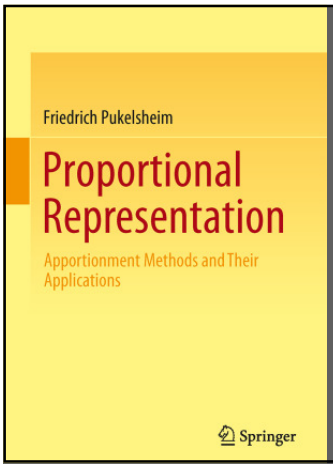
Proportional Representation

Prologue

2014: 234 pp.

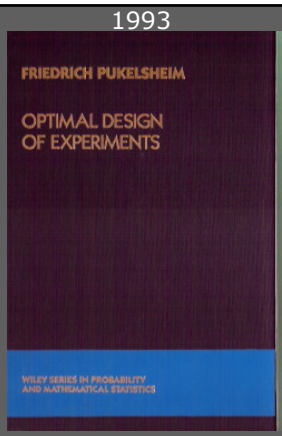


2015: 123 pp.

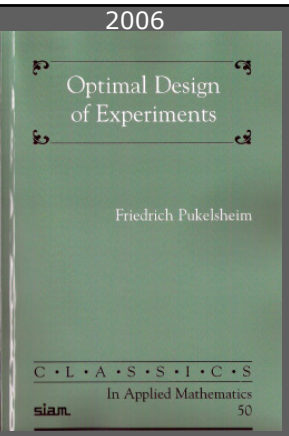
www.uni-augsburg.de/bazi
Berechnung von Anzahlen mit Zuteilungsmethoden im Internet

www.uni-augsburg.de/llull
Ramon Llull 1232–1316
Nikolaus von Kues 1401–1464

1993



2006



Chapter 12: Efficient designs for finite sample sizes

Design of experiments

Experimental conditions:

-1	-0.8	-0.3	0.3	0.8	1

Proportion of observations, in percent:

7.4	18.0	24.6	24.6	18.0	7.4

Number of observations, for sample size n = 60:

4	11	15	15	11	4

Proportional representation

Political parties:

PS	AL	CDU	MPT	B.E.	L

Proportion of votes, in per cent:

35.7	28.9	10.2	9.1	8.4	7.7

Number of seats, for house size h = 230:

82	67	23	21	19	18

8

1975 World population
(from *Kopfermann* 1991, p. 109)

Continent	*Population*	*Proportion*	*%*
1. Asia	2 295 000 000	0.57289	57
2. Europe	734 000 000	0.18323	18
3. Americas	540 000 000	0.13480	13
4. Africa	417 000 000	0.10409	10
5. Australia	20 000 000	0.00499	0
Total	4 006 000 000	1.00000	98

Discrepancy = −2% = 80mio. "rounding victims"?

9

Theorem: Individual roundings do not suffice.

Corollary: Apportionment *methods* are needed.

10

The 8th European Parliament 2014 – 2019

has 750 members plus the president, that is,

751 MEPs

according to Art. 14 TEU = Treaty of Lisbon

12

Chapter 1
PR Examples

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TABLE 1 *EP 2014 Election, Germany.* The divisor method with standard rounding is used. Every 298 900 votes justify roughly one seat. That is, a party's "Votes" and the divisor 298 900 yield an interim "Quotient" that is rounded downwards when its fractional part is below one half, and upwards when above. The resulting seat apportionment is shown in column "DivStd".

EP2014DE	Votes	Quotient	DivStd
CDU	8 812 653	29.48	29
SPD	8 003 628	26.8	27
GRÜNE	3 139 274	10.503	11
DIE LINKE	2 168 455	7.3	7
AFD	2 070 014	6.9	7
CSU	1 567 448	5.2	5
FDP	986 841	3.3	3
FREIE WÄHLER	428 800	1.4	1
PIRATEN	425 044	1.4	1
TIERSCHUTZPARTEI	366 598	1.2	1
NPD	301 139	1.0	1
FAMILIE	202 803	0.7	1
ÖDP	185 244	0.6	1
DIE PARTEI	184 709	0.6	1
11 Others	512 442	—	0
Sum (Divisor)	29 355 092	(298 900)	96

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TABLE 2 *EP 2014 Election, Germany, CDU Sub-apportionment.* The 29 CDU seats are sub-apportioned among States since the party tenders a separate list for each state where it stands. Again the divisor method with standard rounding is used; every 300 000 votes justify roughly one seat. No more sub-apportionments are needed as all other parties submit a single, countrywide list of nominees.

EP2014DE-CDU	Votes	Quotient	DivStd
CDU Sub-apportionment			
Schleswig-Holstein	334 121	1.1	1
Mecklenburg-Vorpommern	210 268	0.7	1
Hamburg	135 780	0.45	0
Lower Saxony	1 174 739	3.9	4
Bremen	43 353	0.1	0
Brandenburg	233 468	0.8	1
Saxony-Anhalt	245 010	0.8	1
Berlin	232 274	0.8	1
North Rhine-Westphalia	2 439 979	8.1	8
Saxony	559 899	1.9	2
Hessen	564 294	1.9	2
Thuringia	290 703	1.0	1
Rhineland-Palatinate	661 339	2.2	2
Baden-Württemberg	1 542 244	5.1	5
Saarland	145 182	0.48	0
Sum (Divisor)	8 812 653	(300 000)	29

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TABLE 3 *EP 2014 Election, Czech Republic.* The divisor method with downward rounding is used. Every 50 000 votes justify roughly one seat. That is, a party's "Votes" and the divisor 50 000 yield a "Quotient" that is rounded downwards. The resulting seat apportionment is shown in column "DivDwn". The list position of those elected is given in the column "MEPs' list positions".

EP2014CZ	Votes	Quotient	DivDwn	MEPs' list positions
ANO 2011	244 501	4.9	4	1*, 2, 3, 4
TOP 09+STAN	241 747	4.8	4	2*, 1*, 5*, 3
ČSSD	214 800	4.3	4	1*, 2*, 3, 4
KSČM	166 478	3.3	3	1*, 4*, 2*
KDU-ČSL	150 792	3.02	3	2*, 1*, 3
ODS	116 389	2.3	2	1*, 2*
SVOBODNI	79 540	1.6	1	1*
Sum (Divisor)	1 214 247	(50 000)	21	

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TABLE 4 *EP 2014 Election, Bulgaria.* The Hare-quota method with residual fit by greatest remainders is used. The Hare-quota is the votes-to-seats ratio, 110 280.05. A party's "Votes" and the quota 110 280.05 yield a "Quotient" that is rounded downwards when its fractional part lies below the split .5, and upwards when above. The resulting seat apportionment is shown in column "HaQgrR".

EP2014BG	Votes	Quotient	HaQgrR	MEPs' list positions
GERB	680 838	6.174	6	1*, 2, 3, 4, 5, 6
BSP	424 037	3.845	4	15*, 1*, 2, 3
DPS	386 725	3.507	4	1*, 3, 4, 5
BWC et al.	238 629	2.164	2	1*, 2*
RB	144 532	1.311	1	2*
Sum (Split)	1 874 761	(.5)	17	

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TABLE 5 *EP 2014 Election, Belgium, District Magnitudes.* The 21 Belgian seats are allocated to three districts. The German Language Community is guaranteed one seat due to its minority status. The remaining 20 seats are allocated by referring the population figures to the national quota 548 429. The quota is obtained via $(11\,044\,712 - 76\,141)/20 = 548\,428.55 \to 548\,429$.

EP2014BE-DistrictMagnitudes	2012 Population	Quotient	Seats
Dutch Electoral College	6 484 459	11.824	12
French Electoral College	4 484 112	8.176	8
German Language Community	76 141	–	1
Sum (Split)	11 044 712	(.5)	21

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TABLE 6 *EP 2014 election, Belgium.* In Belgium the apportionment of seats to parties is executed separately within a district, in each case using the divisor method with downward rounding. The assignment of a party's seats to its candidates distinguishes between those candidates who reach the party's eligibility figure (in italics), and the others.

EP2014BE	Votes	Quotient	DivDwn	MEPs' list positions
District 1: Dutch Electoral College				
N-VA	1 123 355	4.005	4	1*, 2*, 3*, 4
OPEN VLD	859 099	3.1	3	1*, 2, 12
CD&V	840 783	2.997	2	1*, 2*
SP.A	555 348	1.98	1	1*
GROEN	447 391	1.6	1	1*
VLAAMS BELANG	284 856	1.02	1	1*
1 Other	101 237	—	0	
Sum (Divisor)	4 212 069	(280 500)	12	
District 2: French Electoral College				
PS	714 645	3.6	3	1*, 2*, 3
MR	661 332	3.3	3	1*, 2*, 3*
ECOLO	285 196	1.4	1	1*
CDH	277 246	1.4	1	1*
8 Others	501 627	—	0	
Sum (Divisor)	2 440 046	(200 000)	8	
District 3: German Language Community				
CSP	11 710	1.2	1	1*
5 Others	26 886	—	0	
Sum (Divisor)	38 596	(10 000)	1	

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Apportionment methodology involves

- the allocation of seats between districts
- the apportionment of seats among parties
- the assignment of seats to candidates

Chapter 2
Seat Apportionment Methods

Divisor Methods

Find a (flexible) divisor d such that, with given rounding function $[\cdot]$, the seat allocations

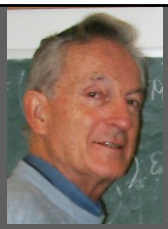
$$x_j = \left[\frac{v_j}{d} \right]$$

sum to the preordained house size h, that is,

$$x_1 + \cdots + x_\ell = h$$

Motto: *Divide and round!*

But: *Which rounding function?*



Fair Representation

Meeting the Ideal of One Man, One Vote

SECOND EDITION

MICHEL L. BALINSKI AND

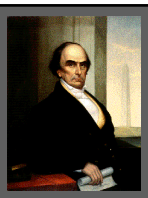
H. PEYTON YOUNG

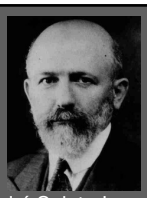
2001 (first edition 1982)

Divisor method with standard rounding (DivStd)

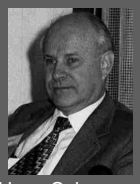
$$x_j = \left\langle \frac{v_j}{d} \right\rangle$$

has signposts 0.5, 1.5, 2.5, 3.5, and so on


Daniel Webster
1782-1852


André Sainte-Laguë
1882-1950


Ladislaus von Bortkiewicz
1868-1931


Hans Schepers
*1928

26

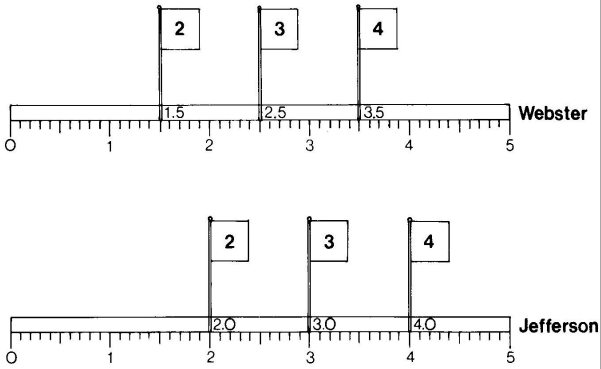


Figure 7.2 Signposts for the Five Historical Divisor Methods

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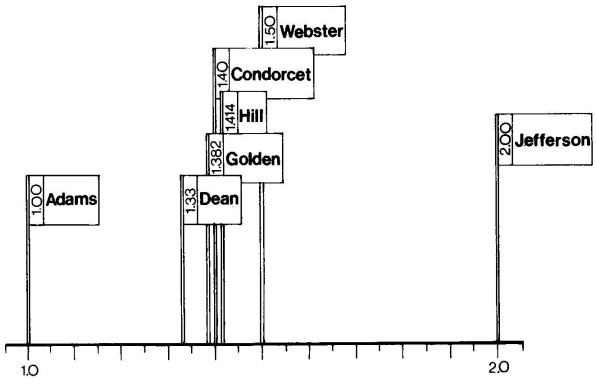

Figure 7.4 A Forest of Signposts

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Rounding functions $[\cdot]$ are induced by "signpost sequences":

$$s(n) \in [n-1;n] \quad (\text{"general"})$$

$$s_r(n) = (n-1) + r \quad (\text{"stationary"})$$

$$s_p(n) = \left(\frac{(n-1)^p + n^p}{2} \right)^{1/p} \quad (\text{"power - mean"})$$

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Divisor methods $D(h; v_1, \ldots, v_l) = (x_1, \ldots, x_l)$ are

A. anonymous
B. balanced
C. concordant
D. decent (homogeneous of degree zero)
E. exact

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Max-Min Inequality. *Suppose $x_1, \ldots, x_\ell$ are whole numbers house size h. Then the seat vector $x = (x_1, \ldots, x_\ell)$ is a solution,*

$$D(h; v_1, \ldots, v_\ell) = x$$

if and only if the "Max-Min Inequality" holds true,

$$\max_{j \le \ell} \frac{v_j}{s(x_j + 1)} \le \min_{j \le \ell} \frac{v_j}{s(x_j)}.$$

Proof.

$$\left[\frac{v_j}{d}\right] = x_j \iff s(x_j) \le \frac{v_j}{d} \text{ and } \frac{v_j}{d} \le s(x_j+1) \iff \frac{v_j}{s(x_j+1)} \le d \le \frac{v_j}{s(x_j)}.$$

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Jump-and-Step Procedure

Init.: $y_j := [v_j / d_{initial}]$ with $d_{initial} := \frac{v_+}{h}$

Discrepancy removal:

$$y_+ = h \Rightarrow x \leftarrow y$$
$$y_+ < h \Rightarrow d \downarrow \text{ until } y_+ = h$$
$$y_+ > h \Rightarrow d \uparrow \text{ until } y_+ = h$$

Rank-order of parties by vote shares: $w_1, \geq \cdots \geq w_\ell \geq t.$

Seat Bias Formula. *For a stationary divisor method with split r and for a given threshold t, the seat bias of the kth-strongest party is given by the formula*

$$\left(r - \tfrac{1}{2}\right)\left(\tfrac{1}{k} + \tfrac{1}{k+1} + \cdots + \tfrac{1}{\ell - 1} + \tfrac{1}{\ell} - 1\right)(1 - \ell t).$$

Pólya (1918) for DivDwn (i.e. r=1) and three parties:

$$\frac{5}{12}, \quad -\frac{1}{12}, \quad -\frac{4}{12}$$

Vote Shares for Given Seat Numbers. *For a stationary divisor method with split r, the lowest and highest vote shares for x_j seats are given by the formulas*

$$a(x_j) = \frac{x_j - (1-r)I}{h - (1-r)I + r(\ell - 1)}, \qquad b(x_j) = \frac{x_j + r}{h + r - (1-r)M},$$

where $I = 1$ when $x_j \geq 1$ and $I = 0$ when $x_j = 0$, and where $M = \min\{\ell - 1, h - x_j\}$.

Natural threshold: $b(0) = \dfrac{r}{h + r - (1-r)(\ell - 1)}$

Majority preservation: $h = 2n + 1 \Rightarrow \left(b(n) \leq \frac{1}{2} \Leftrightarrow r = 1, \text{i.e., DivDwn}\right)$

-> Majority clauses

Chapter 3 Proportionality and Personalization

Minimum-maximum restricted variants of divisor methods

$$[q]_{a_j}^{b_j} := \begin{cases} \{b_j\} & \text{in case } q > b_j, \\ [q] & \text{in case } q \in [a_j; b_j], \\ \{a_j\} & \text{in case } q < a_j. \end{cases}$$

$$x_1 = \left[\frac{v_1}{d}\right]_{a_1}^{b_1}, \quad \ldots \quad , x_\ell = \left[\frac{v_\ell}{d}\right]_{a_\ell}^{b_\ell}.$$

Unproportionality index

With $D_a^b(h; v_1, \ldots, v_\ell) = (x_1, \ldots, x_\ell)$
and $D(h; v_1, \ldots, v_\ell) = (z_1, \ldots, z_\ell)$
define $u(x) = \frac{1}{2}\left(|x_1 - z_1| + \cdots + |x_\ell - z_\ell|\right)$

Article 14(2) TEU-Lisbon:

… Representation of citizens shall be degressively proportional, with a minimum threshold of six members per Member State. No Member State shall be allocated more than ninety-six seats. …

Andrew Duff MEP
(AFCO rapporteur on electoral procedure)

“durable and transparent formula that is impartial to politics”

Cambridge Meeting January 2011

EP2014Composition	Seats 2014	Population 2013	5+Min..Max	5+Quotient	5+DivUpw•
DE Germany	96	80 523 700	5+89..91	5+94.3•	96
FR France	74	65 633 200	5+67..91	5+76.9	82
UK United Kingdom	73	63 730 100	5+66..91	5+74.6	80
IT Italy	73	59 685 200	5+66..91	5+69.9	75
ES Spain	54	46 704 300	5+47..91	5+54.7	60
PL Poland	51	38 533 300	5+44..91	5+45.1	51
RO Romania	32	20 057 500	5+25..91	5+23.5•	30
NL Netherlands	26	16 779 600	5+19..91	5+19.6	25
BE Belgium	21	11 161 600	5+14..91	5+13.1	19
EL Greece	21	11 062 500	5+14..91	5+12.95•	19
CZ Czech Republic	21	10 516 100	5+14..91	5+12.3•	19
PT Portugal	21	10 487 300	5+14..91	5+12.3•	19
HU Hungary	21	9 908 800	5+14..91	5+11.6•	19
SE Sweden	20	9 555 900	5+13..91	5+11.2•	18
AT Austria	18	8 451 900	5+11..91	5+9.9•	16
BG Bulgaria	17	7 284 600	5+10..91	5+8.5•	15
DK Denmark	13	5 602 600	5+6..91	5+6.6	12
FI Finland	13	5 426 700	5+6..91	5+6.4	12
SK Slovakia	13	5 410 800	5+6..91	5+6.3	12
IE Ireland	11	4 591 100	5+4..91	5+5.4	11
HR Croatia	11	4 262 100	5+4..91	5+4.99	10
LT Lithuania	11	2 971 900	5+4..91	5+3.5	9
SI Slovenia	8	2 058 800	5+1..91	5+2.4	8
LV Latvia	8	2 023 800	5+1..91	5+2.4	8
EE Estonia	6	1 324 800	5+1..91	5+1.6	7
CY Cyprus	6	865 900	5+1..91	5+1.01	7
LU Luxembourg	6	537 000	5+1..91	5+0.6	6
MT Malta	6	421 400	5+1..91	5+0.5	6
Sum (Divisor)	751	505 572 500	—	(854 000)	751

TABLE 8 *Election of the 18. German Bundestag 2013, assignment of 299 constituencies to sixteen States.* The divisor method with standard rounding is used. Every 250 000 Germans justify roughly one constituency. Reporting date of the population figures is 31. December 2009.

18BT2013-Constituencies	Population 2009	Quotient	DivStd
SH Schleswig-Holstein	2 687 425	10.7	11
MV Mecklenburg-Vorpommern	1 612 879	6.45	6
HH Hamburg	1 534 853	6.1	6
NI Lower Saxony	7 406 139	29.6	30
HB Bremen	578 445	2.3	2
BB Brandenburg	2 446 621	9.8	10
ST Saxony-Anhalt	2 314 050	9.3	9
BE Berlin	2 969 466	11.9	12
NW North Rhine-Westphalia	16 003 993	64.0	64
SN Saxony	4 054 656	16.2	16
HE Hessen	5 389 333	21.6	22
TH Thuringia	2 202 259	8.8	9
RP Rhineland-Palatinate	3 706 222	14.8	15
BY Bavaria	11 346 304	45.4	45
BW Baden-Württemberg	9 480 946	37.9	38
SL Saarland	937 752	3.8	4
Sum (Divisor)	74 671 343	(250 000)	299

TABLE 9 *Election of the 18. German Bundestag 2013, super-apportionment of 631 seats among parties.* The divisor method with standard rounding is used. Every 58 420 list votes (Zweitstimmen) justify roughly one seat. The Bundestag size of 631 seats is determined beforehand in a separate step.

18BT2013-Super-app.	List Votes	Quotient	DivStd
CDU	14 921 877	255.4	255
SPD	11 252 215	192.6	193
LINKE	3 755 699	64.3	64
B90/GRÜNE	3 694 057	63.2	63
CSU	3 243 569	55.52	56
Sum (Divisor)	36 867 417	(58 420)	631

18BT2013-Sub-app.	Dir	List Votes	Quotient	DivStd•	Dir	List Votes	Quotient	DivStd•
	CDU Sub-apportionment				*SPD Sub-apportionment*			
SH	9	638 756	10.7	11	2	513 725	8.8	9
MV	6	369 048	6.2	6	0	154 431	2.6	3
HH	1	285 927	4.8	5	5	288 902	4.9	5
NI	17	1 825 592	30.6	31	13	1 470 005	25.1	25
HB	0	96 459	1.6	2	2	117 204	2.0	2
BB	9	482 601	8.1•	9	1	321 174	5.49	5
ST	9	485 781	8.1•	9	0	214 731	3.7	4
BE	5	508 643	8.52	9	2	439 387	7.51	8
NW	37	3 776 563	63.3	63	27	3 028 282	51.8	52
SN	16	994 601	16.7	17	0	340 819	5.8	6
HE	17	1 232 994	20.7	21	5	906 906	15.503	16
TH	9	477 283	8.0•	9	0	198 714	3.4	3
RP	14	958 655	16.1	16	1	608 910	10.4	10
BY	—	—	—	—	0	1 314 009	22.46	22
BW	38	2 576 606	43.2	43	0	1 160 424	19.8	20
SL	4	212 368	3.6	4	0	174 592	3.0	3
Sum (Divisor)	191	14 921 877	(59 700)	255	58	11 252 215	(58 500)	193
	Dir	List Votes	Quotient	DivStd•	Dir	List Votes	Quotient	DivStd•
	LINKE Sub-apportionment				*B90/GRÜNE Sub-apportionment*			
SH	0	84 177	1.4	1	0	153 137	2.53	3
MV	0	186 871	3.1	3	0	37 716	0.6	1
HH	0	78 296	1.3	1	0	112 826	1.9	2
NI	0	223 935	3.7	4	0	391 901	6.47	6
HB	0	33 284	0.6	1	0	40 014	0.7	1
BB	0	311 312	5.2	5	0	65 182	1.1	1
ST	0	282 319	4.7	5	0	46 858	0.8	1
BE	4	330 567	5.51	6	1	220 737	3.6	4
NW	0	582 925	9.7	10	0	760 642	12.6	13
SN	0	467 045	7.8	8	0	113 916	1.9	2
HE	0	188 654	3.1	3	0	313 135	5.2	5
TH	0	288 615	4.8	5	0	60 511	1.0	1
RP	0	120 338	2.0	2	0	169 372	2.8	3
BY	0	248 920	4.1	4	0	552 818	9.1	9
BW	0	272 456	4.54	5	0	623 294	10.3	10
SL	0	56 045	0.9	1	0	31 998	0.53	1
Sum (Divisor)	4	3 755 699	(60 000)	64	1	3 694 057	(60 600)	63

Sh2012DistrictMagnitudes	Population	Min.	Quotient	HaQgrR
Schaffhausen	34 943	1	27.458	28
Klettgau	15 453	1	12.143	12
Neuhausen	10 185	1	8.003	8
Reiat	8 986	1	7.061	7
Stein	5 222	1	4.103	4
Buchberg-Rüdlingen	1 567	1	1.231	1
Sum (Split)	76 356	6	(.3)	60

TABLE 11 *District magnitudes, Schaffhausen 2012.* The 60 seats are allocated to districts proportionately to the census figures as 31 December 2010 using the Hare-quota method with residual fit by greatest remainders. The minimum restriction of at least one seat per district remains dormant. The minimum restricted variant of the divisor method with standard rounding yields the same result.

Sh2012Super-app.	Voter number	Quotient	DivStd
SVP	6 740	16.1	16
SP	5 314	12.7	13
FDP	3 778	9.0	9
AL	1 886	4.51	5
ÖBS	1 878	4.49	4
CVP	1 232	2.9	3
JSVP	1 117	2.7	3
EDU	889	2.1	2
JFSH	827	2.0	2
SVP Sen.	618	1.48	1
EVP	551	1.3	1
JUSO	384	0.9	1
Sum (Divisor)	25 214	(418)	60

TABLE 12 *Super-apportionment, Schaffhausen 2012.* The determination of the overall party-seats is based on the parties' canton-wide totals of the per-district voter numbers. Every 418 voters justify roughly one seat. Since the divisor method with standard rounding is used the resulting overall party-seats realize practically equal success values for all voters in the whole canton.

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Sh2012Sub-app.		SVP	SP	FDP	AL	ÖBS	CVP	District divisor
		16	13	9	5	4	3	
Schaffhausen	28	55 905-5	70 837-6	46 656-4	34 800-4	27 243-2	12 596-1	10 700
Klettgau	12	23 901-4	11 871-2	11 980-2	2 802-1	3 431-1	2 350-0	5 400
Neuhausen	8	4 493-2	5 252-3	3 309-2	781-0	1 003-0	2 054-1	2 000
Reiat	7	8 749-2	4 380-1	3 493-1	968-0	2 087-1	443-0	3 100
Stein	4	2 519-2	1 681-1	464-0	301-0	782-0	1 064-1	1 400
Buchberg-Rüdlingen	1	309-1	92-0	85-0	98-0			400
Party divisor		1.16	1.05	1	0.9	1.2	1	

(*continued*)		JSVP	EDU	JFSH	SVP Sen.	EVP	JUSO	District divisor
		3	2	2	1	1	1	
Schaffhausen	28	8 214-1	9 204-1	11 126-1	5 031-1	7 178-1	5 617-1	10 700
Klettgau	12	5 650-1	3 952-1	1 336-0	1 348-0	3 006-0	917-0	5 400
Neuhausen	8	644-0	457-0	377-0	820-0	348-0	292-0	2 000
Reiat	7	1 241-1	936-0	1 106-1	1 033-0		318-0	3 100
Stein	4	201-0	159-0	202-0	149-0		100-0	1 400
Buchberg-Rüdlingen	1	45-0		63-0	38-0			400
Party divisor		0.8	1	0.7	0.9	1.2	1	

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"Biproportional" methods
(Balinski/Demange 1989)
account for

- vote counts of parties
- population figures of districts

Debut performances in Zurich,
on 12 February 2006

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Gaffke/P (2006):

$$\text{Minimize} \quad F(x) = \prod_{(i,j):v_{ij}>0} \prod_{n=1}^{x_{ij}} \frac{n-0.5}{v_{ij}}$$

$$\text{where} \quad x_{i+} = r_i \qquad \forall i \le k$$

$$\text{and} \quad x_{+j} = c_j \qquad \forall j \le l$$

$$\text{among} \quad x \in N^{k\times l}$$

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$$\text{Let} \quad g_v(E,F) := \left(\prod_{i\le k} E_i^{z_{i+}-r_i}\right)\left(\prod_{j\le \ell} F_j^{z_{+j}-s_j}\right) f_v(z),$$

$$\text{where} \quad z_{ij} := [v_{ij}/E_iF_j]_r \qquad \forall i \le k, \quad j \le \ell.$$

Gaffke/P (2008):

$$\min_{x\in \mathrm{N}^{k\times\ell}(r,s)} f_v(x) = \max_{E\in(0,\infty)^k,\, F\in(0,\infty)^\ell} g_v(E,F)$$

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Iterative algorithms:
in floating point arithmetic,
in integer arithmetic

AS	alternating scaling
TT	tie-and-transfer
AS|TT	hybrid version

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