Axiomatic Foundations of Voting Theory (part II)

William S. Zwicker
Mathematics Department, Union College

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Exercises

- This section contains precise versions of problems mentioned on slides
- Only do the ones you find interesting (there are too many for you to do all right now)
- Most of the tutorial is based on Chapter 2 of the ***Handbook of Computational Social Choice***, Cambridge University Press, 2016. You may find the chapter helpful for these problems.
- Free PDF of the book at http://www.cambridge.org/download_file/898428
- To open the PDF use password: cam1CSC

Exercises

1) Copeland scoring

- Recall ***symmetric Copeland score*** is given by $Cop(x) = |\{y | x >^{\mu} y\}| - |\{y | y >^{\mu} x\}|$
- ***Asymmetric Copeland score*** is given by $Cop^{Ass.}(x) = |\{y | x >^{\mu} y\}|$
- ***Asymmetric+ Copeland score*** is given by $Cop^{Ass.+}(x) = |\{y | x >^{\mu} y\}| + (½)|\{y | y =^{\mu} x\}|$ *

Are these three rules all the same? All different? Answer as completely as possible.

* *We write $y =^{\mu} x$ when $Net_P(x>y) = 0$. You will need to consider profiles for an even number of voters, making $y =^{\mu} x$ possible.*

Exercises

2) Scoring weights and affine equivalence

- Scoring vectors $w = w_1, \ldots, w_m$ and $v = v_1, \ldots, v_m$ are ***affinely equivalent*** if there exist constants γ, δ with $\gamma > 0$ such that $v_i = \gamma w_i + \delta$ for each i.
- Prove that two scoring vectors w, v induce the same scoring rule iff they are affinely equivalent.
- Prove that any two evenly spaced vectors are affinely equivalent.
- Prove that ***symmetric*** Borda weights m-1, m-3, . . ., –m+1 yield a total score of $\beta(x)$ for each alternative x.

Recall that $\beta(x) = \boldsymbol{\Sigma}_{y \in A} \, Net_P(x > y)$

Exercises

3) Reversal Manipulation We saw Copeland can be ***manipulated via reversal***: a profile P exists for which some voter i can, by completely reversing her ranking, switch the winning alternative from x to some alternative y whom she sincerely prefers (she ranked y over x before reversing)

- Prove that Borda cannot be manipulated via reversal (the same argument shows all scoring rules are similarly immune)
- Prove that Simpson-Kramer can be manipulated via reversal
- **Difficult:** Prove that every resolute Condorcet extension for 4 or more alternatives can be manipulated via reversal

Recall . . . 3 large classes of SCFs

6) More Rules: 3 Important Classes

I Scoring rules

Like Borda, they use a vector of scoring weights

$w_1 \geq w_2 \geq \ldots \geq w_m; \ w_1 > w_m$

to award points.

Each voter awards w_1 points to top-ranked, w_2 to 2nd, etc.

Winner is the alternative with most points.

Examples include Borda,

Plurality: w = (1,0,0, ... , 0)

Anti-Pl: w = (1,1, ..., 1,0) OR

w = (0,0, ..., 0,-1)

Formula 1 racing champ:

w = (25,18,15,12,10,8,6,4, 1, 0, 0, ... , 0) [since 2010]

k-approval:

w = (1, ..., 1, 1, 0, ..., 0, 0) with k 1s

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II Condorcet Extensions

Recall: A ***Condorcet alternative*** a satisfies $a >^{\mu} b$ for each alternative $b \neq a$

A SCF f is a ***Condorcet Extension*** if f(P) = the Cond. alt. (for each P having a Cond. alt.)

Examples include Copeland, ***Maximin (Minimax, Simpson-Kramer)***:

Simpson Score SS(a) = $\text{Min}\left\{\text{Net}_P(a>x) \,\middle|\, x \in A\setminus\{a\}\right\}$

S-K rule chooses the $x \in A$ maximizing SS(x): it's a Condorcet Extension

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Top Cycle: A subset $X \subseteq A$ is a ***dominating set*** if x $>^{\mu}$ y holds for each $x \in X$, $y \notin X$

TC(P) = the smallest dominating set (which is unique)

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<u>III Scoring Elimination Rules</u>

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Single Transferrable Vote
(STV, alternative vote, Hare, Instant Run-off)

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STV used:

- when John Major replaced Margaret Thatcher as conservative party head
- briefly in Burlington Vermont (USA)
- 2011 U.K. referendum: use STV for Parliamentary elections . . . failed.

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Nanson voting rule

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Voter i’s ballot:

$\mathbf{b_i}$
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x
y
z
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b_i	b_i^*
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⋮	⋮
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- $b_i \mapsto b_i^*$

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- Then f(P*) = **???**

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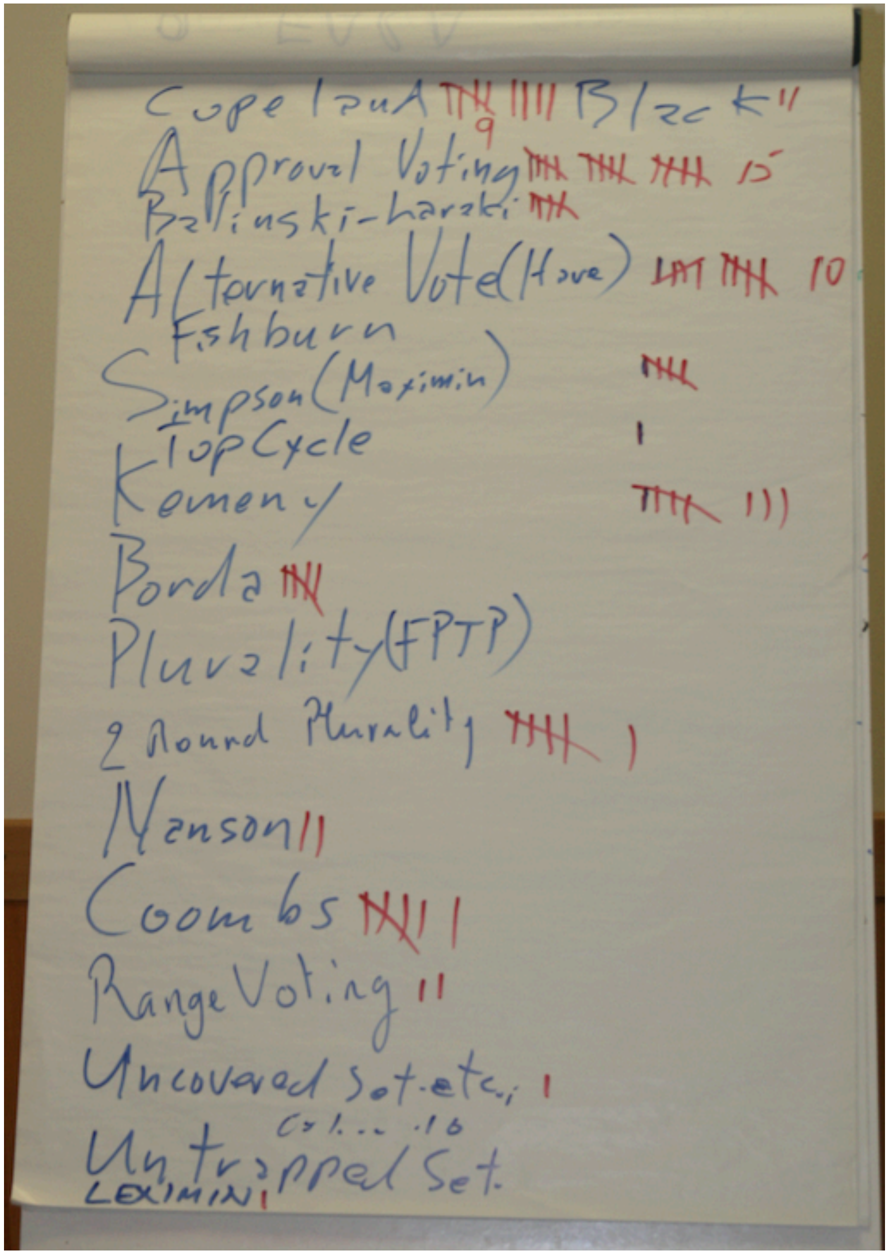
- P is a profile
- i is one of the voters
- f(P) = {**z**}
- **y** lies immediately above **z** on i's ballot
- Voter i moves **z** over **y** (no other changes)
- $b_i \mapsto b_i^*$; $P \mapsto P^*$
- Then f(P*) = **z**

This is the resolute version of the monotonicity axiom.

What is the "correct" reformulation for irresolute voting rules? ***Not what you might first guess . . .***

All voting rules discussed so far satisfy monotonicity . . . except STV and Nanson !

Surprising? **Fatal?**



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- What question should you be asking me . . .?
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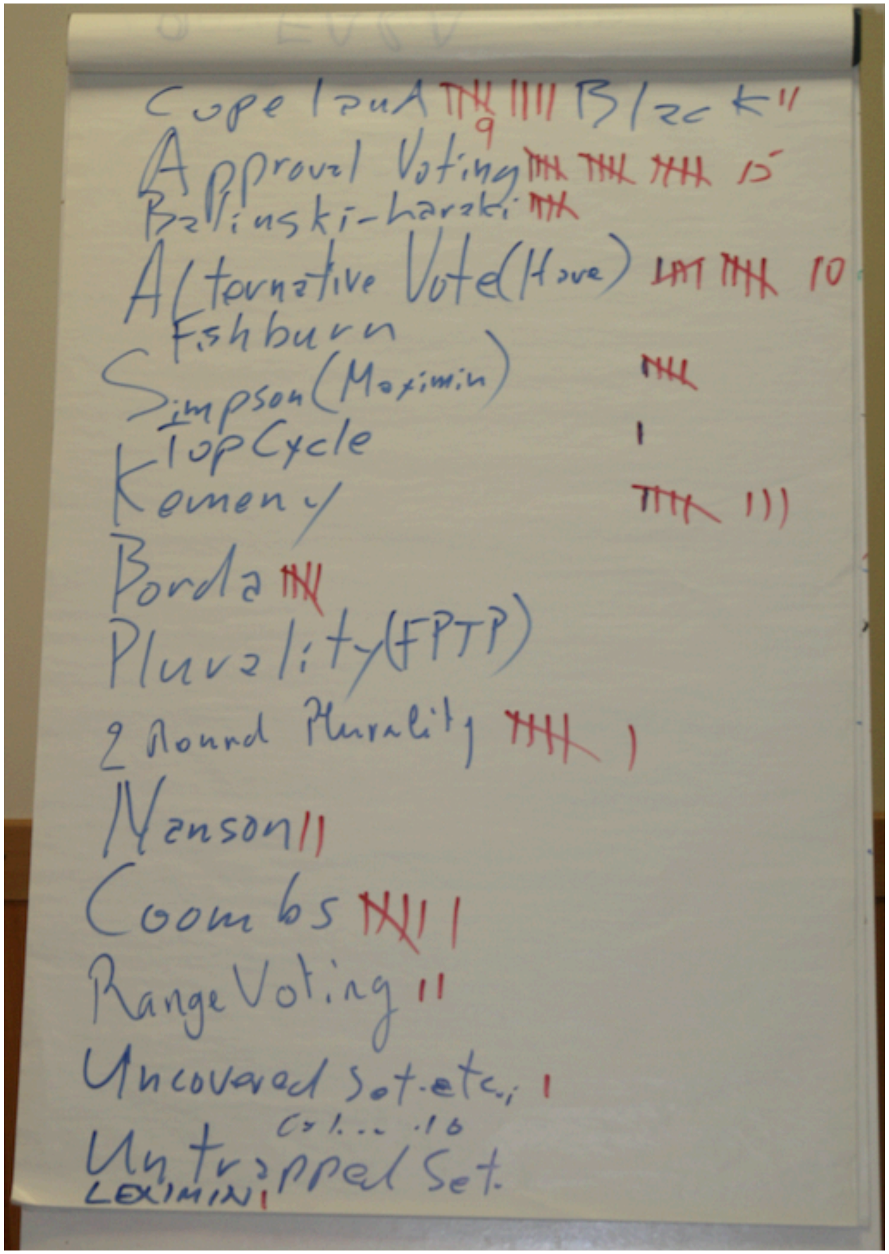

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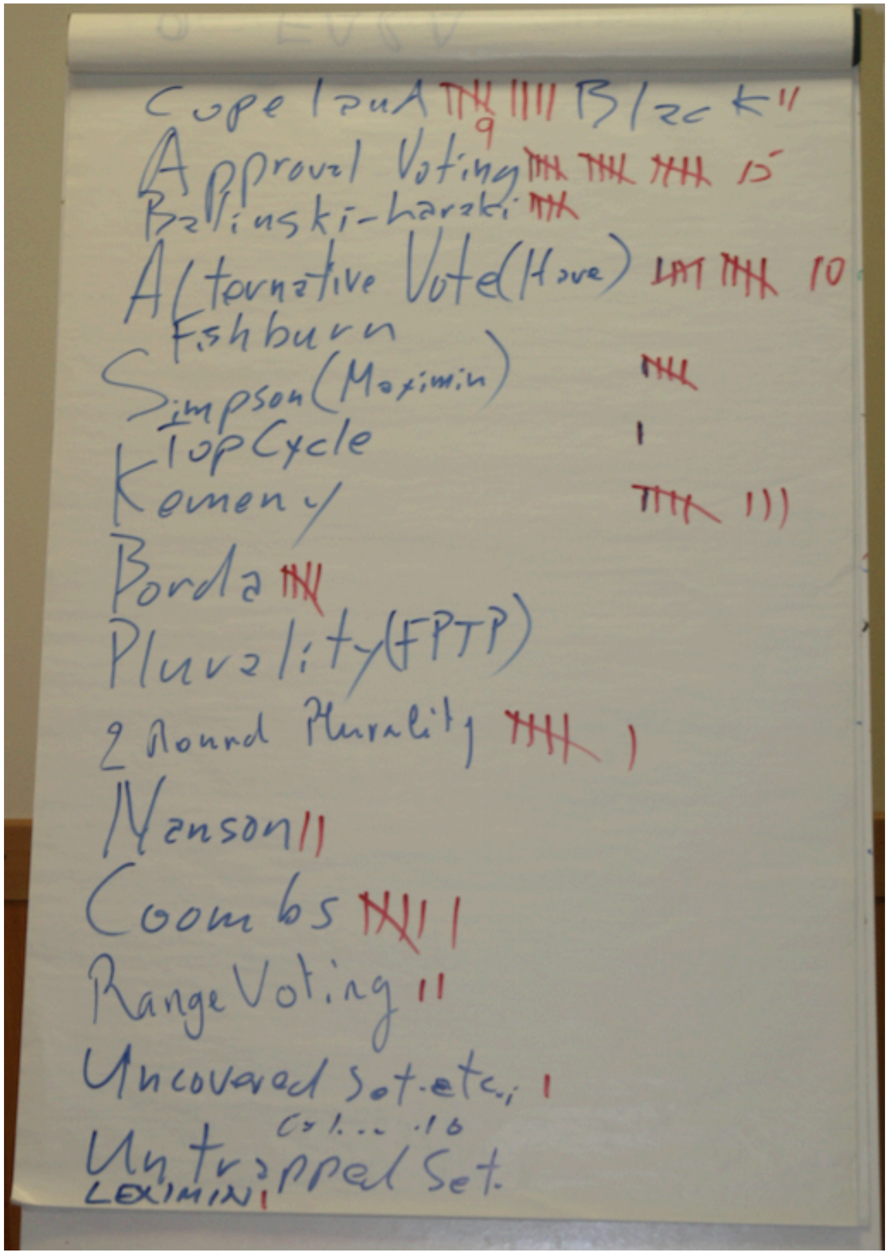

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- "Alternative vote" (= STV) came in 2nd (10 votes) after Approval voting (15)
- Probably 2 of the 10 were from the Electoral Reform Society.
- Discounting those, STV came in 3rd after Copeland . . . not bad!

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← Hare mechanism "seems fair" . . . but behaves oddly

Interlude: Voting with Rubber Bands and Strings

http://www.math.union.edu/research/mediancenter/evolver.html

http://www.math.union.edu/locate/voting-simulation

Click on the link: **Voting with rubber bands, weights, and strings**

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- P_2: **b** higher than **a**
- $P_1 + P_2$: add P_1, P_2 scores

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Another triumph of the axiomatic method!

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Back to voting simulator: “McBorda” rule

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D. Greater acceptance of Approval Voting as a "compromise candidate" among voting rules.

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Exercises

4) Nanson's Rule

- Prove that Nanson's Rule is a Condorcet Extension.

  Hint: Using $\beta(x) = \mathbf{\Sigma}_{y \in A} \text{Net}_P(x{>}y)$ to generate Borda scores, show that the average Borda score of all alternatives is 0. Then show that $\beta(z) > 0$ holds for a Condorcet alternative z.

If $x >^{\mu} z$ holds for each alternative x other than z itself, we say that z is a ***Condorcet loser***. Condorcet losers exist for some profiles, but not for others.

- Prove that Nanson's Rule will never elect a Condorcet loser.