The Complexity of Greedy Matching

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Abstract

- Motivated by the fact that in several cases a matching in a graph is stable if and only if it is produced by a greedy algorithm, we study the problem of computing a *maximum weight greedy matching* on weighted graphs, termed GreedyMatching. We prove that GreedyMatching is *strongly* **NP**-*hard* and **APX**-*complete*, and thus it does not admit a PTAS unless **P=NP**. Moreover we consider natural parameters of the problem, for which we establish a *sharp threshold* behavior between NP-hardness and tractability.
- On the positive side, we present a randomized approximation algorithm (Rgma) for GreedyMatching on a special class of weighted graphs, called *bush graphs*. We highlight an unexpected connection between Rgma and the approximation of maximum cardinality matching in unweighted graphs via randomized greedy algorithms.

Greedy Matching Procedure

Input: Graph $G = (V, E)$, with $w_1 > \ldots > w_\ell$ edge weight values
Output: Greedy matching $\mathcal{M}$

$\mathcal{M} \leftarrow \emptyset$
for $i = 1 \ldots \ell$ **do**
  while there is an $e \in E$ such that $w(e) = w_i$ **do**
    Pick an edge $e^* \in E$ with $w(e^*) = w_i$ and add it to $\mathcal{M}$;
    Remove all edges adjacent to e^* from E;

The problem

GreedyMatching
Instance: Graph $G = (V, E)$ with positive edge weights.
Task: Compute a maximum weight greedy matching $\mathcal{M}$ for G.

Hardness result

Unless $\mathbf{P} = \mathbf{NP}$, GreedyMatching admits no PTAS:

- even on graphs with maximum degree 3 and
- with at most three different integer weight values on their edges.

Parameters

1. **Number of different weights values.**
   - One weight value $\longrightarrow$ Maximum cardinality matching.
   - GreedyMatching is **NP**-hard
     - even on graphs with maximum degree 4,
     - with at most two different weight values,
     - and the graph is bipartite or planar.
2. **Minimum ratio between weights values.**
   We define $\lambda_i = \frac{w_i}{w_{i+1}}$ for every $i \in [\ell - 1]$ and $\lambda_0 = \min_i \lambda_i$
   - GreedyMatching can be solve in polynomial time if $\lambda_0 \geq 2$.
   - GreedyMatching is strongly **NP**-hard
     - for any constant $\lambda_0 < 2$,
     - even on graphs with maximum degree 3,
     - with at most three different integer weight values.
3. **Maximum edge cardinality.**
   a) $G(w_i)$ is the subgraph of G spanned by the edges of weight w_i.
   b) μ_i is the maximum edge cardinality of the connected components of $G(w_i)$.
   c) $\mu = \max_i \mu_i$.
   - $\mu = 1 \Rightarrow$ A unique solution for GreedyMatching.
   - GreedyMatching is strongly **NP**-hard and **APX**-complete
     - for $\mu \geq 2$,
     - even on graphs with maximum degree 3,
     - with at most five different integer weight values.

A Randomized Algorithm

Bush graph An *edge-weighted* graph $G = (V, E)$ with ℓ edge weight values $w_1 > w_2 > \ldots > w_\ell$ is a *bush graph* if, for every $i \in \{1, 2, \ldots, \ell\}$, the edges of $G(w_i)$ form a *star*, which we call the i-*th bush*.

Rgma Algorithm

Input: Bush Graph G with edge weight values $w_1 > w_2 > \ldots > w_\ell$.
Output: A greedy matching $\mathcal{M}_{RG}$.

$\mathcal{M}_{RG} \leftarrow \emptyset$
for $i = 1 \ldots \ell$ **do**
  if $G(w_i) \neq \emptyset$
    Select uniformly at random an edge $e_i \in G(w_i)$
    Add e_i to $\mathcal{M}_{RG}$
    Remove from G the endpoints of e_i

Maximum Cardinality Matching and Greedy Algorithms

Randomly pick the next (unweighted) edge in the matching $\Rightarrow$ approximation ratio:

- $\frac{1}{2}$: [Korte and Hausmann, 1978]
- $\frac{1}{2} + \frac{1}{400{,}000}$: [Aronson, Dyer, Frieze and Suen, *RSA***1995**]
- $\frac{1}{2} + \frac{1}{256}$: [Poloczek and Szegedy, *FOCS'***12**]

But: Experiments indicate a ratio close to $\frac{2}{3}$.
Apply Rgma on unweighted graphs

Bush Decomposition

Input: Unweighted graph $G = (V, E)$ and $\epsilon \ll \frac{1}{|V|^3}$.
Output: A (weighted) bush graph G^*.

Set $k \leftarrow 0$
while $E \neq \emptyset$ **do**
  Chose a random vertex $u \in V$
  For every $v' \in S := \{v' \in V : (u, v') \in E\}$ set $w(u, v') = 1 - k \cdot \epsilon$
  Remove the edges of S from E
  $k \leftarrow k + 1$

Theorem
Let ρ be the approximation guarantee of Rgma algorithm on bush graphs. Then, for every $\epsilon < 1$, Rgma computes a $(\rho - \epsilon)$ approximation of the maximum cardinality matching for unweighted graphs.

- We conjecture that $\rho = \frac{2}{3}$.
- Bush graphs offer new approach to a well studied problem.
- Might be helpful as bush graphs impose a *fixed* ordering of matching the vertices.

Open Questions

- Approximation guarantee of Rgma:
  - on bush graphs
  - on general weighted graphs
- Complexity of GreedyMatching for:
  - graphs of maximum degree 2
  - other parameters?
- *Deterministic* approximation algorithm for GreedyMatching

References

[1] Argyrios Deligkas, George Mertzios, and Paul Spirakis (2016).
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