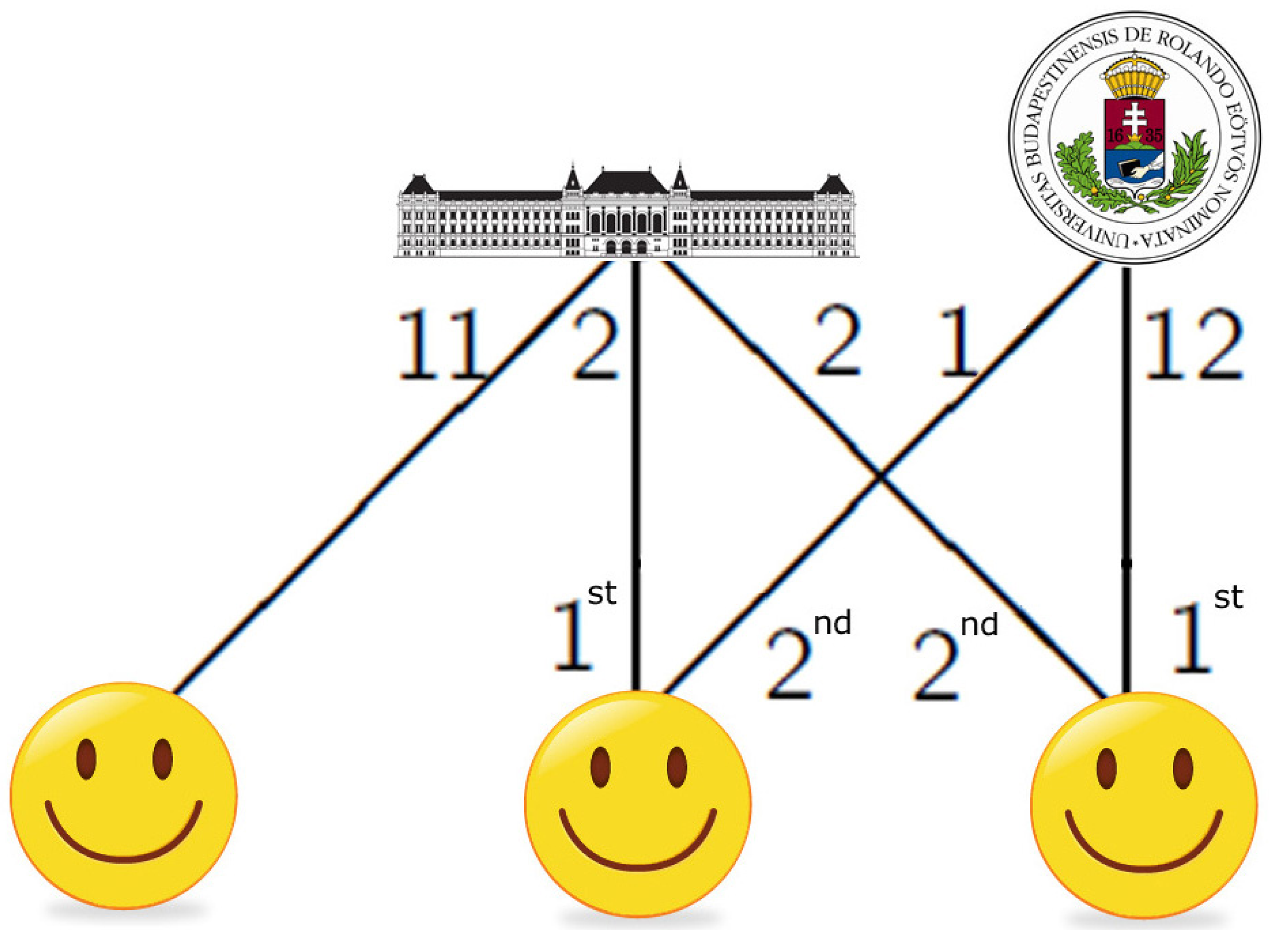
Choice Function-Based Two-Sided Markets

Tamás Fleiner, Zsuzsanna Jankó
Budapest University of Technology and Economics, Eötvös Loránd University

Motivation

Stable matching between colleges and applicants, based on scores on the entrance exams.



This stability notion is based on the Hungarian college admission scheme. There are n applicants $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and m colleges $\mathcal{C} = \{c_1, c_2, \ldots c_m\}$. The set of all possible applications (contracts) is E. Assume that each application $e = a_i c_j$ has score $s(e)$ (an integer, $1 \le s(e) \le M$) which is the score college c_j assigned to applicant a_i on the entrance exam. Each college has a quota.

Each college c_j declares a score limit t_j, and each student goes to the best school where she is accepted. The *score-vector* $\underline{t} = (t_1, t_2, \ldots, t_m)$ is *valid*, if no college is oversubscibed. The score vector $\underline{t}$ is *critical* if for any $1 \le j \le m$, the new score vector $(t_1, t_2, \ldots, t_{j-1}, t_j - 1, t_{j+1}, \ldots, t_m)$ is not valid for college c_j, or $t_j = 0$. We call $\underline{t}$ *stable* if it is both valid and critical. A contract-set $S \subseteq E$ is *score-stable*, if there exist a stable score-vector $\underline{t}$ such that the set of realized applications is S.

Choice functions

We will describe the preferences of each side in the market with choice functions: Set function $\mathcal{F} : 2^E \to 2^E$ is called a *choice function* if $\mathcal{F}(A) \subseteq A$ holds for any set of contracts $A \subseteq E$. I.e., when the set of possibilities is A, the agent picks contract-set $\mathcal{F}(A)$.

- A choice function, $\mathcal{F} : 2^E \to 2^E$, is *substitutable* if $(A \setminus \mathcal{F}(A)) \subseteq (B \setminus \mathcal{F}(B))$ for any $A \subseteq B$ sets of contracts.
- A choice function, $\mathcal{F} : 2^E \to 2^E$, satisfies *irrelevance of rejected contracts (IRC)*, if $\mathcal{F}(A) \subseteq B \subseteq A$ implies $\mathcal{F}(A) = \mathcal{F}(B)$.
- When contracts have scores, the choice function of colleges, $\mathcal{G}$, is called *loser-free* if any rejected contract has a lower score than any accepted contract.

For a choice function $\mathcal{F} : 2^E \to 2^E$, we can define the so-called determinant on the same ground set. The *canonical determinant* of $\mathcal{F}$ is defined as $\mathcal{D}_{\mathcal{F}}(X) := \{e | e \in \mathcal{F}(X \cup e)\}$. Here, $\mathcal{F}(Y) = Y \cap \mathcal{D}(Y)$ for every $Y \subseteq E$, and $\mathcal{D}_{\mathcal{F}}$ is the minimal such function.

Stability definitions

Consider a two-sided market, where the two sides have choice functions $\mathcal{F}$ and $\mathcal{G}$. There are various ways to define stability of a contract-set.

1. A set of contracts $S \subseteq E$ is *dominating stable* (or *pairwise stable*), if for every $x \notin S$, $x \notin \mathcal{F}(S \cup \{x\})$ or $x \notin \mathcal{G}(S \cup \{x\})$ i.e. one side of the market doesn't accept x if it is offered alongside S.
2. Subset S of E is *three-stable*, if there exists subsets A and B of E, such that $\mathcal{F}(A) = S = \mathcal{G}(B)$ and $A \cup B = E$, $A \cap B = S$. Pair (A, B) is called a *three-stable pair*, and S is a *three-stable set*.
3. Subset S of E is *four-stable*, if there exists subsets A and B of E, such that $A \cap B = S$ and $\mathcal{D}_{\mathcal{F}}(A) = B, \mathcal{D}_{\mathcal{G}}(B) = A$. We call the (A, B) pair fulfilling this property a *four-stable pair*.

Existence and lattice property

Statement 1 *If $\mathcal{F}$ and $\mathcal{G}$ are substitutable choice functions, a dominating stable set may not exist, but three-stable and four-stable sets always do. If $\mathcal{G}$ is also loser-free, there exist a score-stable set.*

We can define a partial order on contract-set-pairs, let $(A', B') \le (A, B)$, if $A' \subseteq A$ and $B' \supseteq B$.

Theorem 2 *If $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ are substitutable choice functions, then three-stable pairs form a nonempty complete lattice for partial order $\le$. The same is true for four-stable pairs.*

For choice function $\mathcal{F}$, let $S' \le_{\mathcal{F}} S$ if $\mathcal{F}(S \cup S') = S$. If $\mathcal{F}$ is substitutable and IRC, this gives a partial ordering.

Theorem 3 (Blair) *[1] If $\mathcal{F}, \mathcal{G} : 2^E \to 2^E$ are substitutable, IRC choice functions, then the dominating stable sets form a lattice for partial order $\le_{\mathcal{F}}$.*

If $\mathcal{F}$ and $\mathcal{G}$ are both IRC, the dominating stability, three-stability and four-stability are equivalent, so Blair's theorem holds for each of these notions.

Theorem 4 (Generalization of Blair's theorem) *If $\mathcal{F}$ and $\mathcal{G}$ are substitutable choice functions and $\mathcal{F}$ is IRC, then the four-stable sets form a non-empty lattice for partial order $\le_{\mathcal{F}}$.*

Theorem 5 *If choice functions $\mathcal{F}$ and $\mathcal{G}$ are substitutable and $\mathcal{G}$ is loser-free, then the score-stable sets form a non-empty lattice.*

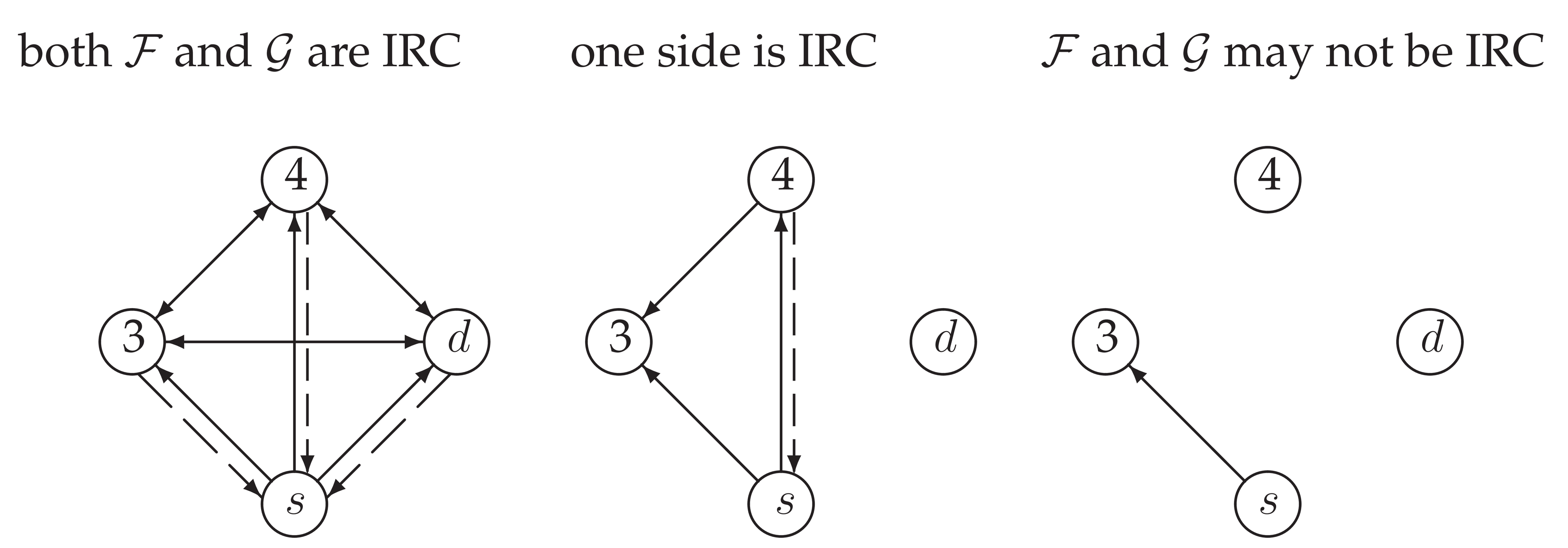
Connection between Stability Notions

We say the market is *simple* is there is only one possible contract between a given student-college pair. So, the underlying graph is simple.

Theorem 6 *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are substitutable choice functions. The implications between stability definitions can be described as in the figure below:*

- *If $\mathcal{F}$ and $\mathcal{G}$ are IRC, then three-stability, four-stability and dominating stability are equivalent.*
- *If $\mathcal{F}$ is IRC, then every four-stable set is three-stable.*
- *If $\mathcal{F}$ IRC and $\mathcal{G}$ is loser-free, then every score-stable set is also four-stable. Furthermore, if we require the market is simple, then score-stability is equivalent with four-stability.*
- *If $\mathcal{G}$ is loser-free, every score-stable solution is three-stable.*

In the notations, 3 stands for three-stable, 4 for four-stable, d for dominating stable and s for score-stable sets.



The solid lines denote implications that are true even if the market is not simple. The dashed lines denote the extra implications when the graph of possible contracts is simple. For all the implications that are not showed in the above picture, we can show a counterexample.

References

[1] Blair, C. The lattice structure of the set of stable marriages with multiple partners. *Math. Oper. Res.* **1988**, *13*, 619–628.

Acknowledgement

The authors kindly acknowledge the support of the EGRES Research Group. Research was supported by Hungarian Scientific Research Fund OTKA grant K 108383.