# Searching for the "least" and "most" dictatorial voting rules


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**Abstract**. Distance rationalizability of voting rules is based on minimization of the distance to some plausible creiterion, such as the unanimity or Condorcet critrerion. We propose a new alternative: maximization of the distance to an undesirable voting rule, namely, dictatorial voting rule. Applying a reasonable metric between social choice functions, we obtain two results: (i) the anti-plurality voting rule is the farthest away from the dictatorial rule; and (ii) the common plurality rule is the closest to the dictatorial rule.


## Framework

A set of alternatives
$A = \{1, \ldots, m\}$, $m \geq 3$

A set of voters
$N = \{1, \ldots, n\}$, $n \geq 3$

Voters have preferences over the set of alternatives:
$\mathcal{P}$ is the set of all linear orderings on $A$

$\mathcal{P}^n$ is the set of all preference profiles.

If $\succ \in \mathcal{P}^n$ and $i \in N$,

then $\succ_i$ is the preference ordering of voter $i$ over $A$.

A *voting rule* (or *social choice function*, SCF)

$f : \mathcal{P}^n \to A$

is aimed to select the „best" alternative,
taking as an input each voter´s preference profile.
Ties are broken by a fixed anonymous tie-reaking rule
$\tau$: $\mathcal{P}^n \to \mathcal{P}$

winning alternative
$f(\succ)$

There is a great variety of voting rules (SCFs), $\mathcal{F} = A^{\mathcal{P}}$ is the set of SCFs (plurality, Borda, etc).

Two ways of characterizing voting rules:

*Normative approach:*

to evaluate the voting rules in terms of certain desirable properties (axioms) they should satisfy (universal domain, Pareto efficiency, IIA, non-dictatorship)

A long list of impossibility results arises stating that
the only voting rule that satisfies simultaneously several natural desiderata
is **dictatorial voting rule**

Let $\mathcal{D} \subseteq \mathcal{F}$ be the set of dictatorial voting rules, where $\mathcal{D} = \{d_1, \ldots, d_n\}$

and $d_i$ is the dictatorial voting rule with voter $i$ as the dictator.

We aim to get away from the undesirable dictatorial rule, using the *distance function* between social choice functions.

We are searching for
the *least-dictatorial voting rule*

*Metric* or *distance-based approach*:

a voting rule can be characterized in terms of a *goal state* and a *metric* used in measuring the distance between the observed state (given voters' preferences) and the goal state.

e.g. the goal state is the election with a single, clear winner. To reach it, we define the distance function (metric) and search for the closest to the goal state election.

### Least-dictatorial voting rule

Domain restriction: $S \subseteq \mathcal{P}^n$ is the set of preference profiles with a clear winner (e.g. majority winner).

Then the values of SCFs have to be specified only on $\hat{S} = \mathcal{P}^n \setminus S$.

In order to define the *least dictatorial rule*, we use the following the *distance function* between SCFs:

$$\rho_S(f, g) = \#\{\succ \in \hat{S} \mid f(\succ) \neq g(\succ)\},$$

where $f$, $g$ are SCFs and $\rho_S(f, g)$ is the number of profiles on which $f$ and $g$ choose different alternatives.

We specify the set of least dictatorial rules by those ones which are the furthest away from the closest dictatorial rule:

$$\mathcal{F}_{ld}(S) = \{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F}: \min_{g \in \mathcal{D}} \rho_S(f, g) \geq \min_{g \in \mathcal{D}} \rho_S(f', g)\}$$

where $\mathcal{F}_{ld}(S)$ is the set of least-dictatorial voting rules for domain restriction $S$.

$g$ is the closest dictatorial rule, $f$ and $f'$ are two SCFs.

### Most-dictatorial voting rule

From an opposite point of view, we want to see if SCF that lets the voters to be a dictator in as many cases as possible could result in a desirable SCF. Thus, a measure

$$\mu(f, \mathcal{D}) = \Sigma_{\succ \in \mathcal{P}^n} \#\{i \in N \mid f(\succ) = d_i(\succ)\},$$

appears as a natural candidate, which we call a measure of conformity. Considering all profiles, $\mu(f, \mathcal{D})$

counts the number of cases in which a voter's top alternative is chosen.

Introducing the notation $\mu(f, g) = \Sigma_{\succ \in \mathcal{P}^n} 1_{f(\succ) = g(\succ)}$, where $1_{f(\succ) = g(\succ)}$ indicates whether the two chosen alternatives equal, we can obtain the following relationship between $\mu$ and $\rho$:

$$\mu(f, \mathcal{D}) = \Sigma_{\succ \in \mathcal{P}^n} \Sigma_{i \in N} 1_{f(\succ) = d_i(\succ)} = \Sigma_{i \in N} \mu(f, d_i) = n(m!)^n - \Sigma_{i \in N} \rho(f, d_i).$$

The set of *most-dictatorial voting rules* is defined as:

$$\mathcal{F}_{md} = \{f \in \mathcal{F} \mid \forall f' \in \mathcal{F}: \mu(f, \mathcal{D}) \geq \mu(f', \mathcal{D})\}$$
$$= \{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F}: \Sigma_{i \in N} \rho(f, d_i) \leq \Sigma_{i \in N} \rho(f', d_i)\}$$

## Results

Let $\tau$ be a fixed anonymous tie-breaking rule. Then the SCF $f^*_\tau$ is defined as follows: If there is a unique alternative being the fewest times on the top, then that alternative is the chosen one.
If not, disregard all alternatives that are not the fewest times on the top, and select the chosen one based on the given tie-breaking rule.
Clearly, this rule can only be just taken on a subset of profiles $\hat{S}$ in case of a domain restriction $S$.

**Proposition 1.** *Assume that $S$ is anonymous subdomain of $\mathcal{P}^n$. Then $f^*_\tau \in \mathcal{F}_{ld}(S)$. For any anonymous $f \in \mathcal{F}_{ld}(S)$, there exists a tie-breaking rule $\tau$, such that $f = f^*_\tau$ on $\hat{S}$.*

*Proof.* First, observe that $\Sigma_{i \in N} \rho_S(f, d_i) = \Sigma_{i \in N} \#\{\succ \in \hat{S} \mid f(\succ) \neq d_i(\succ)\} = \#\{(i, \succ) \in N \times \hat{S} \mid f(\succ) \neq d_i(\succ)\}$

$$= \Sigma_{\succ \in \hat{S}} \#\{i \in N \mid f(\succ) \neq d_i(\succ)\} \quad (1.1)$$

for any SCF $f$.
By the definition of $f^*_\tau$ we have

$$\forall \succ \in \mathcal{P}^n: \#\{i \in N \mid f^*_\tau(\succ) \neq d_i(\succ)\} \geq \#\{i \in N \mid f(\succ) \neq d_i(\succ)\} \quad (1.2)$$

Now taking the sums over $\hat{S}$ of (1.2) and then combining it with (1.1), we get

$$\Sigma_{i \in N} \rho_S(f^*_\tau, d_i) \geq \Sigma_{i \in N} \rho_S(f, d_i),$$

from which for any $i \in N$ it follows that

$$\rho_S(f^*_\tau, d_i) = 1/n\, \Sigma_{i \in N} \rho_S(f^*_\tau, d_i) \geq 1/n\, \Sigma_{i \in N} \rho_S(f, d_i) \geq \min_{i \in N} \rho_S(f, d_i)$$

since $f^*_\tau$ and $S$ are anonymous and the average is larger than the minimum; meaning that $f^*_\tau \in \mathcal{F}_{ld}(S)$.

Let $\tau$ be a fixed anonymous tie-breaking rule. Then the SCF $f^t_\tau$ is defined as follows:
If there is a unique alternative being the most times on the top, then that alternative is the chosen one.
If not, disregard all alternatives that are not the most times on the top, and select the chosen one based on the given tie-breaking rule.

The above specified rule is the plurality rule.

**Proposition 2.** *$f^t_\tau \in \mathcal{F}_{md}$. For any anonymous $f \in \mathcal{F}_{md}$, there exists a tie-breaking rule $\tau$ such that $f = f^t_\tau$.*

*Proof.* By the definition of $f^t_\tau$ we have

$$\forall \succ \in \mathcal{P}^n: \#\{i \in N \mid f^t_\tau(\succ) = d_i(\succ)\} \geq \#\{i \in N \mid f(\succ) = d_i(\succ)\} \quad (1.3)$$

For any $f \in \mathcal{F}$. Now summing (1.3) over $\mathcal{P}^n$, we get

$$\mu(f^t_\tau, \mathcal{D}) \geq \mu(f, \mathcal{D}) \quad (1.4)$$

from which follows that $f^t_\tau \in \mathcal{F}_{md}$.

For the second statement observe that if $f$ selects for at least one profile in $\mathcal{P}^n$ the alternative that is not the most times on the top, the inequality in (1.4) will be strict.

The tie-breaking rule $\tau$ can be selected in line with $f$.

## Concluding remarks

In this work we were interested in getting away from an undesirable dictatorial voting rule, by constructing the least-dictatorial voting rule.
We obtained two findings:
(i) the anti-plurality (the least-dictatorial) rule is the furthest away from the dictatorial rule, implying that being away from a „bad" rule is not necessary a sensible property as we end up with a very undesirable voting rule.
(ii) the common plurality (the most-dictatorial) rule is the closest to dictatorial rule, implying that the common plurality rule has a questionable property.

We considered a metric which did not take into account the whole preference profile.
A possible extension of the metric could be $\rho_{S,w}(f, g) = \Sigma_{\succ \in \hat{S}}\, w(\succ)\, 1_{f(\succ) \neq g(\succ)}$,

where the weight function $w$ could take into account the homogeneity of profile $\succ$ and $1_{f(\succ) \neq g(\succ)}$ indicates whether the two chosen alternatives differ.

We also plan to consider a social welfare functions instead of social choice functions, i.e. we care about the whole social ranking and not only about the socially best alternative.