Competitive fair division of bads, hairy ball theorem and concentration effects

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Fair division of divisible goods

Example: divorcing partners divide assets

Setting: divisible goods & additive utilities

- the set A of divisible goods is to be divided between the set of agents N
- x_a^i is a share of a good a obtained by agent i
  - each good is completely allocated: $\Sigma_{i\in N} x_a^i = 1$ for all a
- the total utility of agent i is

$$U^i = \Sigma_{a\in A} u_a^i x_a^i$$

Division rule: assigns an allocation x (or a feasible vector of total utilities $U = (U^i)_{i\in N}$) to a utility profile u.

- we do not distinguish x and x' if $U(x) = U(x')$

Competitive Equilibrium with Equal Incomes (CEEI rule)

Allocation x is CEEI iff there exists a vector of prices p s.t.

$$x^i = \text{argmax}_{z:\ \Sigma_a p_a z_a = 1} \Sigma_a u_a^i z_a,$$

i.e., all agents have equal budgets and each agent maximizes his total utility given prices and budget constraints.

- **Eisenberg-Gale optimization problem:** CEEI maximizes the Nash product $\Pi_{i\in N} U^i$ over all allocations.

CEEI rule is

1. Efficient
2. Envy-Free (every agents weakly prefers his allocation to the allocation of any other agent)
3. Single-valued (utilitywise)

The case of bads: so similar and so different

Example: substitutable workers get tasks

- The same formalization as for goods. But now $U^i = \Sigma_{a\in A} u_a^i x_a^i$ is the *disutility* obtained by agent i (he wants to minimize it)

A.Bogomolnaia, H.Moulin (2016):

1. CEEI can be defined in a similar way and always exists;
2. CEEI is Efficient and Envy-free;
3. **CEEI becomes multivalued** (utilitywise);
4. Negative results:
   - No single-valued rule is Efficient + Continuous + Envy-Free;
   - No single-valued rule is Efficient + Fair Share Guaranteed + Resource-Monotonic.

What do we do?

- Find the origin of multiplicity of CEEI allocation for bads
- Count the number of different CEEI mod 2
- Show that for large number of random bads multiplicity disappears with high probability

Multiplicity of CEEI

Extending Eisenberg-Gale result

CEEI for goods or for bads are the critical points of the Nash product $\Pi_{i\in N} U^i$

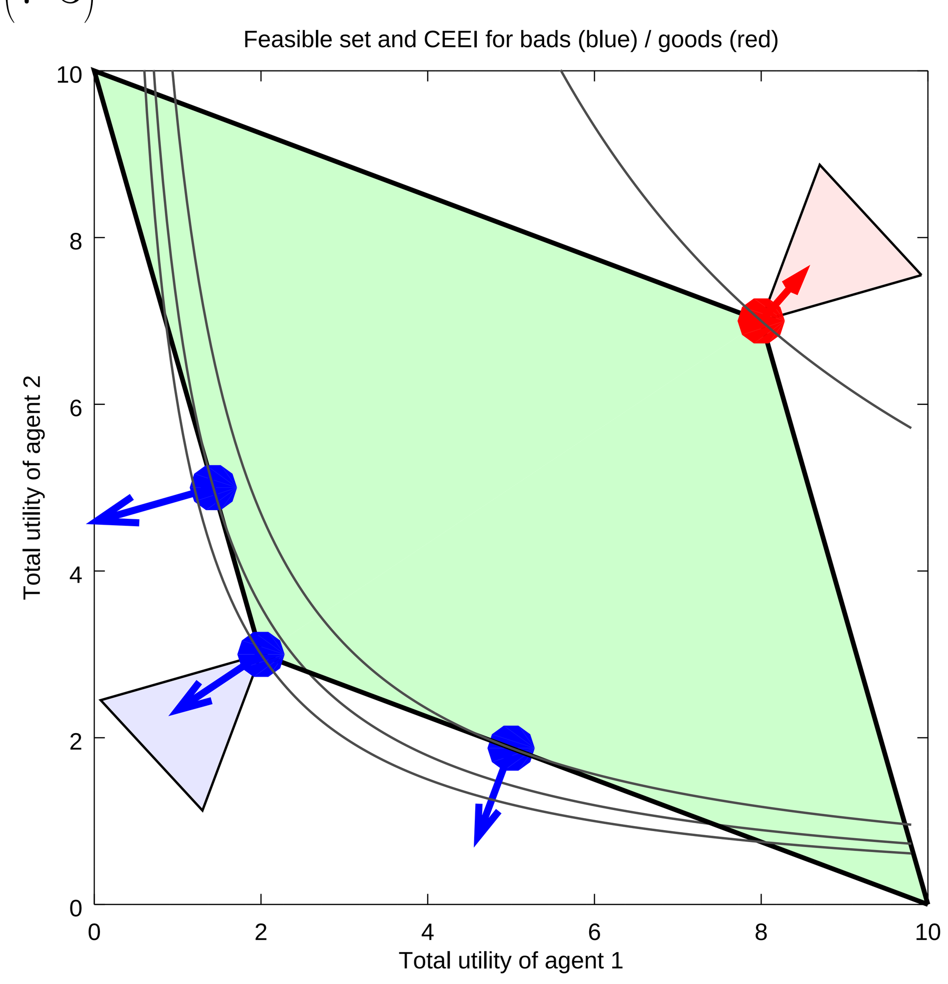
1. CEEI for goods is the global maximum
2. CEEI for bads are *local non-zero minima*

Remarks:

- concave function on a convex set can have many local minima but only one maximum;
- the global minimum $\Pi_{i\in N} U^i = 0$ corresponds to giving no bads to some agent.

Example: 2 agents & 2 objects (goods/bads)

$u = \begin{pmatrix} 2 & 8 \\ 7 & 3 \end{pmatrix} \Longrightarrow$ 3 CEEI (bads) +1 CEEI (goods)



Counting CEEI modulo 2

Typical oddness

In case of bads the number of different CEEI is odd for almost all utility profiles u (w.r.t. the Lebesgue measure over $\mathbb{R}_+^{N\times A}$).

- **Corollary:** In case of two agents, there is a natural median selector of CEEI correspondence.

Idea of the proof:

- CEEI for goods/bads $\Longleftrightarrow$ points of the feasible set such that the gradient of the Nash product $\Pi_{i\in N} U^i$ is orthogonal to the boundary.
- **Hairy ball (Poincare-Hopf) "theorem":** *if you comb a hairy ball, you produce an even number of cowlicks.*
- Interpret the gradient projected to the tangent space as an attempt to comb, then cowlicks are CEEI for goods/bads. □

Large number of random bads

- two agents and m bads, $m \to \infty$
- u_a^i are given by i.i.d. random variables uniformly distributed on $[0, 1]$ normalized to sum up to one

Concentration effects

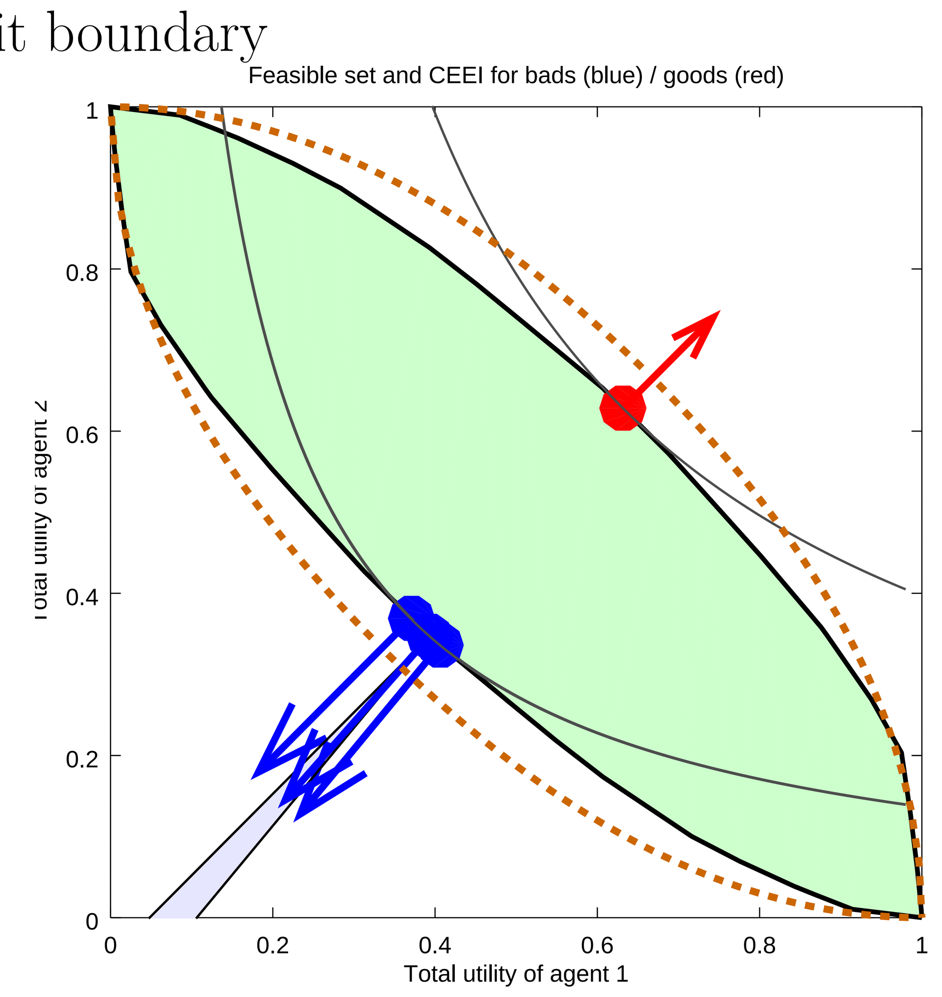
With probability that tends to 1, as $m \to \infty$:

1. for any $\varepsilon > 0$ the boundary B_m of the feasible set lies in ε-neighborhood of the limit boundary B_∞
   - anti-Pareto part of B_∞ is given by

$$U^2 = \frac{3}{4}\left(1 - U^1\right)^2 \quad \text{and} \quad U^1 = \frac{3}{4}\left(1 - U^2\right)^2;$$

2. all CEEIs for bads are concentrated in ε-neighborhood of the point $(1/3, 1/3)$, the equilibrium point of the limit cake-cutting problem.
   - **Interpretation:** in case of large number of small bads CEEI is essentially-unique.

Example of concentration effect: 2 agents & $m = 20$ objects; dotted line is the theoretical limit boundary



Conclusion

1. Similarly to the case of goods, CEEI for bads can be computed as a solution of Eisenberg-Gale-like optimization problem
2. But this problem is no longer convex (as in the case of goods) and one seeks for local extrema $\Longrightarrow$ multiplicity of CEEIs.
3. In a typical problem with bads the number of different CEEIs is odd.
4. In a typical problem with large number of small bads all CEEIs lie in a small ball, i.e., CEEI becomes essentially single-valued.

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