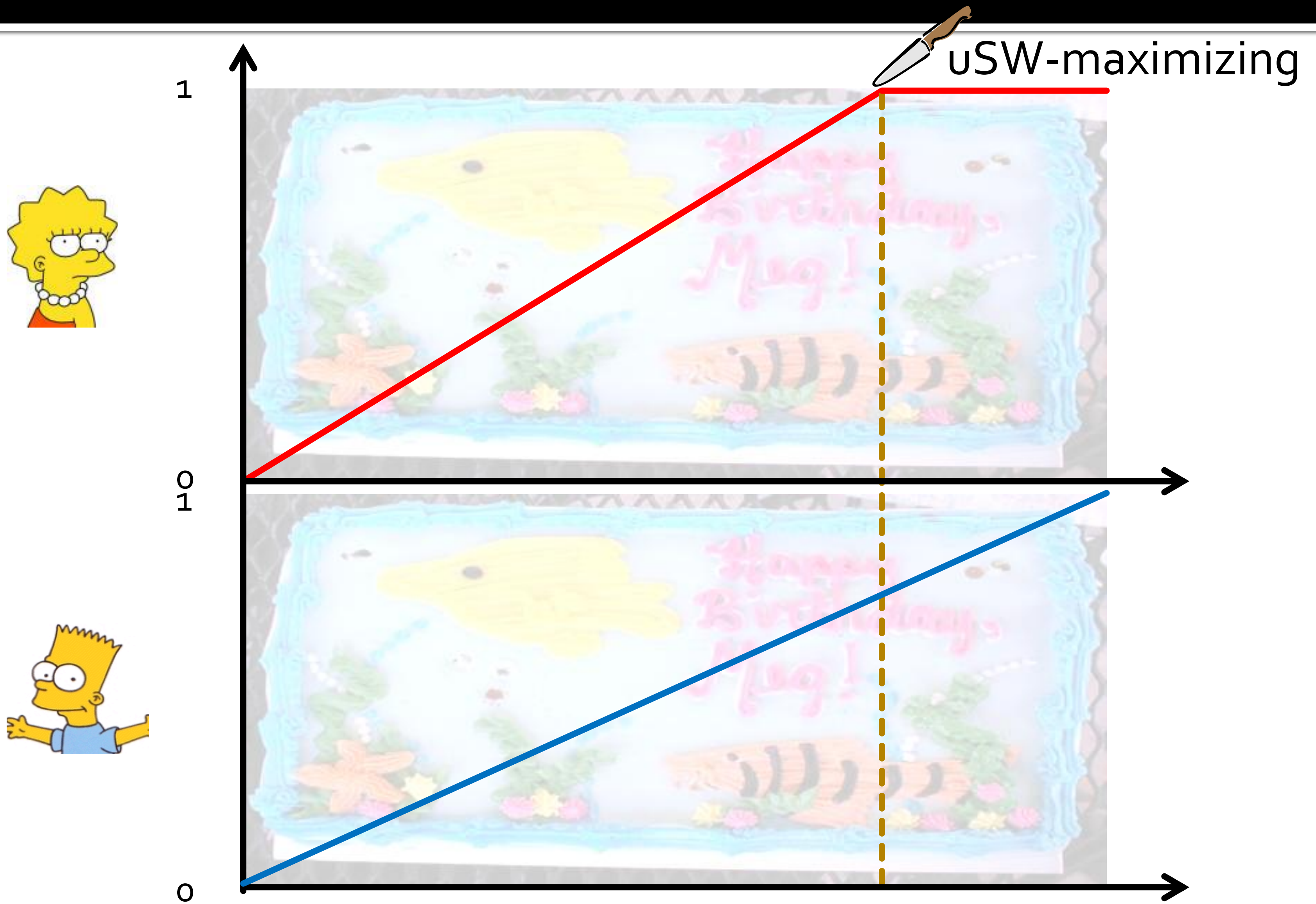


- What does an EF/uSW-maximizing allocation look like?
  - EF: Lisa cuts, Bart chooses
  - uSW-maximizing: give each trimming to the agent with the highest utility slop

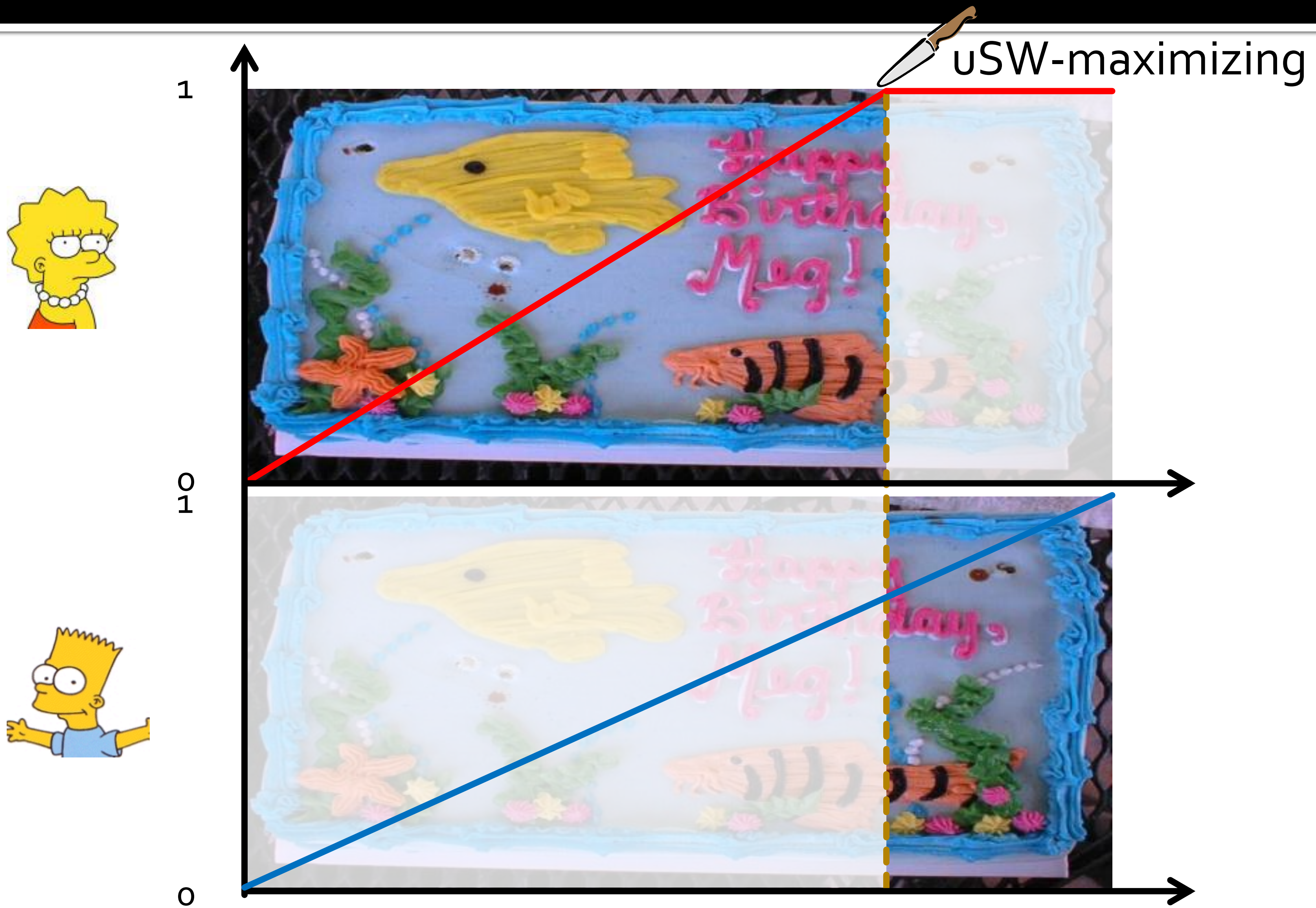
PoP/EF: what can go wrong?



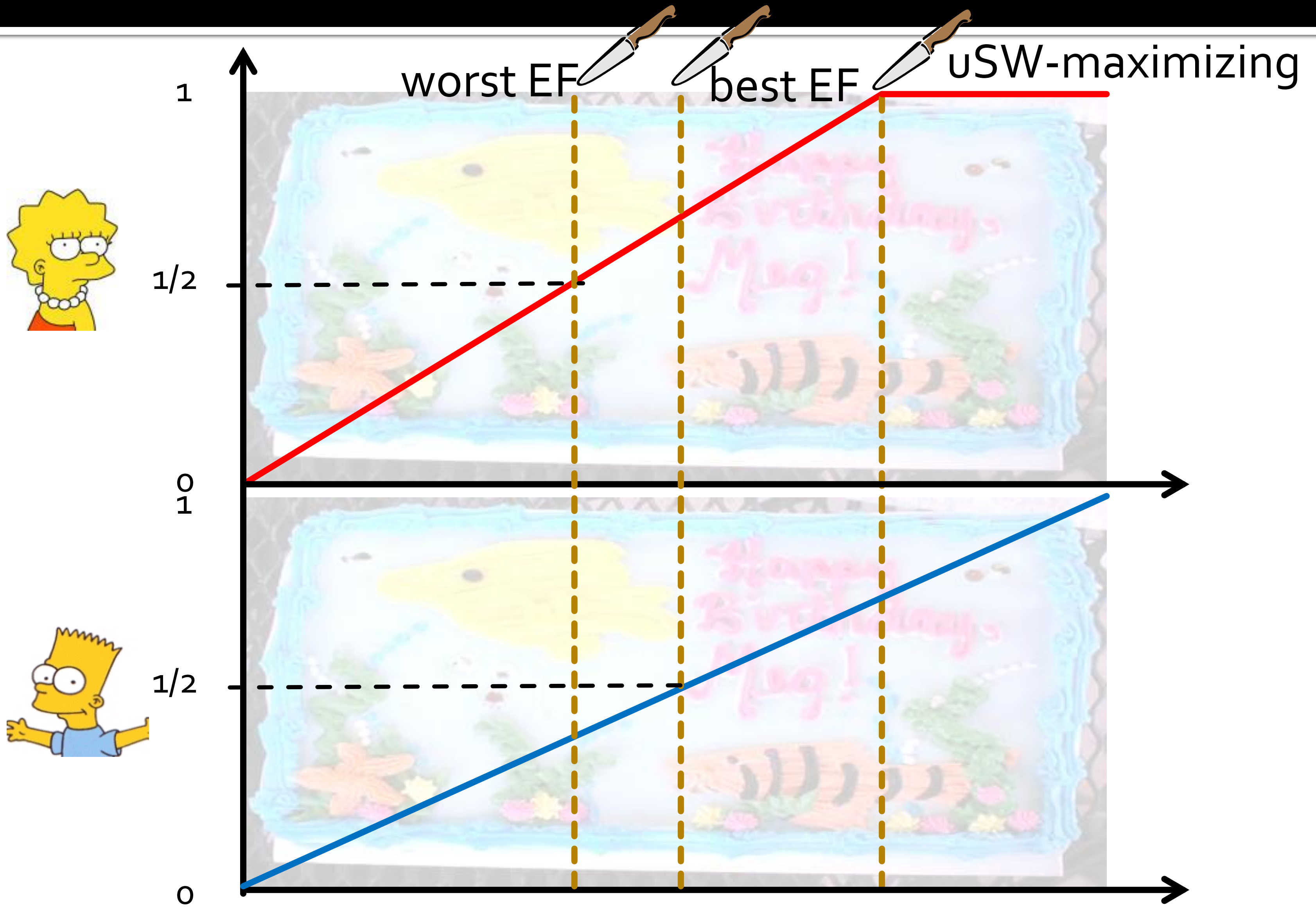
PoP/EF: what can go wrong?



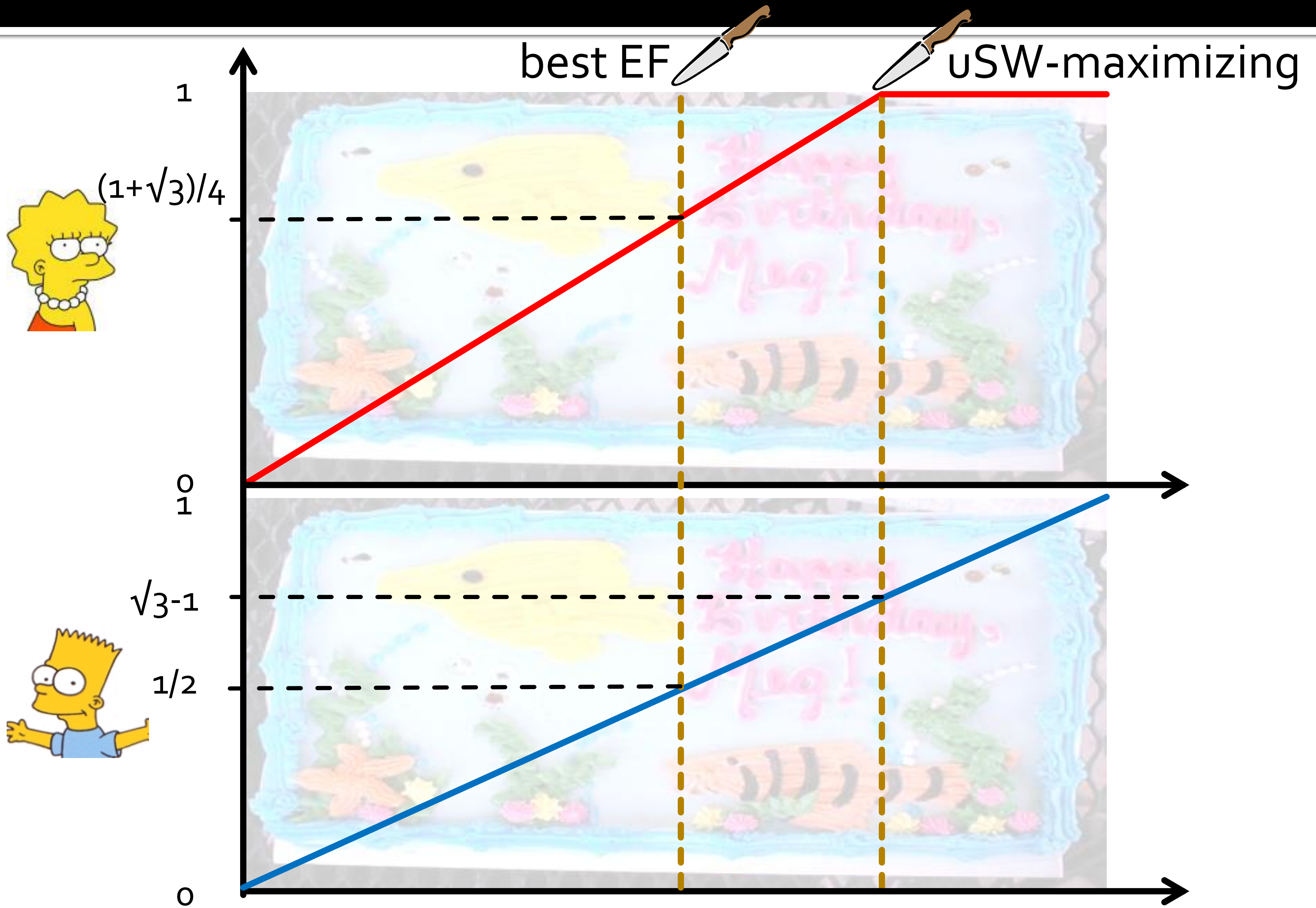
PoP/EF: what can go wrong?



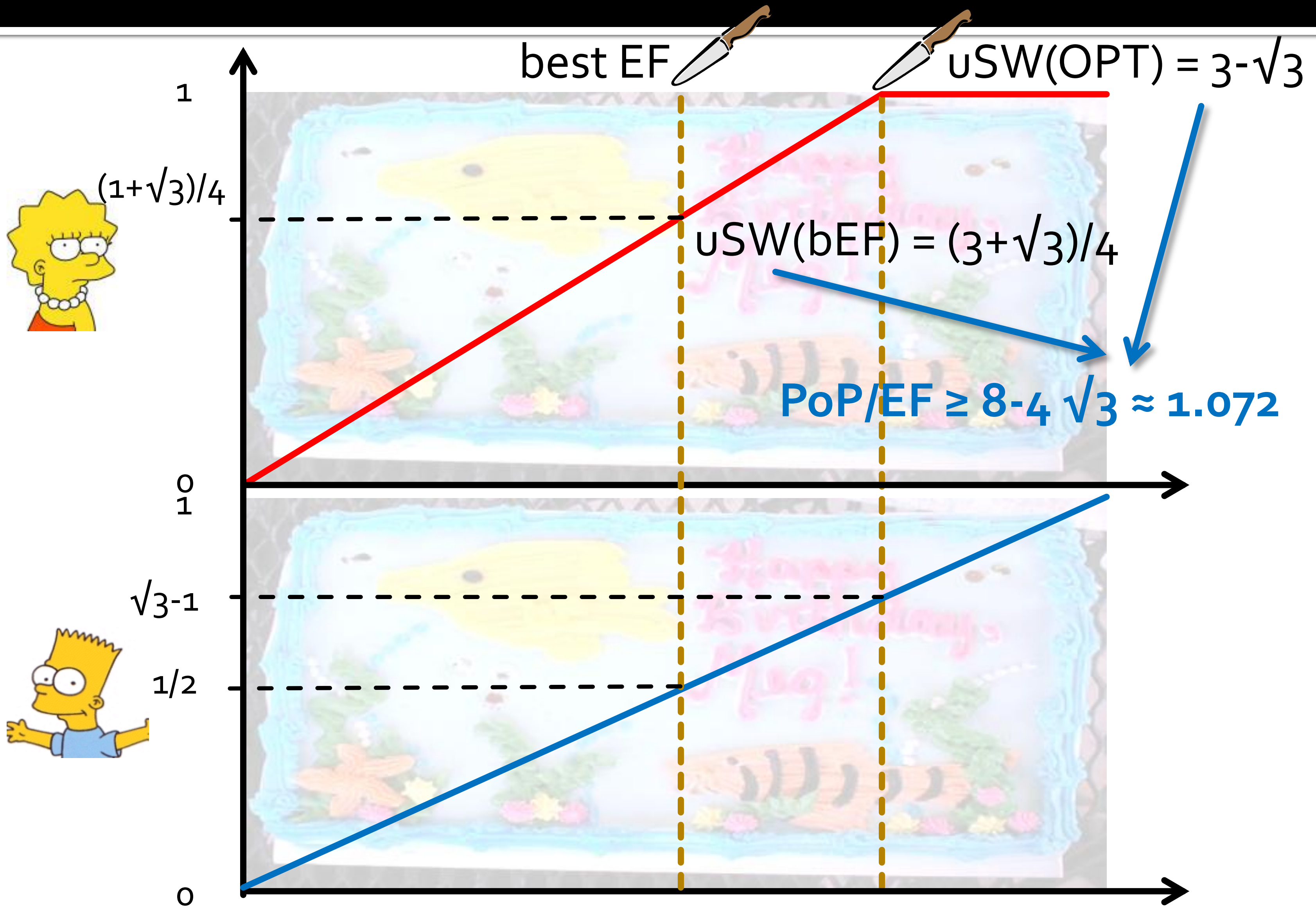
PoP/EF: what can go wrong?



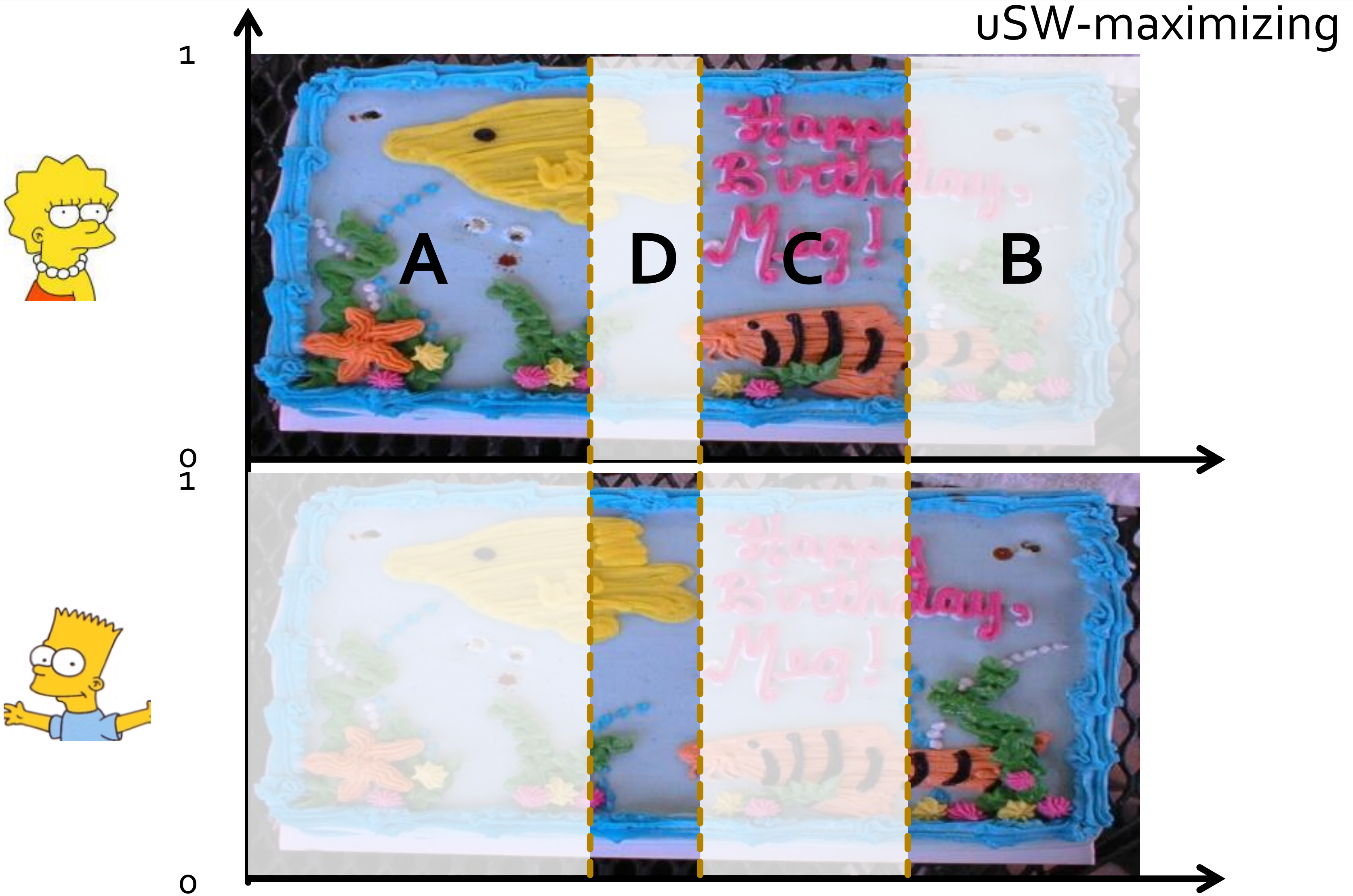
PoP/EF: what can go wrong?



PoP/EF: what can go wrong?

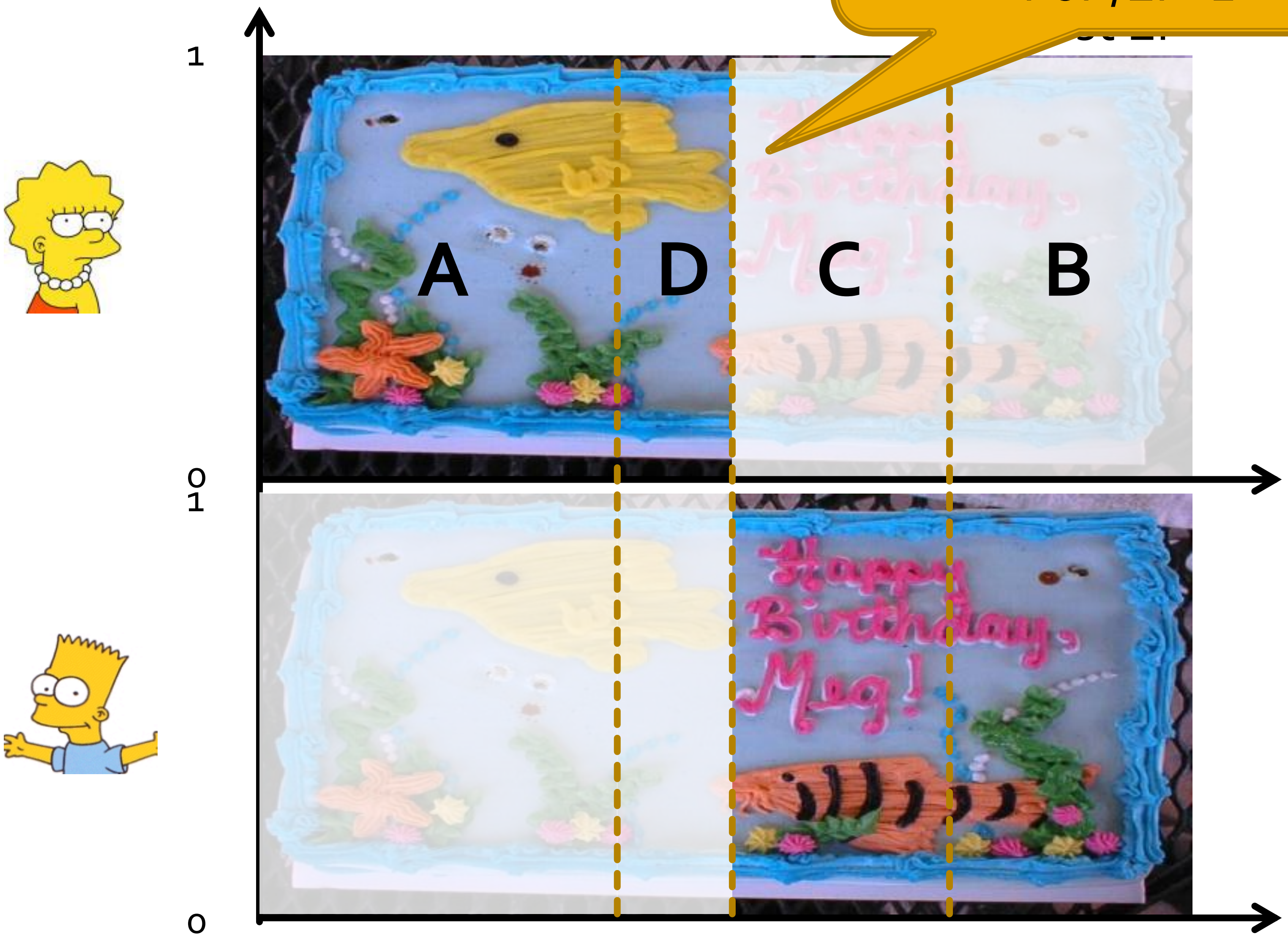


PoP/EF: upper bound for 2 agents



PoP/EF: upper bound

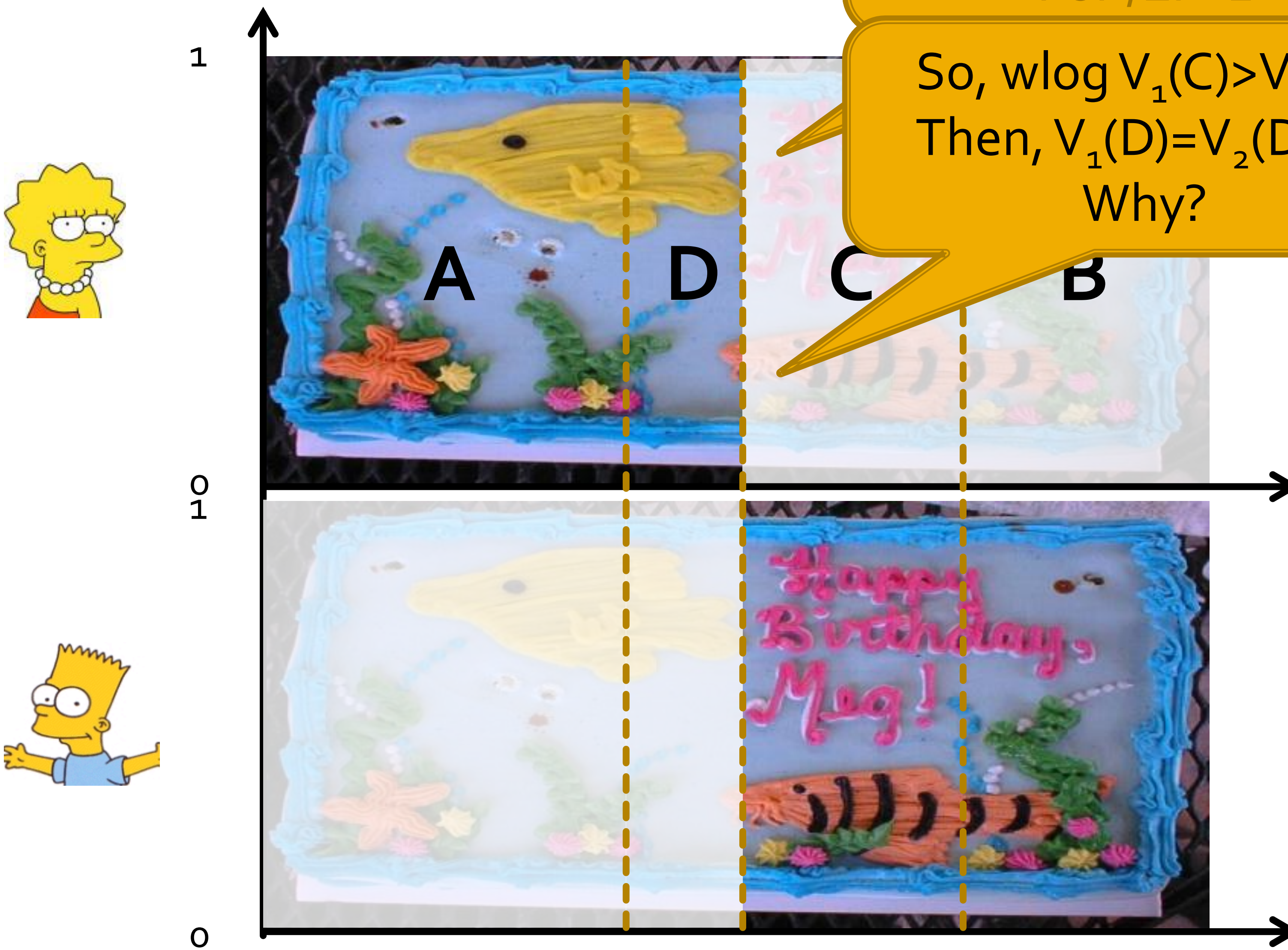
If $V_1(C)=V_2(C)$ and $V_1(D)=V_2(D)$, then PoP/EF=1



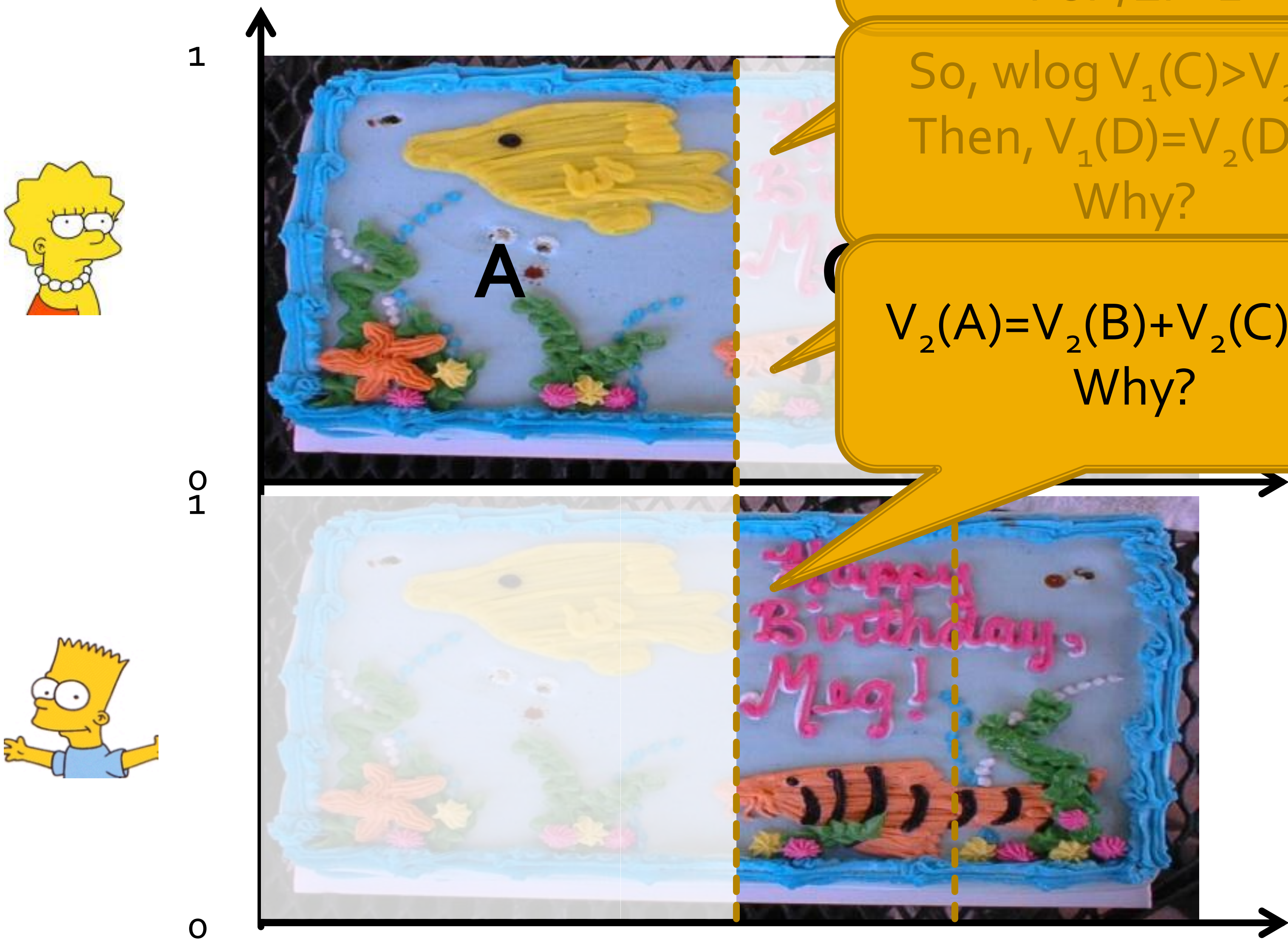
PoP/EF: upper bound

If $V_1(C)=V_2(C)$ and $V_1(D)=V_2(D)$, then PoP/EF=1

So, wlog $V_1(C)>V_2(C)$
Then, $V_1(D)=V_2(D)=0$
Why?



PoP/EF: upper bound

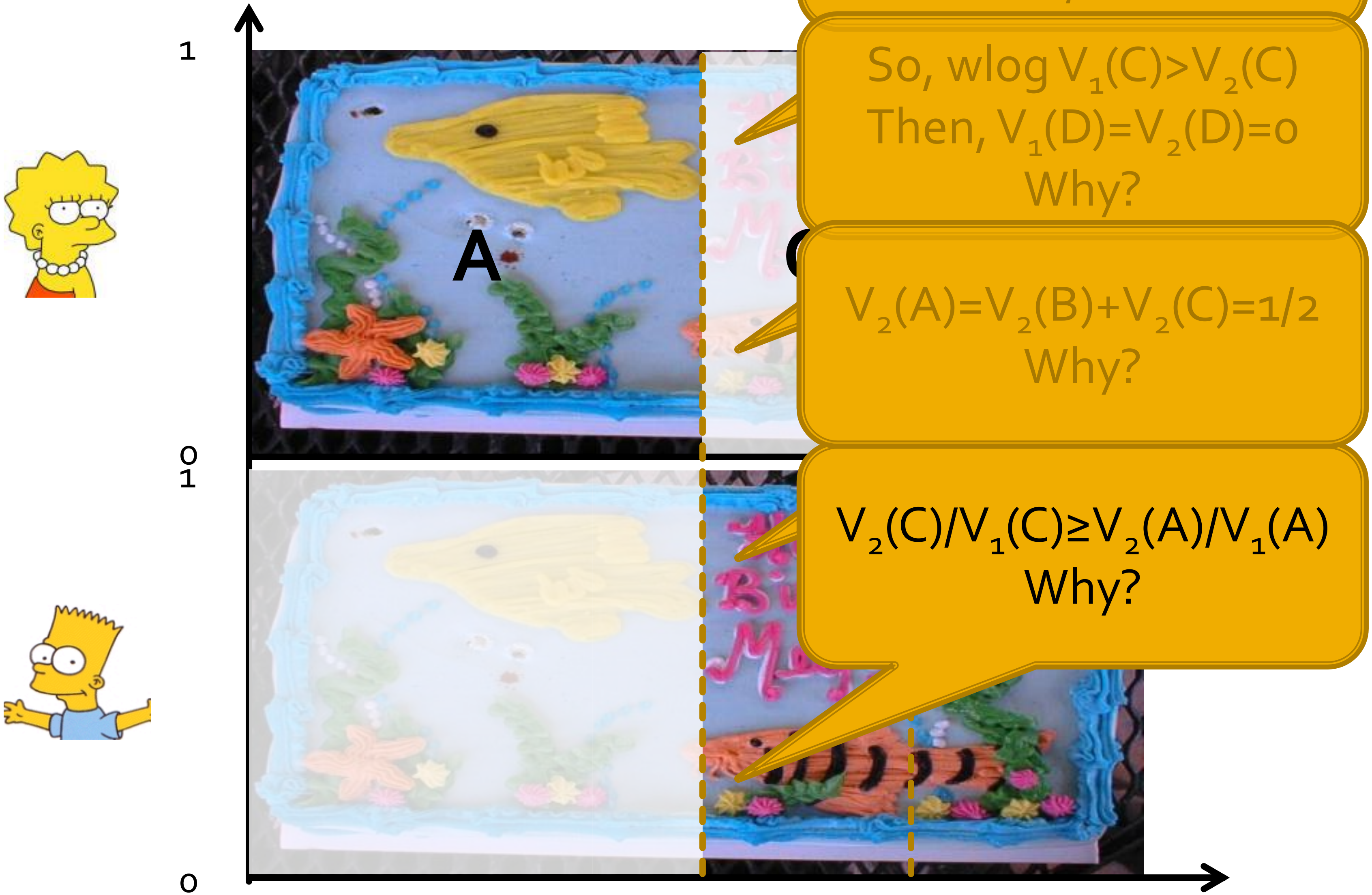


If $V_1(C)=V_2(C)$ and $V_1(D)=V_2(D)$, then PoP/EF=1

So, wlog $V_1(C)>V_2(C)$
Then, $V_1(D)=V_2(D)=0$
Why?

$V_2(A)=V_2(B)+V_2(C)=1/2$
Why?

PoP/EF: upper bound



PoP/EF: upper bound for 2 agents

$$\text{PoP/EF} = \frac{V_1(A) + V_2(B) + V_1(C)}{V_1(A) + V_2(B) + V_2(C)}$$

PoP/EF: upper bound for 2 agents

$$\text{PoP/EF} = \frac{V_1(A) + V_2(B) + V_1(C)}{V_1(A) + V_2(B) + V_2(C)} = \frac{V_1(A) + 1/2 - V_2(C) + V_1(C)}{V_1(A) + 1/2}$$

PoP/EF: upper bound for 2 agents

$$\text{PoP/EF} = \frac{V_1(A) + V_2(B) + V_1(C)}{V_1(A) + V_2(B) + V_2(C)} = \frac{V_1(A) + 1/2 - V_2(C) + V_1(C)}{V_1(A) + 1/2}$$

$$\leq \frac{V_1(A) + 1/2 - \dfrac{V_1(C)}{2V_1(A)} + V_1(C)}{V_1(A) + 1/2}$$

PoP/EF: upper bound for 2 agents

$$\text{PoP/EF} = \frac{V_1(A) + V_2(B) + V_1(C)}{V_1(A) + V_2(B) + V_2(C)} = \frac{V_1(A) + 1/2 - V_2(C) + V_1(C)}{V_1(A) + 1/2}$$

$$\leq \frac{V_1(A) + 1/2 - \dfrac{V_1(C)}{2V_1(A)} + V_1(C)}{V_1(A) + 1/2} = \frac{V_1(A) + 1/2 + V_1(C)\left(1 - \dfrac{1}{2V_1(A)}\right)}{V_1(A) + 1/2}$$

PoP/EF: upper bound for 2 agents

$$\text{PoP/EF} = \frac{V_1(A)+V_2(B)+V_1(C)}{V_1(A)+V_2(B)+V_2(C)} = \frac{V_1(A)+1/2-V_2(C)+V_1(C)}{V_1(A)+1/2}$$

$$\leq \frac{V_1(A)+1/2-\dfrac{V_1(C)}{2V_1(A)}+V_1(C)}{V_1(A)+1/2} = \frac{V_1(A)+1/2+V_1(C)\left(1-\dfrac{1}{2V_1(A)}\right)}{V_1(A)+1/2}$$

$$\leq \frac{V_1(A)+1/2+\left(1-V_1(A)\right)\left(1-\dfrac{1}{2V_1(A)}\right)}{V_1(A)+1/2}$$

PoP/EF: upper bound for 2 agents

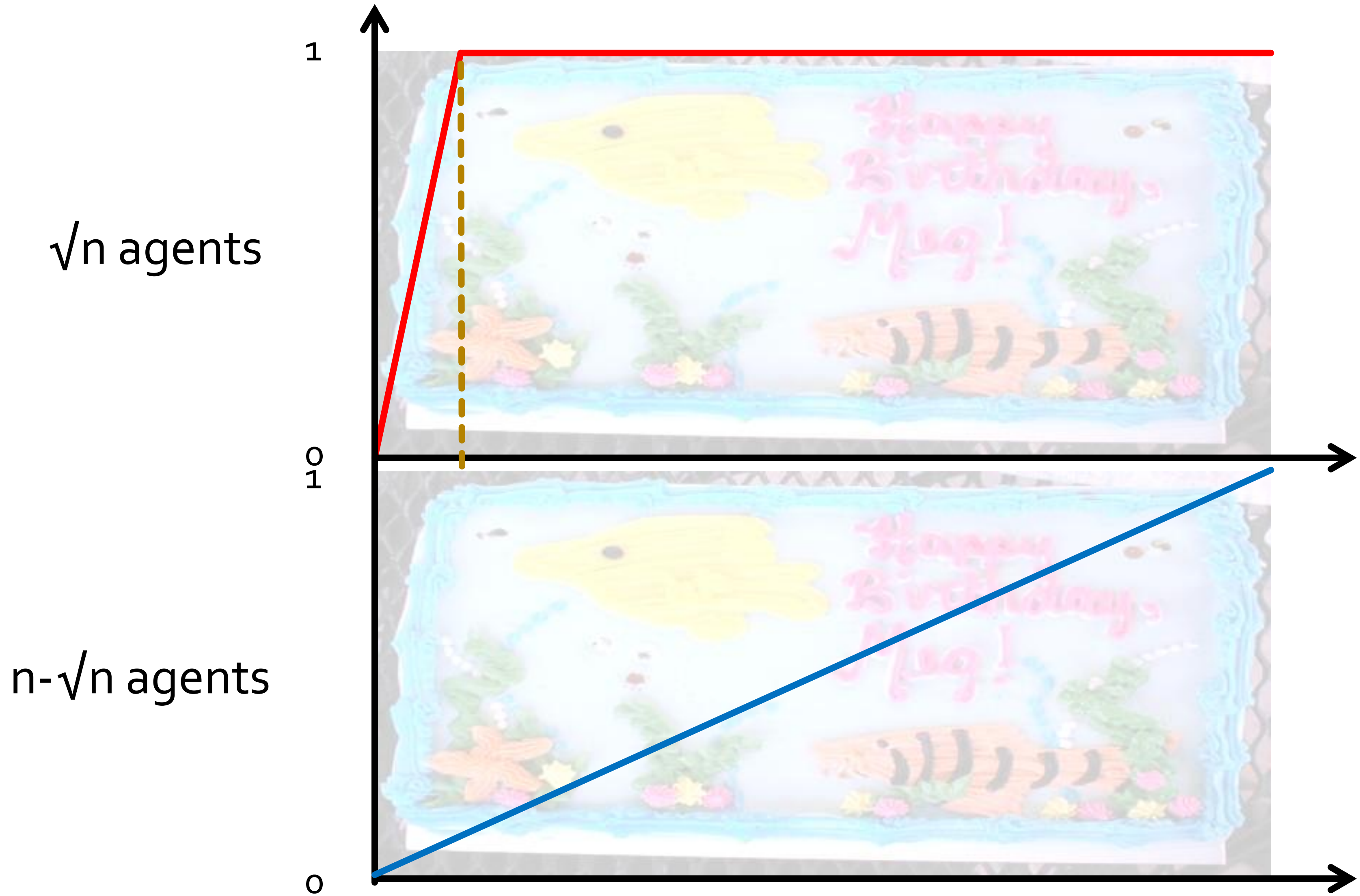
$$\text{PoP/EF} = \frac{V_1(A)+V_2(B)+V_1(C)}{V_1(A)+V_2(B)+V_2(C)} = \frac{V_1(A)+1/2-V_2(C)+V_1(C)}{V_1(A)+1/2}$$

$$\le \frac{V_1(A)+1/2-\frac{V_1(C)}{2V_1(A)}+V_1(C)}{V_1(A)+1/2} = \frac{V_1(A)+1/2+V_1(C)\left(1-\frac{1}{2V_1(A)}\right)}{V_1(A)+1/2}$$

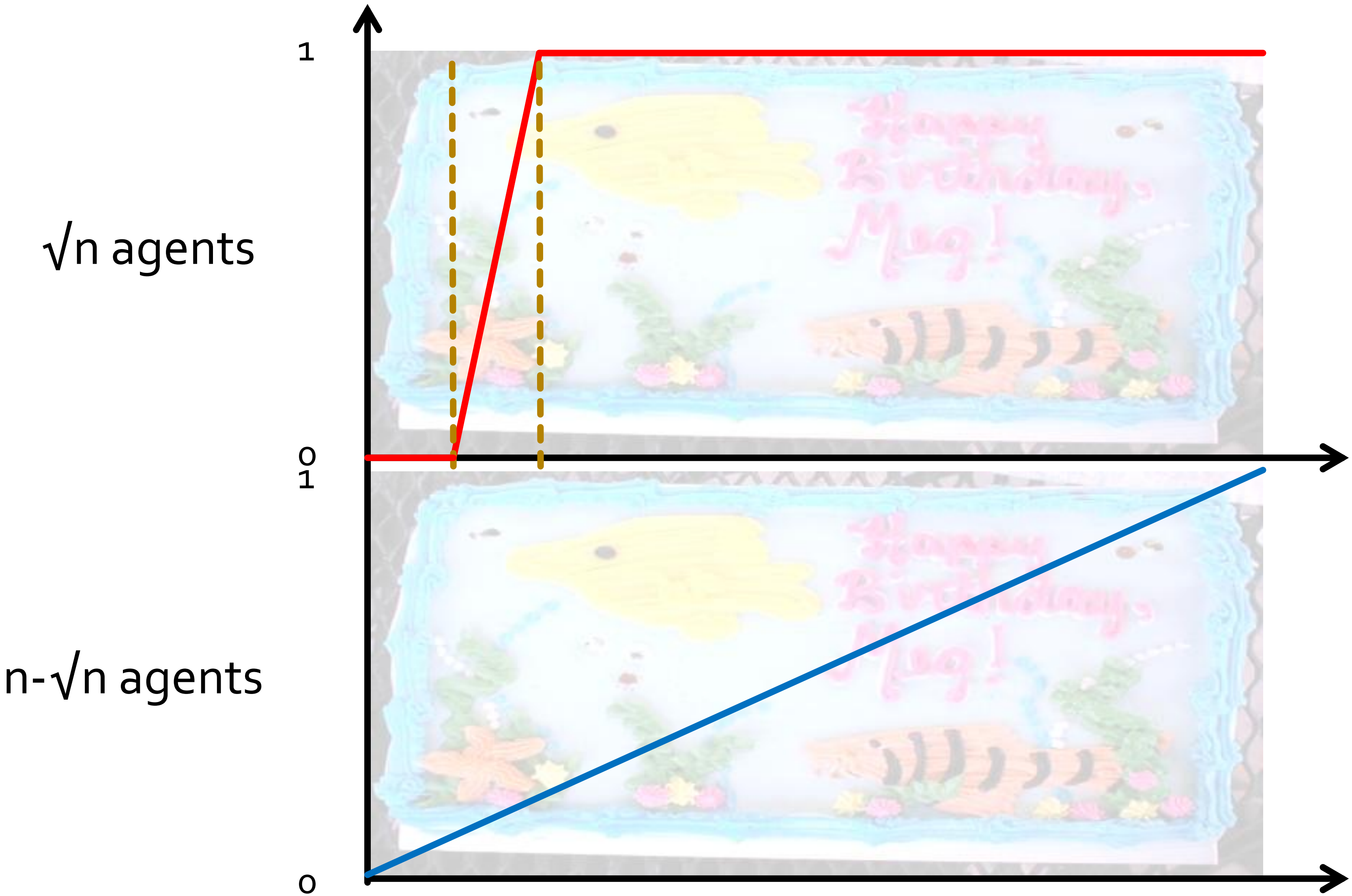
$$\le \frac{V_1(A)+1/2+\left(1-V_1(A)\right)\left(1-\frac{1}{2V_1(A)}\right)}{V_1(A)+1/2}$$

which is maximized for $V_1(A) = (1+\sqrt{3})/4$ to to $8-4\sqrt{3}$

A lower bound for PoP (n agents)

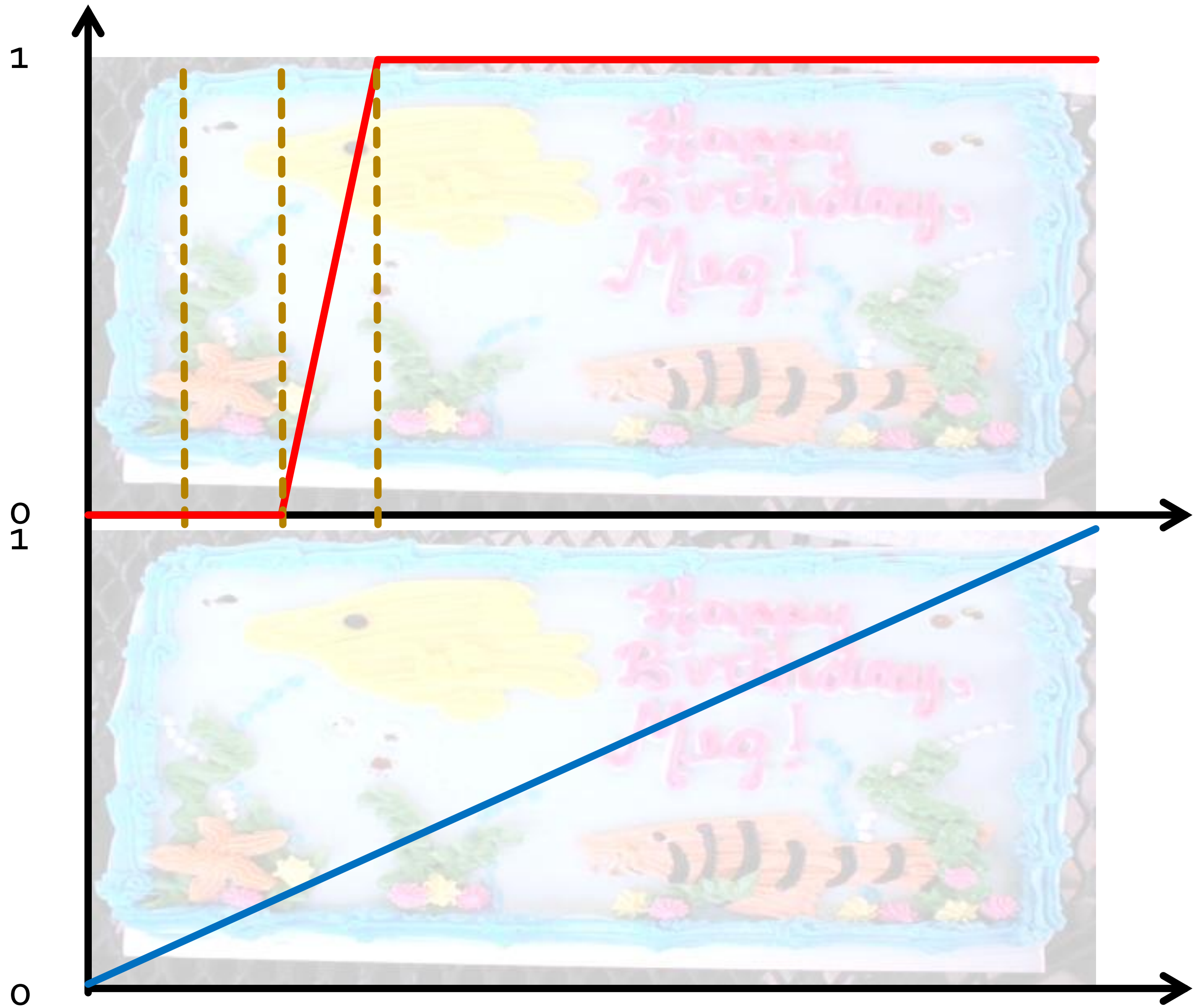


A lower bound for PoP (n agents)



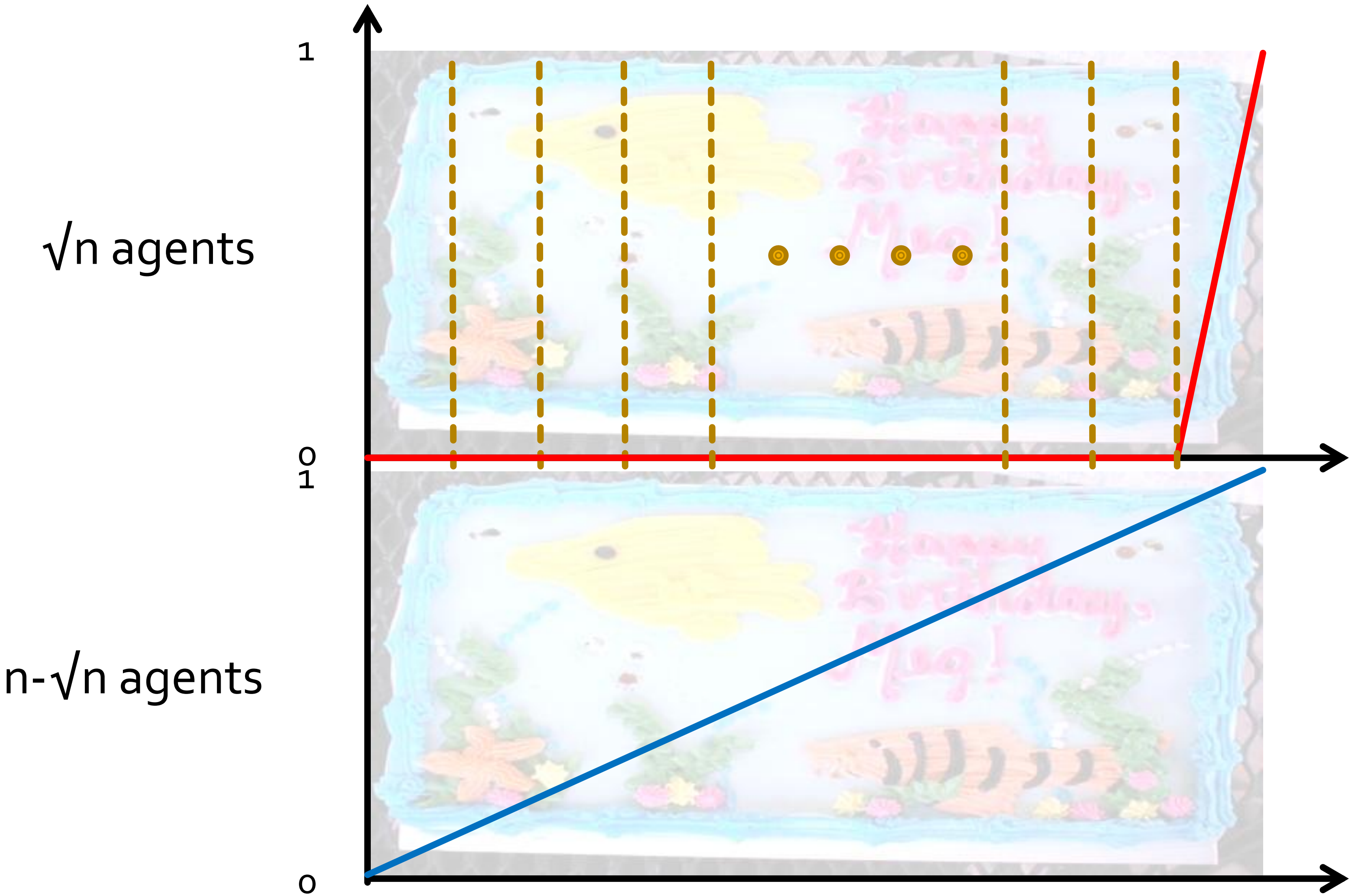
A lower bound for PoP (n agents)

√n agents

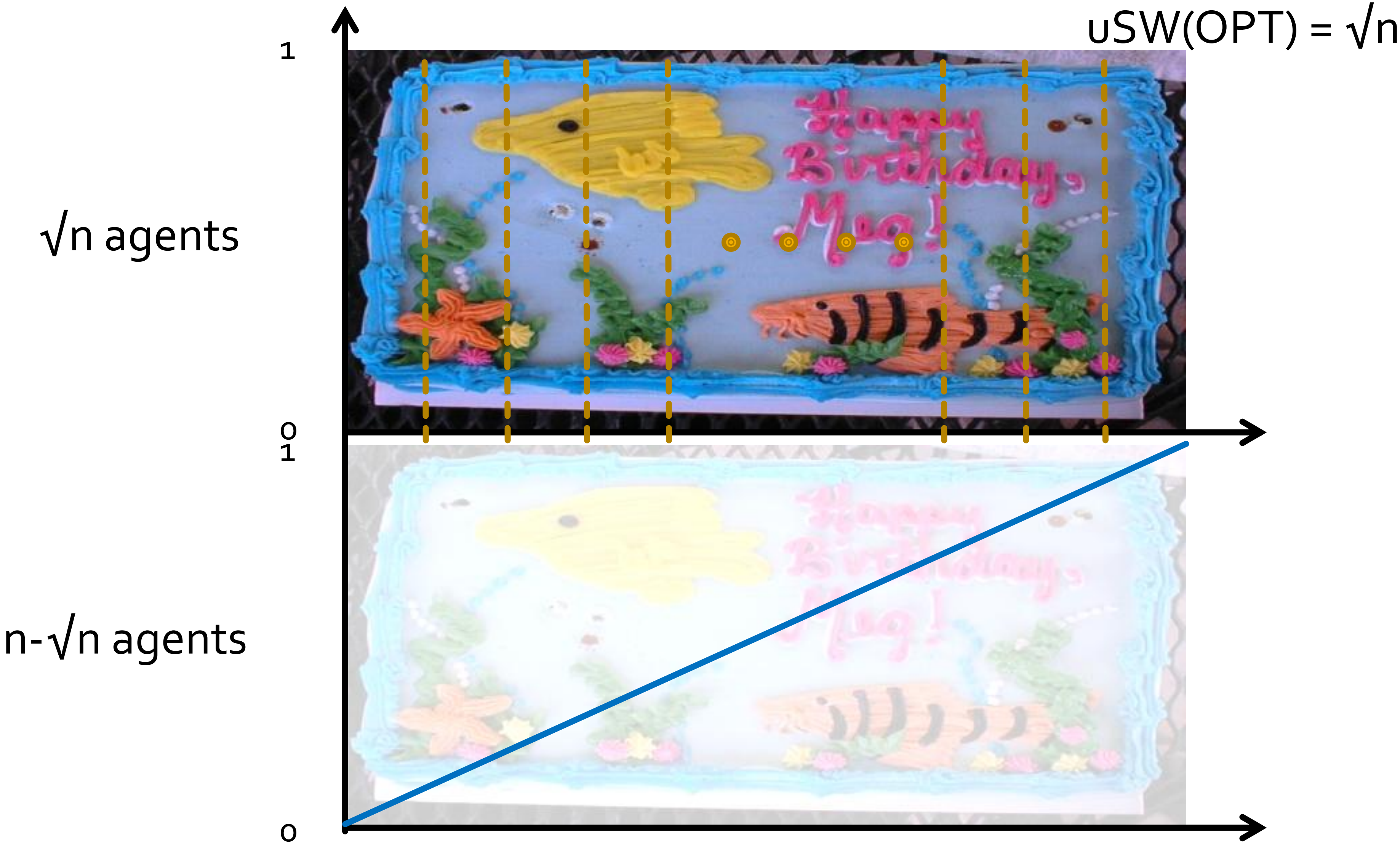


n-√n agents

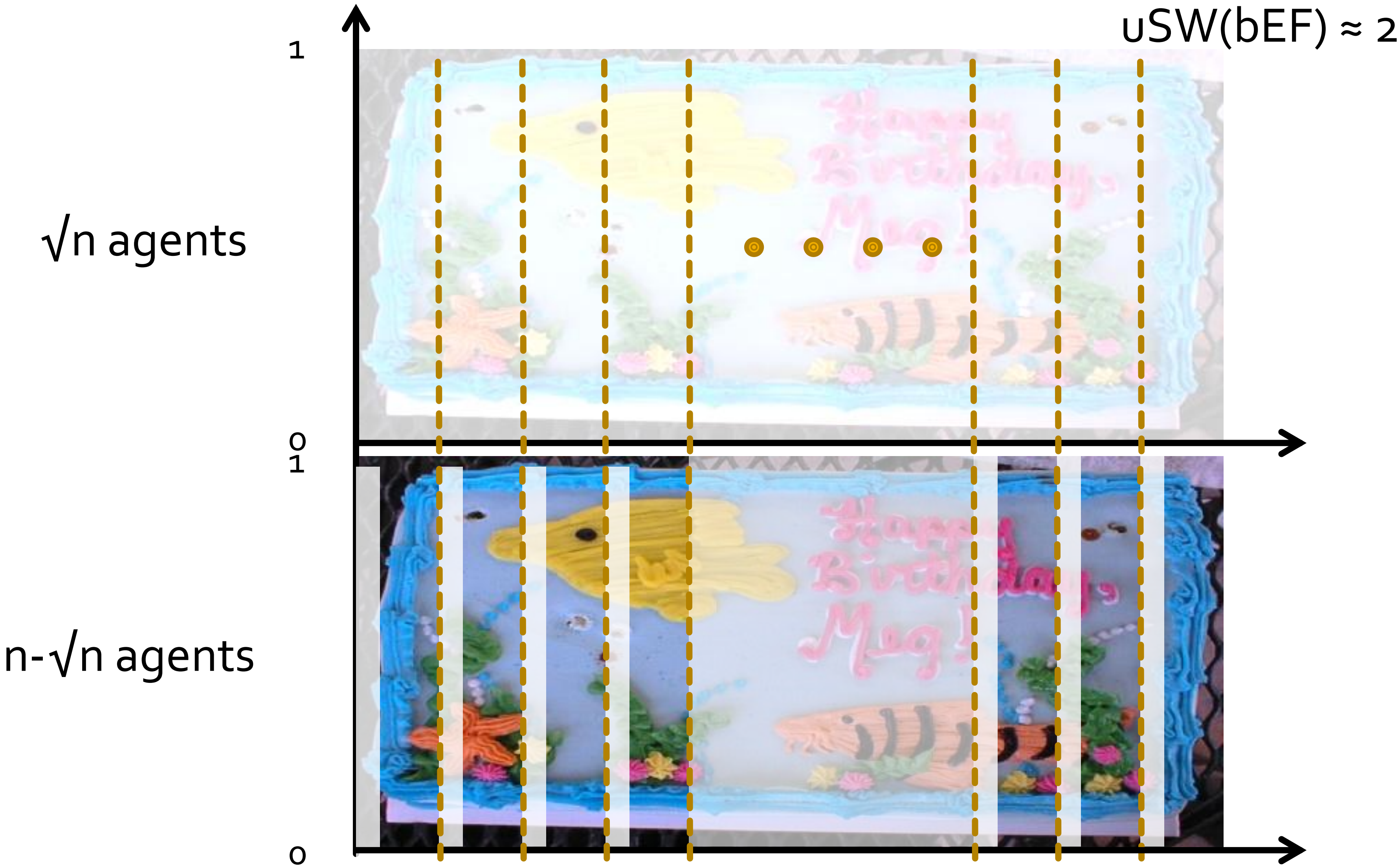
A lower bound for PoP (n agents)



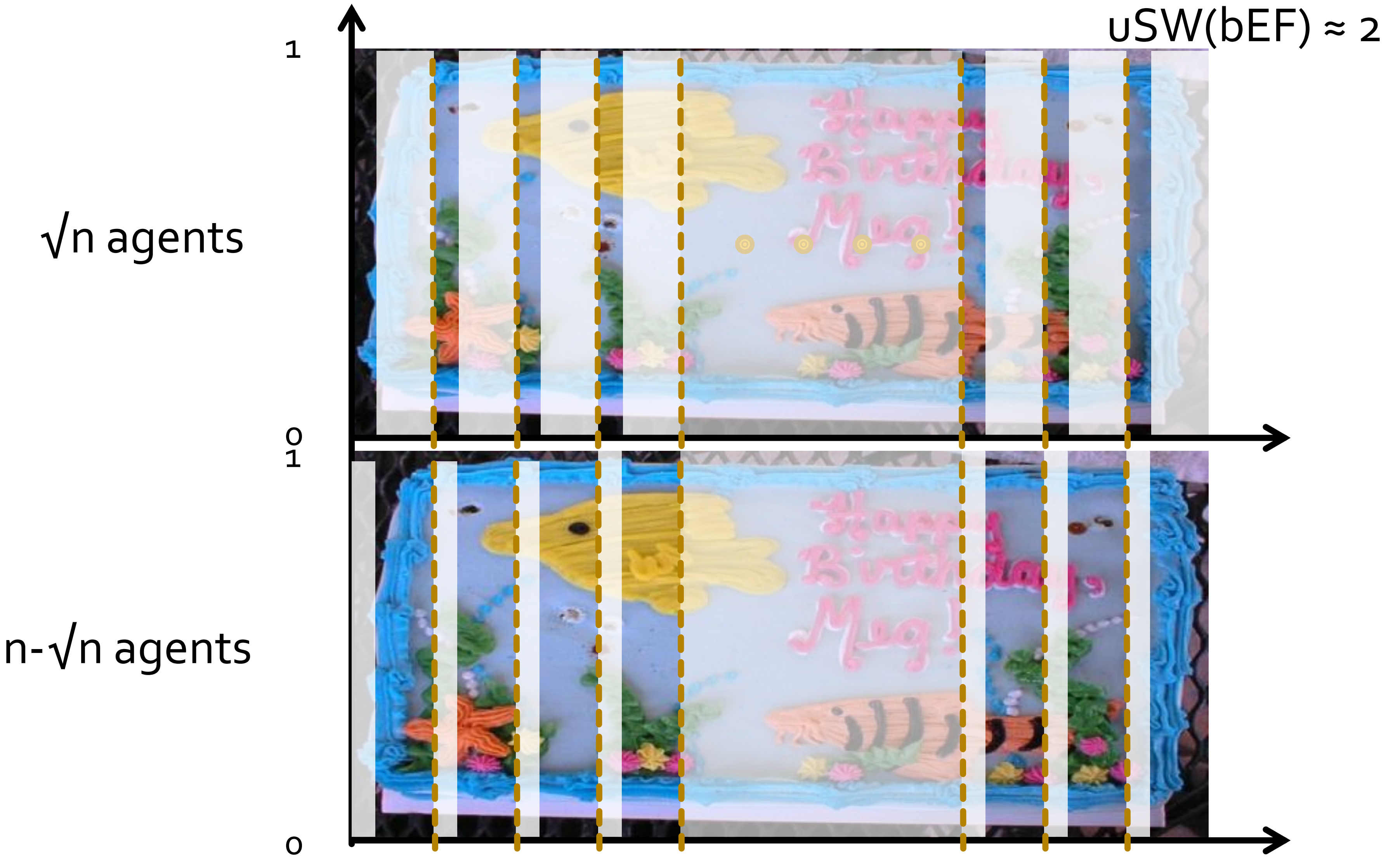
A lower bound for PoP (n agents)



A lower bound for PoP (n agents)



A lower bound for PoP (n agents)



An upper bound for PoP (n agents)

- The PoP wrt the utilitarian social welfare in n-agent instances is at most $O(\sqrt{n})$
- Proof: structure of a uSW-maximizing allocation
  - set L of large agents (utility higher than $1/\sqrt{n}$)
  - set S of small agents (utility smaller than $1/\sqrt{n}$)
- Easy case: $L < \sqrt{n}$
  - Then $uSW(OPT) < L+S/\sqrt{n} < 2\sqrt{n}$ and $SW(bEF) \geq 1$
- Difficult case: $L \geq \sqrt{n}$
  - ...

An upper bound for PoP: $L \geq \sqrt{n}$

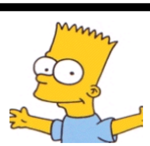
- For each small agent i, re-allocate A_i to all small agents
- For each large agent i, re-allocate A_i to $\sqrt{n}$ copies of agent i and all small agents
- Why proportional?
  - Each small agent gets either $1/(|S|+\sqrt{n})$-th or $1/|S|$-th of each piece in A
  - Each large agent gets at least $1/\sqrt{n}$ of her optimal piece
- Why $O(\sqrt{n})$?

Complexity of achieving fairness

- How hard is it to compute a proportional/EF allocation in the indivisible item setting?
  - just 2 agents
  - the number m of items is part of the input

m indivisible items

agents

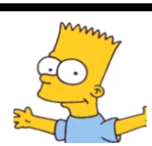
				
	3	0	5	12
	0	2	2	1

Complexity of achieving fairness

- Partition: Given m items with values v_1, v_2, …, v_m, is there a partition of the items in two disjoint sets S_1 and S_2 of the same total value? I.e., $\sum_{j \in S_1} v_j = \sum_{j \in S_2} v_j$

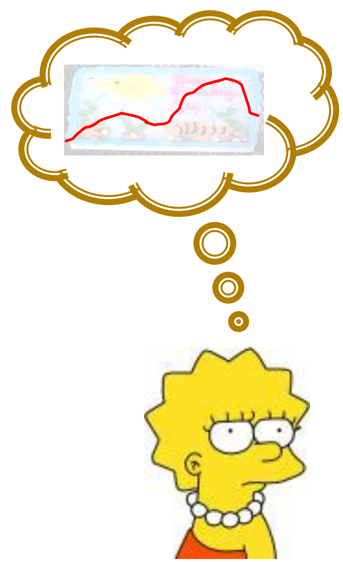
- Reduction:

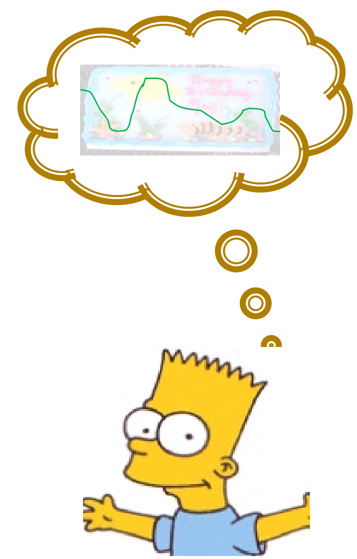
m indivisible items

agents	item 1	item 2	.....	item m
	v_1	v_2	.....	v_m
	v_1	v_2	.....	v_m

Query complexity

- Equitability:
  - an allocation A is equitable if $V_1(A_1) = V_2(A_2)$
  - How can we come up with an equitable allocation in the query model?





Further reading

- C., Kaklamanis, Kanellopoulos, & Kyropoulou (Theory of Computing Systems, 2012)
- Bertsimas, Farias, & Trichakis (Operations Research, 2011)
- Aumann & Dombb (WINE 2010) and follow-up work by Aumann et al.
- Bouveret & Lemaitre (AAMAS 2014)
- Surveys by Procaccia (COMSOC Handbook, 2015; CACM, 2013), Bouveret, Chevaleyre, & Maudet (COMSOC Handbook, 2015)