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“Proportional Representation”

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CHAPTER 1

Introductory Session: Proportional Representation Examples

1.1 The Mathematics of Electoral Systems

Many parliaments in this world are elected by means of a proportional representation electoral system. To be precise, it is the Members of Parliament who get elected. In contemporary democracies the candidates who stand at an election are organized in political parties. People cast their votes for the parties, the institutions, as much as they cast them for the candidates, the individuals. The attribute “proportional” aims at the institutional worth of the votes. The electoral system should grant every political party a number of seats in parliament that fairly reflects the strength of the party as manifested by the count of votes in favor of this party.

Electoral systems are rather complex systems. The law for the election of the German parliament comprises fifty-five sections and an appendix. It is supplemented by regulatory instructions with another ninety-three sections, and by further electoral provisions. Setting up such a system is a challenging task, and it is similarly demanding to understand the system and predict its political consequences. A good source to study the complexities of electoral systems and the intricacies of national provisions is the compendium *The Politics of Electoral Systems* of Gallagher / Mitchell (2008).

The present module is less broad and confines attention to “the mathematics” of electoral systems: How is the number of votes cast for a party translated into the number of seats apportioned to the party? The rules for translating votes into seats often reflect the past of a country, how an electoral system came into being and how it grew to what it is today. In order to evade national peculiarities, we use the 2014 elections of the European Parliament as a common denominator. We start out by explaining the electoral systems of four Member States: Germany, Czech Republic, Bulgaria, and Belgium. For a discussion of the systems for the election of the European Parliament in all twenty-eight Member States see Oelbermann / Pukelsheim (2016).

The 2014 European Parliament comprises 751 seats. They are allocated between the twenty-eight Member States well before the elections. Currently the allocation is arrived at by negotiations, but use of a formula is being discussed (see Sect. 3.4).

Thus every Member State commands a fixed contingent of seats to fill. The electoral systems within the Member States have to follow some common principles as laid down in the European Electoral Act of 2002. The Act demands that the electoral systems must be proportional representation systems. There are many proportional systems, though. Naturally States have tended to use the existing system to elect their national parliament as a template for the system to elect their due seats in the European Parliament.

As a matter of fact no two of the twenty-eight electoral systems are the same. This gives rise to the extraordinary situation that there is a single political body, the European Parliament, which is elected by means of twenty-eight distinct electoral systems. For this reason we continue to refer to the European Parliament elections in the plural: elections, rather than in the singular: election. Nevertheless the setting offers a common denominator which helps us to focus on the topic of this module. Our theme is the multitude of seat apportionment methods, and the assignment of a party's seats to the party's candidates.

1.2 Germany: The Divisor Method with Standard Rounding

Germany is entitled to ninety-six seats in the European Parliament. In the polling stations voters are handed a ballot sheet showing the labels of the parties competing. All parties except two submit a single list of candidates for all of Germany. The first ten names are spelled out on the ballot sheet. The exceptions are CDU and CSU. CDU tenders fifteen distinct lists of nominees, one list for each of the fifteen States (all but Bavaria) where it stands. CSU, only standing in the State of Bavaria, presents a list of candidates that is restricted to this state. Altogether there are sixteen distinct ballot sheets, one for each State. The choices they offer are identical for all parties except CDU and CSU, for whom it varies from state to state. On the ballot sheets voters are supposed to mark the party of their choice.

The total number of valid ballots is 29 355 092. The distribution of votes among parties is shown in Table 1. The first column contains the parties' tabs. They are taken from the Internet site

www.europarl.europa.eu/elections2014-results/en/seats-member-state-absolut.html

The site contains a sub-site where the tabs are expanded into the full names of the parties. Eleven parties turn out to be too weak to obtain a seat, they are subsumed in the table's line "11 Others".

The table's second column exhibits the aggregate number of "Votes" cast for a party throughout all of Germany. These vote counts are converted into seat numbers by means of the "divisor method with standard rounding". We abbreviate the method with the acronym "DivStd". The method calculates a quantity called "divisor". The way in which the divisor is calculated makes sure that the sum of all seat numbers exhausts the given seat total, 96. The divisor can be though of as an electoral key. Given the key, everybody can confirm the seat number of a party quite easily. In the present case a suitable key is 298 900. Thus the method is captured by the phrase: *Every 298 900 votes justify roughly one seat.*

TABLE 1 *EP 2014 Election, Germany.* The divisor method with standard rounding is used. Every 298 900 votes justify roughly one seat. That is, a party's "Votes" and the divisor 298 900 yield an interim "Quotient" that is rounded downwards when its fractional part is below one half, and upwards when above. The resulting seat apportionment is shown in column "DivStd".

EP2014DE	Votes	Quotient	DivStd
CDU	8 812 653	29.48	29
SPD	8 003 628	26.8	27
GRÜNE	3 139 274	10.503	11
DIE LINKE	2 168 455	7.3	7
AFD	2 070 014	6.9	7
CSU	1 567 448	5.2	5
FDP	986 841	3.3	3
FREIE WÄHLER	428 800	1.4	1
PIRATEN	425 044	1.4	1
TIERSCHUTZPARTEI	366 598	1.2	1
NPD	301 139	1.0	1
FAMILIE	202 803	0.7	1
ÖDP	185 244	0.6	1
DIE PARTEI	184 709	0.6	1
11 Others	512 442	—	0
Sum (Divisor)	29 355 092	(298 900)	96

More precisely, a party's vote count is divided by the divisor 298 900. The ensuing "Quotient" is shown in the third column of Table 1. The interim quotient is rounded to a whole number in the standard way: downwards when its fractional part is below one half, and upwards when above. The table displays as many digits as are needed to clearly determine whether fractional parts are below or above one half. (The event that a quotient is exactly equal to one half is called a "tie". Ties complicate the theory, but are irrelevant in practice. We simply neglect the occurrence of ties, in this module.)

The whole number that results from the rounding operation indicates the number of seats apportioned to the party. It is given in the last column of Table 1, labeled "DivStd". The label is indicative of the apportionment method used: the divisor method with standard rounding.

To which candidates are the seats assigned? In Germany voters must accept the ordering of the candidates as given by the party lists. This type of electoral system is termed a "closed list" system. Most parties present a single list of candidates for the entire country. Thus the 27 SPD seats are assigned to the top 27 nominees on the SPD list. The other parties proceed similarly. The CSU stands at the election only in the State of Bavaria, with a list of candidates restricted to this state. Hence the five CSU seats are filled with the top 5 candidates on the Bavarian CSU list.

The CDU pursues a different strategy, by tendering a separate list in each of the fifteen States where the party stands. Therefore the 29 CDU seats need to be sub-apportioned to the fifteen CDU state-lists. The sub-apportionment is carried out again using the divisor method with standard rounding. The details are shown in Table 2.

The example teaches us some instant lessons. Take a look at the Quotient column of Table 1. The CDU quotient, 29.48, is stopping short of the critical value 29.5 where it would become eligible to be rounded upwards to 30. The CDU quotient would hit the value 29.5 if the divisor, a say, would satisfy the relation

$$\frac{8\,812\,653}{a} = 29.5, \qquad \text{that is,} \qquad a = \frac{8\,812\,653}{29.5} = 298\,734.$$

TABLE 2 *EP 2014 Election, Germany, CDU Sub-apportionment.* The 29 CDU seats are sub-apportioned among States since the party tenders a separate list for each state where it stands. Again the divisor method with standard rounding is used; every 300 000 votes justify roughly one seat. No more sub-apportionments are needed as all other parties submit a single, countrywide list of nominees.

EP2014DE-CDU	Votes	Quotient	DivStd
CDU Sub-apportionment			
Schleswig-Holstein	334 121	1.1	1
Mecklenburg-Vorpommern	210 268	0.7	1
Hamburg	135 780	0.45	0
Lower Saxony	1 174 739	3.9	4
Bremen	43 353	0.1	0
Brandenburg	233 468	0.8	1
Saxony-Anhalt	245 010	0.8	1
Berlin	232 274	0.8	1
North Rhine-Westphalia	2 439 979	8.1	8
Saxony	559 899	1.9	2
Hessen	564 294	1.9	2
Thuringia	290 703	1.0	1
Rhineland-Palatinate	661 339	2.2	2
Baden-Württemberg	1 542 244	5.1	5
Saarland	145 182	0.48	0
Sum (Divisor)	8 812 653	(300 000)	29

Hence the next seat would go to the CDU, and for this to happen the divisor would have to fall below 298 734. On the other hand the GRÜNE quotient, 10.503, just made it passed the critical value 10.5, thus seizing the last seat available. The GRÜNE quotient would hit the value 10.5 provided the divisor b were to satisfy the relation

$$\frac{3\,139\,274}{b} = 10.5, \qquad \text{that is,} \qquad b = \frac{3\,139\,274}{10.5} = 298\,978.47.$$

Therefore the GRÜNE party would lose the last seat as soon as the divisor raises above 298 978.47. From this discussion we may draw three conclusions.

Firstly, once we know the definitive seat apportionment for a total of 96 seats, the interim quotients tell us how to hand out one seat more than 96, or one seat less. They identify the party who is eligible to gain the next, ninety-seventh seat (in the example: CDU). Similarly they point at the party who would have to give up the last, ninety-sixth seat (here: GRÜNE).

Secondly, in order to determine a suitable divisor we start from an educated guess, the votes-to-seats ratio. In the example it amounts to $29\,355\,092/96 = 305\,782.2$. This initial divisor yields quotients which, when rounded in a standard fashion, sum to a seat total of 94 seats only. Now the first lesson helps. We need to increment the initial apportionment by the ninety-fifth seat (which goes to SPD), and by the last, ninety-sixth seat (GRÜNE, as we have seen already).

Thirdly, the discussion of the cases (a) which party would receive the next seat, and (b) which party did gain the last seat delimits the "divisor interval" $[a; b]$. This interval contains all values that may play the role of a divisor. Different divisors yield different quotients of course. But the variation of the quotients is so minute that the concluding rounding step remains unaffected, for every divisor between a and b. For this reason a divisor is also called a "flexible" electoral key.

TABLE 3 *EP 2014 Election, Czech Republic.* The divisor method with downward rounding is used. Every 50 000 votes justify roughly one seat. That is, a party's "Votes" and the divisor 50 000 yield a "Quotient" that is rounded downwards. The resulting seat apportionment is shown in column "DivDwn". The list position of those elected is given in the column "MEPs' list positions".

EP2014CZ	Votes	Quotient	DivDwn	MEPs' list positions
ANO 2011	244 501	4.9	4	1*, 2, 3, 4
TOP 09+STAN	241 747	4.8	4	2*, 1*, 5*, 3
ČSSD	214 800	4.3	4	1*, 2*, 3, 4
KSČM	166 478	3.3	3	1*, 4*, 2*
KDU-ČSL	150 792	3.02	3	2*, 1*, 3
ODS	116 389	2.3	2	1*, 2*
SVOBODNI	79 540	1.6	1	1*
Sum (Divisor)	1 214 247	(50 000)	21	

We take advantage of the flexibility by always communicating a user-friendly "select divisor". It is obtained from the midpoint $(a + b)/2$ of the divisor interval by reducing it to as few significant digits and as many trailing zeros as the interval permits. In the example the midpoint 298 856.23 is reduced via 298 856 and 298 860 to 298 900. The latter is the select divisor quoted in Table 1.

What does the divisor interval look like in Table 2? The district closest to the next rounding point and prone to receive the next seat is Saarland, whose quotient 0.48 is just below 0.5. The district that gained the last seat turns out to be Nordrhein-Westfalen. Hence the divisor interval has left and right endpoints

$$a = \frac{145\,182}{0.5} = 290\,364 \qquad \text{and} \qquad b = \frac{2\,439\,979}{7.5} = 325\,330.53.$$

Within this interval, the select divisor is 300 000.

1.3 Czech Republic: The Divisor Method with Downward Rounding

The Czech seat contingent comprises 21 seats. Every political party, movement and coalition has a separate ballot sheet. Voters receive a full collection of ballot sheets, of which they take out one. On the ballot sheet chosen, voters may mark up to two preference votes for candidates of the party of their choice.

There are 1 515 492 valid votes. The Czech electoral law imposes an electoral threshold of five percent of the valid votes, that is, 75 775 votes. This means that votes, though valid, are discarded when cast for parties which draw less than 75 775 votes. This cuts out thirty-one parties; the 301 245 votes cast for them turn ineffective. The effective votes remaining are 1 214 247; they are cast for seven parties and coalitions. They enter into the seat apportionment calculations.

The seat apportionment is carried out using the "divisor method with downward rounding", denoted by "DivDwn". The select divisor turns out to be 50 000. Thus the method is captured by the phrase: *Every 50 000 votes justify roughly one seat.* More precisely, a party's vote count is divided by the divisor 50 000. The ensuing "Quotient", in the third column of Table 3, is rounded downwards. "Downward rounding" means that fractional parts are simply neglected.

TABLE 4 *EP 2014 Election, Bulgaria.* The Hare-quota method with residual fit by greatest remainders is used. The Hare-quota is the votes-to-seats ratio, 110 280.05. A party's "Votes" and the quota 110 280.05 yield a "Quotient" that is rounded downwards when its fractional part lies below the split .5, and upwards when above. The resulting seat apportionment is shown in column "HaQgrR".

EP2014BG	Votes	Quotient	HaQgrR	MEPs' list positions
GERB	680 838	6.174	6	1*, 2, 3, 4, 5, 6
BSP	424 037	3.845	4	15*, 1*, 2, 3
DPS	386 725	3.507	4	1*, 3, 4, 5
BWC et al.	238 629	2.164	2	1*, 2*
RB	144 532	1.311	1	2*
Sum (Split)	1 874 761	(.5)	17	

For the assignment of seats to candidates, party lists remain essential, but may be modified by way of preference votes. Candidates whose preference votes amount to at least five percent of their party's total of preference votes, bypass the list ranking and are placed at the top of their party list. We refer to this use of preference votes as the "five percent bypass rule".

In Table 3 the column "MEPs' list position" quotes the list position of the elected candidates. List position are starred (*) to indicate that candidates are moved to the top of the list by the five percent bypass rule. For example the TOP 09+STAN entry "2*, 1*, 5*, 3" says that list positions 2, 1, and 5 move to the top of the list, on the ground of their preference votes, in this order. They receive the first three seats. The fourth seat is allotted to the top list candidate still unseated, on position 3. Aside from occasional transpositions of the list orderings the lasting effects of the five percent bypass rule are meager. For TOP 09+STAN, list position 5 displaces list position 4. For KSČM, list position 4 supersedes list position 3. The five percent bypass rule upsets the assignment of seats to candidates only marginally, in this example.

1.4 Bulgaria: The Hare-Quota Method with Residual Fit by Greatest Remainders

Bulgaria's due number of seats is 17. On the ballot sheet voters mark either a party or an independent candidate. A voter may adjoin a preference vote by ticking a box with a numeral $1, 2, \ldots, 17$, thereby endorsing the nominee who has this rank number on the corresponding party list.

The total number of valid ballot sheets is 2 239 430. There is an electoral threshold, the votes-to-seat ratio $2\,239\,430/17 = 131\,731.2$. The threshold of 131 732 votes applies to parties as well as to independent candidates. Twenty-five parties miss the threshold, their 364 669 votes are dismissed and become ineffective. The effective votes remaining, 1 874 761, are cast for five parties. They participate in the apportionment process.

Interestingly, relative to ballots cast—of which there are 2 361 966—the threshold constitutes $131\,732/2\,361\,966 = 5.6$ percent. This percentage violates the five percent lid decreed in Art. 3 of the European Electoral Act 2002.

The apportionment of seats among parties is carried out by means of the "Hare-quota method with residual fit by greatest remainders", tagged "HaQgrR". The method builds on the votes-to-seats ratio where here the term "votes" means effective votes, $1\,874\,761/17 = 110\,280.05$. In the context of electoral systems the ratio is also known as the "Hare-quota". The Hare-quota method with residual fit by greatest remainders divides a party's vote count by the Hare-quota 110 280.05. The ensuing "Quotient" is shown in the third column of Table 4.

The method evaluates these quotients in two stages, called "main apportionment" and "residual fit". The main apportionment apportions every party as many seats as indicated by the integral part of its quotient. In Table 4 the main apportionment hands out $6 + 3 + 3 + 2 + 1 = 15$ seats. In view of the targeted 17 seats, a residual of two seats is left over. They are taken care of by the residual fit. The quotients' fractional parts are ranked from largest to smallest: .845, .507, .311, .174, and .164. The two parties with the largest remainders get one seat each: The sixteenth seat is awarded to BSP (remainder .845), the seventeenth to DPS (.507). The job is done.

Actually, Table 4 summarizes the job in a way more efficient than the preceding description is suggesting. The key to the end result is *not* the ranking of remainders from largest to smallest. True, the ranking looks innocuous when there are just five players as in Table 4. However, it becomes rather cumbersome for an apportionment between the Member States of the European Union of which there are twenty-eight, or between the States of the USA of which there are fifty. Rather, the key to the final result is the split that separates the remainders that get rounded downwards, as in the main apportionment which hence is persisting, from those that get rounded upwards and thereby add one of the residual seats to the main apportionment. Table 4 quotes a feasible split in the bottom line, .5. Now individual seat numbers can be double-checked simply by relating the pertinent remainder to the publicized split, without repeating the awkward job of ranking all remainders.

For the assignment of seats to candidates Bulgaria practices a fifteen percent bypass rule, akin to the Czech five percent bypass rule. Candidates bypass the ordering of their party lists and are placed at the top when their preference votes exceed fifteen percent of their party's preference vote total. Table 4 marks the bypassers by a star (*). As for DPS, list position 2 is missing because the candidate rejected the seat offered and instantly was substituted by list position 5.

Incidentally, the Hare-quota is named after the English barrister and proportional representation proponent *Thomas Hare* (1806–1891). To be honest Hare used "his" quota to introduce a proportional system of a slightly different type, a single transferable vote (STV) scheme. In the course of time, however, ignorance and negligence linked his name to the apportionment method under discussion. History is not always fair to those who contribute great ideas.

1.5 Belgium: Subdivision into Electoral Districts

Belgium has 21 seats to fill. Domestic provisions allocate the seats to three districts. The Dutch Electoral College is allotted twelve seats, the French Electoral College eight, and the German Language Community one. The German Language Community is allotted a guaranteed seat out of the legislator's intention to protect a recognized minority population. For the allocation of the remaining twenty seats the relevant population $11\,044\,712 - 76\,141 = 10\,968\,571$ is divided by 20. The resulting quotient 548 428.55 is rounded to obtain the "national quota" 548 429. The population figures of the Dutch and French Electoral Colleges are divided by the quota, and the ensuing interim quotients are rounded by largest remainders. See Table 5.

TABLE 5 *EP 2014 Election, Belgium, District Magnitudes.* The 21 Belgian seats are allocated to three districts. The German Language Community is guaranteed one seat due to its minority status. The remaining 20 seats are allocated by referring the population figures to the national quota 548 429. The quota is obtained via $(11\,044\,712 - 76\,141)/20 = 548\,428.55 \to 548\,429$.

EP2014BE-DistrictMagnitudes	2012 Population	Quotient	Seats
Dutch Electoral College	6 484 459	11.824	12
French Electoral College	4 484 112	8.176	8
German Language Community	76 141	–	1
Sum (Split)	11 044 712	(.5)	21

On the ballot sheets voters may mark a party, or one candidate or more from the same party list, or both. When no party is marked the ballot is attributed to the party to which the candidate belongs. When no candidate is marked the ballot is thought to express some support for the party list as is, in a peculiar way to be detailed below. Each of the three districts is evaluated separately. Within a district the apportionment of seats to parties is carried out using the divisor method with downward rounding.

The assignment of seats to candidates follows an intricate directive using the notion of an “eligibility figure”. In Table 6 candidates who reach the eligibility figure are marked by a star (*). The eligibility figure amalgamates a candidate’s preference votes with pure list votes. We demonstrate the details by assigning in the Dutch Electoral College the three OPEN VLD seats to list positions 1*, 2, and 12.

To begin with OPEN VLD’s eligibility figure is determined. The party’s valid votes are divided by its number of seats plus one, the ensuing quotient is rounded upwards. This number is also known as a Droop-quota. Since $859\,099/(3+1) = 214\,774.8$, the eligibility figure is 214 775. Candidates get a seat if they draw 214 775 preference votes or more. This rule grants the first OPEN VLD seat to Guy VERHOFSTADT, on list position 1, with 531 030 preference votes. No other OPEN VLD candidate has sufficiently many preference votes to qualify for the eligibility figure, at this juncture.

Now the law injects a supportive action for the upper echelons, in an involved manner. The underlying rationale claims that a voter who casts a pure party ballot factually approves of the party’s list of nominees in general, and of the top nominees in particular. The rationale is converted into a numerical procedure by decreeing that half of the pure party ballots are reused to support the party’s top nominees, as follows.

OPEN VLD has 261 855 pure list votes, creating $261\,855/2 = 130\,927.5 \to 130\,928$ support votes. Annemie NEYTS, on list position 2, has 79 494 preference votes. If there were plenty of support votes, 430 928 say, then 135 281 of them would lift NEYTS to the eligibility figure ($79\,494 + 135\,281 = 214\,775$) and secure her a seat. The remaining $430\,928 - 135\,281 = 295\,647$ support votes would benefit subsequent list positions 3, 4, etc. Alas, there are only 130 928 support votes. They lift NEYTS’ preference-plus-support votes to $79\,494 + 130\,928 = 210\,422$, but still fail the eligibility figure.

Now the procedure draws to a close. The eligibility figure is ignored, but the amenities afforded by the support votes are preserved. List positions 2–12 are rearranged in decreasing order, of the preference-plus-support votes for NEYTS, and of the bare preference votes for the others. The rearrangement yields the ranking 2, 12, 3, 4, 7, 9, 5, 10, 11, 6, 8. NEYTS stays top. Next is Karel DE GUCHT, from the very last list position 12. With 88 779 preference votes he bypasses list positions 3–11. Now the two remaining seats go to NEYTS and, finally, to DE GUCHT.

TABLE 6 *EP 2014 election, Belgium.* In Belgium the apportionment of seats to parties is executed separately within a district, in each case using the divisor method with downward rounding. The assignment of a party's seats to its candidates distinguishes between those candidates who reach the party's eligibility figure (in italics), and the others.

EP2014BE	Votes	Quotient	DivDwn	MEPs' list positions
District 1: Dutch Electoral College				
N-VA	1 123 355	4.005	4	1*, 2*, 3*, 4
OPEN VLD	859 099	3.1	3	1*, 2, 12
CD&V	840 783	2.997	2	1*, 2*
SP.A	555 348	1.98	1	1*
GROEN	447 391	1.6	1	1*
VLAAMS BELANG	284 856	1.02	1	1*
1 Other	101 237	—	0	
Sum (Divisor)	4 212 069	(280 500)	12	
District 2: French Electoral College				
PS	714 645	3.6	3	1*, 2*, 3
MR	661 332	3.3	3	1*, 2*, 3*
ECOLO	285 196	1.4	1	1*
CDH	277 246	1.4	1	1*
8 Others	501 627	—	0	
Sum (Divisor)	2 440 046	(200 000)	8	
District 3: German Language Community				
CSP	11 710	1.2	1	1*
5 Others	26 886	—	0	
Sum (Divisor)	38 596	(10 000)	1	

1.6 Some General Terminology

While the technical handling of variables does not depend on their interpretation and meaning, we find it nevertheless helpful to adapt our language to the type of prospective application. For the conversion of votes into seats we distinguish three tasks:

the allocation of seats between districts,

the apportionment of seats among parties,

the assignment of seats to candidates.

Apportionment methods are instrumental to resolve the first two tasks, the allocation of seats between electoral districts, and the apportionment of seats among political parties. The third task, the assignment of seats to list candidates, has a slightly different character. Yet it interacts with the apportionment methods that precede it.

A general apportionment method converts large numbers (such as population figures or vote counts that may reach into the millions) into whole numbers much smaller (such as district magnitudes or seat numbers that are limited to a few hundred or less). All methods start in the same way: The large input numbers are scaled down to the smaller output level by means of a division. All methods finish in the same way: The interim quotients thus obtained are rounded to whole numbers. Start and finish, scaling and rounding, are no big deal by themselves.

The true challenge comes only now: The whole numbers that constitute the output seat numbers must sum to a preordained total. In Table 1 the seat numbers must sum to the seat contingent of Germany (96), in Table 2 to the countrywide CDU seats (29), and so on. While maintaining the preordained total we cannot vary just one component. When raising one component we must lower another. Better even, we keep an eye on the apportionment in its entirety.

It is helpful to refer the discussion of apportionment methods to a concrete setting. The seminal monograph *Fair Representation – Meeting the Ideal of One Man, One Vote* of Balinski / Young (2001) revolves around the allocation of the seats of the US House of Representatives between the States of the Union proportionately to population figures. In this module we prefer the other option, the language of a parliamentary election. The seats of a parliament are to be apportioned among political parties proportionately to the parties' vote counts at the end of a popular election.

Envisaging a parliament with h seats the preordained seat total is called "house size", h. Parties are plainly numbered from the first to the last, $1, \ldots, \ell$. Our generic party tag is the letter j. The vote count of party j is denoted by v_j. The unknown quantities we wish to determine are the seat numbers x_j for the parties $j = 1, \ldots, \ell$. They must be whole numbers collectively exhausting the given house size,

$$x_1 + \cdots + x_\ell = h.$$

When applicable, electoral districts are labeled $i = 1, \ldots, k$. The tracking of districts adds another subscript to the notation. In district i party j enters the apportionment process with vote count v_{ij}, and leaves it with seat number x_{ij}.

CHAPTER 2

Intermediate Session: Seat Apportionment Methods

2.1 Divisor Methods

The Introductory Session has acquainted us with two divisor methods that are popular for the purpose of seat apportionment, the divisor method with standard rounding (DivStd) and the divisor method with downward rounding (DivDwn). While they are the most important and most prominent divisor methods, there are plenty of others. This Intermediate Session tells us a bit more about the family of divisor methods, and its like-minded kin, the family of quota methods.

The distinctive characteristic of a divisor method is its underlying rounding rule. As indicated by the name, the divisor method with standard rounding relies on the rounding rule that is standard in business and science: A number is rounded downwards when its fractional part is smaller than one half, and upwards otherwise. For decimal numbers the rule checks the first digit after the decimal point. If the digit is a 0, 1, 2, 3, 4, then the number is rounded downwards. If the digit is a 5, 6, 7, 8, 9, then the number is rounded upwards.

The divisor method with downward rounding uses the rounding rule which always rounds a number downwards, to its integral part. In mathematics this is achieved by the "floor function", in computer science by the "truncation operator". Accordingly, depending on the field of science, notational conventions vary. Downward rounding often uses "floor brackets", such as $\lfloor 5.8 \rfloor = \mathrm{trunc}(5.8) = 5$. Standard rounding is occasionally expressed by "angle brackets", such as $\langle 5.8 \rangle = \mathrm{round}(5.8) = 6$.

We find it convenient to subsume the different rounding rules into a unified symbolism, square brackets. Hence, given a quotient q, the expression $[q]$ is a whole number next to it—either the whole number below q, or the whole number above—obtained by means of the rounding rule that is specified by the context. If standard rounding is the theme, then $[5.8] = \langle 5.8 \rangle = 6$. If downward rounding is the topic of the day, then $[5.8] = \lfloor 5.8 \rfloor = 5$. Or we may contemplate upward rounding, or a strange rounding rule not met yet. This puts us in a position to define divisor methods in a proper and general fashion.

Definition. *The "divisor method with rounding rule* $[\cdot]$*" maps the house size* h *and the vote weights* $v_1, \ldots, v_\ell$ *into the seat numbers that are given by*

$$x_1 = \left[\frac{v_1}{d}\right], \qquad \ldots, \qquad x_\ell = \left[\frac{v_\ell}{d}\right],$$

where the "divisor" $d > 0$ *is determined in such a way that the seat numbers collectively sum to the given house size,* $x_1 + \cdots + x_\ell = h$*.*

Go back to the Introductory Session and verify that the definition describes exactly what is done there, when the rounding rule is specified to be standard rounding (DivStd), or downward rounding (DivDwn)!

To make things simple we are ignoring a complication which is practically irrelevant and theoretically cumbersome: Ties. For instance what happens when all five Bulgarian parties in Table 4 would have the same vote count, 374 321 each, say? They should be treated equally, of course. However, nobody can divide seventeen seats equally between five claimants. Fifteen seats would be okay: three seats each. Or twenty: four each. But seventeen seats cannot but bring forward two lucky winners and leave behind three unlucky losers. A proper handling of ties would force us to introduce a *solution set* $\{x, y, \ldots, z\}$, not just a single solution vector $x = (x_1, \ldots, x_\ell)$. To keep things simple, we neglect ties in the sequel.

2.2 Authorities

We find a systematic nomenclature for apportionment methods instructive and informative, such as divisor method with standard rounding, or Hare-quota method with residual fit by greatest remainders. Though longish the names are indicative of the steps that the apportionment process needs to go through.

Experts often prefer a cryptic jargon by naming a seat apportionment method after an authority who fought for it. Regrettably the scientific community does not agree who deserves the honor most. The divisor method with standard rounding is called "Webster method" in the USA, "Sainte-Laguë method" in most of Europe, and "Sainte-Laguë / Schepers method" specifically in Germany. The divisor method with downward rounding is termed "Jefferson method" in the USA, "D'Hondt method" in most of Europe, and "Hagenbach-Bischoff method" specifically in Switzerland.

There are three further divisor methods which, together with the two just mentioned, form the elite club of "the five traditional divisor methods". The "divisor method with upward rounding" (DivUpw) uses the rounding rule where all numbers are rounded upwards to the next integer. The "divisor method with geometric rounding" (DivGeo) is induced by the rounding rule that splits an integer interval $[n-1; n]$ at the geometric mean $\sqrt{(n-1)n}$ into a lower class where numbers are rounded downwards to $n-1$, and into an upper class where they are rounded upwards to n. The "divisor method with harmonic rounding" (DivHar) refers the split point to the harmonic mean, $\big(\{(n-1)^{-1} + n^{-1}\}/2\big)^{-1}$.

Here is a list of celebrities associated with the five traditional divisor methods:

DivStd Daniel Webster (1782–1852), US statesman, Senator from Massachusetts, US Secretary of State

Jean-André Sainte-Laguë (1882–1950), Professor of Mathematics, Conservatoire national des arts et métiers, Paris

Hans Schepers (b. 1928), Physicist, Head of the Data Processing Unit, Scientific Services of the German Bundestag

DivDwn Thomas Jefferson (1743–1826), principal author of the US Declaration of Independence, third US President 1801–1809

Victor D'Hondt (1841–1901), Professor of Law, Ghent University, co-founder of the Belgian L'Association réformiste pour l'adoption de la Représentation Proportionnelle 1881

Eduard Hagenbach-Bischoff (1833–1910), Professor of Physics, University of Basel, and cantonal politician

DivUpw John Quincy Adams (1767–1848), US diplomat and statesman, sixth US President 1825–1829

DivGeo Joseph Adna Hill (1860–1938), Statistician, Assistant Director of the Census, US Bureau of the Census

Edward Vermilye Huntington (1874–1952), Professor of Mathematics, Harvard University, Cambridge, Massachusetts

DivHar James Dean (1776–1849), Professor of Astronomy and Mathematics, University of Vermont, Burlington, Vermont

2.3 Organizing Principles

A divisor method maps its input quantities, house size and vote weights, into an output result, the seat numbers $x_1, \ldots, x_\ell$. Viewed in this way a divisor method is a function. In order to emphasize the functional viewpoint, we denote a divisor method by the letter D and visibly exhibit the input variables h and $v_1, \ldots, v_\ell$,

$$D(h; v_1, \ldots, v_\ell) = (x_1, \ldots, x_\ell).$$

The dependence between input and output should be in line with the meanings that the variables acquire for the intended application of apportioning parliamentary seats among parties. Here are five properties which probably you and everybody else consider indispensable: Every divisor method is anonymous, balanced, concordant, decent, and exact. We discuss the five principles in alphabetical order, one after the other.

Anonymity. A divisor method D is "anonymous". That is, every rearrangement of the vote weights induces the same rearrangement of the seat numbers. This is obvious from the definition of divisor methods.

Whether a party is listed first or last has no effect on its seat number. The Introductory Session makes use of anonymity in that parties are ranked by decreasing vote counts. However, districts usually follow a fixed institutional ordering. Whichever order applies, the resulting seat numbers are carried along and stay the same.

Balancedness. A divisor method D is "balanced". That is, the seat numbers of equally strong parties differ by at most one seat. Formally, the relation $D(h; v_1, \dots, v_\ell) = (x_1, \dots, x_\ell)$ guarantees

$$v_j = v_k \qquad \Longrightarrow \qquad \big|x_j - x_k\big| \le 1, \qquad \text{for all parties} \quad j, k = 1, \dots, \ell.$$

Balancedness is of concern only when admitting ties. However, we have promised to neglect ties, whence balancedness is dispensable, in this module.

Concordance. A divisor method D is "concordant". That is, of two parties the stronger party gets at least as many seats as the weaker party. Formally, the relation $D(h; v_1, \dots, v_\ell) = (x_1, \dots, x_\ell)$ entails

$$v_j > v_k \qquad \Longrightarrow \qquad x_j \ge x_k, \qquad \text{for all parties} \quad j, k = 1, \dots, \ell.$$

Concordance is easy to check visually provided parties are listed by decreasing vote counts. Then the corresponding seat numbers must be non-increasing, too. When vote counts are ordered otherwise, as with districts, a visual check of concordance is unreliable and needs to be double-checked by a computer program.

Decency. A divisor method D is "decent", or in mathematical parlance, "positively homogeneous of degree zero". That is, its result for vote weights $v_1, \dots, v_\ell$ is identical to the result for the scaled vote weights $v_1/c, \dots, v_\ell/c$,

$$D(h; v_1, \dots, v_\ell) = D(h; \tfrac{1}{c}v_1, \dots, \tfrac{1}{c}v_\ell), \qquad \text{for all constants} \quad c > 0.$$

Indeed, in the definition of divisor methods we only have to scale the divisor d into d/c. Since the constant c cancels out, $(v_j/c)/(d/c) = v_j/d$, the result remains the same.

As an application we divide the vote counts v_j by the vote total $v_+ = v_1 + \dots + v_\ell$. This introduces the "vote shares" $w_j = v_j/v_+$. Because of decency the resulting seat numbers stay the same. Hence it does not matter whether we start from the raw vote counts v_j, or from the induced vote shares w_j. Therefore we prefer the neutral term "vote weight", leaving it open whether the weights are vote counts, or vote shares.

Exactness. A divisor method D is "exact". That is, every sequence of vote vectors that converges to a seat vector x whose zeros are also zeros of the vote vectors, is mapped to sequences of solutions which converges to the same limit x. Formally, all vote vectors $v(k)$ and all solutions $y(k) = D\big(h; v(k)\big)$, $k \ge 1$, satisfy

$$\lim_{k\to\infty} v(k) := x \ \text{ and } \ \Big(x_j = 0 \Rightarrow v_j(k) = 0 \text{ for all } j \le \ell, k \ge 1\Big) \quad \Longrightarrow \quad \lim_{k\to\infty} y(k) = x. \tag{1}$$

The proof that divisor methods are exact is a bit involved, and hence omitted.

The notion of exactness simplifies in two situations in which the zeros of x are of no concern. Firstly, if all components of the limiting seat vector x are positive, as practically happens more often than not, then (1) simply says that all sequences of vote vectors which tend to x produce sequences of seat vectors which also tend to x.

Secondly, if the sequence of vote vectors is constant, $v(k) = x$ for all $k \geq 1$, then (1) says that every seat vector x, when construed as an input vote vector, is reproduced as the unique solution,

$$D(h; x) = x. \tag{2}$$

Indeed, there are only finitely many seat vectors. Hence the sequence $y(k)$ in (1) converges if and only if it is eventually constant to some seat vector y. Then (1) entails the implication $y \in D(h; x) \Rightarrow y = x$, that is, (2). The message of (2) is pleasing. The result of divisor method D cannot be changed, let alone be improved, by repeatedly applying D to its solutions.

The stronger property (1) merges the discrete nature of the grid of seat vectors and the continuum character of the quadrant of vote vectors in a more sophisticated manner than (2).

2.4 Max-Min Inequality

While it is pleasing to condense a divisor method to the line $D(h; v_1, \ldots, v_\ell) = (x_1, \ldots, x_\ell)$, the functional notation does not tell the full story. It hides the underlying rounding rule $[\cdot]$, and it misses out on the divisor d that is an essential ingredient in the definition of divisor methods (Section 2.1). In the present section we transform the functional representation into the important "Max-Min Inequality". In the next section we use the Max-Min Inequality to determine a feasible divisor d.

A rounding rule is determined by the value $s(n)$ in the interval $[n-1; n]$ where the rounding results advance from $n-1$ to n, for $n \geq 1$. That is, $s(n)$ is the decision point where the rule changes its mode from downward rounding to upward rounding. Following Balinski / Young (2001 [62]) the change point $s(n)$ is called "signpost". This notion nicely conforms with *Sprungstelle* in German, and *seuil* in French. We refer to $s(n)$ as the nth signpost. Thus the first signpost $s(1)$ lies in the first interval $[0; 1]$, the second signpost $s(2)$ in the second interval $[1; 2]$, etc. It transpires to be convenient to also introduce the "zeroth signpost" $s(0) = 0$.

Here are some examples. Standard rounding, downward rounding, and upward rounding have signposts $s(0), s(1), s(2), s(3)$ etc. as follows:

Standard rounding:	0,	0.5,	1.5,	2.5,	etc.
Downward rounding:	0,	1,	2,	3,	etc.
Upward rounding:	0,	0,	1,	2,	etc.

Now we turn to our subject proper, divisor methods. A quotient $q = v_j/d$ is rounded to the seat number x_j if and only (a) if q lies in the interval $[s(x_j); x_j]$ where it is rounded upwards, or (b) if q lies in the interval" $[x_j; s(x_j+1)]$ where it is rounded downwards. With reference to signposts the position of q requires $q \geq s(x_j)$ in the first interval. In the second interval we must have $q \leq s(x_j+1)$. Hence the state of affairs is expressed by the "fundamental relation":

$$\left[\frac{v_j}{d}\right] = x_j \quad \Longleftrightarrow \quad s(x_j) \leq \frac{v_j}{d} \text{ and } \frac{v_j}{d} \leq s(x_j+1) \quad \Longleftrightarrow \quad \frac{v_j}{s(x_j+1)} \leq d \leq \frac{v_j}{s(x_j)}.$$

That is, a quotient is rounded to the seat number x_j if and only if it lies between the signposts $s(x_j)$ and $s(x_j+1)$. The Max-Min Inequality now follows immediately.

Max-Min Inequality. *Suppose $x_1, \dots, x_\ell$ are whole numbers summing to the house size h. Then the seat vector $x = (x_1, \dots, x_\ell)$ is a solution,*

$$D(h; v_1, \dots, v_\ell) = x$$

if and only if the "Max-Min Inequality" holds true,

$$\max_{j \le \ell} \frac{v_j}{s(x_j+1)} \le \min_{j \le \ell} \frac{v_j}{s(x_j)}.$$

Proof. The fundamental relation yields $v_j/s(x_j+1) \le d$ and $d \le v_j/s(x_j)$. The direct part of the proof follows from $a := \max_{j \le \ell} v_j/s(x_j+1) \le d \le \min_{j \le \ell} v_j/s(x_j) =: b$. As for the converse part, the max-min inequality renders the interval $[a; b]$ nonempty. Every number d in the interval $[a; b]$ furnishes a feasible divisor to verify the desired functional relationship. $\square$

The Max-Min Inequality is at the heart of divisor methods. It identifies the "divisor interval" $[a; b]$ which contains all divisors d viable to verify the solution, $[v_j/d] = x_j$. The width of the interval allows us to pick a user-friendly "select divisor". We opt for reducing the midpoint $(a+b)/2$ to as few significant digits as is possible without leaving the interior of the interval (Section 1.2).

The Max-Min Inequality examines quotients of votes and signposts, $v_j/s(n)$. Specifically, the divisor method with downward rounding has signposts $s(n) = n$. Some authors (mis-)interpret these quotients as averages, of v_j votes per n seats, and refer to the maximum as the "highest average". Generalizing, they call any divisor method a "highest average method". The wording is ill-conceived and a conceptual hindrance. The figures to be compared are quotients of votes and signposts, not of votes and seats.

2.5 Jump-and-Step Procedure

There is no one-step formula to compute a feasible divisor d and the seat numbers x_j. We must go on a multi-step trip. The length of the trip depends on where we start. A starting point close to the goal promises a short trip. With a starting point further away the trip becomes longer. Different starting points do not entail different apportionment methods, the final result is one and the same. Nor is somebody who starts out far away and progresses laboriously to the finish more serious about the problem than somebody else who uses a clever initialization and finishes sooner.

Ideal proportionality would equate the share of seats of party j to its share of votes, $x_j/h = v_j/v_+$. We would get $x_j = v_j/(v_+/h)$. Hence a promising initial divisor is the votes-to-seats ratio, v_+/h. With it, we jump to the initial seat numbers $y_j := [(v_j/v_+)\, h]$. The initial seat vector $y = (y_1, \dots, y_\ell)$ obeys one of the cases a–c:

a. The component sum of y exhausts the house size, $y_+ = h$. In this case the initial seat vector is a solution, $x = y$. The job is done.

b. The component sum of y stays below the house size, $y_+ < h$. In this case a further seat is given to some party i for which $v_i/s(y_i+1) = \max_{j \le \ell} v_j/s(y_j+1)$. Seatwise incrementation continues until the eventual seat vector x satisfies $x_+ = h$.

c. The component sum of y exceeds the house size, $y_+ > h$. In this case a seat is retracted from some party k for which $v_k/s(y_k) = \min_{j \le \ell} v_j/s(y_j)$. Seatwise decrementation continues until the eventual seat vector x satisfies $x_+ = h$.

We conclude the trip by determining the select divisor d, as described above. The solution is captured by the phrase: *Every d votes justify roughly one seat.* The formulation "roughly one seat" emphasizes that the interim quotients v_j/d are subject to an inevitable rounding step.

Nowadays we use computers to crunch numbers. Write a little code for your machine to apportion the seats in one (or all) of the instances in the Introductory Session! Then go to the internet site `www.uni-augsburg.de/bazi` and download the program "BAZI – Calculation of Allocations by Apportionment Methods in the Internet". BAZI asks you for the input and responds with the output. Compare your results with those of BAZI. BAZI also includes a large data base with a lot of other examples.

2.6 Seat Biases

Having mastered the concept of a divisor method and the mechanics how to carry it out, this section turns to one of its most important properties: seat biases. Legislators do not like to reform electoral laws all too often and amend the apportionment method. Once a method is wired into a law, it is doomed to stay there for quite a while. What can we say when it is used repeatedly over many elections? From the political viewpoint we can ask a more precise question. Does the method favor stronger parties at the expense of weaker parties, consistently and predictably? Or perhaps the other way round: Would it favor weaker parties at the expense of stronger parties?

From the viewpoint of electoral systems the notion of "party strength" does not allude to the financial strength of a party, or the charisma of its leaders. In this module party strength means voter support as substantiated by vote counts v_j or vote shares $w_j = v_j/v_+$. The Introductory Session shows that some countries grant seats only to parties whose vote share exceeds an electoral threshold. For example in the Czech Republic a party's vote share must exceed five percent of the valid votes (Section 1.3). In our exposition we designate a general electoral threshold by the letter t. Moreover we rank-order parties from strongest to weakest. Hence a typical situation has vote shares

$$w_1, \geq \cdots \geq w_\ell \geq t.$$

As a visual reminder for switching from vote counts v_j to vote shares w_j, we substitute the letter k for the otherwise favored party tag j. The first party, $k = 1$, is the strongest party. The second party, $k = 2$, is second-strongest, and so on until the weakest party which comes last, $k = \ell$. Note that a party's rank-score may vary from one election to the next, as repeated elections may feature distinct parties finishing strongest.

The notion of a "seat bias of the k-strongest party" aims at the difference between the seats actually apportioned, x_k, and the ideal share of seats, $w_k h$, which a party with vote share w_k would claim if seats were divisible quantities that could be handed out in continuous fractions. Practically, in a single instance, the "seat excess" $x_k - w_k h$ is nonzero simply because the seat number x_k is a whole number and the ideal share $w_k h$ is not. However, the term "bias" signifies the *average* of all conceivable values of the seat excess. Thus, in the long run, a positive bias predicts that the party can expect more seats than warranted by ideal proportionality. If the bias is negative, then the party has to make do with fewer seats than strict proportionality would promise.

Seat bias formulas vary with the apportionment methods under investigation. To obtain a single formula that applies to many divisor methods, we embed the divisor methods with standard rounding, downward rounding, and upward rounding into the family of "stationary divisor methods". They are induced by "stationary rounding rules". A stationary rounding rule depends on a parameter r between zero and one, called "split" parameter. In the interval $[n-1; n]$ the split parameter defines the signpost $s_r(n) := n - 1 + r$. To its left we round downwards, to its right, upwards. The family of stationary divisor methods begins with the divisor method with upward rounding ($r = 0$), passes through the divisor method with standard rounding ($r = 1/2$), and ends with the divisor method with downward rounding ($r = 1$). For this family the seat bias formula sends a clear and informative message.

Seat Bias Formula. *For a stationary divisor method with split r and for a given threshold t, the seat bias of the kth-strongest party is given by the formula*

$$\left(r - \frac{1}{2}\right)\left(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{\ell - 1} + \frac{1}{\ell} - 1\right)\left(1 - \ell t\right).$$

Proof. The formula results from a transition to large house sizes, $h \to \infty$. The seat excesses $x_k - w_k h$ are averaged under the assumption that all conceivable vote shares $w_1, \ldots, w_\ell$ are equally likely. The detailed arguments are too advanced to fit into this module. If you want to know, you find the details in Chapter 7 of Pukelsheim (2014). □

All good things come in threes, and so do the terms of the Seat Bias Formula. The first "method factor" $(r - 1/2)$ measures the influence of the stationary divisor method with split r. The second "rank-order factor" $(1/k + \cdots + 1/\ell - 1)$ reflects the contribution of the party's ranking as the kth strongest in a party system of size ℓ. The third "threshold factor" $(1 - \ell t)$ mirrors the effect of the electoral threshold t.

The seat bias formula does *not* involve the house size h. The formula is established assuming huge house sizes ($h \to \infty$). Yet the formula yields true predictions also for small house sizes. Extensive empirical studies confirm the validity of the formula whenever the house size meets or exceeds twice the number of participating parties, $h \geq 2\ell$. In other words, the seat bias formula is applicable for all practical purposes.

Most notably the divisor method with standard rounding ($r = 1/2$) is seen to be *unbiased*. Since the method factor vanishes, all of its seat biases are zero. Every other stationary divisor method is biased. Specifically, the divisor method with downward rounding ($r = 1$) is biased. It favors stronger parties at the expense of weaker parties.

The first scholar who investigated seat biases in a rigorous manner was George Pólya (1918). For the divisor method with downward rounding, in a three-party system with no threshold, he found the seat biases of the strongest party, of the medium party, and of the weakest party to be

$$\frac{5}{12}, \qquad -\frac{1}{12}, \qquad -\frac{4}{12}$$

Thus, averaged over twelve elections, the strongest party can expect a bonus of five seats beyond its ideal shares. The medium party loses one seat, the weakest party four seats. Use the Seat Bias Formula to verify Pólya's result!

2.7 Vote Shares for Given Seat Numbers

The functional relation $D(h; w_1, \dots, w_\ell) = (x_1, \dots, x_\ell)$ is usually read from left to right. The divisor method D maps the house size h and the vote shares $w_1, \dots, w_\ell$ into the seat numbers $x_1, \dots, x_\ell$. However, the reverse direction is also of interest. Given some seat number x_j, which range of variation of the vote shares w_j is feasible? This question is asked in the present section.

Whether a given seat number x_j is small or large can be judged only against the seat total, h. Therefore our precise question is this: Given a house size h and a seat number x_j, and assuming that the other vote shares, w_i for $i \neq j$, are free to vary, what is the range of vote shares w_j that may result in x_j seats?

If a vote share w_j is too small, it entails fewer than x_j seats. If it is too large, it produces more than x_j seats. Hence the range sought forms an interval $[a(x_j); b(x_j)]$. The interval extends from the "lowest vote share for x_j seats", $a(x_j)$, to the "highest vote share for x_j seats", $b(x_j)$. For stationary divisor methods the lowest and highest vote shares for x_j seats attain a succinct form.

Vote Shares for Given Seat Numbers. *For a stationary divisor method with split r, the lowest and highest vote shares for x_j seats are given by the formulas*

$$a(x_j) = \frac{x_j - (1-r)I}{h - (1-r)I + r(\ell - 1)}, \qquad b(x_j) = \frac{x_j + r}{h + r - (1-r)M},$$

where $I = 1$ when $x_j \geq 1$ and $I = 0$ when $x_j = 0$, and where $M = \min\{\ell - 1, h - x_j\}$.

Proof. For stationary divisor methods the range of variation of the seat excess $x_j - w_j h$ is found to be $-(1 - w_j)r - w_j(1 - r)M \leq x_j - w_j h \leq (1 - w_j)(1 - r)I + w_j r(\ell - 1)$. By solving for w_j we obtain $a(x_j) \leq w_j \leq b(x_j)$. For details see Chapter 11 in Pukelsheim (2014). □

Both values are almost equal to x_j/h. This is no surprise, since for the apportionment of a seat share x_j/h a proportional representation system should require almost the same vote share. The precise formulas have numerators and denominators shifted by the split parameter and by the number of competitors. The formulas cover the boundary cases when party j is unseated, $x_j = 0$, or when one of the competitors is unseated, $M = h - x_j < \ell - 1$. For the non-boundary cases, when $I = 1$ and $M = \ell - 1$, the structure of the two formulas looks rather similar. Check!

The formulas have practical consequences. Two applications are of particular interest. The first relates to constitutional law. The highest vote share for no seat, $b(0)$, indicates the threshold above which a party is certain to get at least one seat. In other words, representation in parliament is guaranteed. The value $b(0)$ is called the method's "natural threshold", or "quorum". It provides orientation for courts when contemplating whether an electoral system has a house size h so small that weak parties are unduly excluded. The formula of the natural threshold for the divisor method with standard rounding (DivStd) is $0.5/(h + 1 - 0.5\ell)$. For the divisor method with downward rounding (DivDwn) it is $1/(h + 1)$.

The second application relates to political practice. An apportionment method is called "majority preserving" when a party with an absolute majority of votes is guaranteed an absolute majority of seats. It is a sobering fact that none of the prevalent apportionment methods is majority preserving.

The sole result in this direction is the following. If the house size is odd, then the divisor method with downward rounding is majority preserving, and it is the only stationary divisor method with this property.

Indeed, for an odd house size, $h = 2n+1$, an absolute seat majority calls for more than n seats. The highest vote share for n seats is $b(n)$. Hence the assertion finds its formal expression through the inequality $b(n) \leq 1/2$. In fact, if a method is majority preserving, then the highest vote share for missing an absolute seat majority cannot possibly be larger than one half, meaning $b(n) \leq 1/2$. Conversely, if $b(n) \leq 1/2$, then an absolute majority of votes, $w_j > 1/2$, entails $b(n) < w_j$, whence the party gets $n+1$ seats or more. For stationary divisor methods with split r the highest vote share for n seats is given by

$$b(n) = \frac{n+r}{2(n+r) - (\ell-2)(1-r)}.$$

Hence the inequality $b(n) \leq 1/2$ holds true if and only if $\ell = 2$ or $r = 1$. For party systems of arbitrary size, $\ell \geq 2$, we are left with split $r = 1$, that is, with the divisor method with downward rounding.

Electoral laws that want to make sure that their provisions are majority preserving need to include a "majority clause". The clause takes effect only in the (rare) instance when the actual seat apportionment fails to be majority preserving. One option is the "house size augmentation clause". It creates extra seats for the majority party until a straight majority of seats is reached.

Another option is the "majority-minority partition clause". In an instance when majority preservation has failed, the clause starts afresh. It handles the seat apportionment for the majority party separately from the seat apportionment for the minority parties. The majority party is allotted the smallest possible absolute majority of seats, $\lfloor h/2 \rfloor + 1$. The remaining seats are apportioned among the remaining parties using the apportionment method decreed by the applicable law.

2.8 Quota Methods

An apportionment method manages its way from large vote counts to small seat numbers by starting with a scaling step and finishing with a rounding step (Section 1.6). These steps require some scaling constant and some rounding rule. Divisor methods employ a flexible scaling constant—called divisor—and a fixed rounding rule. There is a rival family of apportionment methods called quota methods. Every quota method uses a fixed scaling constant—called quota—and a flexible rounding rule.

The juxtaposition of divisor methods versus quota methods suggests a kind of symmetric role of the two families. As a matter of fact the opposite is true. Divisor methods outperform quota methods in almost every respect. For this reason the module devotes to divisor methods much space, and to quota methods little.

The most prominent quota method is the Hare-quota method with residual fit by greatest remainders. Let us recall its operations (Section 1.4). First we calculate the

Hare-quota, $Q = v_+/h$, the votes-to-seats ratio. Next party j's vote count v_j is divided by the quota. The integral part of the quotient v_j/Q signifies the number of seats y_j allotted to party j in the main apportionment, $y_j := \lfloor v_j/Q \rfloor$. Altogether the main apportionment hands out y_+ seats. This leaves $h - y_+$ residual seats to be attended to in the residual fit. Specifically, the residual fit "by greatest remainders" gives the residual seats, one by one, to the parties featuring the greatest remainders $v_j/Q - y_j$.

Using the language of rounding rules, the residual fit may be described in another way. It determines a split r^* separating the parties that receive one of the residual seats, from the parties that do not. Hence the applied rounding rule is a stationary rule, with split parameter r^*. However, the split r^* is data-dependent, which is why we flag it with an asterisk (*). It depends on the vote counts $v_1, \ldots, v_\ell$. A different set of vote counts generally entails a different split parameter. The rounding rule varies from one instance to another and is flexible, in this sense.

For divisor methods, the one-parameter family of stationary divisor methods helps to unify the view and to relieve the notation (Section 2.6). For quota methods, the introduction of a one-parameter family is similarly helpful. To this end we introduce the "shifted quota" $Q(s) = v_+/(h+s)$. Restricting the shift parameter s to the half-open interval $[-1; 1)$, the main apportionment leaves at most ℓ seats for the residual fit. The proof is straightforward. Thus the main apportionment may be combined with a residual fit by greatest remainders.

The result is the family of "shifted quota methods with residual fit by greatest remainders", with shift parameter s. For these methods the seat bias of the kth-strongest party turns out to be

$$\frac{s}{\ell}\left(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{\ell-1} + \frac{1}{\ell} - 1\right)(1 - \ell t).$$

The lowest and highest vote shares for x_j seats are given by the formulas

$$a(x_j) = \frac{x_j - 1 + (1+s)/\ell}{h+s}, \qquad b(x_j) = \frac{x_j + 1 - (1-s)/\ell}{h+s}.$$

In particular we see (setting $s = 0$) that the Hare-quota method with residual fit by greatest remainders is unbiased, and has natural threshold $b(0) = (1 - 1/\ell)/h$.

Quota methods occasionally exhibit inconsistencies that some experts refer to as paradoxes. But philosophical depth is out of place, the strange behavior is elementary. It can happen that, when the house size grows, a party loses a seat ("house size paradox"). It can happen that, when a party grows faster than another party, the fast-growing party loses a seat to the slow-growing party ("vote ratio paradox"). It can happen that, when a weak party joins the field but remains unseated, successful parties are forced to swap a seat ("new party paradox"). Irritating instances of this sort are so rare that other experts shrug it off as irrelevant. What is really harming quota methods is the fact that divisor methods are immune against these "paradoxes".

The true weakness of quota methods is that they are too rigid to adapt to the advanced needs imposed by practical electoral systems. An electoral system may decree minimum requirements, or maximum restrictions. Because of the quota's rigidity, quota methods have difficulties to incorporate such side conditions in a natural way. For the study of some more advanced electoral systems, in the last session, we therefore return to divisor methods.

CHAPTER 3

Advanced Session: Proportionality and Personalization

3.1 Proportional Representation and the Election of Persons

Many electoral systems aim at more goals than just proportionality between votes and seats. The intention is to reinforce the bond between those electing, the voters, and those elected, the representatives. For example a flexible list system with preference votes allows voters to support a particular candidate (Sections 1.3–1.5: Czech Republic, Bulgaria, Belgium). Or a subdivision into districts makes sure that voters and representatives share the same provenance (Section 1.5: Belgium).

Whether constitutionally mandated or just politically desirable, further goals add further complications to the system. Here we discuss two approaches that reach beyond pure proportionality. More demanding as they are they call for modified variants of the prevalent apportionment methods.

The first approach imposes minimum restrictions or maximum restrictions for the seat numbers. That is, the number of seats apportioned to a party (or allocated to a district) ranges between a fixed minimum and maximum, and not simply between no seat and the seat total. The second approach, double proportionality, is tailored to situations when the whole electoral region is subdivided into districts. It maintains proportionality between districts according to their population figures, as well as proportionality between parties according to their vote counts.

3.2 Minimum–maximum Restricted Variants of Divisor Methods

Divisor methods are readily modified to obey restrictions stipulating minimum or maximum seat numbers. Suppose party j is to receive a minimum of a_j seats and a maximum of b_j seats. Generally, such restrictions must be "compatible". First, they must leave space for some feasible seat numbers in-between, $a_j \leq b_j$. Second, the aggregated minima can bind at most all seats, $a_+ \leq h$. The aggregated maxima must allow to apportion all seats, $h \leq b_+$. The minimum–maximum restrictions that are to be considered in the sequel are always compatible.

The rounding rule $[\cdot]$ of the divisor method under investigation is modified so that it yields values not falling below the minimum restrictions a_j, nor raising above the maximum restrictions b_j. That is, a quotient q is rounded to

$$[q]_{a_j}^{b_j} := \begin{cases} \{b_j\} & \text{in case } q > b_j, \\ [q] & \text{in case } q \in \left[a_j; b_j\right], \\ \{a_j\} & \text{in case } q < a_j. \end{cases}$$

The modified rounding rule induces the “minimum–maximum restricted variant” of the given divisor method, with seat numbers

$$x_1 = \left[\frac{v_1}{d}\right]_{a_1}^{b_1}, \qquad \ldots \qquad , x_\ell = \left[\frac{v_\ell}{d}\right]_{a_\ell}^{b_\ell}.$$

The divisor d is determined so as to exhaust the preordained house size, $x_1+\cdots+x_\ell = h$. The solution phrase must be expanded to acknowledge the presence of restrictions: *Every d votes justify roughly one seat, except when a minimum restriction warrants more seats or a maximum restriction imposes fewer seats.*

For a succinct notation we assemble the minimum restrictions into the vector $a = (a_1, \ldots, a_\ell)$, and the maximum restrictions into the vector $b = (b_1, \ldots, b_\ell)$. Then the minimum–maximum restricted variant of the given divisor method is expressed through the functional relationship $D_a^b(h; v_1, \ldots, v_\ell) = (x_1, \ldots, x_\ell)$.

3.3 Unproportionality Index

What is the cost of incorporating restrictions? Which measure is appropriate to assess the amount by which the two seat vectors differ, the restricted proportionality solution $D_a^b(h; v_1, \ldots, v_\ell) = (x_1, \ldots, x_\ell)$, and the pure (that is, unrestricted) proportionality solution $D(h; v_1, \ldots, v_\ell) = (z_1, \ldots, z_\ell)$?

The German Federal Constitutional Court focuses on the concept of *seat relevance.* The constitutional appraisal of methodological differences focuses on those seats that are apportioned differently. This view motivates the definition of the “unproportionality index” $u(x)$ of the seat vector $x = (x_1, \ldots, x_\ell)$ through

$$u(x) = \frac{1}{2}\Big(\left|x_1 - z_1\right| + \cdots + \left|x_\ell - z_\ell\right|\Big).$$

That is, $u(x)$ is half the sum of the absolute values of the differences of the seat numbers x_j and z_j. The factor one half is included because every transfer of a seat counts twice, once for the party that receives the transfer seat and once for the party that has to give it up. As an example consider a restricted solution $x = (8, 4, 2, 1, 1, 0)$ and an unrestricted solution $z = (7, 4, 2, 1, 1, 1)$. The difference $x - z = (1, 0, 0, 0, 0, -1)$ tells us that the two vectors differ by transferring a seat between the first and the last parties. Accordingly the unproportionality index of x is found to be one seat, $u(x) = 1$.

3.4 Composition of the European Parliament

The intricacies of designing a practical apportionment method are illustrated with the composition of the European Parliament. "Composition" is the parliamentary term for the allocation of Parliament's 751 seats between the Union's twenty-eight Member States. In the past the composition resulted from negotiations and bartering. For future decisions the Parliament contemplates the adoption of a formula-based procedure. The procedure should be durable, transparent, and impartial to politics.

The principles that govern Parliament's composition are spelled out in the Union's primary law, the Treaty of Lisbon. Article 14(2) decrees that there are at most 751 seats, and that every Member State is allocated at least six seats and at most ninety-six.

There is yet another principle. It demands that the *representation of citizens shall be degressively proportional*. This principle is more a declaration of intent than a manageable restriction. The notion of "degressive proportionality" is a neologism of opaque content. Taken literally it is a contradiction in terms. There is no degressive proportionality nor progressive proportionality. There is but degressive representation, proportional representation, and progressive representation. The distinction parallels the notions of degressive taxation, proportional taxation, and progressive taxation.

Degressive proportionality, when interpreted in the sense of degressive representation, demands that more populous Member States are allocated relatively fewer seats than less populous Member States. In the words of this module, degressive proportionality imposes seat biases favoring less populous Member States at the expense of more populous Member States. The seat bias formula reveals which apportionment method to look for (Section 2.6). The desired biases materialize with the divisor method with upward rounding (DivUpw). In particular, even a tiny state gets at least one seat.

Parliament's composition must be in line with the Union's constitutional construction. Countries with a federal system often install two chambers, a parliament for the representation of the people, and a second chamber for the representation of the federal states. In Germany the parliament is the Bundestag, the second chamber is the Bundesrat. In Italy the parliament is the Camera, the second chamber is the Senate. However, the Treaty of Lisbon establishes just a single chamber, the European Parliament. Yet the Treaty acknowledges two types of constitutional subjects, the Member States and the European citizens. The challenge is to merge the representation of the two constitutional subjects in a single chamber, the European Parliament.

Grimmett and coauthors (2011) propose an apportionment method tailored to the present needs, the "Cambridge Compromise". Every Member State receives five "base seats". This initial step honors the principle of equality among states that often underlies the establishment of second chambers, *one state, one vote*. The remaining $751 - 140 = 611$ seats are apportioned between the Member States proportionately to their population figures. This follow-up step obeys the motto *one person, one vote*. In view of degressive proportionality the divisor method with upward rounding is used. Thus every Member State is allocated at least six seats: five base seats plus at least one seat that is owed to the divisor method with upward rounding. Finally a maximum restriction of 96 seats is imposed. Regrettably, the Cambridge Compromise was not well received. It deviated too much from what the Members of the European Parliament considered their vested rights, the status quo. Some Member States would have lost three or four seats, which deemed totally unacceptable to those concerned.

TABLE 7 *Limited loss variant of the Cambridge Compromise.* Every Member State gets five base seats. The remaining $751 - 140 = 611$ seats are allocated with the minimum–maximum restricted variant of the divisor method with upward rounding, thereby securing a sixth seat for every Member State. The minimum restrictions guarantee that Member States give up at most two seats of their status quo “Seats 2014”. The final seat numbers are listed in column “5+DivUpw•”.

EP2014Composition	Seats 2014	Population 2013	5+Min..Max	5+Quotient	5+DivUpw•
DE Germany	96	80 523 700	5+89..91	5+94.3•	96
FR France	74	65 633 200	5+67..91	5+76.9	82
UK United Kingdom	73	63 730 100	5+66..91	5+74.6	80
IT Italy	73	59 685 200	5+66..91	5+69.9	75
ES Spain	54	46 704 300	5+47..91	5+54.7	60
PL Poland	51	38 533 300	5+44..91	5+45.1	51
RO Romania	32	20 057 500	5+25..91	5+23.5•	30
NL Netherlands	26	16 779 600	5+19..91	5+19.6	25
BE Belgium	21	11 161 600	5+14..91	5+13.1	19
EL Greece	21	11 062 500	5+14..91	5+12.95•	19
CZ Czech Republic	21	10 516 100	5+14..91	5+12.3•	19
PT Portugal	21	10 487 300	5+14..91	5+12.3•	19
HU Hungary	21	9 908 800	5+14..91	5+11.6•	19
SE Sweden	20	9 555 900	5+13..91	5+11.2•	18
AT Austria	18	8 451 900	5+11..91	5+9.9•	16
BG Bulgaria	17	7 284 600	5+10..91	5+8.5•	15
DK Denmark	13	5 602 600	5+6..91	5+6.6	12
FI Finland	13	5 426 700	5+6..91	5+6.4	12
SK Slovakia	13	5 410 800	5+6..91	5+6.3	12
IE Ireland	11	4 591 100	5+4..91	5+5.4	11
HR Croatia	11	4 262 100	5+4..91	5+4.99	10
LT Lithuania	11	2 971 900	5+4..91	5+3.5	9
SI Slovenia	8	2 058 800	5+1..91	5+2.4	8
LV Latvia	8	2 023 800	5+1..91	5+2.4	8
EE Estonia	6	1 324 800	5+1..91	5+1.6	7
CY Cyprus	6	865 900	5+1..91	5+1.01	7
LU Luxembourg	6	537 000	5+1..91	5+0.6	6
MT Malta	6	421 400	5+1..91	5+0.5	6
Sum (Divisor)	751	505 572 500	—	(854 000)	751

Table 7 presents a version of the Cambridge Compromise called “limited loss variant”. The variant guarantees that no Member State loses more than two seats compared to her status quo. To this end minimum restrictions are imposed based on the status quo “Seats 2014” column. For example Germany has 96 seats. With five base seats and a loss of at most two seats, her seat number must not fall below $96-2-5 = 89$. Hungary’s 2014 due is 21 seats. Hence her minimum restriction is $21 - 5 - 2 = 14$. Thus the limited loss variant of the Cambridge Compromise reads: *Every Member State is allocated five base seats. Of the 611 remaining seats every 854 000 citizens of the Union justify roughly one seat, except when a minimum restriction warrants more seats or a maximum restriction imposes fewer seats.* See Table 7.

The population figures in Table 7 are a bureaucratic malice. They are taken from the *Official Journal of the European Union*, Volume L 333 of 12.12.2013, pages 77–78. Greece happened to have a 2013 population that was a straight multiple of a hundred, no other Member State did. As a matter of fact EuroStat, the Union’s statistical office, counts citizens one by one and records population figures in whole numbers, not as multiples of anything. However, when communicated to the European Commission anonymous bureaucrats intervened. They fancied that 80 523 746 Germans are a nuisance, but that 80 523.7 thousand Germans are to the point.

TABLE 8 *Election of the 18. German Bundestag 2013, assignment of 299 constituencies to sixteen States.* The divisor method with standard rounding is used. Every 250 000 Germans justify roughly one constituency. Reporting date of the population figures is 31. December 2009.

18BT2013-Constituencies		Population 2009	Quotient	DivStd
SH	Schleswig-Holstein	2 687 425	10.7	11
MV	Mecklenburg-Vorpommern	1 612 879	6.45	6
HH	Hamburg	1 534 853	6.1	6
NI	Lower Saxony	7 406 139	29.6	30
HB	Bremen	578 445	2.3	2
BB	Brandenburg	2 446 621	9.8	10
ST	Saxony-Anhalt	2 314 050	9.3	9
BE	Berlin	2 969 466	11.9	12
NW	North Rhine-Westphalia	16 003 993	64.0	64
SN	Saxony	4 054 656	16.2	16
HE	Hessen	5 389 333	21.6	22
TH	Thuringia	2 202 259	8.8	9
RP	Rhineland-Palatinate	3 706 222	14.8	15
BY	Bavaria	11 346 304	45.4	45
BW	Baden-Württemberg	9 480 946	37.9	38
SL	Saarland	937 752	3.8	4
Sum (Divisor)		74 671 343	(250 000)	299

3.5 Election to the German Bundestag

The electoral system for the German Bundestag combines proportional representation of political parties with the election of individual candidates. It does so by establishing single-seat constituencies, and by accounting for the ensuing constituency winners by way of minimum restrictions. Furthermore parties nominate their candidates not via a federal list, but via sixteen “state lists” (Landeslisten). This multi-purpose multi-layer system is highly praised in the literature, see Shugart / Wattenberg (2001). The multi-stage calculations needed are described in the sequel.

The Federal Electoral Law demands that every constituency must lie entirely within a State. Therefore the 299 constituencies are assigned to the sixteen States, a year or two before an election. The assignment is based on the States’ population figures, and uses the divisor method with standard rounding. For the 2013 election every 250 000 Germans justify roughly one constituency. Table 8 summarizes the results.

The design of the ballot sheets invites voters to mark two votes: a “constituency vote” (Erststimme, first vote) in the left column printed in black, and a “list vote” (Zweitstimme, second vote) in the right column printed in blue. An explanatory text in the header of the right column points out that the list vote is the *decisive vote for the distribution of the seats altogether among the distinct parties* (maßgebende Stimme für die Verteilung der Sitze insgesamt auf die einzelnen Parteien).

At the end of the election day the list votes are aggregated across the whole country. A list vote becomes effective (zuteilungsberechtigt), that is, enters into the apportionment process, provided it is valid and cast for a party which (1) gains at least five percent of the countrywide total of valid list votes, or (2) wins at least three constituencies, or (3) represents a national minority.

In 2013 the total count of effective list votes was 36 867 417, they were cast for five parties. The 631 Bundestag seats are apportioned among the five parties proportionately to their countrywide list votes, using the divisor method with standard rounding. Every 58 420 effective list votes justify roughly one Bundestag seat. This step is called the “super-apportionment” (Oberzuteilung); see Table 9.

TABLE 9 *Election of the 18. German Bundestag 2013, super-apportionment of 631 seats among parties.* The divisor method with standard rounding is used. Every 58 420 list votes (Zweitstimmen) justify roughly one seat. The Bundestag size of 631 seats is determined beforehand in a separate step.

18BT2013-Super-app.	List Votes	Quotient	DivStd
CDU	14 921 877	255.4	255
SPD	11 252 215	192.6	193
LINKE	3 755 699	64.3	64
B90/GRÜNE	3 694 057	63.2	63
CSU	3 243 569	55.52	56
Sum (Divisor)	36 867 417	(58 420)	631

The last step assigns a party's countrywide seats to this party's candidates. Now the constituency votes come into play. In every constituency the candidate with a plurality of constituency votes wins the constituency's "direct seat" (Direktmandat).

The seat assignment for CSU is straightforward. This party stands in just one State, Bavaria. The super-apportionment says that the CSU is entitled to 56 Bundestag seats. On the other hand the CSU wins 45 direct seats in Bavaria. Hence the 56 CSU seats are filled by its 45 constituency winners plus the 11 list nominees who rank highest (when skipping nominees who win a direct seat).

The other four parties stand in several States, the CDU in the fifteen States without Bavaria, and the SPD, LINKE and BÜNDNIS90/DIE GRÜNEN in all sixteen States. For each party a "sub-apportionment" (Unterzuteilung) distributes the countrywide seats of the party among its various State lists. Each sub-apportionment is proportionate to the list votes per State while simultaneously guarding the direct seat wins in the States. To this end the sub-apportionments use the "direct seat restricted variant" of the divisor method with standard rounding. As it turns out, the SPD, LINKE and BÜNDNIS90/DIE GRÜNEN have direct seat restrictions that remain inactive. The sub-apportionments stay the same with and without direct seat restrictions and hence have unproportionality index zero.

The CDU sub-apportionment has three direct seat restrictions that are active. In Table 10 they are highlighted by a trailing dot (•). Every 59 700 CDU votes justify roughly one of the 255 CDU seats, except in Brandenburg, Saxony-Anhalt and Thuringia where the direct seat wins warrant more seats. In each of the three States the number of direct seat wins happens to be the same, nine, and overrules the number of "proportionality seats", eight. See Table 10.

Without restrictions the 255 CDU seats would have been allotted differently. Brandenburg, Saxony-Anhalt and Thuringia would lose a seat each, while North Rhine-Westphalia would gain two seats and Baden-Württemberg one. Hence the unproportionality index of the direct seat restricted variant is three seats. The unrestricted apportionment fails to accommodate all direct seats though. Instead three "overhang seats" would be brought to life. Formerly overhang seats were found troublesome, now they are a relic of the past. The direct seat restricted variant makes do without them.

A question we have ignored so far is this: Where does the 2013 Bundestag size of 631 seats come from? The answer is that it is a result of an advance calculation. Its aim is to determine a Bundestag size large enough so that the direct seat restrictions in the eventual per-party sub-apportionments will be compatible (Section 3.2). The advance calculation is awkward and boring, and not worth to be reproduced here.

TABLE 10 *Election of the 18. German Bundestag 2013, sub-apportionments of a party's seats to its state lists.* The direct-seat restricted variant of the divisor method with standard rounding is used (DivStd•). Every state list is allotted at least as many seats as indicated by its direct seat wins (Dir). The restrictions are active for the CDU in BB, ST und TH (•) and inactive otherwise.

18BT2013-Sub-app.	Dir	List Votes	Quotient	DivStd•	Dir	List Votes	Quotient	DivStd•
	CDU Sub-apportionment				*SPD Sub-apportionment*			
SH	9	638 756	10.7	11	2	513 725	8.8	9
MV	6	369 048	6.2	6	0	154 431	2.6	3
HH	1	285 927	4.8	5	5	288 902	4.9	5
NI	17	1 825 592	30.6	31	13	1 470 005	25.1	25
HB	0	96 459	1.6	2	2	117 204	2.0	2
BB	9	482 601	8.1•	9	1	321 174	5.49	5
ST	9	485 781	8.1•	9	0	214 731	3.7	4
BE	5	508 643	8.52	9	2	439 387	7.51	8
NW	37	3 776 563	63.3	63	27	3 028 282	51.8	52
SN	16	994 601	16.7	17	0	340 819	5.8	6
HE	17	1 232 994	20.7	21	5	906 906	15.503	16
TH	9	477 283	8.0•	9	0	198 714	3.4	3
RP	14	958 655	16.1	16	1	608 910	10.4	10
BY	—	—	—	—	0	1 314 009	22.46	22
BW	38	2 576 606	43.2	43	0	1 160 424	19.8	20
SL	4	212 368	3.6	4	0	174 592	3.0	3
Sum (Divisor)	191	14 921 877	(59 700)	255	58	11 252 215	(58 500)	193

	Dir	List Votes	Quotient	DivStd•	Dir	List Votes	Quotient	DivStd•
	LINKE Sub-apportionment				*B90/GRÜNE Sub-apportionment*			
SH	0	84 177	1.4	1	0	153 137	2.53	3
MV	0	186 871	3.1	3	0	37 716	0.6	1
HH	0	78 296	1.3	1	0	112 826	1.9	2
NI	0	223 935	3.7	4	0	391 901	6.47	6
HB	0	33 284	0.6	1	0	40 014	0.7	1
BB	0	311 312	5.2	5	0	65 182	1.1	1
ST	0	282 319	4.7	5	0	46 858	0.8	1
BE	4	330 507	5.51	6	1	220 737	3.6	4
NW	0	582 925	9.7	10	0	760 642	12.6	13
SN	0	467 045	7.8	8	0	113 916	1.9	2
HE	0	188 654	3.1	3	0	313 135	5.2	5
TH	0	288 615	4.8	5	0	60 511	1.0	1
RP	0	120 338	2.0	2	0	169 372	2.8	3
BY	0	248 920	4.1	4	0	552 818	9.1	9
BW	0	272 456	4.54	5	0	623 294	10.3	10
SL	0	56 045	0.9	1	0	31 998	0.53	1
Sum (Divisor)	4	3 755 699	(60 000)	64	1	3 694 057	(60 600)	63

3.6 Double Proportionality

When an electoral area is subdivided into regional districts the electoral system often is expected to honor the subdivision of the electoral region by geographical districts in a similar way as it honors the division of the electorate by political parties. Historically, the idea that a Member of Parliament represents a local district is older than the view that parliamentary representation provides a mirror image of the division along party lines. Procedures that successfully meet the double challenge are double-proportional divisor methods. Because of the dual objective, the notational requirements and abstract analysis of double-proportional apportionment methods are more elaborate than those of simple-proportional apportionment methods.

We present an illustrative example from Pukelsheim (2014, Chapter 14), the September 2012 election of the parliament of the Swiss Canton of Schaffhausen. The Cantonal Parliament (Kantonsrat) comprises sixty seats. For its election the canton is subdivided into six districts. The seat apportionment uses the double-proportional variant of the divisor method with standard rounding. We show how the sixty seats are allocated to the six districts, how the sixty seats are apportioned among the twelve parties that stood at the 2012 election, and how the double-proportional sub-apportionment finalizes the seat assignments.

Prior allocation of seats to districts: District magnitudes. The number of seats allocated to a district is called the "district magnitude". The district magnitudes for the 2012 election are obtained from the census figures of 31 December 2010, using the Hare-quota method with residual fit by greatest remainders and guaranteeing all districts at least one seat. The seat guarantees hold without further ado, without given rise to any ambiguities. See Table 11.

The minimum restricted variant of the divisor method with standard rounding yields the same allocation. Every 1 250 citizens justify roughly one seat. The district magnitudes range from twenty-eight seats in the largest district, the City of Schaffhausen, to a single seat in the smallest district, the exclave Buchberg-Rüdlingen. The smallest district is a single-seat constituency where formerly the representative was elected by plurality vote. Votes cast for candidates other than the constituency winner were wasted. The volatility of district magnitudes loses its importance when a double-proportional system is used.

In Schaffhausen the district magnitude also fixes the number of candidates that may be marked on the ballot sheet. Since parties, candidates, and election officials need time to make appropriate preparations, the district magnitudes were publicized in January 2012 well ahead of the September 2012 election.

Super-apportionment of seats to parties: Overall party-seats. In every district the votes of all candidates of a party are aggregated into the *party votes* (Parteistimmen). Since a voter in the City of Schaffhausen may mark up to twenty-eight candidates while a voter in Buchberg-Rüdlingen can mark only one, different districts yield party votes on different scales. However, electoral equality pertains to human beings, not to marks on the ballot sheets. Therefore party votes are converted into "voter numbers". Party votes are divided by the district magnitude, and the resulting quotient is rounded to a whole number by means of standard rounding:

$$\text{voter number} = \left\langle \frac{\text{party votes}}{\text{district magnitude}} \right\rangle.$$

See Table 12. Party votes and district magnitudes are taken from Table 13. In the Schaffhausen district, 55 905 SVP party votes yield voter number $55\,905/28 = 1\,996.6 \to 1\,997$. In Buchberg-Rüdlingen, the 309 SVP votes stay put (since $309/1 = 309$). The sum of the SVP voter numbers over the six districts is 6 740. The resulting seat number of a party is referred to as the *overall party-seats*. For example, the strongest party SVP is apportioned 16 overall party-seats, the weakest party JUSO one.

Sh2012DistrictMagnitudes	Population	Min.	Quotient	HaQgrR
Schaffhausen	34 943	1	27.458	28
Klettgau	15 453	1	12.143	12
Neuhausen	10 185	1	8.003	8
Reiat	8 986	1	7.061	7
Stein	5 222	1	4.103	4
Buchberg-Rüdlingen	1 567	1	1.231	1
Sum (Split)	76 356	6	(.3)	60

TABLE 11 *District magnitudes, Schaffhausen 2012.* The 60 seats are allocated to districts proportionately to the census figures as 31 December 2010 using the Hare-quota method with residual fit by greatest remainders. The minimum restriction of at least one seat per district remains dormant. The minimum restricted variant of the divisor method with standard rounding yields the same result.

Sh2012Super-app.	Voter number	Quotient	DivStd
SVP	6 740	16.1	16
SP	5 314	12.7	13
FDP	3 778	9.0	9
AL	1 886	4.51	5
ÖBS	1 878	4.49	4
CVP	1 232	2.9	3
JSVP	1 117	2.7	3
EDU	889	2.1	2
JFSH	827	2.0	2
SVP Sen.	618	1.48	1
EVP	551	1.3	1
JUSO	384	0.9	1
Sum (Divisor)	25 214	(418)	60

TABLE 12 *Super-apportionment, Schaffhausen 2012.* The determination of the overall party-seats is based on the parties' canton-wide totals of the per-district voter numbers. Every 418 voters justify roughly one seat. Since the divisor method with standard rounding is used the resulting overall party-seats realize practically equal success values for all voters in the whole canton.

Sh2012Sub-app.		SVP	SP	FDP	AL	ÖBS	CVP	District divisor
		16	13	9	5	4	3	
Schaffhausen	28	55 905-5	70 837-6	46 656-4	34 800-4	27 243-2	12 596-1	10 700
Klettgau	12	23 901-4	11 871-2	11 980-2	2 802-1	3 431-1	2 350-0	5 400
Neuhausen	8	4 493-2	5 252-3	3 309-2	781-0	1 003-0	2 054-1	2 000
Reiat	7	8 749-2	4 380-1	3 493-1	968-0	2 087-1	443-0	3 100
Stein	4	2 519-2	1 681-1	464-0	301-0	782-0	1 064-1	1 400
Buchberg-Rüdlingen	1	309-1	92-0	85-0	98-0			400
Party divisor		1.16	1.05	1	0.9	1.2	1	

(*continued*)		JSVP	EDU	JFSH	SVP Sen.	EVP	JUSO	District divisor
		3	2	2	1	1	1	
Schaffhausen	28	8 214-1	9 204-1	11 126-1	5 031-1	7 178-1	5 617-1	10 700
Klettgau	12	5 650-1	3 952-1	1 336-0	1 348-0	3 006-0	917-0	5 400
Neuhausen	8	644-0	457-0	377-0	820-0	348-0	292-0	2 000
Reiat	7	1 241-1	936-0	1 106-1	1 033-0		318-0	3 100
Stein	4	201-0	159-0	202-0	149-0		100-0	1 400
Buchberg-Rüdlingen	1	45-0		63-0	38-0			400
Party divisor		0.8	1	0.7	0.9	1.2	1	

TABLE 13 *Sub-apportionment, Schaffhausen 2012.* The Schaffhausen SVP party votes (55 905) are divided by the Schaffhausen divisor (10 700) and SVP divisor (1.16). The resulting quotient 4.504 justifies 5 seats. The other seat numbers are obtained similarly. The published divisors guarantee that each district meets its district magnitude and that each party exhausts its overall party-seats.

Sub-apportionment: Assignment of seats to districts and parties. The final sub-apportionment consists of the allocation of all 60 seats to the lists of nominees presented to the electorate in the six districts by the twelve parties. The maximum number of potential lists would be $6 \times 12 = 72$. But some parties do not stand in some districts, and so only 65 lists materialize. The sub-apportionment delivers an assignment of seats to districts and parties aiming at three goals:

(1) Each district meets its district magnitude.
(2) Each party exhausts its overall party-seats.
(3) Proportionality is observed among parties within a given district, as well as among districts within a given party.

The goals are achieved by the double-proportional variant of the divisor method with standard rounding. Two sets of divisors are needed. The first consists of a "district divisor" $c_i > 0$ for every district i. The second set contains a "party divisor" $d_j > 0$ for every party j. The divisors ensure that goals (1) and (2) are satisfied meticulously. Goal (3) is realized in that district divisors scale the party votes within a given district, while party divisors scale the party votes within a given party. The divisors are calculated by repeated applications of the Jump-and-Step Procedure.

Once the divisors are obtained and published it is rather easy to determine the seat numbers. The party votes v_{ij} which in district i are cast for party j are divided by the two associated divisors to obtain the interim quotient $v_{ij}/(c_i d_j)$. Standard rounding yields the "double-proportional seat number" x_{ij}:

$$x_{ij} = \left\langle \frac{v_{ij}}{c_i d_j} \right\rangle .$$

Table 13 shows the double-proportional solution. In the inner box, party votes and seat numbers are separated by a hyphen "-", for all districts i and for all parties j. Appreciation and verification of the solution is aided by the information arranged on the outside: district magnitudes on the left, overall party-seats along the top, district divisors on the right, and party divisors along the bottom.

For instance, the Schaffhausen SVP's 55 905 party votes are divided by the Schaffhausen divisor (10 700) and the SVP divisor (1.16). The resulting interim quotient, 4.504, is rounded to 5 seats. Therefore the Schaffhausen list of the SVP is allocated five seats. Similarly the JUSO's 5 617 party votes in Schaffhausen are divided by the Schaffhausen divisor and the JUSO divisor (1). The interim quotient is 0.52 and justifies one seat. Across the whole table the seat numbers x_{ij} sum rowwise to the district magnitude, and columnwise to the overall party-seats. The seat apportionment of the 2012 Cantonal Parliament election in the Canton of Schaffhausen is complete.

The merits of double proportionality vividly surface in the single-seat district Buchberg-Rüdlingen. Formerly the election was by plurality vote, for the last time in 2004. Voter turnout in 2004 amounted to $580/1\,068 = 54$ percent. The 2012 turnout of $730/1\,136 = 64$ percent is a significant increase of ten percentage points. The increased turnout is owed to people who voted for somebody else than the prospective winner. Presumably these people are appreciative of the fact that their votes contribute to the canton-wide success of the party of their choice, even though their votes cannot overturn the traditional winner in the district.