Fair Division of Indivisible Items

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COST Summer School | Computational Social Choice

Recap: fairness notions

- (PROP) proportionality
- (MFS) maxmin fair share
- (EF) envy-freeness
- (ESW) egalitarian social welfare
- (NSW) Nash social welfare
- ($e^{sum,max,bool}$) number of envious agents
- ($e^{max,max,raw}$) max envy between any pair of agents
- envy up to one (some/any) good

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- in practice: MIP formulations allowing efficient solving
- in practice: envy-free allocations always satisfied from a certain number of goods
- in practice: existence of MFS allocations always satisfied

Two open problems we mentioned:

- complexity of deciding whether an MFS allocation exist
- existence of envy-freeness up to any good

One question we left pending

☞ *With Borda utilities, any EF allocation must be egalitarian-optimal*

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☞ *With Borda utilities, any EF allocation must be egalitarian-optimal*

False. Easy to see (similar to what we showed in general) : 2 agents, 4 resources, reversed preferences. Both agents can get they top and third preferred items, this is EF, but it would be better (and possible) for both to get they two preferred items.

One question we left pending

How about this stronger statement?

☞ *With Borda utilities, some EF allocation (if there exist) must be egalitarian-optimal*

False. Counter-example:

	r_0	r_1	r_2	r_3	r_4	r_5
agent 1	6	5	3	4	2	1
agent 2	6	5	3	4	2	1
agent 3	4	5	6	1	2	3

There is an allocation with maxmin value = 7 which is envy-free, but egalitarian-optimal is 8 and none of the allocation are envy-free.

More about this interplay between egalitarian social welfare

protocols

Why do we need (nice) protocols

There are many reasons why protocols often have to be used in practice:

- lack of access to (or trust in) a central authority,
- agents prefer to take part in the allocation process,
- interesting compromise between communication burden and efficiency/fairness guarantees

Communication is often a real bottleneck in resource allocation problems, and in principle protocols can make a difference.

Example: protocols for allocating one good

Consider the following situation:

There are two agents (A and B); and one object to allocate.
Each agent x has a valuation $v_x \in \{0, 1, 2, 3\}$ *for the object.*
Goal: assign the object to the agent who values it the most.

Can we design efficient protocols to achieve this goal?

I. Segal. *Communication in Economic Mechanisms.* CES-2006.

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Protocol π_0: "One-sided Revelation"	bits
A gives her valuation	2
B computes the allocation, and send it	1
	total $\Rightarrow$ 3

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Protocol π_1: "English Auction"	bits
$p \leftarrow 0$, $X \leftarrow A$	
while not(stop):	
ask X "stop"' or "raise"	1
$p \leftarrow p + 1$	
$X \leftarrow \overline{X}$	
allocate to $\overline{X}$	
	total $\Rightarrow$ 1, 2, or 3

Example: protocols for allocating one good

Consider the following situation:

> *There are two agents (A and B); and one object to allocate.*
> *Each agent x has a valuation $v_x \in \{0, 1, 2, 3\}$ for the object.*
> *Goal: assign the object to the agent who values it the most.*

Can we design efficient protocols to achieve this goal?

Protocol π_2: "High/Low Bisection"	bits
A says whether her valuation $\{0, 1\}$ (low) or $\{2, 3\}$ (high)	1
B computes the allocation	
(if low (if $v_B = 0$ then give to A else give to B))	
(if high (if $v_B = 3$ then give to B else give to A))	
and send it	1
	total $\Rightarrow$ 2

Communication issues

- for additive utilities, centralized protocols require $O(nm \log K)$ for full elicitation
- in general, communication complexity arguments show that you cannot hope to get more frugal protocols
- but some protocols offer interesting compromises

the adjusted winner

The Adjusted Winner

The protocol is designed for two agents, who initially have the same amount of points to assign to items.
It runs in two phases:

1. winning phase: allocate goods efficiently, ie. assign each good to the agent who values it most
2. adjusting phase: goods are transferred from the “high” agent to the “low” agent in increasing order of the ratio

$$\frac{u_h(r)}{u_l(r)}$$

until the poorest become the richest (or they enjoy the same utility).

Brams and Taylor. *The Win-win Solution. Guaranteeing Fair Shares to Everybody*. 2000.

The Adjusted Winner

But the protocol may require the last resource r to be splitted.

The idea is to split precisely so as to attain exactly the same utility for both agents :

$$\frac{u_l(r) + u_l(\pi \setminus \{r\}) - u_h(\pi \setminus \{r\})}{u_h(r) + u_l(r)}$$

However, without knowing in advance which resource may be splitted, it must be assumed that all are. Under this assumption:

☞ *Adjusted Winner returns an envy-free Pareto-optimal allocation, and both agents enjoy the same utility.*

The Adjusted Winner

Example:

	r_0	r_1	r_2	r_3	r_4
agent 1	1	2	5	3	8
agent 2	2	3	8	1	5

The Adjusted Winner

Example:

	r_0	r_1	r_2	r_3	r_4
agent 1	1	2	5	**3**	**8**
agent 2	**2**	**3**	**8**	1	5

winning phase

agent 1 enjoys utility 11

agent 2 enjoys utility 13

Example:

	r_0	r_1	r_2	r_3	r_4
agent 1	1	2	5	**3**	**8**
agent 2	**2**	3	**8**	1	5

adjusting phase

$((r_1, \frac{3}{2}), (r_2, \frac{8}{5}), (r_0, \frac{2}{1}))$

r_1 must be transferred

The Adjusted Winner

Example:

	r_0	r_1	r_2	r_3	r_4
agent 1	1	2	5	3	8
agent 2	2	3	8	1	5

adjusting phase

agent 2 must get (of r_1):

(2+11-10)/(2+3)=3/5

lipton et al.

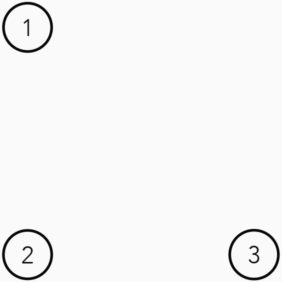
Lipton *et al.*

We first present informally the approach, based on a simple sequential allocation of resources.
For each resource r_k to be allocated:

- build the envy graph $G = (\mathcal{N}, E)$, where $(i, j) \in E \times E$ if agent i envies agent j
- while the graph has cycles, pick one $C = (c_1, c_2, \ldots c_q)$, and reallocates the bundle of c_i to c_{i-1} (and of c_1 to c_q).
- allocate r_k to an agent that no one envies.

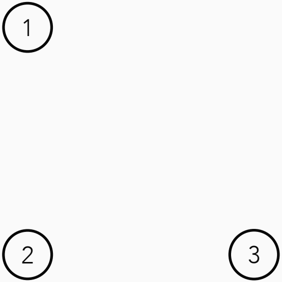
Lipton et al. *On approximately fair allocations of divisible goods.* EC-04.

Lipton *et al.*



	r_0	r_1	r_2	r_3	r_4	r_5
agent 1	1	2	5	3	7	2
agent 2	2	6	8	1	1	2
agent 3	5	4	4	3	2	2

Lipton *et al.*



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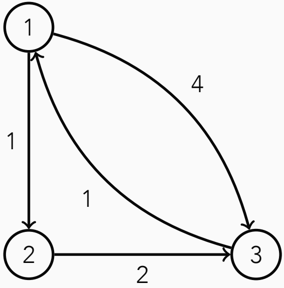
No object is allocated yet.

Lipton *et al.*



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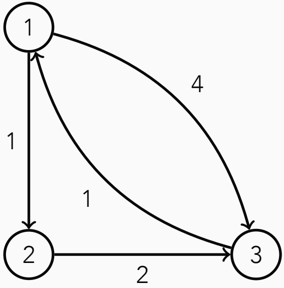

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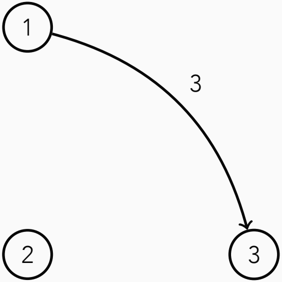
agent 1	1	2	5	3	7	2
agent 2	2	6	8	1	1	2
agent 3	5	4	4	3	2	2

There are two cycles: (1,3) or (1,2,3)

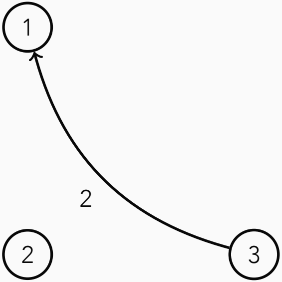
Lipton *et al.*



	r_0	r_1	r_2	r_3	r_4	r_5
agent 1	1	2	5	3	7	2
agent 2	2	6	8	1	1	2
agent 3	5	4	4	3	2	2

Suppose we chose cycle (1,2,3). After a single rotation, agent 1 and agent 2 are not envied any longer.

Lipton *et al.*



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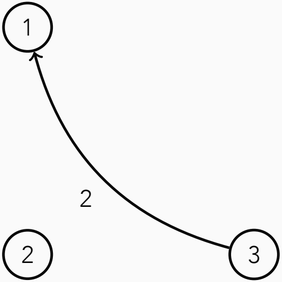

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agent 1	1	2	5	3	7	2
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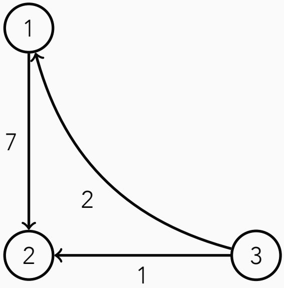
agent 3	5	4	4	3	2	2

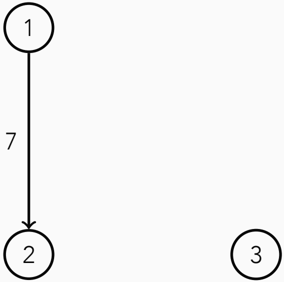
We can give r_3 to agent 1. There are no cycle, agent 2 and agent 3 are not envied.

Lipton *et al.*



	r_0	r_1	r_2	r_3	r_4	r_5
agent 1	1	2	5	3	7	2
agent 2	2	6	8	1	1	2
agent 3	5	4	4	3	2	2

We can give r_4 to agent 2. There are no cycles but only agent 3 is not envied.



	r_0	r_1	r_2	r_3	r_4	r_5
agent 1	1	2	5	3	7	2
agent 2	2	6	8	1	1	2
agent 3	5	4	4	3	2	2

We finally give r_5 to agent 3. The final allocation is not envy-free, as agent 1 envies agent 2.

Lipton *et al.*: analysis

Cycle reallocation step: $C = (c_1, c_2, \ldots, c_q)$

☞ Envy must have decreased.

- any agent in the cycle has increased its utility.
- bundles are unaffected

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☞ The number of edges in the envy graph has decreased.

- edges between agents $\notin C$ are not affected
- edges from agents $\notin C$ to C now point to previous agent in C
- edges from agents $\in C$ to agents $\notin C$ may only decrease
- (original) edges between agents $\in C$ are deleted

Lipton et al. *On approximately fair allocations of divisible goods.* EC-04.

Lipton *et al.*: envy is bounded

Let α be the max value that any agent gives to a good.

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Base case:

A_0: allocate first resource randomly. Clearly $e(A_0) \leq \alpha$.

Induction step:

Suppose A with $\{r_1, \ldots, r_k\}$ allocated, and $e(A) \leq \alpha$.

By repeatedly applying cycle reallocation in the envy graph, we must get an acyclic graph.

Hence at least an agent j is not envied: she gets r_{k+1}.

Envy among agents $\neq j$ is not affected.

Envy of agents $i \neq j$ towards j is $\leq \alpha$, since j was not envied.

Lipton et al. *On approximately fair allocations of divisible goods.* EC-04.

Lipton *et al.*: complexity

- computational complexity : cycle detection $O(n^2)$ and edge removing. Number of edges to remove is at most n^2. This takes place m times (for each resource), hence $O(mn^4)$.
- the communication requirement of the protocol is, for each agent, to say whether she envies the other ones (n^2). This occurs for each resource allocation, giving overall mn^2 bits.
- observe that the protocol as presented never requires agents to communicate utilities

picking sequences

Picking sequences

We fix beforehand a sequence of agents, eg. ($n = 3, m = 6$)

$$[123231]$$

- agents pick one resource at a time, at their turn
- if they do so sincerely they pick the best resource available to them at that stage of the protocol

Only requires to communicate m times which resource to pick ($\log(m)$ bits), hence overall $m \log m$ bits.

Picking sequences

Sequence = [123231]

	r_0	r_1	r_2	r_3	r_4	r_5
agent 1	1	2	5	3	7	2
agent 2	2	6	8	1	1	2
agent 3	5	4	4	3	2	2

Picking sequences

Assuming for the moment that $k = m \mod (n)$, ie. we can ensure that each agent gets the same number of resources.

Take a permutation of agents:

$$p = [p(1), p(2), \ldots, p(n)]$$

Let p^{-1} be the "mirror" sequence of p.

- round robin: the subsequence p is repeated k times
- balanced: $(p \circ p^{-1})$ is repeated $k/2$ times

(When there are only two agents, it is common to talk about strict alternation or balanced alternation)

Brams and Taylor. *The Win-win Solution. Guaranteeing Fair Shares to Everybody*. 2000.

Bouveret and Lang. *A general elicitation-free protocol for allocating indivisible goods*. IJCAI-11.

Picking sequences: Round-Robin

Round-robin sequences are arguably the simplest ones (they are also called draft mechanisms).

When $n = m$ similar to serial dictatorship.

Do they have interesting properties?

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☞ *Round-robin picking sequences satisfy envy-freeness up to one good*

Intuition: during each k phase, when picking its resource r, agent i prefers r over the $n-1$ ones subsequently chosen by other agents. Envy towards j can result from resource chosen by j before his first pick (during the first phase). Removing this resource from bundle of j removes envy.

Picking sequences: Round-Robin

What about Pareto-efficiency?

Picking sequences: Round-Robin

What about Pareto-efficiency?

	r_0	r_1	r_2	r_3	r_4	r_5
agent 1	18	8	1	1	1	1
agent 2	4	6	5	5	5	5

$\Rightarrow$ Round-robin gives r_0 to agent 1, r_1 to agent 2, and two other resources each among $\{r_2, r_3, r_4, r_5\}$: utilities = $(20, 16)$.

Picking sequences: Round-Robin

What about Pareto-efficiency?

	r_0	r_1	r_2	r_3	r_4	r_5
agent 1	18	8	1	1	1	1
agent 2	4	6	5	5	5	5

$\Rightarrow$ But exchanging r_1 that agent 2 got against the two items among $\{r_2, r_3, r_4, r_5\}$ that agent 1 obtained gives $(24, 20)$.

Picking sequences: Round-Robin

What about Pareto-efficiency?

	r_0	r_1	r_2	r_3	r_4	r_5
agent 1	18	8	1	1	1	1
agent 2	4	6	5	5	5	5

☞ *Round-robin picking sequences are not guaranteed to satisfy Pareto-optimality*

However, for two agents, for Borda utilities and under assumption of uniform distribution, they maximizes the expected utilitarian social welfare.

Kalinowski *et al. A social welfare optimal sequential allocation procedure.*. IJCAI-13.

Picking sequences: designing fair sequences

Can we design sequences such that they are fair?

Let us make the assumption that utilities are Borda.

- under uniform preferences, what are the sequences which maximize egalitarian social welfare?

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- can we design sequences which guarantee proportionality? (or maximize the likelihood to get a proportional allocation?)

Picking sequences: proportionality

☞ *For even k, when $m = k \cdot n$, the balanced picking sequence returns a proportional allocation*

Intuition: Worst case is when agents have same preferences. Possible to analyse the situation in that case.

Darmann and Klamler. *Proportional Borda Allocations.* COMSOC-2016.

Picking sequences: proportionality

☞ *For odd n, when $m = k \cdot n$, there exists a picking sequence which returns a proportional allocation*

The following picking sequence can be used:

- for the first $3n$ picks, follow the sequence

$$
\begin{array}{llll}
[1, & \ldots & \ldots & , n, \\
n, n-2, n-4, \ \ldots & , 1, & n-1, n-3, \ \ldots & , 2, \\
n-1, n-3, \ \ldots & , 2, n, & n-2 \ \ldots & , 1]
\end{array}
$$

- for the remaining picks use the balanced sequence

Darmann and Klamler. *Proportional Borda Allocations*. COMSOC-2016.

Picking sequences: proportionality

Note that this leaves some cases where proportional allocations are not guaranteed to exist.

- when $m = n$ a proportional allocation may not exist (consider two agents, two resources, same preferences).
- or some odd k, even n, eg. for $n = 2$ and $m = 6$ (4 problematic cases)

rational local exchanges

Local Exchanges

We conclude with a fully distributed approach:

- resources are initially held by agents
- agents agree on local rational deals
- agents may have restrictions on the types of deals they can perform
- agents may not be able too see/deal with any other agents

This approach relies on a dynamics, with agents encountering each others and (potentially) agreeing on deals. The final allocation is when no more deals are possible.

Sandholm. *Contract types for satisficing task allocation.* AAAI Spring Symposium.

Endriss *et al.* *Negotiating socially optimal allocations of resources.* JAIR-06.

Local Exchanges

The notions of fairness/efficiency behave differently wrt. this distributed setting:

Intuitively:

- if some agents perform a deal which increase locally the sum of utilities, then globally the sum of utility will increase

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Local Exchanges

The notions of fairness/efficiency behave differently wrt. this distributed setting:

Intuitively:

- if some agents perform a deal which increase locally the sum of utilities, then globally the sum of utility will increase
- if some agents perform a deal which increase locally the min of utility, then globally the min of utility cannot decrease
- if some agents perform a deal which decrease locally envy, then globally envy may very well increase

This has consequences on convergence guarantees that can be given.

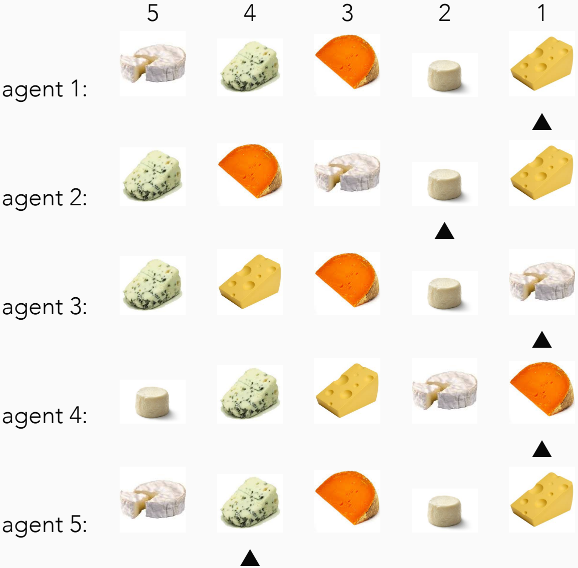
Local Exchanges

Let us illustrate this approach on a simple scenario:

- same number of resources as agents
- each agent can hold only hold one resource
- TTC is the method of choice with nice properties
- but suppose agents can simply perform rational swap deals

Damamme *et al. The power of swap deal in distributed resource allocation.* AAMAS-15.

Example



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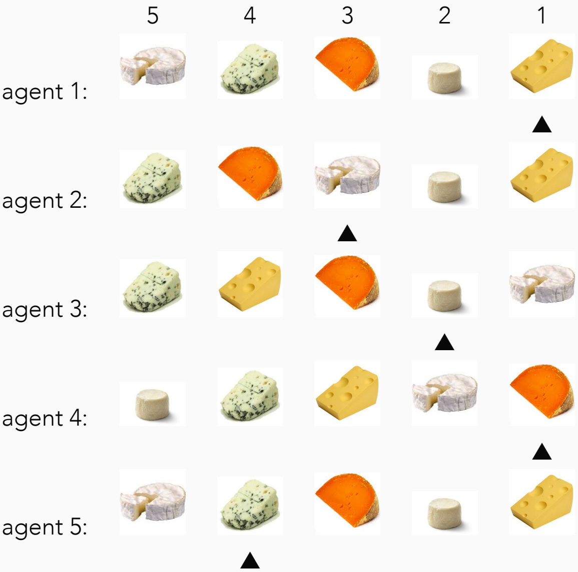

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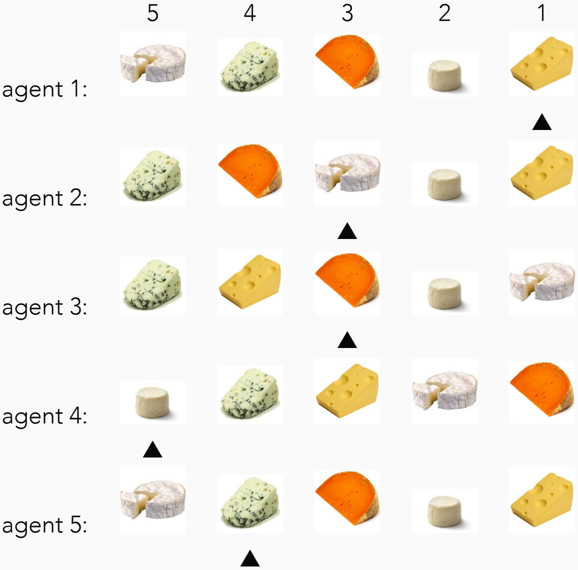

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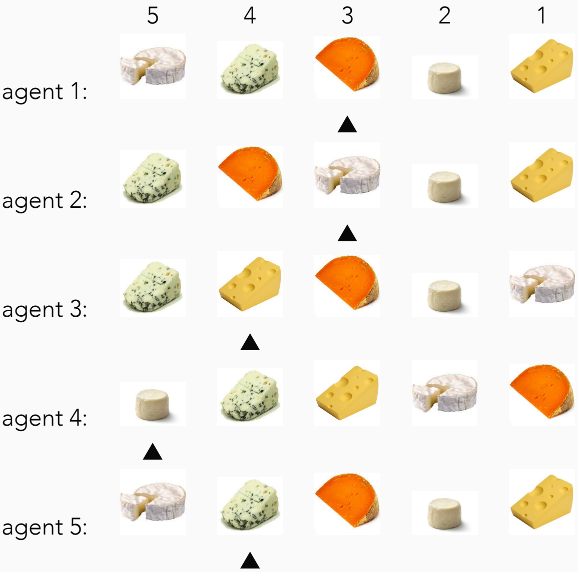

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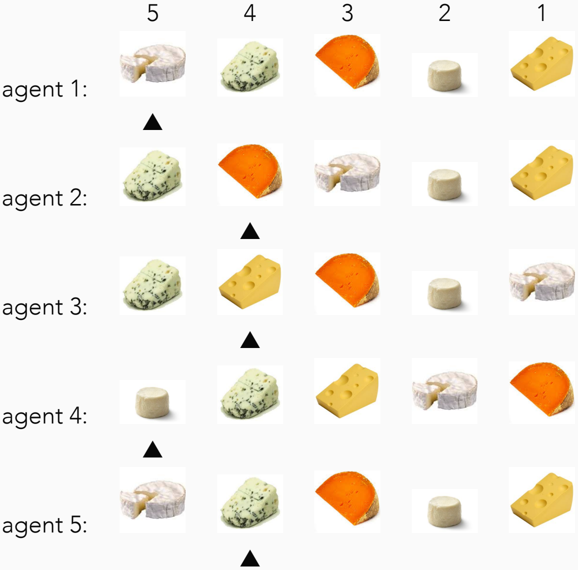

$\Rightarrow sw_u(A) = 22$

$\Rightarrow sw_e(A) = 4$

Local Exchanges

What are the properties of such a protocol?

- Is Pareto-optimality guaranteed?
- What is the "price" of using this protocol wrt. egalitarian social welfare?
- What is the "price" of using this protocol wrt. number of pairwise envies? (ie. utilitarian social welfare in this case...)
- What is the complexity of the reachability question?

Local Exchanges

Is Pareto-optimality guaranteed?

Local Exchanges

Is Pareto-optimality guaranteed? No.

Local Exchanges

Is Pareto-optimality guaranteed? No.

Domain restriction guaranteeing Pareto-optimal outcomes?

Local Exchanges

Is Pareto-optimality guaranteed? No.

Domain restriction guaranteeing Pareto-optimal outcomes?

☞ *In a single-peaked domain, any sequence of rational swap deals reaches a Pareto-optimal allocation.*

Local Exchanges

Price for egalitarian social welfare:

$$
\begin{array}{lllllllll}
1 & : \boxed{r_1} & \succ & \boxed{r_n} & \succ & \dots & & & \\
2 & : \boxed{r_2} & \succ & \boxed{r_1} & \succ & \dots & & & \\
3 & : \boxed{r_3} & \succ & \boxed{r_2} & \succ & \dots & & & \\
4 & : \boxed{r_4} & \succ & \boxed{r_3} & \succ & \dots & & & \\
\vdots & & & & & & & & \\
n-1 & : \boxed{r_{n-1}} & \succ & \boxed{r_{n-2}} & \succ & \dots & & & \\
n & : \boxed{r_n} & \succ & \dots & \succ & \dots & \succ & \boxed{r_{n-1}} & \succ \; r_{n-2}
\end{array}
$$

Local Exchanges

Price for utilitarian social welfare / number of pairwise envies:

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Price is at most 2. Take a swap-stable allocation A: for each pair of agents (x, y), at least one agent ranks the resource of the other below her current. Hence overall at least $n(n-1)/2$ resources ranked below.

Local Exchanges

Price for utilitarian social welfare / number of pairwise envies:

Price is at most 2. Take a swap-stable allocation A: for each pair of agents (x, y), at least one agent ranks the resource of the other below her current. Hence overall at least $n(n-1)/2$ resources ranked below.

Price can be 2:

$$
\begin{array}{lllllllll}
a_1 & : & \colorbox{yellow}{1} & \succ 2 \succ & \boxed{3} & \succ 4 & \succ & 5 \\
a_2 & : & \colorbox{yellow}{2} & \succ 3 \succ & \boxed{4} & \succ 5 & \succ & 1 \\
a_3 & : & \colorbox{yellow}{3} & \succ 4 \succ & \boxed{5} & \succ 1 & \succ & 2 \\
a_4 & : & \colorbox{yellow}{4} & \succ 5 \succ & \boxed{1} & \succ 2 & \succ & 3 \\
a_5 & : & \colorbox{yellow}{5} & \succ 1 \succ & \boxed{2} & \succ 3 & \succ & 4
\end{array}
$$

More on distributed settings

Other typical results in such settings:

- allowing the use of money and characterizing convergence properties under various protocols/preference constraints
- accounting for the underlying visibility/deal graph
- communication complexity (typically in terms of number of deals) of such protocols

Chevaleyre *et al.* *Allocating Goods on a Graph to Eliminate Envy.* AAAI-07.

Dunne. *Extremal behaviour in multiagent contract negotiation.* JAIR-05.

Going further

More general preferences than cardinal additive utilities:

- first note that the additivity assumption is not used in Lipton's et al. approach. In that case the maximum marginal utility becomes :

$$\alpha = \max_{i,r,S \subseteq \mathcal{O} \setminus \{r\}} [u_i(S \cup \{g\}) - u_i(S)]$$

- many other protocols available: the descending demand procedure, the undercut procedure, ...

Slides of the COST Summer School on Fair Division. *Grenoble*. 2015.