New Results on Strategic Candidacy

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Example: Choosing the location of our next COST meeting

The Management Committee has to decide about the location of the next meeting. 4 candidates: Andorra (a), Monaco (m), San Marino (s), Vaduz (v). The local organizers of these cities have the following preferences regarding the outcome:

- $s : s \succ m \succ v \succ a$
- $a : a \succ m \succ v \succ p$
- $v : v \succ a \succ p \succ m$
- $m : m \succ p \succ v \succ a$

The committee uses plurality. The votes of the committee members are known (not unrealistic...).
Andorra should be elected (4 points).

2	1	2	2	2
a	a	s	v	m
m	s	v	p	a
v	m	m	a	s
s	v	a	m	v

One week before the deadline for candidacy, the four towns are candidates.

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San Marino realize it cannot win, but it can prevent Andorra from being elected. San Marino withdraws its candidacy (Vaduz should now be elected).

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Andorra likes Monaco better than Vaduz.
Andorra withdraws its candidacy.

2	1	2	2	2
a	a	s	v	m
m	s	v	s	a
v	m	m	a	s
s	v	a	m	v

Our next MC meeting will be located in Monaco.

Outline of the Talk

The framework

4 candidates

More than 4 candidates

The game-theoretical interpretation

We define a candidacy game as a normal form game $\Gamma = \langle X, P, r, P^X \rangle$ with m players, where:

- $X = \{x_1, \ldots, x_m\}$ is a set of candidates;
- $P = \langle \succ_1, \ldots, \succ_n \rangle$ is the profile expressing the voters' preferences over candidates;
- r is a deterministic voting rule (defined for an arbitrary number of candidates)
- $P^X = \langle \succ_{x_1}, \ldots, \succ_{x_m} \rangle$ is a profile expressing the candidates' preferences over candidates.

The strategy set available to each player is either 1 (running for the election) or 0 (not running).

Note: We use $adf \mapsto d$ to say that candidate d wins in the (restricted) profile consisting of candidates adf.

B. Dutta, M. Le Breton, M. O. Jackson. *Strategic candidacy and voting procedures*. Econometrica-2001.

Assumptions

1. each candidate may choose to run or not for the election;
2. each candidate has a preference ranking over candidates;
3. each candidate ranks himself on top of his ranking;
4. the candidates' preferences are common knowledge among them;
5. the outcome of the election as a function of the set of candidates who choose to run is common knowledge among the candidates.

Strategies and equilibria

- The set of strategy profiles is $S = \{0,1\}^m$; for instance, $s = (1,0,1)$ means that candidates x_1 and x_3 run and x_2 does not.
- For every strategy profile $s = (s_1, \ldots, s_n)$, let $X[s] = \{x_i \mid s_i = 1\}$, $P[s]$ be the restriction of P to $X[s]$, and $s^{\leftrightarrow i} = (s_1, \ldots, s_{i-1}, 1 - s_i, s_{i+1}, \ldots, s_m)$.
- A strategy s is an Nash equilibrium if

  there is no $i = 1, \ldots, n$ such that $r(P[s \leftrightarrow i]) \succ_{x_i} r(P[s])$

  [No running candidate would get a better outcome by withdrawing, and no non-running candidate would get a better outcome by running.]
- For $K \subseteq \{1, \ldots, n\}$, let $s^{\leftrightarrow K} = (s'_1, \ldots, s'_m)$, where $s'_i = s_i$ if $i \notin K$ and $s'_i = s_i$ if $i \in K$. A strategy s is a strong Nash equilibrium if

  there is no K such that $r(P[s \leftrightarrow K]) \succ_{x_i} r(P[s])$ for every $i \in K$.

  [No coalition of candidates can change their decisions so that they get an outcome that is better for all of them.]

Strategies and equilibria: example

- $X = \{a, b, c\}$
- $r =$ scoring rule associated with scoring vector $(5, 4, 0)$
- $P = \langle 2 : a \succ b \succ c, 2 : c \succ b \succ a, c \succ a \succ b\rangle$
- $ab \mapsto a$; $ac \mapsto c$; $bc \mapsto c$; $abc \mapsto b$;
- P^X:
  - $a : a \succ b \succ c$;
  - $b : b \succ c \succ a$;
  - $c : c \succ a \succ b$
- is $\{a, b, c\}$ an NE?

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- is $\{a, c\}$ an NE? no: it is in the interest of b to join;
- is $\{a, b\}$ an NE? yes: neither a nor b have an interest to leave, c has no interest to join.

Research questions

Natural research questions include:

- for which rules can $(1, \ldots, 1)$ be guaranteed to be an NE?
- for which rules can the existence of an NE be guaranteed?
- do natural (*e.g.* best response) dynamics converge?

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What is known so far...

- no non-dictatorial voting rule satisfying unanimity is candidacy-strategy-proof, that is, for every non dictatorial voting rule, there exists a candidacy game for which $(1, \ldots, 1)$ is not an NE.
- for the particular case of voting trees, there exist candidacy games with no NE.

B. Dutta, M. Le Breton, M. O. Jackson. *Strategic candidacy and voting procedures*. Econometrica-2001.

B. Dutta, M. Le Breton, M. O. Jackson. *Voting by successive elimination and voting procedures*. JET-2002.

Preliminary observations

First observe that with $m = 2$, $(1, 1)$ is a NE (the winner does not leave by narcissim, and the other would not affect the outcome by leaving).

- With $m \leq 3$ candidates, any candidacy game has an NE.
  $m = 2$: trivial. $m = 3$: Assume that $(1, 1, 1)$ is not NE. Then one candidate has an interest to leave. But then the two remaining candidates must be in NE
- If r is Condorcet-consistent and P has a Condorcet winner c, then $Y \subseteq X$ is an (S)NE if and only if $c \in Y$.
  c remains a Condorcet winner in any subprofile, no agent $x \neq c$ has an interest to join or leave X. And c has no interest to leave.

Preliminary observations

Does the result about 3 candidates still hold for **strong** Nash equilibria?

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Group manipulations:

- is $\{a, b, c\}$ an SNE?

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- is $\{a\}$ an NE?

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- is $\{b\}$ an NE? no (b joins);
- is $\{c\}$ an NE? no (a and b join).

Outline of the Talk

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4 candidates

More than 4 candidates

Scoring rules

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We make use of a (powerful!) result by Saari:

> *For "almost" all scoring rules, any conceivable choice function can result from a voting profile*

- Our problem thus boils down to check whether there exists at least one feasible choice function which, taken along with some candidates' preferences, exhibits no NE in our candidacy setting.

D. Saari. *A dictionary of voting paradoxes.* JET-1987.

Scoring rules

From one such feasible choice function, we can work out a profile for almost all scoring rules.

- For plurality and $m = 4$, there may be no NE.

3	1	1	1	1	1	1	2	2	a	b	c	d
d	d	d	a	a	a	b	b	c	a	b	c	d
c	b	a	b	c	d	c	a	b	b	a	d	a
a	c	b	c	b	b	d	c	d	c	c	a	b
b	a	c	d	d	c	a	d	a	d	d	b	c

... however Borda's voting rule stands out as an exception (basically, the sole exception) to Saari's result. (There is no such guarantee for Borda).

- How can we make sure that feasible choice functions can or cannot be rationalizable for Borda?

D. Saari. *A dictionary of voting paradoxes*. JET-1987.

Scoring rules: Borda

How can we deal with Borda?

- the idea is to exploit the fact that the Borda rule can be represented by a weighted majority graph—and we try to construct a feasible weighted majority graph corresponding to the choice function;
- we build an integer linear program with the following constraints:
  - in any state, there is be one winner;
  - in any state, one agent at least must deviate to another state.
  - a candidate deviating must prefer the winner in the new state;
  - preferences of candidates are transitive, irreflexive, narcissistic.
  - we include additional constraints into the ILP to account for the feasibility of a corresponding weighted majority graph.

$$\begin{aligned}&\forall s \in S, \forall i \in A(s), \forall j \in A(s) \setminus \{i\} :\\ &(1 - w_{s,i}) \times M + \sum_{j \in A(s) \setminus \{i\}} N_{i,j} \geq 1 + \sum_{j \in A(s) \setminus \{k\}} N_{k,j}\end{aligned}$$

- the infeasibility of the ILP tells us that there must always be an NE

For 4 candidates and the Borda rule, there is always an NE

Condorcet-consistent rules

We just assume Condorcet-consistency (CC).
Only 4 different tournaments, making case by case analysis possible.

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr] & b \arrow[d] \arrow[dl] \\ c \arrow[r] & d \end{tikzcd}$$

G_1

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr] & b \arrow[dl] \\ c \arrow[r] & d \arrow[u] \end{tikzcd}$$

G_2

$$\begin{tikzcd} a & b \arrow[l] \arrow[dl] \\ c \arrow[u] \arrow[r] & d \arrow[u] \arrow[ul] \end{tikzcd}$$

G_3

$$\begin{tikzcd} a \arrow[d] \arrow[dr] & b \arrow[l] \\ c \arrow[r] \arrow[ur] & d \arrow[u] \end{tikzcd}$$

G_4

- G_1 and G_2: a Condorcet-winner, any X containing a is NE
- G_3: Condorcet-loser a and a cycle $bcdb$. Wlog, $bcd \mapsto b$, but then take bc. No candidate wants to leaves, a does not join ($abc \mapsto b$ by CC), d does not join (since $bcd \mapsto b$). Hence bc is a NE.
- G_4: more tedious but can be shown in a similar manner.

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4 candidates

More than 4 candidates

More candidates

Do counter-examples transfer to a larger number of candidates?

More candidates

Do counter-examples transfer to a larger number of candidates?

- They do under a very mild assumption

Insensitiveness to bottom-ranked candidates (IBC)

Suppose a rule r elects x in a profile, then r must elect x if we add a new candidate at the bottom of every votes.

Satisfied by most usual voting rules (veto is an exception).

For any rule satisfying IBC, a profile without NE with m candidates can be extended to a profile with $m+1$ candidates without NE either.

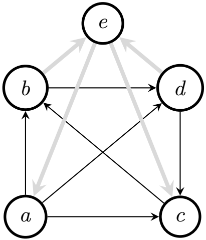
- So any negative (no NE) result for m transfer to $m' > m$.

What about positive results?

Copeland

- Copeland^0: the score of a candidate is the number of candidate it strictly beats in a pairwise comparison; winner: candidate with maximal score.
- Copeland^1: the score of a candidate is the number of candidate it beats or ties in a pairwise comparison; winner: candidate with maximal score.
- For any number m of candidates, and for $r =$ Copeland^0 and $r =$ Copeland^1, there always exists an NE.

Proof sketch. Let a be the Copeland^0 winner (tie-breaking winner among candidates with maximal Copeland^0 scores). Let M_P be the majority graph associated with P. Let $Dom(a)$ be the set of candidates beaten by a in M_P.
Claim: $Y = \{a\} \cup Dom(a)$ is a NE.



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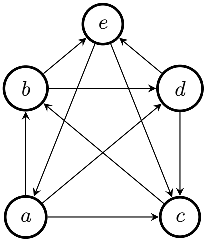

a is Condorcet winner in Y.
Noone wants to leave Y.

Assume $y \notin Y$ joins.
a maximizes the Copeland^0 score (=|Dom(a)|).
The score of a is unaffected by y.
Best score other candidates can hope to achieve: score of a but then they would be beaten by a due to tie-breaking.

Uncovered Set

- x is in the uncovered set if for any z that beats x in M_P there exists y such that x beats y and y beats z.
- uncovered set as a deterministic rule: winner = most prioritary candidate in the uncovered set.
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- Proof similar to the proof for Copeland

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We do not know any other rules than Copeland and the uncovered set for which the existence of an NE is guaranteed.

Maximin

For maximin with $m = 5$ there may be no NE.

Take the following weighted majority graph:

	a	b	c	d	e	a	b	c	d	e
a	$-$	1	4	2	3	a	b	c	d	e
b	4	$-$	1	4	3	c	e	d	a	b
c	1	4	$-$	2	2	b	c	a	c	a
d	3	1	3	$-$	0	e	a	e	b	d
e	2	2	3	5	$-$	d	d	b	e	c

We have $abcde \mapsto e$, the maximin winner. Furthermore, $abcd \mapsto a$, $abde \mapsto b$, $abce \mapsto e$, $acde \mapsto a$, $bcde \mapsto c$, ...

Other remarks

- importance of narcissism—relaxing the constraint of narcissism can have a huge impact on the results
  Example for Borda, with 9 voters. Only b is not narcissistic.

	a	b	c	d	a	b	c	d
a	–	5	6	3	a	c	c	d
b	4	–	8	5	b	d	a	a
c	3	1	–	6	c	b	d	b
d	6	4	3	–	d	a	b	c

- best response dynamics—even for those rules enjoying stability, we could exhibit cycles (no convergence) in best-responses dynamics.

Conclusion

- 3 candidates: always an NE;
- 4 candidates, Condorcet-consistent: always an NE;
- 4 candidates, Borda: always an NE;
- m candidates, almost all scoring rules (but not Borda): there may be no NE;
- m candidates, Copeland and uncovered set: always an NE;
- m candidates, maximin: there may be no NE;
- m candidates, any non dictatorial rule satisfying unanimity: (1,. . . , 1) may not be an NE [Dutta et al., 2001];
- m candidates, voting trees: there may be no NE [Dutta et al., 2002].

Relationship to control by adding/deleting candidates, but with consenting agents.