

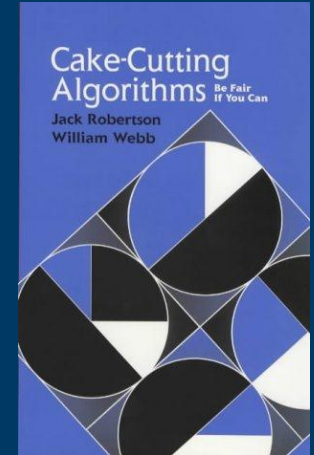
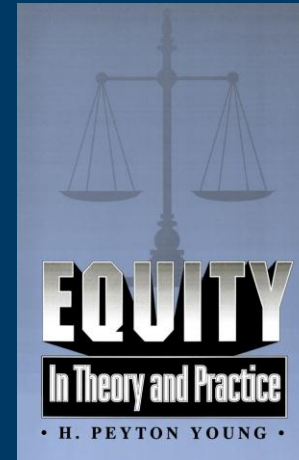
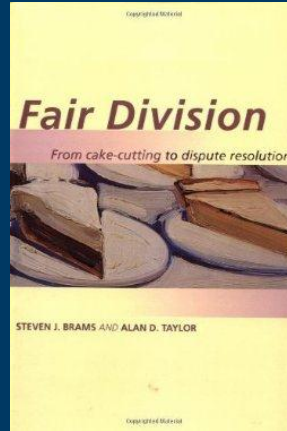
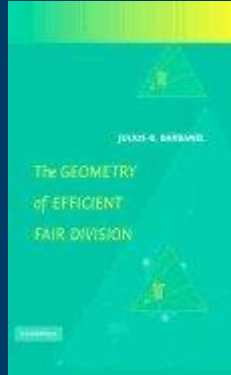
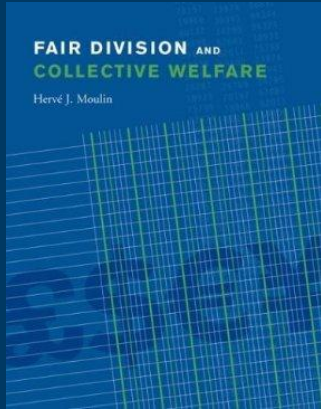
Dynamic Fair Division of Multiple Resources

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Fair Division



Homogeneous Divisible Goods



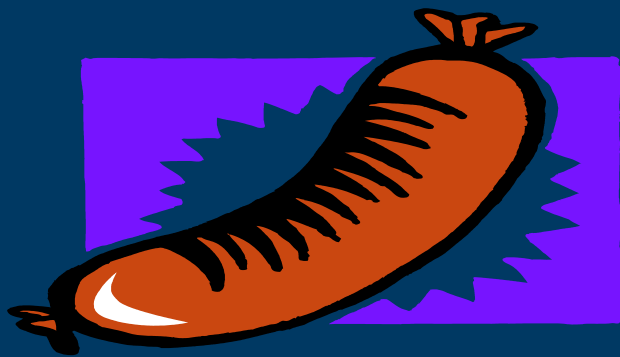
Axiomatic Approach

- 1) Sharing Incentives (SI) – Everyone gets $1/n$
- 2) Envy Freeness (EF) – Everyone prefers his own
- 3) Strategyproofness (SP) – Truth-telling is optimal
- 4) Pareto Optimality (PO) – Nothing wasted

General Impossibilities

- Theorem [Barberà and Jackson '95]: On domain of continuous, strictly quasiconcave and increasing utility functions, there **does not exist** an allocation mechanism that satisfies SI, PO, and SP.
- Theorem [Zhou '91]: With two agents and strictly convex monotonic utility functions, any allocation mechanism that satisfies PO + SP is **dictatorial** (not SI)

Leontief Utilities



Leontief Utilities



Leontief Utilities

$$U(x) = \min_r \frac{x_r}{d_r}$$

Results on Leontief

- 1) [Nicolò '04] – SI+EF+SP+PO for 2 agents+2 goods
- 2) [Dolev *et al.* '12] – “No Justified Complaints”
- 3) [Cole *et al.* '13] – SP + Approximate fairness
- 4) [Ghodsi *et al.* '11] – SI+EF+SP+PO via “DRF”

Dominant Resource Fairness

“Everyone gets the same share of his dominant resource”

Dominant Resource Fairness



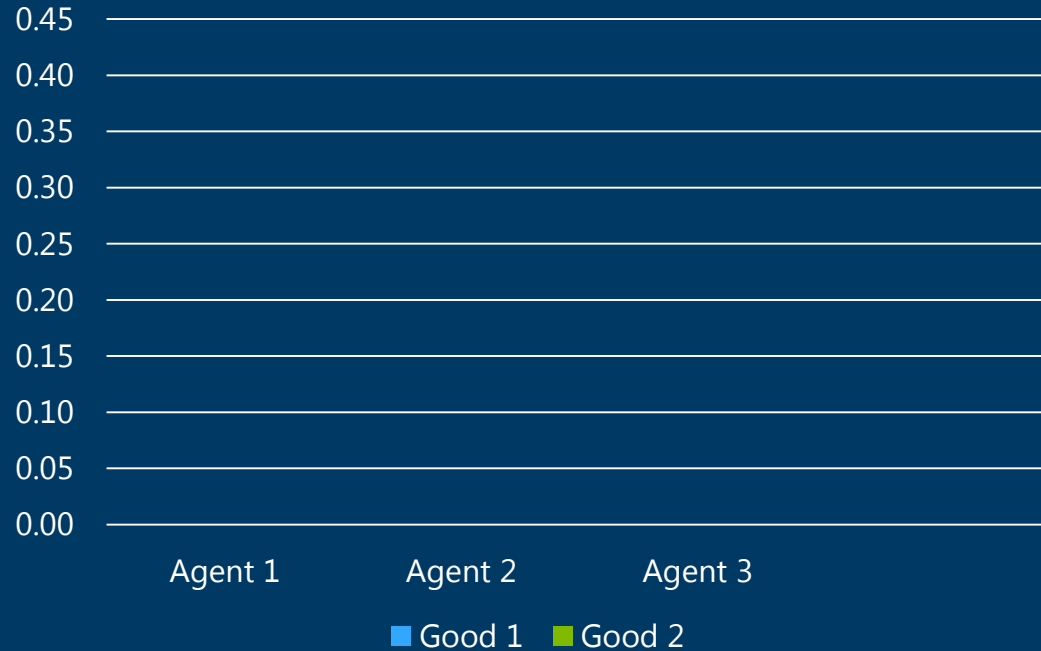
Dominant Resource Fairness



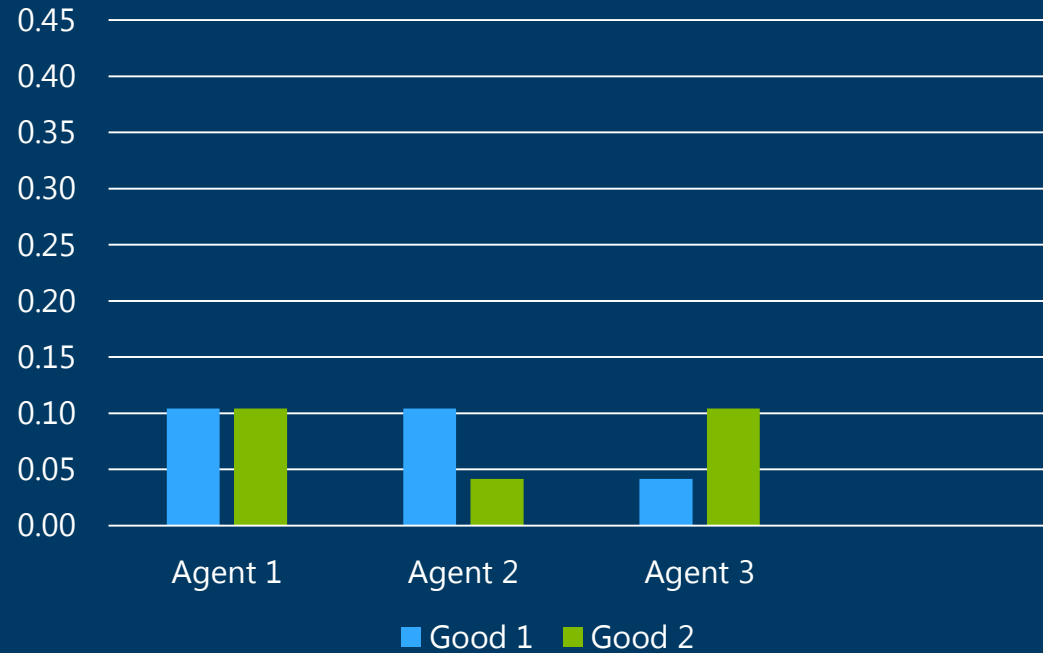
Dominant Resource Fairness



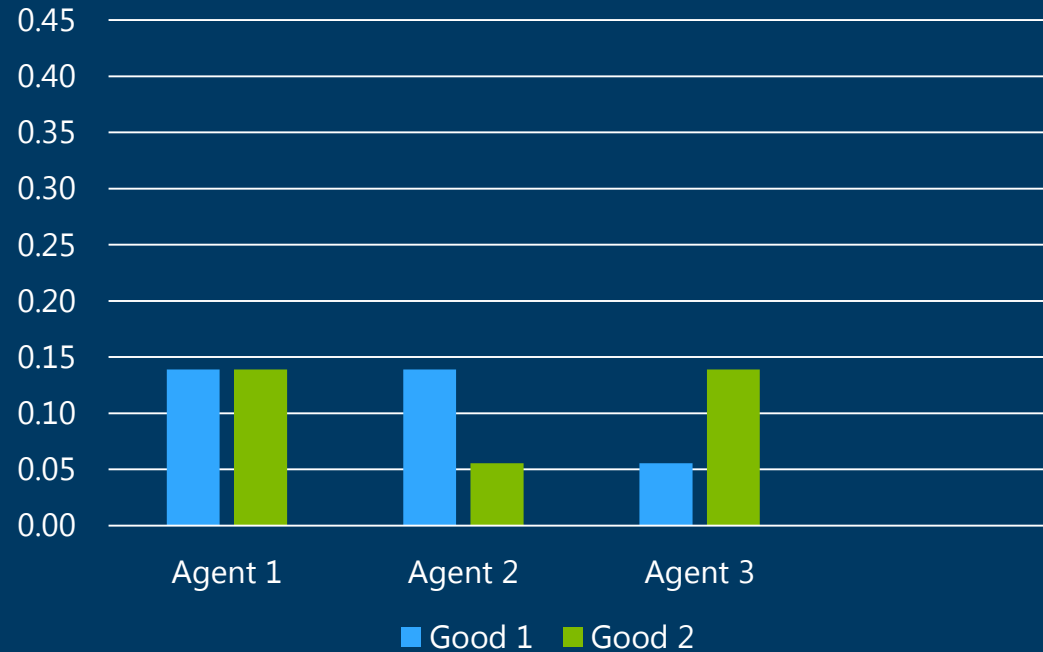
Dominant Resource Fairness



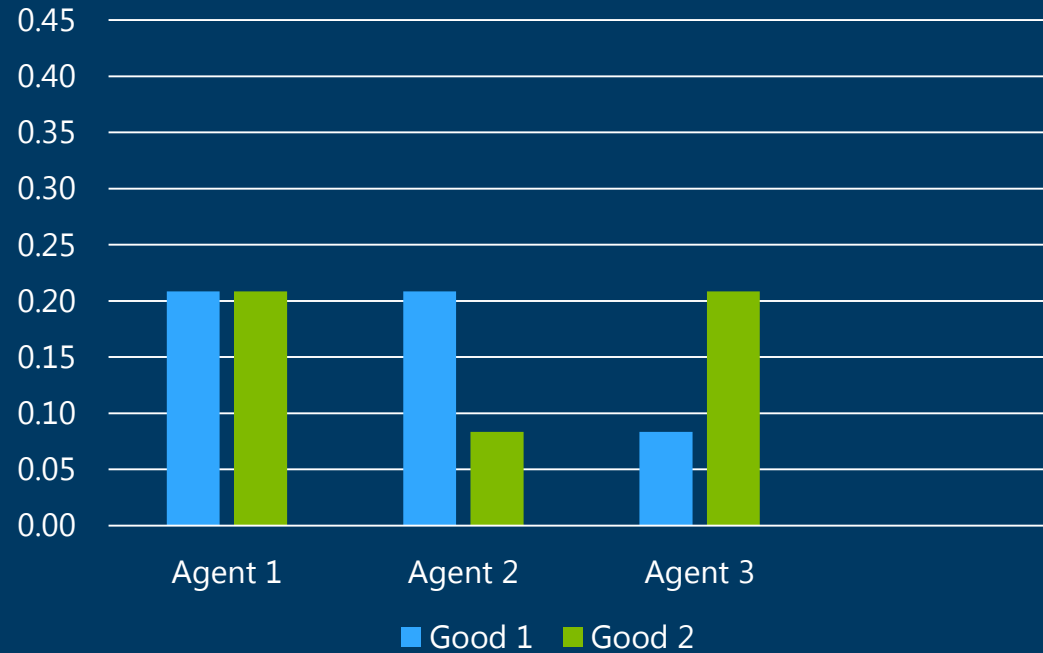
Dominant Resource Fairness



Dominant Resource Fairness



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Dominant Resource Fairness



Dominant Resource Fairness

$\max x$

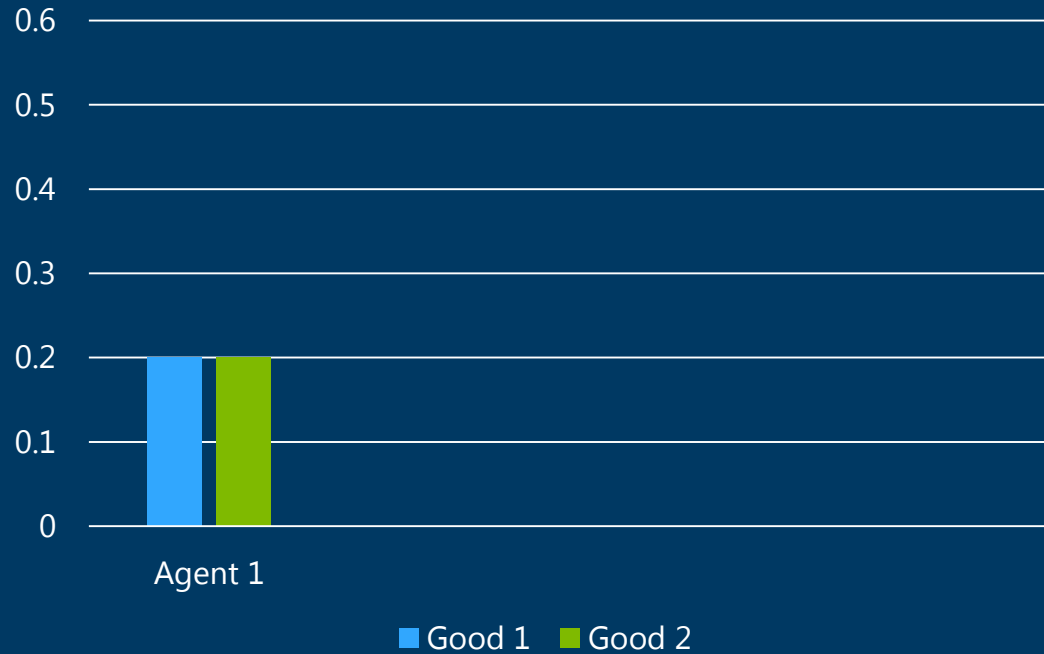
Subject to

$$\sum_i x d_{ir} \leq 1 \forall r$$

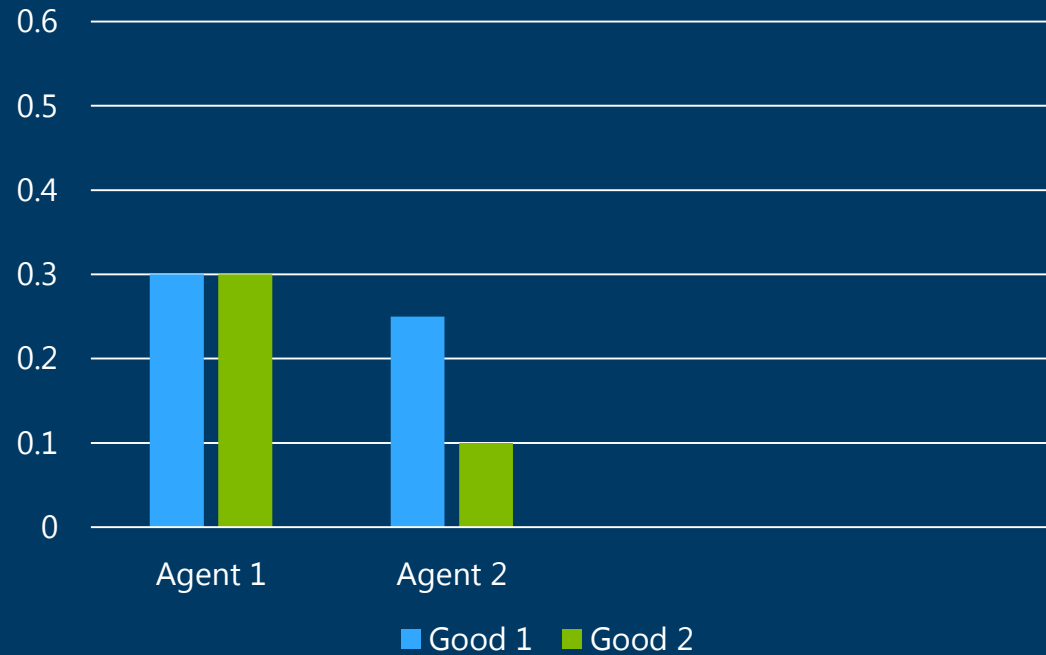
Dominant Resource Fairness

Theorem: DRF satisfies $SI + EF + SP + PO$

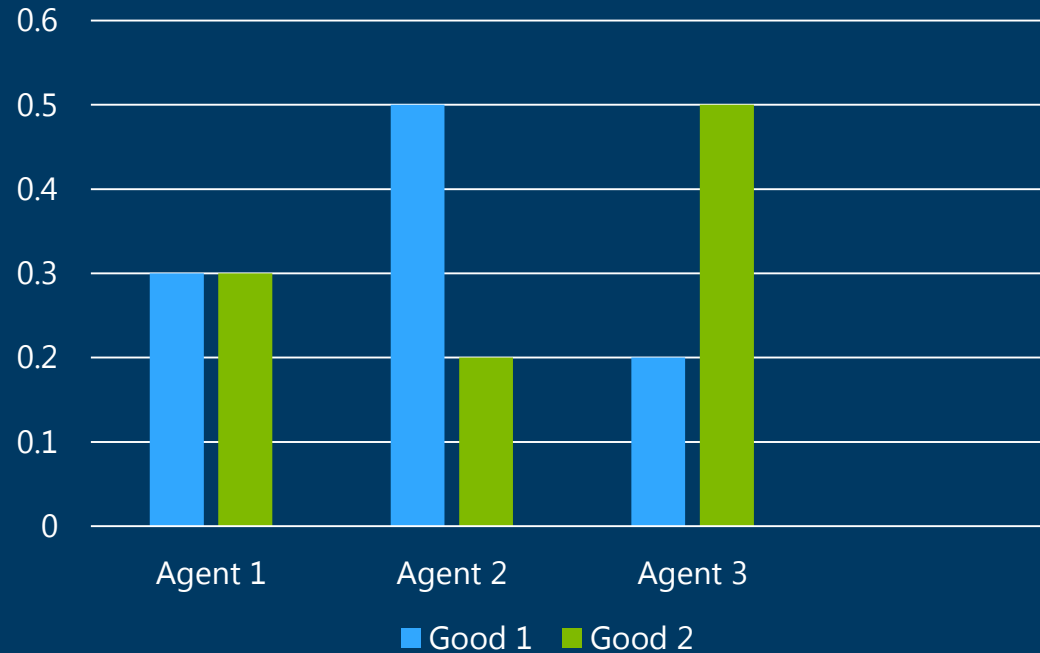
Dynamic Fair Division



Dynamic Fair Division



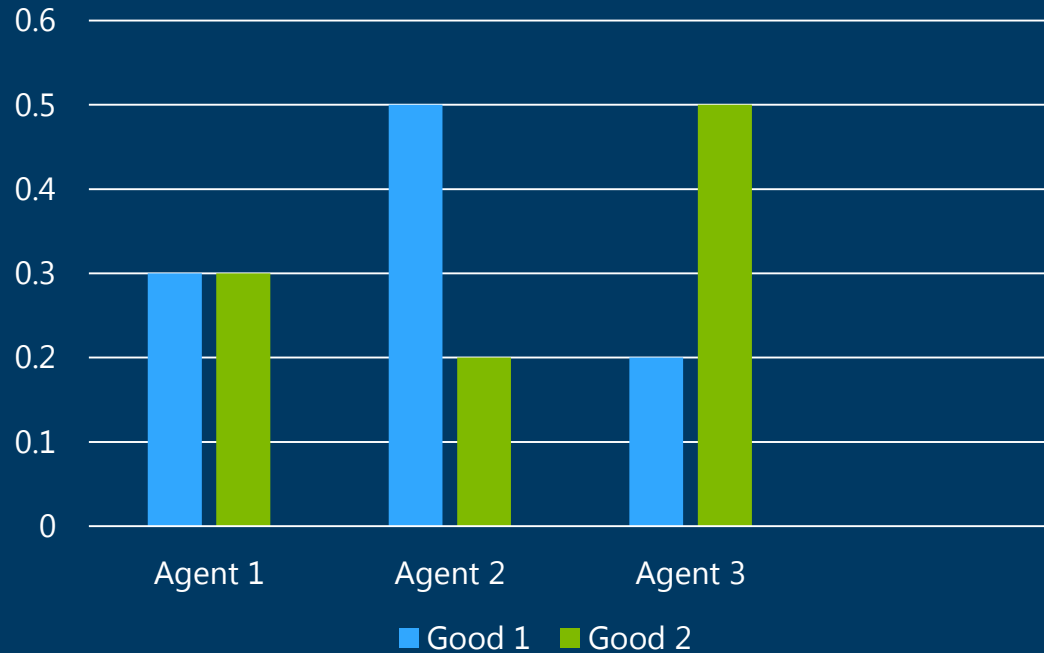
Dynamic Fair Division



Dynamic Fair Division

Assumptions:

- No departures
- Cannot reclaim resources
- Strictly Positive Demands



Axioms for Dynamic Fair Division

- 1) Sharing Incentives (SI) – Everyone gets $1/n$
- 2) Envy Freeness (EF) – Everyone prefers his own
- 3) Strategyproofness (SP) – Truth-telling is optimal
- 4) Pareto Optimality (PO) – Nothing wasted

Axioms for Dynamic Fair Division

- 1) Sharing Incentives (SI) – Everyone gets $1/n$
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- 3) Strategyproofness (SP) – Truth-telling is optimal
- 4) Dynamic PO (DPO) – k / n used PO in step k

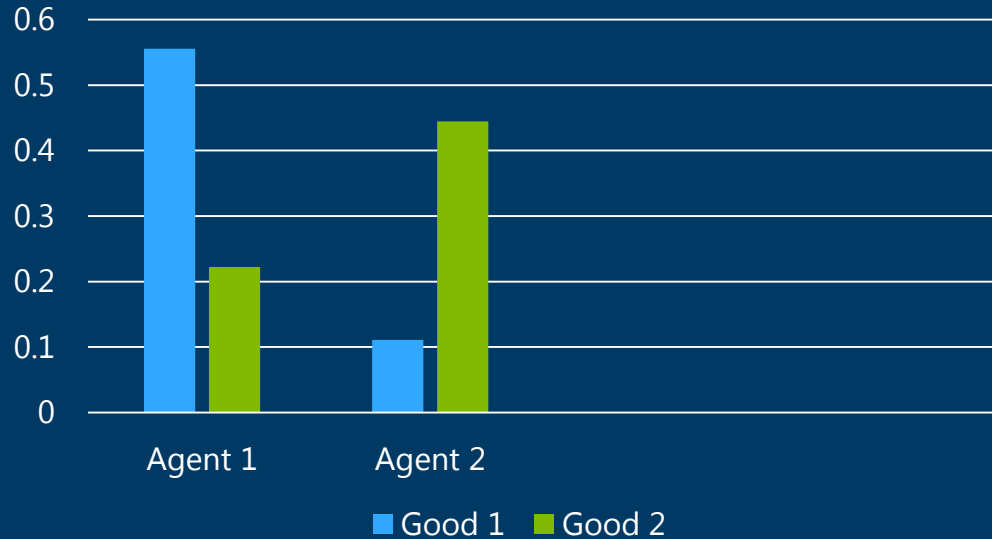
Another Impossibility

Theorem: With at least 3 agents and 2 resources, no dynamic mechanism satisfies EF + DPO.

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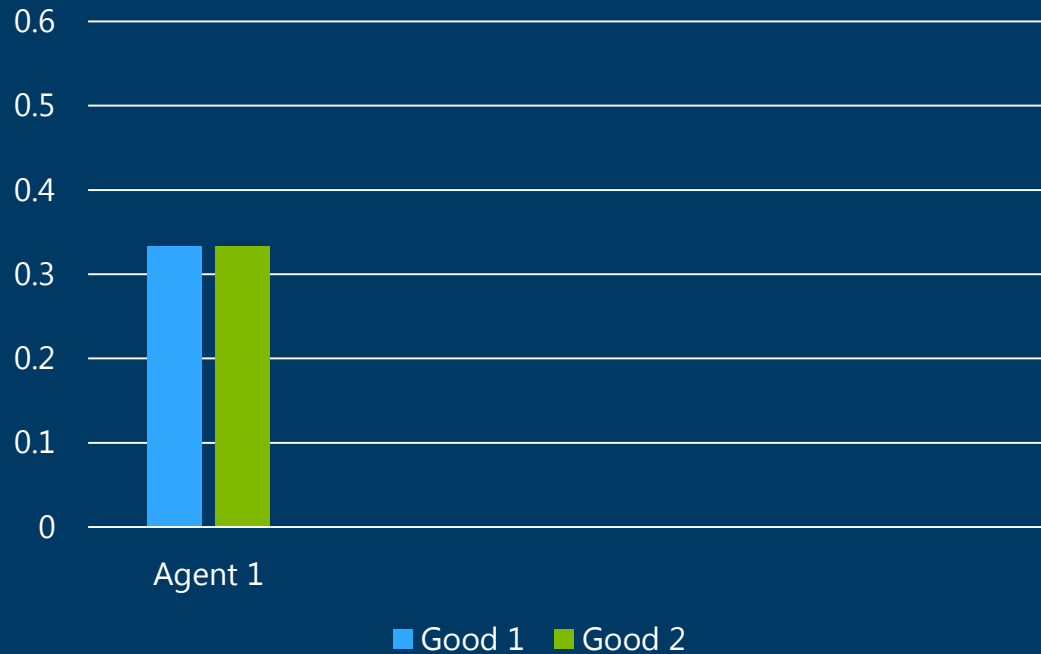
Can we do without?

- SI + EF + SP: "Equal Split"



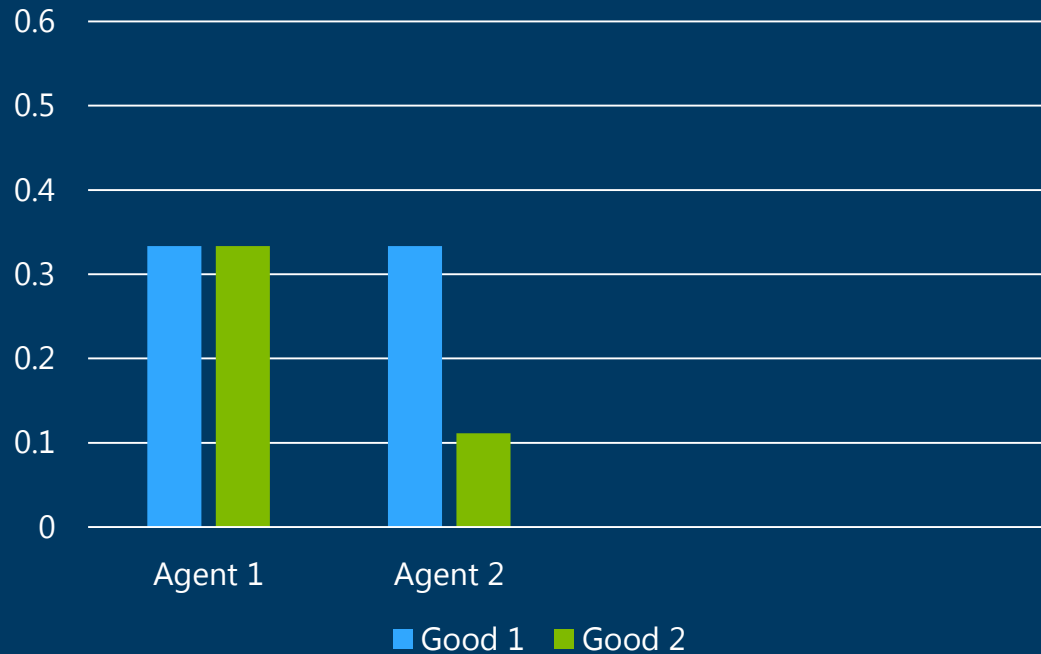
Can we do without?

- SI + DPO + SP: "Dynamic Dictatorship"



Can we do without?

- SI + DPO + SP: "Dynamic Dictatorship"



Can we do without?

- SI + DPO + SP: "Dynamic Dictatorship"

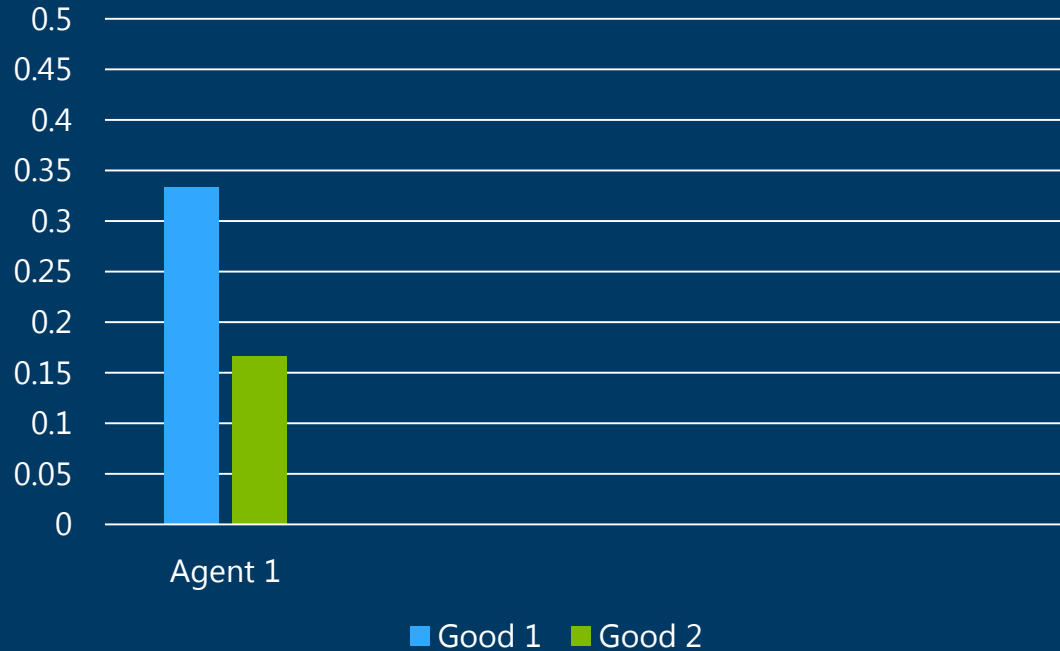


Dynamic Envy Freeness (DEF)

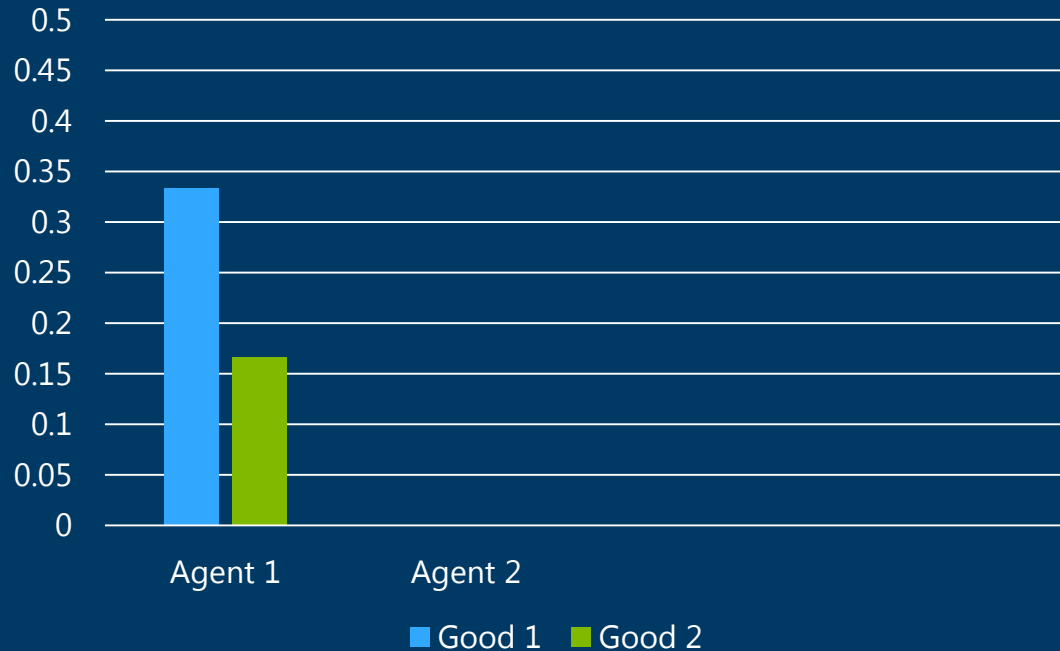
If agent i envies agent j then:

- 1) Agent j arrived before agent i
- 2) Agent j has not been allocated resources since i arrived

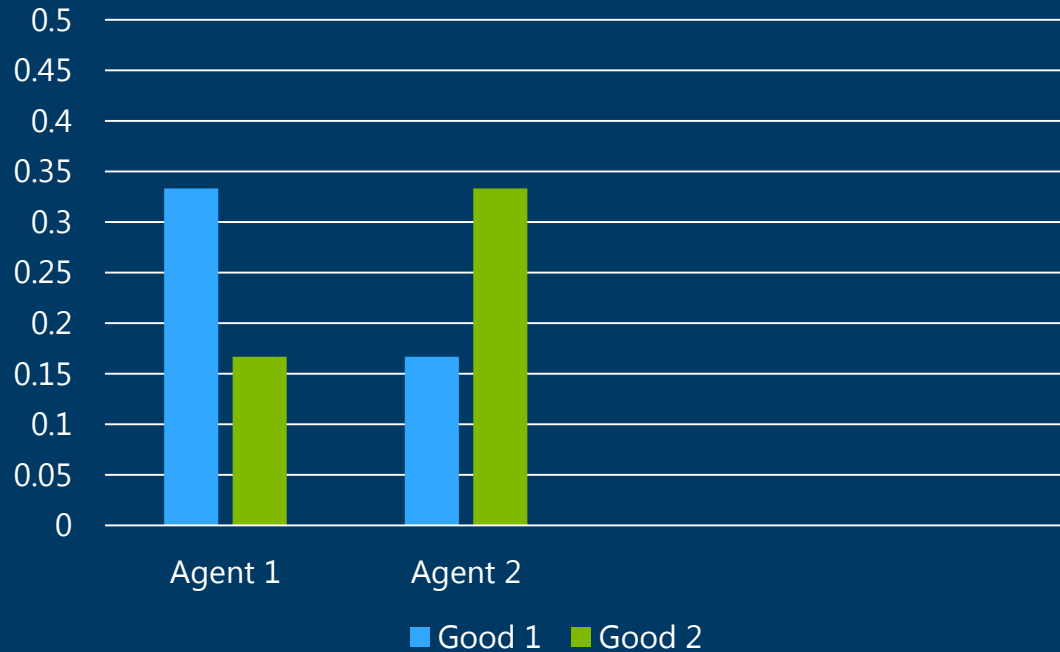
Dynamic DRF



Dynamic DRF



Dynamic DRF



Dynamic DRF



Dynamic DRF



Dynamic DRF



Dynamic DRF

Theorem: Dynamic DRF satisfies $SI + DEF + DPO + SP$

What if we really want fairness?

- Wait until end and run DRF?
- Can we do better?
- Yes, give each person $1/n$ of dominant resource on arrival.
- Can we do better yet?

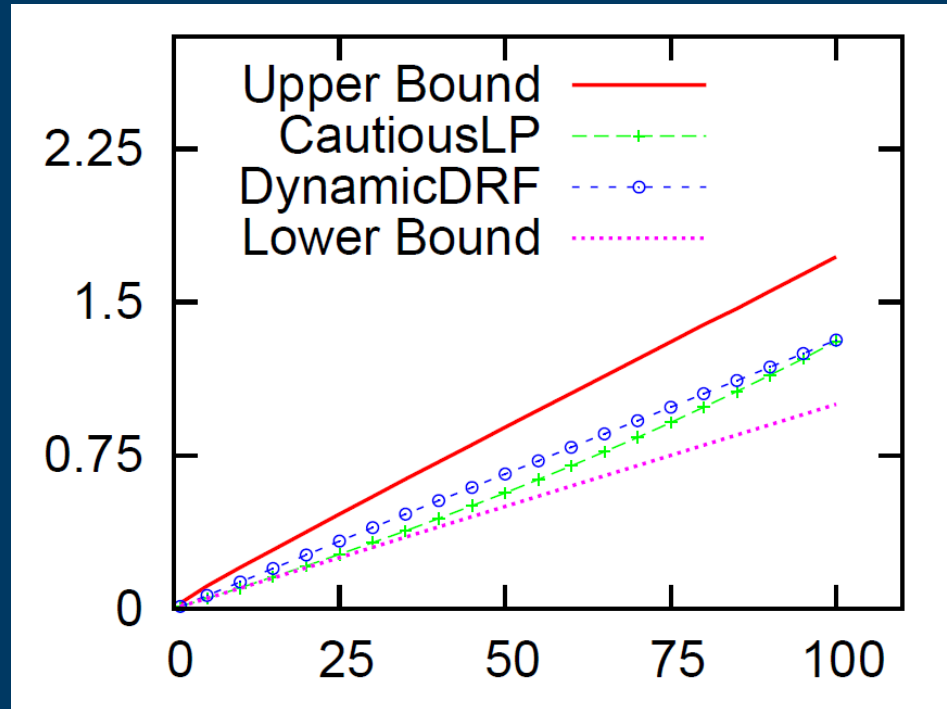
Cautious Dynamic Pareto Optimality

At every step, the mechanism returns an allocation that is not Pareto dominated by any allocation that can be **extended to an EF allocation** regardless of future demands.

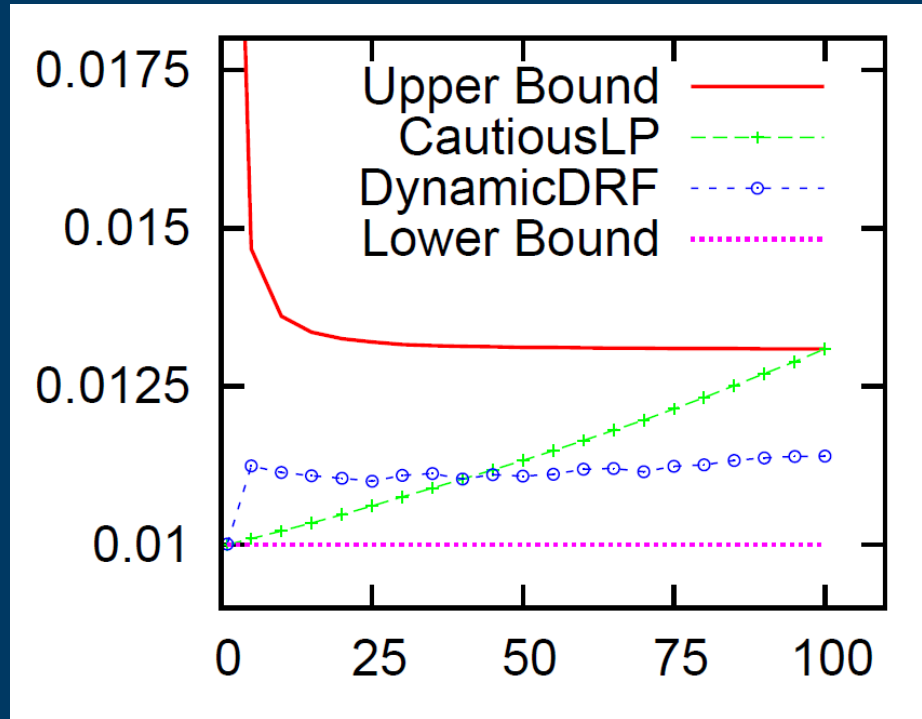
Envy-free extensions

Lemma: An allocation can always be extended to an envy-free allocation if and only if it can be extended when all future agents have the **same utility function** and this function is that of one of the **existing agents**.

Simulations



Simulations



Recap

- 1) Can't have EF + DPO.
- 2) Cautious LP guarantees EF + CDPO, but has low initial resource utilization.
- 3) Dynamic DRF guarantees DEF + DPO. It has higher initial resource utilization but is unfair.

Open Questions

- Zero Demands
- Indivisible Tasks
- Departures
- Reallocating Resources

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Thank You!

