On envy-free allocations in large fair division problems with indivisible goods

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Fair division of indivisible goods

- m indivisible goods to allocate among n agents
- allocation $\mathcal{A} = (\mathcal{A}_i)_{i=1\ldots n}$ is a partition of a set of goods
- u_g^i is agent i's satisfaction of a good g
- utilities of agents are additive:

$$u^i(\mathcal{A}_i) = \sum_{g\in\mathcal{A}_i} u_g^i$$

- *For example, division of inheritance*

Fairness criterion

- Envy-freeness:

Allocation $\mathcal{A}$ is *envy-free* iff

$$u^i(\mathcal{A}_i) \geq u^i(\mathcal{A}_j)\ \forall\, i,j = 1 \ldots n$$

Assumptions

- u_g^i are i.i.d. random variables uniformly distributed on $[0;1]$
- the number of goods is large

The question is:

How often does envy-free allocation of indivisible goods exist?

Previous results:

J.P. Dickerson et al. show that the probability of envy-free allocation existence tends to 1 as the number of goods becomes large.

Remark:

They do not obtain an explicit estimate on this probability in terms of n and m.

Our aim:

- obtain explicit estimate
- introduce measure-concentration tools in large fair division problems

Result

The probability of existence of envy-free allocation is greater, than

$$1 - n^2 \exp\left(\frac{-m}{4(n+1)^3}\right).$$

Note, that probability increases with increasing of number of goods.

The main ideas of the proof

- **Utilitarian maximum**
  Consider an allocation $\mathcal{A}^{UT}$ in which each good g is given to an agent i who desires it most, in other words, agent i gets g iff $u_g^i \geq u_g^j\ \forall\, i,j = 1 \ldots n.$

  $\mathcal{A}^{UT}$ can be really unfair.
- We use **measure concentration** tools to estimate the probability that agent i envies agent j in $\mathcal{A}^{UT}$. Measure concentration theory says that the "macroscopic" properties (that depends on a large number of random parameters) of large random objects are non-random, i.e., they are close to expected values with high probability.

McDiarmid's inequality

- $\xi_1, \xi_2, \ldots, \xi_N$ are independent random variables
- function f is such that for any fixed $x_1, \ldots, x_N$ the random variable $f(x_1, \ldots, x_{i-1}, \xi_i, x_{i+1}, \ldots, x_N)$ belongs to interval of length c_i

Then for any $\varepsilon > 0$

$$\mathbb{P}(|f(\xi) - E(f(\xi))| \geq \varepsilon) \leq 2\exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^N c_i^2}\right).$$

Corollary: the law of large numbers.

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References

- J.P. Dickerson, J. Goldman, J. Karp, A.D. Procaccia, T. Sandholm, The Computational Rise and Fall of Fairness, Proceedings of the Twenty-Eighth AAAI Conference on Articial Intelligence, 2014.
- C. McDiarmid, On the Method of Bounded Differences, Surveys in Combinatorics 141: 148–188, 1989.