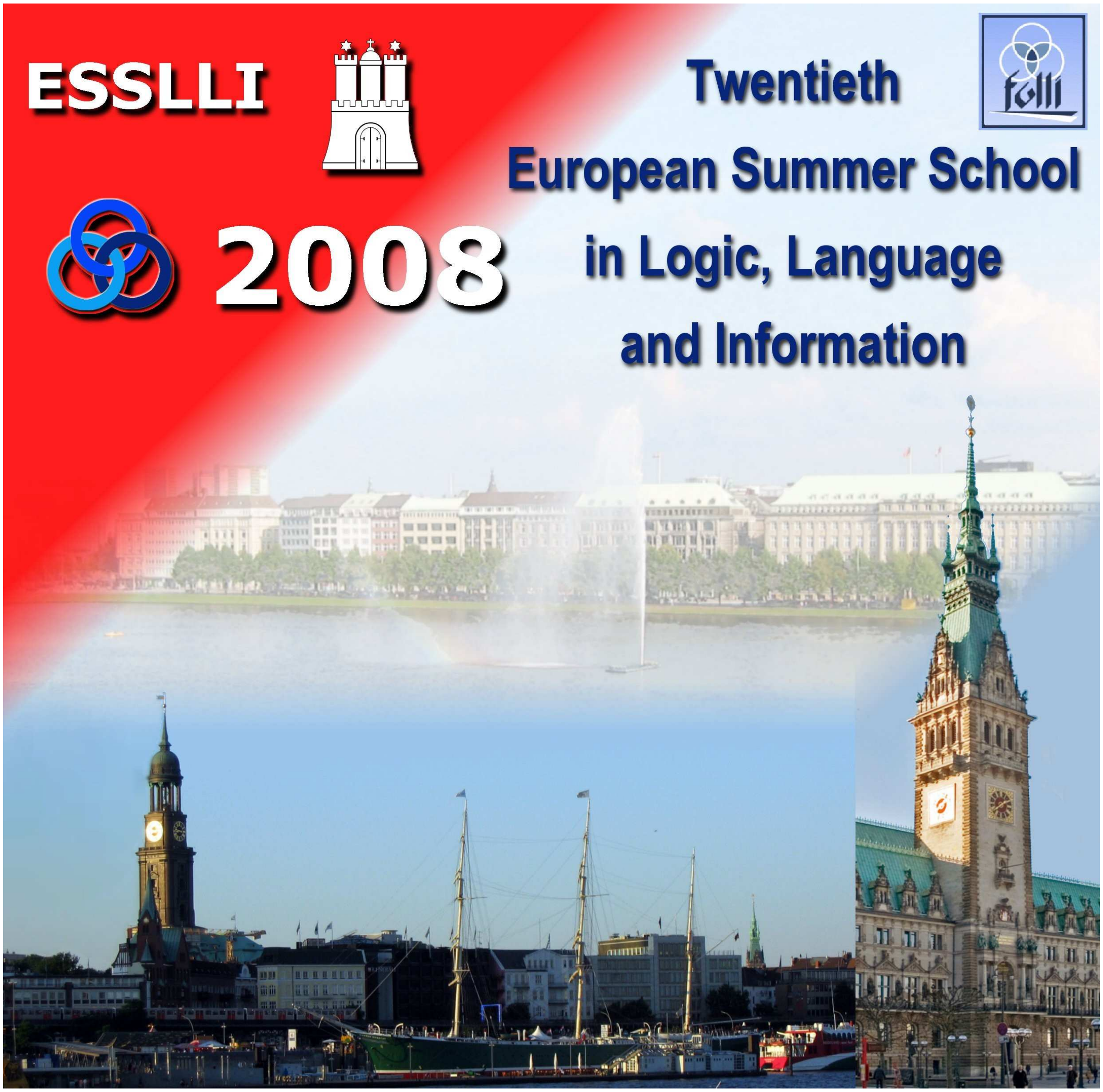

Model theory for extensions of modal logic

Balder ten Cate and David Gabelaia

ESSLLI 2008
20th European Summer School in Logic, Language and Information
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Balder ten Cate and David Gabelaia

Model theory for extensions of modal logic

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ESSLLI 2008

Table of content

About this reader

The main purpose of this reader is to provide a convenient source of reference for some of the things that will be discussed in the course. We do not intend to cover all of this material in the course.

This reader has been compiled out of material from the following three sources:

- Balder ten Cate (2005). Model theory for extended modal languages. PhD thesis, University of Amsterdam. Available from `http://staff.science.uva.nl/~bcate`
- David Gabelaia (2001). Modal definability in topology. Master's thesis, University of Amsterdam. Available from `http://www.science.uva.nl/pub/theory/illc/researchreports/MoL-2001-11.text.pdf`
- Balder ten Cate, David Gabelaia, and Dmitry Sustretov (2006). Modal languages for topology: expressivity and definability. Available from `http://arxiv.org/abs/math/0610357`

Part I

Modal model theory

Chapter 1

Background

1.1 Basics of model theory

This section reviews a number of important results on the model theory of first order logic that are used in proofs throughout these course notes. For a more detailed treatment, cf. [17, 9]. We assume that the reader is familiar with the syntax and semantics of first-order logic. We will only consider first-order languages with constants and relation symbols and without function symbols of arity greater than zero. We will denote first-order models (or, structures) as pairs $\mathfrak{M} = (D, I)$ consisting of a domain D and an interpretation function I that assigns relations of the appropriate arity to the relation symbols and that assigns elements of D to constants. Given such a structure $\mathfrak{M}$ and a first-order formula $\varphi(x_1, \ldots, x_n)$, we will write $\mathfrak{M} \models \varphi\ [d_1, \ldots, d_n]$ if $d_1, \ldots, d_n$ are elements of the domain of $\mathfrak{M}$, such that φ holds in $\mathfrak{M}$ interpreting $x_1, \ldots, x_n$ as $d_1, \ldots, d_n$.

The first three results are easily stated.

1.1.1. Theorem (Compactness). *Let Σ be a set of first-order formulas. If every finite subset of Σ has a model, then Σ has a model.*

1.1.2. Theorem (Löwenheim-Skolem). *Let Σ be a countable set of first-order formulas. If Σ has a model then Σ has a countable model.*

1.1.3. Theorem (Craig Interpolation). *Let φ, ψ be first-order formulas, such that $\models \varphi \to \psi$. Then there is a formula ϑ such that $\models \varphi \to \vartheta$, $\models \vartheta \to \psi$ and all constants, relation symbols and function symbols occurring in ϑ occur both in φ and in ψ.*

For the remaining results we need to introduce some terminology. A model $\mathfrak{M}$ is a *submodel* of a model $\mathfrak{N}$ if the domain of $\mathfrak{M}$ is a subset of the domain of $\mathfrak{N}$ and the interpretations of every non-logical symbol in $\mathfrak{M}$ is simply the restriction of its interpretation in $\mathfrak{N}$ with respect to the domain of $\mathfrak{M}$. It follows that if an element of the domain of $\mathfrak{N}$ is named by a constant, then it is also in the domain of $\mathfrak{M}$. We say that $\mathfrak{M}$ is an *elementary submodel* of $\mathfrak{N}$ if it is a submodel, and for all first-order formulas $\varphi(x_1, \ldots, x_n)$ and elements $d_1, \ldots, d_n$ of the domain of $\mathfrak{M}$, $\mathfrak{M} \models \varphi\ [d_1, \ldots, d_n]$ iff $\mathfrak{N} \models \varphi\ [d_1, \ldots, d_n]$. In this case, we also say that $\mathfrak{N}$ is an *elementary extension* if $\mathfrak{M}$.

Given a set of models $\{\mathfrak{M}_i \mid i \in I\}$ for a relational language (i.e., without constants or function symbols), the union $\mathfrak{N} = \bigcup_{i \in I} \mathfrak{M}_i$ is defined in the natural way: the domain of $\mathfrak{N}$ is the union of the domains of $\mathfrak{M}_i$ $(i \in I)$, and the same holds for the interpretation of the relation symbols. In general, this notion can only be applied to models for relational languages. However, there are circumstances in which it can also be applied to models for languages containing constants and function symbols. An example of this is the following situation.

1.1.4. Theorem (Unions of elementary chains). *Let $(\mathfrak{M}_k)_{k \in \omega}$ be a sequence of models, such that $\mathfrak{M}_k$ is an elementary submodel of $\mathfrak{M}_{k+1}$ for all $k \in \omega$, and let $\mathfrak{M}_\omega$ be the union $\bigcup_{k \in \omega} \mathfrak{M}_k$. Then for each $k \in \omega$, $\mathfrak{M}_k$ is an elementary submodel of $\mathfrak{M}_\omega$.*

NB: $\bigcup_{i \in I} \mathfrak{M}_i$ should not be confused with the *disjoint union* of the models $\mathfrak{M}_i$ $(i \in I)$. In fact, for the above result crucially relies on the non-disjointness of the models in question.

An *ultrafilter* over a set W is a set $U \subseteq \wp(W)$ satisfying three conditions:

1. $W \in U$
2. For all $X \subseteq W$, $X \in U$ iff $(W \setminus X) \notin U$
3. For all $X \in U$ and $Y \in U$, $X \cap Y \in U$

An ultrafilter is *principal* if has a singleton element.

1.1.5. Definition (Ultraproducts). *Given a collection of models $\{\mathfrak{M}_a = (D_a, I_a) \mid a \in A\}$ and an ultrafilter U over the set A, the following defines the ultraproduct $\Pi_U \mathfrak{M}_a = (D, I)$.*

Let $\sim$ be the equivalence relation $\sim$ on the product $\Pi_{a \in A} D_a$ given by

$$f \sim g \text{ iff } \{a \in A \mid f(a) = g(a)\} \in U$$

Let D be the quotient $(\Pi_{a \in A} D_a)/\sim$. For each constant c, let

$$I(c) = [\langle I_a(c) \rangle_{a \in A}]_\sim$$

Finally, for each k-ary relation R and $[f_1], \ldots, [f_k] \in D$, let

$$([f_1], \ldots, [f_k]) \in I(R) \text{ iff } \{a \in A \mid (f_1(a), \ldots, f_k(a)) \in I_a(R)\} \in U$$

If all factor models $\mathfrak{M}_a$ are the same, then $\Pi_U \mathfrak{M}_i$ is called an *ultrapower.* Every model $\mathfrak{M}$ is isomorphic to a submodel of the ultrapower $\Pi_U \mathfrak{M}$, the isomorphism being the function that sends every element d to the equivalence class $[\langle d, d, \ldots \rangle]_\sim$.

1.1.6. Theorem (Łos). *For all models $\mathfrak{M}$, ultrafilters U and first-order sentences φ, $\Pi_U \mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \varphi$*

Related to ultraproducts are the simpler notions of products and subdirect products, which will also play a role in this thesis.

1.1.7. Definition (Products and subdirect products). *The product of a collection of models $\{\mathfrak{M}_a = (D_a, I_a) \mid a \in A\}$, (also called* cartesian product *or* direct product, *notation: $\Pi_{a \in A} \mathfrak{M}_a$) is the model (D, I), where D is the cartesian product $\Pi_{a \in A} D_a$, and for each n-ary relation R,*

$$I(R) = \{\langle d_1, \ldots, d_n \rangle \in D^n \mid \langle d_1(a), \ldots, d_n(a) \rangle \in I_a(R) \textit{ for each } a \in A\}$$

A subdirect product *of $\{\mathfrak{M}_a \mid a \in A\}$ is any submodel $\mathfrak{N}$ of the product $\Pi_{a \in A} \mathfrak{M}_a$ for which it holds that the natural projection functions from the domain of $\mathfrak{N}$ to the domains of the models $\mathfrak{M}_a$ ($a \in A$) are surjective.*

The next notion we introduce is that of *ω-saturatedness.* A *1-type* is a set of formulas in one free variable. A 1-type $\Gamma(x)$ is *realized* in a model $\mathfrak{M}$ if there is an element d of the domain of $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma\ [x : d]$. A model is said to be *1-saturated* if for all 1-types $\Gamma(x)$, if every finite subset of $\Gamma(x)$ is realized in $\mathfrak{M}$, then $\Gamma(x)$ itself is realized in $\mathfrak{M}$. One can think of 1-saturatedness as a sort of *compactness within a model.*

Given a model $\mathfrak{M}$ and a finite sequence $d_1, \ldots, d_n$ of elements of the domain of $\mathfrak{M}$, we use $(\mathfrak{M}, d_1, \ldots, d_n)$ to denote the expansion of $\mathfrak{M}$ in which the elements $d_1, \ldots d_n$ are named by additional constants $c_1, \ldots, c_n$ (each new constant c_k denotes the corresponding element d_k in the expanded model). A model $\mathfrak{M}$ is *ω-saturated* if every such expansion $(\mathfrak{M}, d_1, \ldots, d_n)$ (with $n \in \omega$) is 1-saturated. Note that we use ω and $\mathbb{N}$ interchangably to denote the set of non-negative integers.

1.1.8. Theorem (ω-Saturation). *Every model $\mathfrak{M}$ has an ω-saturated elementary extension $\mathfrak{M}^+$. In fact, $\mathfrak{M}^+$ can be constructed such that it is isomorphic to an ultrapower of $\mathfrak{M}$.*

It should be noted that this result holds regardless of the cardinality of the language (i.e., the number of non-logical symbols) [8, Theorem 6.1.4 and 6.1.8].

We say that two models, $\mathfrak{M}, \mathfrak{N}$ are *elementarily equivalent* (notation: $\mathfrak{M} \equiv_{FO} \mathfrak{N}$) if they satisfy the same first-order sentences.

One, rather trivial, sufficient condition for elementary equivalence is the existence of an *isomorphism*. An isomorphism between models $\mathfrak{M}$ and $\mathfrak{N}$ is a bijection f between the domains of $\mathfrak{M}$ and $\mathfrak{N}$ such that for all atomic formulas $\varphi(x_1, \ldots, x_n)$ and elements $d_1, \ldots, d_n$ for the domain of $\mathfrak{M}$, $\mathfrak{M} \models \varphi\ [d_1, \ldots, d_n]$ iff $\mathfrak{N} \models \varphi\ [f(d_1), \ldots, f(d_n)]$. If an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$ exists, then we say that $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic, and that $\mathfrak{N}$ is an isomorphic copy of $\mathfrak{N}$. Clearly isomorphic models satisfy the same first-order formulas. A more interesting sufficient condition for elementary equivalence is the existence of a *potential isomorphism*, a notion that will be defined next.

A *finite partial isomorphism* between models $\mathfrak{M}, \mathfrak{N}$ is a finite relation $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ between the domains of $\mathfrak{M}$ and $\mathfrak{N}$ such that for all atomic formulas $\varphi(x_1, \ldots, x_n)$, $\mathfrak{M} \models \varphi\ [a_1, \ldots, a_n]$ iff $\mathfrak{N} \models \varphi\ [b_1, \ldots, b_n]$. Since equality statements are atomic formulas, every finite partial isomorphism is (the graph of) an injective partial function.

1.1.9. Definition (Potential isomorphisms). *A potential isomorphism between two models $\mathfrak{M}$ and $\mathfrak{N}$ is a non-empty collection F of finite partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$ that satisfies the following conditions:*

- *For all finite partial isomorphisms $Z \in F$ and for each $w \in \mathfrak{M}$, there is a $v \in \mathfrak{N}$ such that $Z \cup \{(w, v)\} \in F$.*
- *For all finite partial isomorphisms $Z \in F$ and for each $v \in \mathfrak{N}$, there is a $w \in \mathfrak{M}$ such that $Z \cup \{(w, v)\} \in F$.*

We write $\mathfrak{M}, w_1, \ldots, w_n \cong_p \mathfrak{N}, v_1, \ldots, v_n$ to indicate the existence of a potential isomorphism F between $\mathfrak{M}$ and $\mathfrak{N}$ such that $\{(w_1, v_1), \ldots, (w_n, v_n)\} \in F$.

It is well known that first-order formulas are invariant under potential isomorphisms. In other words, the existence of a potential isomorphism implies elementary equivalence. The converse does not hold in general, but it holds for ω-saturated models.

1.1.10. Theorem. *If $\mathfrak{M} \cong_p \mathfrak{N}$ then $\mathfrak{M} \equiv_{FO} \mathfrak{N}$. Conversely, if $\mathfrak{M} \equiv_{FO} \mathfrak{N}$ and $\mathfrak{M}$ and $\mathfrak{N}$ are ω-saturated, then $\mathfrak{M} \cong_p \mathfrak{N}$.*

An exact characterization of elementary equivalence can be given in terms of *Ehrenfeucht-Fraïssé games*, which can be seen as finite approximations of potential isomorphisms. The Ehrenfeucht-Fraïssé game of length n on models $\mathfrak{M}$ and $\mathfrak{N}$ (notation: $EF(\mathfrak{M}, \mathfrak{N}, n)$) is as follows. There are two players, Spoiler and Duplicator. The game has n rounds, each of which consists of a move of Spoiler followed by a move of Duplicator. Spoiler's moves consist of picking an element from one of the two models, and Duplicators response consists of picking an element of the opposite model. In this way, Spoiler and Duplicator build up a (finite) binary relation between the domains of the two models: initially, the relation is empty; each round, it is extended with another pair. The winning conditions are as follows: if at some point of the game the constructed binary relation is not a finite partial isomorphism, then Spoiler wins immediately. If after each round the relation is a finite partial isomorphism, then the game is won by Duplicator.

1.1.11. Theorem (Ehrenfeucht-Fraïssé). *Assume a first-order language with only finitely many relation symbols and function symbols.* $\mathfrak{M} \equiv_{FO} \mathfrak{N}$ *iff Duplicator has a winning strategy in the game* $EF(\mathfrak{M}, \mathfrak{N}, n)$ *for each* $n \in \omega$*.*

Observe that, since these games are finite zero-sum perfect information games between two-players, by Zermelo's theorem one of the two players always has a winning strategy.

In fact, Theorem 1.1.11 can be strengthened: equivalence with respect to first-order formulas of quantifier depth n corresponds to Duplicator having a winning strategy in the game of n rounds. Moreover, a winning strategy for spoiler may be constructed from the distinguishing formula, and vice versa [2].

1.2 Basics of computability theory

We briefly review some notions from complexity theory and recursion theory that are used in this thesis. More information can be found in [7], [26] and [16].

A decision problem may be identified either with a set of strings over the alphabet $\{0, 1\}$, or with a set of natural numbers. In fact, these views can be identified by considering natural numbers as written down in binary notation. Thus, while the length of a string s is simply the number of elements of the sequence, the length of a natural number n will be the length of its binary encoding, which is approximately $\log n$. We will use $|s|$ to refer to the length of s, where s is either a bit-string or a natural number.

Given such a set L of bitstrings, or of natural numbers, the task is then to decide for a given string, or natural number, s whether $s \in L$. A problem L is called *decidable* (or, *recursive*) if there is a deterministic Turing machine that solves this problem in finite amount of time (i.e., for each input s it terminates after finitely many steps and correctly answers the question whether $s \in L$). A problem L is called *recursively enumerable* (r.e.) if there is a (not necessarily halting) deterministic Turing machine that enumerates the elements of L. A problem is *co-recursively enumerable* if its complement is recursively enumerable. Any problem that is neither recursively enumerable nor co-recursively enumerable is called *highly undecidable*.

Complexity classes

Complexity theory classifies decision problems with respect to the amount of time and space a Turing machine needs to solve them.

Table 1.1: Some important complexity classes

$$
\begin{aligned}
\textsc{PTime} &= \bigcup_{k\in\mathbb{N}} \textsc{dtime}(n^k)\\
\textsc{NP} &= \bigcup_{k\in\mathbb{N}} \textsc{ntime}(n^k)\\
\textsc{PSpace} &= \bigcup_{k\in\mathbb{N}} \textsc{space}(n^k)\\
\textsc{ExpTime} &= \bigcup_{k\in\mathbb{N}} \textsc{dtime}(2^{n^k})\\
\textsc{NExpTime} &= \bigcup_{k\in\mathbb{N}} \textsc{ntime}(2^{n^k})\\
\textsc{ExpSpace} &= \bigcup_{k\in\mathbb{N}} \textsc{space}(2^{n^k})\\
\textsc{2-ExpTime} &= \bigcup_{k\in\mathbb{N}} \textsc{dtime}(2^{2^{n^k}})\\
\textsc{2-NExpTime} &= \bigcup_{k\in\mathbb{N}} \textsc{ntime}(2^{2^{n^k}})\\
\textsc{2-ExpSpace} &= \bigcup_{k\in\mathbb{N}} \textsc{space}(2^{2^{n^k}})\\
&\vdots\\
\textsc{Elementary} &= \bigcup_{k\in\mathbb{N}} k\textsc{-ExpTime}
\end{aligned}
$$

Consider a function $f : \mathbb{N} \to \mathbb{N}$. We say that a problem L in $\textsc{dtime}(f)$ if there is a deterministic Turing machine M and natural numbers c, d such that on any input s with $|s| > d$, M terminates after at most $c \cdot f(|s|)$ many steps and correctly answers the question whether $s \in L$. $\textsc{ntime}(f)$ is defined similarly, using non-deterministic Turing machines. A problem L in $\textsc{space}(f)$ if there is a deterministic Turing machine M and natural numbers c, d such that, on any input s with $|s| > d$, M decides in finite amount of time whether $s \in L$, using at most $c \cdot f(|s|)$ many cells of the tape.

These notions can be used to define a number of important classes of decision problems that play a role in this thesis, which are listed Table 1.1. Each of these classes is contained in the classes appearing below it in the list.

Reductions and completeness

A polynomial reduction from a problem L to a problem L' (more precisely, a *polynomial time many-one reduction*) is a deterministic Turing machine that, given input s, terminates after at most $f(|s|)$ many steps and produces output t such that $s \in L$ iff $t \in L'$, for some polynomial function $f : \mathbb{N} \to \mathbb{N}$. All complexity classes listed in Table 1.1 are closed under polynomial reductions. For C a class of decision problems and L a decision problem, L is said to be *C-hard* (more precisely, *C-hard under polynomial reductions*) if every problem in C can be polynomially reduced to L. A decision problem L is said to be *C-complete* if $L \in C$ and L is C-hard.

We will also make use of other types of reductions in this thesis. A *computable reduction* from a problem L to a problem L' is a deterministic Turing machine that, given input s, terminates after finitely many steps and produces output t such that $s \in L$ iff $t \in L'$. Clearly, the class of decidable decision problems is closed under computable reductions. On the other hand, the classes listed in Table 1.1 are not closed under computable reductions.

Finally, a *non-deterministic polynomial conjunctive reduction* of a problem L to a problem L' is a polynomial time non-deterministic Turing machine that, given input s (non-deterministically) produces a sequence $t_1, \ldots, t_n$ of instances of L', such that $s \in L$ iff some run of the non-deterministic Turing machine with input s produces a sequence $t_1, \ldots, t_n$ such that each t_i is in L' $(i \leq n)$. Clearly, non-deterministic polynomial conjunctive reduction generalize the usual polynomial time many-one reductions. With the exception of PTime, all complexity classes listed in Table 1.1 are closed under non-deterministic polynomial conjunctive reductions (the class PTime is not closed under such reductions, unless PTime = NP) [20].

Arithmetical and analytical hierarchy

While complexity theory provides the tools to classify the complexity of decidable problems, recursion theory is the proper framework for the studying and classifying undecidable problems. Recursion theory studies decision problems from the perspective of definability in first-order or second-order arithmetic.

The language of *first-order Peano arithmetic*, $\mathcal{L}^1_{PA}$, is the first-order language over the vocabulary that consists of binary relation $\leq$, function symbols $+$ and $\times$, and equality. Formulas of this language are interpreted over the natural numbers. A set L of natural numbers is called *arithmetical* if it is definable in first-order Peano arithmetic, i.e., if there is a formula $\varphi(x)$ of $\mathcal{L}^1_{PA}$ such that for all $n \in \mathbb{N}$, $n \in L$ iff $(\mathbb{N}, \leq, +, \times) \models \varphi\ [n]$. Arithmetical sets may be further classified in terms of the quantifier patterns occuring in the formulas that define them. More specifically, a set of natural numbers is said to be in Σ^0_k (with $k \geq 1$) if it is defined by a $\mathcal{L}^1_{PA}$-formula of the form $Q_1 x_1 \cdots Q_n x_n . \varphi$, with $Q_1, \ldots, Q_n \in \{\exists, \forall\}$ and φ quantifier-free, such that $Q_1 = \exists$ and the number of quantifier alternations

Table 1.2: Some important classes of problems in recursion theory

Arithmetical hierarchy

$$\begin{array}{ccccccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 & & & & \\
 & \subset & & \subset & & \subset & & \subset & & \subset & \\
\text{decidable} & & & & \Delta^0_2 & & & & \Delta^0_3 & & \cdots \\
(= \Delta^0_1) & \subset & & \subset & & \subset & & \subset & & \subset & \\
 & & \Pi^0_1 & & & & \Pi^0_2 & & & &
\end{array}$$

Analytical hierarchy

$$\begin{array}{ccccccccccc}
 & & \Sigma^1_1 & & & & \Sigma^1_2 & & & & \\
 & \subset & & \subset & & \subset & & \subset & & \subset & \\
\text{arithmetical} & & & & \Delta^1_2 & & & & \Delta^1_3 & & \cdots \\
(= \bigcup_k \Delta^0_k) & \subset & & \subset & & \subset & & \subset & & \subset & \\
 & & \Pi^1_1 & & & & \Pi^1_2 & & & &
\end{array}$$

(i.e., universal quantifiers following existential quantifiers or vice versa) in the sequence $Q_1 \ldots Q_n$ is at most $k-1$. A set of natural numbers is said to be in Π^0_k if its complement is in Σ^0_k, and in Δ^0_k if it is both in Σ^0_k and in Π^0_k. A remarkable result in recursion theory states that the decidable sets of natural numbers are precisely the ones that are in Δ^0_1, and the recursively enumerable sets are the ones in Σ^0_1.

The language of *second-order Peano arithmetic*, $\mathcal{L}^2_{PA}$, is the second-order language over the vocabulary that consists of binary relation $\leq$, function symbols $+$ and $\times$, and equality. A set of natural numbers is called *analytical* if it is defined by a formula of $\mathcal{L}^2_{PA}$. Again, the analytical sets can be classified with respect to the quantifier patterns occuring in the defining formulas. A set of natural numbers is said to be in Σ^1_k (with $k \geq 1$) if it is defined by a $\mathcal{L}^2_{PA}$-formula of the form $Q_1X_1 \cdots Q_nX_n.\varphi$, where $Q_1, \ldots, Q_n \in \{\exists, \forall\}$ are quantifiers over sets and φ contains only first-order quantifiers, such that $Q_1 = \exists$ and the number of quantifier alternations (i.e., universal quantifiers following existential quantifiers or vice versa) in the sequence $Q_1 \ldots Q_n$ is at most $k-1$. A set of natural numbers is said to be in Π^1_k if its complement is in Σ^1_k, and in Δ^1_k if it is both in Σ^0_k and in Π^1_k.

Table 1.2 summarizes some of the above classes, and indicates their relationships. Each of the indicated inclusions is strict. Each of the classes listed in Table 1.2 is closed under computable reductions. A set A of natural numbers is said to be Σ^k_ℓ*-hard* (more precisely, Σ^k_ℓ*-hard under computable reductions*) if for every set B in Σ^k_ℓ there is a computable reduction from B to A. A set of natural numbers is Σ^k_ℓ-complete if it is both in Σ^k_ℓ and Σ^k_ℓ-hard. Likewise for Π^k_ℓ and Δ^k_ℓ. When one speaks of an arbitrary decision problem as being, for instance, Σ^1_1-

hard, then it is implicitly understood that the instances of the decision problem are coded into natural numbers (using a computable encoding).

The set of (codings of) true Σ^1_1 sentences of arithmetic is itself a Σ^1_1-complete set. In fact, this can be strengthened slightly, since the intended interpretation of $+$ and $\times$ in $(\mathbb{N}, \leq)$ can be defined using first-order sentences. In this way, we obtain the following.

1.2.1. Theorem. *The existential second order theory of* $(\mathbb{N}, \leq)$ *is* Σ^1_1*-complete.*

Another example of a Σ^1_1-hard decision problem, due to Harel [16] is the recurrent tiling problem, which can be defined as follows. A *tile* is a tuple $t = \langle t_{left}, t_{right}, t_{top}, t_{bottom} \rangle$ of elements of some set C. A *tiling of* $\mathbb{N} \times \mathbb{N}$ *using a set of tiles* T is a function $f : \mathbb{N} \times \mathbb{N} \to T$ such that for all $n, m \in \mathbb{N}$, $f(n, m)_{right} = f(n + 1, m)_{bottom}$ and $f(n, m)_{top} = f(n, m + 1)_{bottom}$. Now, the recurrent tiling problem is the following problem:

> *given a finite set of tiles* T *and a designated tile* $t \in T$*, is there a tiling* f *of* $\mathbb{N} \times \mathbb{N}$ *using* T *such that* $f(n, 0) = t$ *for infinitely many* $n \in \mathbb{N}$*?*

1.2.2. Theorem ([16]). *The recurrent tiling problem is* Σ^1_1*-complete.*

Here is an example of a decision problem that is not analytical.

1.2.3. Theorem. *Satisfiability of monadic second order formulas over the signature consisting of a single binary relation is highly undecidable, and in fact not analytical.*

Proof: There is a computable satisfiability-preserving translation from arbitrary second-order formulas to monadic second order formulas in one binary relation symbol [18]. By a standard recursion theoretic argument, using the fact that the model $(\mathbb{N}, \leq, +, \times)$ is defined up to isomorphism by a second order formula, the class of satisfiable second-order formulas is not analytical (cf. [10]). The result follows. $\square$

Chapter 2

Modal logic

This chapter serves two purposes. Firstly, it reviews the basic notions and results of modal logic, from a model theoretic perspective. Secondly, we prove the following new results: non-recursive enumerability of the first-order formulas preserved under ultrafilter extensions, an improvement of a general interpolation result for modal logics, and some results concernings hallow modal formulas (i.e., modal formulas in which no occurence of a proposition letter is in the scope of more than one modal operator).

2.1 Syntax and semantics

We will assume a countably infinite set of proposition letters PROP and a finite set of (unary) modalities MOD.[1] A *Kripke frame* is a pair $\mathfrak{F} = (W, (R_\diamond)_{\diamond \in \text{MOD}})$, where W is a set, called the domain of $\mathfrak{F}$, and each $R_\diamond$ is a binary relation over W. The elements of the domain of a frame are often called *worlds, states, points, nodes,* or simply *elements*. The relations $R_\diamond$ are often called *accessibility relations*. A Kripke model is a pair $(\mathfrak{F}, V)$, where $\mathfrak{F}$ is a Kripke frame, and $V : \text{PROP} \to \wp(W)$ is a *valuation* for $\mathfrak{F}$, i.e., a function that assigns to each proposition letter a subset of the domain of $\mathfrak{F}$. We will often drop the qualification "*Kripke*", and simply talk about frames and models.

The basic modal language $\mathcal{M}$ is a language that is used for describing models and frames. Its formulas are given by the following recursive definition.

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \diamond\varphi$$

The other connectives, such as $\Box$, will be considered shorthand notations. Given a model $\mathfrak{M} = (W, (R_\diamond)_{\diamond \in \text{MOD}}, V)$, a world $w \in W$ and a modal formula φ, truth or falsity of φ at w in $\mathfrak{M}$ is defined as follows, where $\mathfrak{M}, w \models \varphi$ expresses that φ

[1] In most parts of this thesis, we restrict attention to a finite set of unary modalities. This is only for presentational reasons, and all results we present can be generalized to infinitely many modalities and k-ary modalities ($k \geq 0$).

is true at w in $\mathfrak{M}$.

$$\begin{array}{lll}
\mathfrak{M}, w \models \top & & \\
\mathfrak{M}, w \models p & \text{iff} & w \in V(p) \\
\mathfrak{M}, w \models \neg\varphi & \text{iff} & \mathfrak{M}, w \not\models \varphi \\
\mathfrak{M}, w \models \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models \Diamond\varphi & \text{iff} & \text{there is a } v \in W \text{ such that } R_\Diamond(w, v) \text{ and } \mathfrak{M}, v \models \varphi
\end{array}$$

We say that $\mathfrak{M}$ *globally satisfies* φ (notation: $\mathfrak{M} \models \varphi$) if $\mathfrak{M}, w \models \varphi$ for all $w \in W$. We say that φ is *valid* on a frame $\mathfrak{F}$ (notation: $\mathfrak{F} \models \varphi$) if $(\mathfrak{F}, V) \models \varphi$ for all valuations V for $\mathfrak{F}$. Dually, φ is *satisfiable* on a frame $\mathfrak{F}$ if there is a valuation V and a world w such that $\mathfrak{F}, V, w \models \varphi$. The *frame class defined by* φ is the class of all frames on which φ is valid. Finally, φ is said to be *valid* (notation $\models \varphi$) if φ is valid on all frames, and φ is said to be *satisfiable* if it is satisfiable on some frame.

The *modal depth* of a formula φ, denoted by $md(\varphi)$, is the maximal nesting of modal operators in φ. One can also give a proper inductive definition:

$$\begin{array}{lll}
md(\top) & = & 0 \\
md(p) & = & 0 \\
md(\neg\varphi) & = & md(\varphi) \\
md(\varphi \wedge \psi) & = & \max\{md(\varphi), md(\psi)\} \\
md(\Diamond\varphi) & = & md(\varphi) + 1
\end{array}$$

In the remainder of this chapter, we review the model theory of the basic modal language $\mathcal{M}$, focusing on expressivity, frame definability, axiomatizations, interpolation, and decidability and complexity.

2.2 Bisimulations and expressivity on models

Bisimulation allow us to tell when two worlds in models can be distinguished by a modal formula.

2.2.1. DEFINITION. *A bisimulation between models $\mathfrak{M} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}}, V)$ and $\mathfrak{N} = (W', (R'_\Diamond)_{\Diamond \in \text{MOD}}, V')$ is a binary relation $Z \subseteq W \times W'$ satisfying the following conditions.*

Atom *If wZv then $\mathfrak{M}, w \models p$ iff $\mathfrak{N}, v \models p$ for all $p \in$ PROP*

Zig *If wZv and $wR_\Diamond w'$, then there is a $v' \in W'$ such that $vR'_\Diamond v'$ and $w'Zv'$.*

Zag *If wZv and $vR'_\Diamond v'$, then there is a $w' \in W$ such that $wR_\Diamond w'$ and $w'Zv'$.*

We say that $\mathfrak{M}, w$ and $\mathfrak{N}, v$ are bisimilar (notation: $\mathfrak{M}, w \leftrightarrows \mathfrak{N}, v$) if there is a bisimulation Z between $\mathfrak{M}$ and $\mathfrak{N}$ such that wZv.

Table 2.1: Standard translation from modal logic to $\mathcal{L}^1$

$$
\begin{array}{lcl}
ST_x(\top) & = & \top \\
ST_x(p) & = & P_p(x) \\
ST_x(\neg\varphi) & = & \neg ST_x(\varphi) \\
ST_x(\varphi \wedge \psi) & = & ST_x(\varphi) \wedge ST_x(\psi) \\
ST_x(\Diamond\varphi) & = & \exists y(R(x,y) \wedge ST_y(\varphi)) \text{ for } y \text{ a variable distinct from } x
\end{array}
$$

Modal formulas cannot distinguish bisimilar points. In other words, if two points are bisimilar, they are modally equivalent. The converse does not hold in general, but it hold on ω-saturated models (cf. Appendix 1.1). Let us write $\mathfrak{M}, w \equiv_{\mathcal{M}} \mathfrak{N}, v$ if for all modal formulas φ, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{N}, v \models \varphi$.

2.2.2. Theorem. *Let $\mathfrak{M}, \mathfrak{N}$ be models and w, v points in these models. If w and v are bisimilar then $\mathfrak{M}, w \equiv_{\mathcal{M}} \mathfrak{N}, v$. Conversely, if $\mathfrak{M}$ and $\mathfrak{N}$ are ω-saturated and $\mathfrak{M}, w \equiv_{\mathcal{M}} \mathfrak{N}, v$ then w and v are bisimilar.*

A proof can be found in [6].

The *first-order correspondence language* $\mathcal{L}^1$ is the first-order language with equality that contains a unary predicate P_p for each proposition letter $p \in$ PROP and a binary relation $R_\Diamond$ for each modality $\Diamond \in$ MOD. Any model $\mathfrak{M} = (W, (R_\Diamond)_{\Diamond\in\mathrm{MOD}}, V)$ can be regarded as a model for the first-order correspondence language. The accessibility relations $R_\Diamond$ are used to interpret the binary relation $R_\Diamond$ and the unary predicates P_p are interpreted as the subsets that V assigns to the corresponding proposition letter. In what follows, we will not distinguish between Kripke models and models for the first-order correspondence language, and we will continue to use the notation $\mathfrak{M} = (W, (R_\Diamond)_{\Diamond\in\mathrm{MOD}}, V)$.

Table 2.1 presents the standard translation ST_x from the modal language to the first-order correspondence language $\mathcal{L}^1$. This translation preserves truth, in the sense that for all modal formulas φ, models $\mathfrak{M}$, and worlds w of $\mathfrak{M}$, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M} \models ST_x(\varphi)\ [x : w]$. In this way, the standard translation shows that modal logic is a fragment of first-order logic. Bisimulations allow one to characterize exactly *which* fragment. Call an $\mathcal{L}^1$-formula $\varphi(x_1, \ldots, x_n)$ *bisimulation invariant* if for all bisimulations Z between models $\mathfrak{M}$ and $\mathfrak{N}$ and for all $(w_1, v_1), \ldots, (w_n, v_n) \in Z$, $\mathfrak{M} \models \varphi\ [w_1, \ldots, w_n]$ iff $\mathfrak{N} \models \varphi\ [v_1, \ldots, v_n]$.

2.2.3. Theorem ([4]). *Let $\varphi(x)$ be a formula of the first-order correspondence language with at most one free variable. Then the following are equivalent:*

1. *$\varphi(x)$ is invariant under bisimulations*
2. *$\varphi(x)$ is equivalent to the standard translation of a modal formula.*

Rosen [24] proved that this result holds also on finite structures.

2.3 Frame definability

When interpreted on frames, modal formulas express second order frame conditions. For instance, the modal formula $p \to \Diamond p$ expresses the frame condition $\forall x.\forall P.(Px \to \exists y.(Rxy \wedge Py))$. At it happens, this particular second order formula is equivalent to the first-order formula $\forall x.Rxx$. However, this is in general not the case. For instance, the modal formula $\Box\Diamond p \to \Diamond\Box p$ expresses a frame condition that is not definable by first-order formulas.

To be a little more precise, given a set of modal formulas Σ, the *frame class defined by* Σ is the class of all frames on which each formula in Σ is valid. A frame class is *modally definable* if there is a set of modal formulas that defines it. A frame class is *elementary* if it is defined by a sentence of the *first order frame correspondence language* $\mathcal{L}^1_{fr}$, which is the first-order language with equality and binary relation symbol for each modality.[2]

In this section, we discuss a number of result concerning the relationship between modally definable frame classes and elementary frame classes. First, we will consider model theoretic characterizations. Then, we will review some attempts at syntactic characterizations.

Model theoretic characterizations

A famous result due to Goldblatt and Thomason characterizes the modally definable elementary frame classes in terms of four operations on frames.

2.3.1. Definition (Generated subframe). *A frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}})$ is a* generated subframe *of a frame $\mathfrak{G} = (W', (R'_\Diamond)_{\Diamond\in\text{MOD}})$ if $W \subseteq W'$ and for all $(w, v) \in R'_\Diamond$ ($\Diamond \in \text{MOD}$), if $w \in W$ then $v \in W$.*

2.3.2. Definition (Disjoint union). *Let $\mathfrak{F}_i = (W_i, (R^i_\Diamond)_{\Diamond\in\text{MOD}})$ ($i \in I$) be a set of frames with disjoint domains. The disjoint union of these frames, denoted by $\biguplus_{i\in I}\mathfrak{F}_i$ is the frame $(\bigcup_{i\in I} W_i, (\bigcup_{i\in I} R^i_\Diamond)_{\Diamond\in\text{MOD}})$.*

2.3.3. Definition (Bounded morphism). *A bounded morphism from a frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}})$ to a frame $\mathfrak{G} = (W', (R'_\Diamond)_{\Diamond\in\text{MOD}})$ is a function $f : W \to W'$ satisfying the following conditions.*

forth *for all $w, v \in W$ and $\Diamond \in \text{MOD}$, if $R_\Diamond(w, v)$ then $R'_\Diamond(f(w), f(v))$*

back *for all $w \in W$, $v \in W'$ and $\Diamond \in \text{MOD}$, if $R'_\Diamond(f(w), v)$ then there is a $u \in W$ such that $R_\Diamond(w, u)$ and $f(u) = v$.*

If there is a surjective bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$, then we say that $\mathfrak{G}$ is a bounded morphic image *of $\mathfrak{F}$.*

[2] Note that, in the literature, a class is sometimes called elementary if it is defined by a set of first-order formulas. Here, we call a class elementary if it is defined by a single first-order sentence.

In order to formulate the fourth operation on frames, we need to introduce a piece of notation. Given a frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}})$, $X \subseteq W$ and $\Diamond \in$ MOD, we will write $m_\Diamond(X)$ for the set $\{w \in W \mid \exists v \in X.wR_\Diamond v\}$. In other words, $m_\Diamond(X)$ is the set of $\Diamond$-predecessors of elements of X.

2.3.4. Definition (Ultrafilter extension). *Given a frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond\in\text{MOD}})$, the* ultrafilter extension *of $\mathfrak{F}$, denoted by $\mathfrak{ue}\mathfrak{F}$, is the frame $(\text{Uf}(W), (R^{\mathfrak{ue}}_\Diamond)_{\Diamond\in\text{MOD}})$, where $\text{Uf}(W)$ is the set of ultrafilters over W (cf. Appendix 1.1), and for $u, v \in \text{Uf}(W)$, $R^{\mathfrak{ue}}_\Diamond(u, v)$ iff for all $X \in v$, $m_\Diamond(X) \in u$.*

Every modally definable frame class is closed under disjoint unions, generated subframes and bounded morphic images. Furthermore, modally definable frame classes reflect ultrafilter extensions, meaning that whenever the ultrafilter extension of a frame is in the class, then the frame itself is in the class. Goldblatt and Thomason proved that the converse holds with respect to elementary frame classes.

2.3.5. Theorem (Goldblatt-Thomason[12]). *An elementary frame class is modally definable iff it is closed under generated subframes, disjoint unions and bounded morphic images, and reflects ultrafilter extensions.*

This tells us which elementary frame classes are modally definable. The opposite question, i.e., which modally definable frame classes are elementary, was answered by Van Benthem.

2.3.6. Theorem ([3]). *Let K be any modally definable frame class. The following are equivalent:*

1. K *is elementary*
2. K *is defined by a set of first-order sentences*
3. K *is closed under elementary equivalence*
4. K *is closed under ultrapowers.*

Syntactic characterizations

The above results do not tell us which modal formulas define an elementary frame class, nor which first-order formulas define a modally definable frame class.

As we will soon see (cf. Theorem 2.6.5), the problem whether a given modal formula defines an elementary frame class is highly undecidable. This implies that a syntactic characterization of the form "*a modal formula defines an elementary class iff it is equivalent to a formula of the form X*" with X a decidable class of formulas cannot be obtained. However, this still leaves open the question whether such a characterization exists if *equivalent* is replaced by *frame-equivalent.*

An important sufficient condition for elementarity was proved by Sahlqvist [25] and Van Benthem [4].

2.3.7. DEFINITION (SAHLQVIST FORMULAS). *A modal formula is* positive *(*negative*) if every occurrence of a proposition letter is under the scope of an even (odd) number of negation signs.*

A Sahlqvist antecedent *is a formula built up from $\top, \bot$, boxed atoms of the form $\Box_1 \cdots \Box_n p$ ($n \geq 0$), and negative formulas using conjunction, disjunction and diamonds.*

A Sahlqvist implication *is a formula of the form $\varphi \to \psi$, where φ is a Sahlqvist antecedent and ψ is positive.*

A Sahlqvist formula *is a formula that is obtained from Sahlvist implications by applying boxes and conjunction, and by applying disjunctions between formulas that do not share any proposition letters.*

2.3.8. THEOREM ([25, 4]). *Every Sahlqvist formula defines an elementary class of frames.*

Likewise, Van Benthem [4] has shown that every modal formula that has modal depth at most one defines an elementary class of frames. Axioms of modal depth at most one were first considered by Lewis [21]. Van Benthem's result may be improved slightly, by considering the following class of formulas.

2.3.9. DEFINITION (SHALLOW FORMULAS). *A modal formula is* shallow *if every occurrence of a proposition letter is in the scope of at most one modal operator.*

2.3.10. THEOREM. *Every shallow formula defines an elementary class of frames.*

Proof: The proof will be given in Section 2.4. □

Typical examples of shallow modal formulas are $p \to \Diamond p$, $\Diamond p \to \Box p$ and $\Diamond_1 p \to \Diamond_2 p$. Furthermore, every closed formula (i.e., formula containing no proposition letters) is shallow. The formula $\Box_1(p \vee q) \to \Diamond_2(p \wedge q)$ is an example of a shallow formula that is not a Sahlqvist formula.

Incidentally, correspondence results like these might also be obtained for languages other than the first-order correspondence language. Recently, [5] and [14] have independently found a generalization of the class of Sahlqvist formulas, with the property that every generalized Sahlqvist formula has a correspondent in LFP(FO), which is the extension of first-order logic with least fixed point operators. By results of [1], there are modal formulas that have no correspondent in LFP(FO), not even with respect to finite frames.

Next, let us address the question which first-order formulas define modally definable frame conditions. Again, no complete syntactic characterization is known.

Let a *p-formula* be a first-order formula obtained from atomic formulas (including equality statements) using conjunction, disjunction, existential and universal quantifiers, and bounded universal quantifiers of the form $\forall x(Rtx \to \cdot)$. A p-sentence is a p-formula that is a sentence. An inductive argument shows that p-sentences are preserved under taking images of bounded morphisms. In fact, the converse holds as well, modulo logical equivalence.

2.3.11. THEOREM (VAN BENTHEM [4]). *A first-order sentence φ is preserved under surjective bounded morphisms iff φ is equivalent to a p-sentence.*

It follows that if a first-order sentence defines a modally definable frame class, then it is equivalent to a p-sentence. We can improve this a bit further. Let a *positive restricted formula* be a first-order formula built up from $\perp$ and atomic formulas, using conjunction, disjunction, and restricted quantification of the from $\exists y.(Rxy \wedge \cdot)$ and $\forall y.(Rxy \to \cdot)$, where x and y are distinct variables.

2.3.12. THEOREM (VAN BENTHEM [4]). *A first-order sentence φ is preserved under surjective bounded morphisms, generated subframes and disjoint unions iff φ is equivalent to $\forall x.\psi(x)$, for some positive restricted formula $\psi(x)$.*

Again, it follows that if a first-order sentence defines a modally definable frame class, it is equivalent to a sentence of the given form. What remains in order to obtain a complete characterization is to characterize anti-preservation under ultrafilter extensions. It is possible to give a preservation result similar to the above, that characterizes the first-order sentences (anti-)preserved under ultrafilter extensions? The answer is *No.*

2.3.13. THEOREM. *Preservation of first-order formulas under ultrafilter extensions is Π^1_1-hard.*

In particular, it follows that the first-order sentences (anti-)preserved under ultrafilter extensions are not recursively enumerable, and cannot be characterized by means of a preservation theorem.

2.4 Completeness via general frames

Given a frame class K, one would like to describe the set of modal formulas valid on K ("the modal logic of K"). For the class of all frames, the axioms and inferences rules given in Table 2.2 constitute a sound and complete axiomatization. We will refer to this axiomatization as $\mathbf{K}_{\mathcal{M}}$. We will write $\vdash_{\mathbf{K}_{\mathcal{M}}} \varphi$ if φ is derivable in $\mathbf{K}_{\mathcal{M}}$.

2.4.1. THEOREM (BASIC COMPLETENESS). *For all modal formulas φ, $\models \varphi$ iff $\vdash_{\mathbf{K}_{\mathcal{M}}} \varphi$.*

Thus, $\mathbf{K}_{\mathcal{M}}$ axiomatizes the set of modal formulas valid on the class of all frames. In order to axiomatize more restricted frame classes, extra axioms (or rules) must be added to $\mathbf{K}_{\mathcal{M}}$. For any set Σ of modal formulas, we will use $\mathbf{K}_{\mathcal{M}}\Sigma$ to denote the axiomatization obtained by adding all formulas in Σ as axioms to $\mathbf{K}_{\mathcal{M}}$. One

Table 2.2: Axioms and inference rules of $\mathbf{K}_{\mathcal{M}}$

(CT)	$\vdash \varphi$, for all classical tautologies φ
$(Dual)$	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$, for $\Box \in$ MOD
(K)	$\vdash \Box(p \to q) \to \Box p \to \Box q$, for $\Box \in$ MOD
(MP)	If $\vdash \varphi \to \psi$ and $\vdash \varphi$ then $\vdash \psi$
(Nec)	If $\vdash \varphi$ then $\vdash \Box\varphi$, for $\Box \in$ MOD
$(Subst)$	If $\vdash \varphi$ then $\vdash \varphi\sigma$, where σ is a substitution that uniformly replaces proposition letters by formulas.

might hope that $\mathbf{K}_{\mathcal{M}}\Sigma$ completely axiomatizes the set of modal formulas valid on the frame class defined by Σ. Unfortunately, this is in general not the case. Nevertheless, there are natural classes of modal formulas, for which such a general completeness result can be obtained.

In order to facilitate the study of completeness and incompleteness, it is convenient to introduce a generalization of the notion of frames. A *general frame* consists a frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}})$ together with a set $\mathbb{A} \subseteq \wp(W)$ satisfying certain regularity conditions, to be spelled out below. The elements of $\mathbb{A}$ are called *admissible subsets*. A modal formula φ containing proposition letters $p_1, \ldots, p_n$ is said to be valid on a such a general frame if it is valid under any valuation that assigns admissible subsets to $p_1, \ldots, p_n$. Note that the ordinary frames, or *Kripke frames*, as we will refer to them in this section, are simply general frames for which the set of admissible subsets is the set of all subsets.

Recall that, given a frame $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}})$, $X \subseteq W$ and $\Diamond \in$ MOD, $m_\Diamond(X) = \{w \in W \mid \exists v \in X. wR_\Diamond v\}$.

2.4.2. Definition (General frames). *A general frame is a pair $(\mathfrak{F}, \mathbb{A})$, where $\mathfrak{F} = (W, (R_\Diamond)_{\Diamond \in \text{MOD}})$ is a frame and $\mathbb{A} \subseteq \wp(W)$, such that $W \in \mathbb{A}$ and $\mathbb{A}$ is closed under complement, finite intersection and $m_\Diamond$ for $\Diamond \in$ MOD.*

In addition, the general frame $(\mathfrak{F}, \mathbb{A})$ is

differentiated *if for all $w, v \in W$ with $w \neq v$ there is an $A \in \mathbb{A}$ such that $w \in A$ and $v \notin A$*

tight *if for all $w, v \in W$ and $\Diamond \in$ MOD such that $(w, v) \notin R_\Diamond$ there is an $A \in \mathbb{A}$ such that $v \in A$ and $w \notin m_\Diamond(A)$*

compact *if every $\mathbb{A}' \subseteq \mathbb{A}$ with the finite intersection property has a non-empty intersection*

refined *if it is differentiated and tight*

descriptive *if it is differentiated, tight and compact*

discrete *if for all $w \in W$, $\{w\} \in \mathbb{A}$*

atomless *if for no $w \in W$, $\{w\} \in \mathbb{A}$*

A valuation for a general frame $\mathfrak{F}$ is admissible if $V(p) \in \mathbb{A}$ for all $p \in$ PROP. Validity with respect to general frames is defines as follows: $\mathfrak{F} \models \varphi$ if for all admissible valuations V and worlds w, $(\mathfrak{F}, V), w \models \varphi$. Every set Γ of modal formulas defines a class of general frames, namely the class consisting of those general frames on which each formula in Γ is valid.

Unlike Kripke frames, general frames offer a fully adequate semantics for modal logics, in the sense that for all sets Γ of modal formulas, $\mathbf{K}_{\mathcal{M}}\Gamma$ completely axiomatizes the set of modal formulas valid on the class of general frames defined by Γ. In fact, this holds even if we restrict attention to descriptive frames. Given a set of modal formulas Γ and a class K of general frames, we say that $\mathbf{K}_{\mathcal{M}}\Gamma$ is complete for K if $\mathbf{K}_{\mathcal{M}}\Gamma$ completely axiomatizes the set of modal formulas valid on K, i.e., for all φ, $\mathsf{K} \models \varphi$ iff $\vdash_{\mathbf{K}_{\mathcal{M}}\Gamma} \varphi$.

2.4.3. Theorem ([13]). *Let Γ be any set of modal formulas. $\mathbf{K}_{\mathcal{M}}\Gamma$ is complete for the class of descriptive general frames defined by Γ.*

Of course, our actual interest is not in general frames but in Kripke frames. Theorem 2.4.3 can be seen as an important first step towards proving Kripke completeness. The second step typically involves persistence, a notion that will be defined next.

2.4.4. Definition. *A modal formula φ is* persistent *with respect to a type of general frames (such as descriptive general frames, etc.) if for all general frames $\mathfrak{F}$ of the relevant type, if $\mathfrak{F} \models \varphi$ then φ is valid on the underlying Kripke frame of $\mathfrak{F}$.*

Persistence with respect to descriptive frame is also called *d-persistence*, or *canonicity.* Persistence with respect to discrete frames is often called *di-persistence.*

Recall the definition of Sahlqvist formulas on page 18. An important result in modal logic is the following.

2.4.5. Theorem ([25]). *Every modal Sahlqvist formula is persistent with respect to descriptive general frames.*

If we put Theorem 2.4.3 and Theorem 2.4.5 together, we obtain the following Kripke completeness result for Sahlqvist formulas.

2.4.6. Corollary ([25]). *If Γ is a set of Sahlqvist formulas, then $\mathbf{K}_{\mathcal{M}}\Gamma$ is complete for the class of Kripke frames defined by Γ.*

A similar result can be proved for shallow formulas. Recall that a modal formula is shallow if every occurrence of a proposition letter is under the scope of at most one modal operator.

2.4.7. Theorem. *Every shallow formula is persistent with respect to refined frames, and hence with respect to descriptive frames and with respect to discrete frames.*

2.4.8. Corollary. *If Γ is a set of shallow formulas, then $\mathbf{K}_{\mathcal{M}}\Gamma$ is complete for the class of Kripke frames defined by Γ.*

In fact, combining Theorem 2.4.3, 2.4.5 and 2.4.7, we obtain completeness of $\mathbf{K}_{\mathcal{M}}\Gamma$ for all sets Γ consisting of shallow and/or Sahlqvist formulas.

Incidentally, every modal formula that is persistent with respect to refined frames defines an elementary frame class [19]. Hence, this also proves Theorem 2.3.10.

To finish this section, we briefly consider discrete general frames. Venema [28] proved the following persistence result with respect to discrete general frames.

2.4.9. Definition (Very simple Sahlqvist formulas). *A* very simple Sahlqvist antecedent *is a modal formula built up from $\top, \bot$ and proposition letters using conjunction and diamonds. A* very simple Sahlqvist formula *is an implication $\varphi \to \psi$, where φ is a very simple Sahlqvist antecedent and ψ is positive.*

2.4.10. Theorem ([28]). *Every very simple Sahlqvist formula is persistent with respect to discrete frames.*

This by itself does not imply completeness for logics axiomatized by very simple Sahlqvist formulas (even though this follows from Theorem 2.4.5). The reason is that $\mathbf{K}_{\mathcal{M}}\Gamma$ might not be complete for the class of discrete general frames defined by Γ. In other words, there is no analogue of Theorem 2.4.3 for discrete general frames. Indeed, Venema [28] proved the following strong incompleteness result.

2.4.11. Theorem ([28]). *There is a modal formula φ such that $\mathbf{K}_{\mathcal{M}}\{\varphi\}$ is consistent and every general frame on which φ is valid is atomless.*

It follows that for the relevant formula φ, $\mathbf{K}_{\mathcal{M}}\{\varphi\}$ is incomplete with respect to the class of discrete frames defined by φ. Incidentally, the formula φ used by [28] contains more than one modality. This is necessarily so: an observation due to Makinson implies that, for all uni-modal formulas φ, if $\mathbf{K}_{\mathcal{M}}\{\varphi\}$ is consistent then it has a general frame whose domain is a singleton set. Clearly every such general frame is discrete.

2.5 Interpolation and Beth definability

Analogues of Craig's interpolation theorem have been proved for many modal logics. For any modal formula φ, let $\textsc{prop}(\varphi)$ is the set of proposition letters occurring in φ. Further, let us say that the basic modal language has interpolation on a frame class K, if for all modal formulas φ, ψ such that $\mathsf{K} \models \varphi \to \psi$, there is a modal formula ϑ such that $\mathsf{K} \models \varphi \to \vartheta$ and $\mathsf{K} \models \vartheta \to \psi$, and $\textsc{prop}(\vartheta) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$. Note that no restriction is made on the modalities occurring in ϑ. It would therefore be more appropriate to talk about *interpolation over proposition letters*, indicating that it is only the proposition letters in the interpolation that must occur both in the antecedent and in the consequent.

2.5.1. DEFINITION. *A bisimulation product of a set of frames $\{\mathfrak{F}_i \mid i \in I\}$ is a subframe $\mathfrak{G}$ of the cartesian product $\Pi_i \mathfrak{F}_i$ such that for each $i \in I$, the natural projection function $f_i : \mathfrak{G} \to \mathfrak{F}_i$ is a surjective bounded morphism.*

Bisimulation products are a special case of subdirect products (for the definition of cartesian products and subdirect products, see Appendix 1.1). Their name is motivated by the following observation:

2.5.2. PROPOSITION ([22]). *Let $\mathfrak{H}$ be a submodel of the product $\mathfrak{F} \times \mathfrak{G}$. Then $\mathfrak{H}$ is a bisimulation product of $\mathfrak{F}$ and $\mathfrak{G}$ iff the domain of $\mathfrak{H}$ is a total frame bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$.*

Here, with a total frame bisimulation between the frames $\mathfrak{F}$ and $\mathfrak{G}$ we mean a binary relation Z between the domains of $\mathfrak{F}$ and $\mathfrak{G}$ satisfying the *zig* and *zag* conditions of Definition 2.2.1, and such that for each world w of $\mathfrak{F}$ there is a world v of $\mathfrak{G}$ such that wZv, and vice versa.

We say that a class of frames K is *closed under bisimulation products* if for all $\mathfrak{F}, \mathfrak{G} \in \mathsf{K}$, all bisimulation products of $\mathfrak{F}$ and $\mathfrak{G}$ are in K. It was proved in [22] that if a frame class K is defined by a set of d-persistent modal formulas and closed under bisimulation products, then the basic modal language has interpolation relative to K. Here, we will slightly strengthen this result.[3]

2.5.3. THEOREM (INTERPOLATION FOR MODAL LOGICS). *Let K be any elementary frame class closed under generated subframes and bisimulation products. Then the basic modal language has interpolation relative to K.*

Proof: Let K be any elementary frame class closed under generated subframes and bisimulation products, let $\mathsf{K} \models \varphi \to \psi$, and suppose for the sake of contradiction that there is no interpolant for this implication. Let $Cons(\varphi)$ be the set of modal formulas χ such that $\mathsf{K} \models \varphi \to \chi$ and $\textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$.

[3] Strictly speaking, Theorem 2.5.3 is not a strengthening of the result of [22], since there are canonical modal formulas that define a non-elementary frame class [11].

Claim 1: There is a model $\mathfrak{M}$ based on a frame in K, with a world w, such that $\mathfrak{M}, w \models Cons(\varphi) \cup \{\neg\psi\}$.

Proof of claim: By Compactness, it suffices to show that every finite subset of $Cons(\varphi) \cup \{\neg\psi\}$ is satisfiable on K. Consider any $\chi_1, \ldots, \chi_n \in Cons(\varphi)$. If $\{\chi_1, \ldots, \chi_n, \neg\psi\}$ wouldn't be satisfiable on K, then $\chi_1 \wedge \cdots \wedge \chi_n$ would be an interpolant for $\varphi \to \psi$. By assumption, $\varphi \to \psi$ has no interpolant, and therefore, $\{\chi_1, \ldots, \chi_n, \neg\psi\}$ is satisfiable on K. $\dashv$

Since K is closed under generated subframes, we may assume that $\mathfrak{M}$ is generated by w. Let $Th(\mathfrak{M}, w)$ be the set of all modal formulas χ such that $\mathfrak{M}, w \models \chi$ and $\textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$.

Claim 2: There is a model $\mathfrak{N}$ based on a frame in K, with a world v, such that $\mathfrak{N}, v \models Th(\mathfrak{M}, w) \cup \{\varphi\}$.

Proof of claim: By Compactness, it suffices to show that every finite subset of $Th(\mathfrak{M}, w) \cup \{\varphi\}$ is satisfiable on K. Consider any $\chi_1, \ldots, \chi_n \in Th(\mathfrak{M}, w)$. Suppose for the sake of contradiction that $\{\chi_1, \ldots, \chi_n, \varphi\}$ is not satisfiable on K. Then $\mathsf{K} \models \varphi \to \neg(\chi_1 \wedge \cdots \wedge \chi_n)$. Hence, $\neg(\chi_1 \wedge \cdots \wedge \chi_n) \in Cons(\varphi)$, and therefore, $\mathfrak{M}, w \models \neg(\chi_1 \wedge \cdots \wedge \chi_n)$. This contradicts the fact that $\chi_1, \ldots, \chi_n \in Th(\mathfrak{M}, w)$. $\dashv$

Again, we may assume that $\mathfrak{N}$ is generated by v. Let $\mathfrak{M}^+$ and $\mathfrak{N}^+$ be ω-saturated elementary extensions of $\mathfrak{M}$ and $\mathfrak{N}$. Since K is elementary, the underlying frames of $\mathfrak{M}^+$ and $\mathfrak{N}^+$ are in K. Define the binary relation Z between the domains of $\mathfrak{M}^+$ and $\mathfrak{N}^+$ by letting dZe if $\mathfrak{M}^+, d \models \chi \Leftrightarrow \mathfrak{N}^+, e \models \chi$ for all modal formulas χ with $\textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$. In other words, dZe if d and e cannot be distinguished by a modal formula in the common language of φ and ψ. Note that, by construction, wZv.

Claim 3: Z is a total bisimulation between $\mathfrak{M}^+$ and $\mathfrak{N}^+$ with respect to the common language of φ and ψ.

Proof of claim: We will show that Z satisfies the *zig* condition of Definition 2.2.1. The proof of the *zag* condition is similar, and that Z respects the proposition letters in $\textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$ is immediate from its definition.

Suppose $w'Zv'$ and $w'R_\Diamond w''$. Let $\Gamma = \{ST_x(\chi) \mid \mathfrak{M}^+, w'' \models \chi \text{ and } \textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)\}$. We need to show that Γ is realized in $\mathfrak{N}^+$ by a $\Diamond$-successors of v'. By ω-saturatedness, it suffices to show that every finite subset of Γ is realized in $\mathfrak{N}^+$ by a $\Diamond$-successors of v'. But this is clearly the case: consider any $ST_x(\chi_1), \ldots, ST_x(\chi_n) \in \Gamma$. Then $\mathfrak{M}^+, w' \models \Diamond(\chi_1 \wedge \cdots \wedge \chi_n)$, and hence $\mathfrak{N}^+, v' \models \Diamond(\chi_1 \wedge \cdots \wedge \chi_n)$.

Finally, it needs to be shown that Z is a *total* bisimulation. Let $w' \in \mathfrak{M}^+$. Let $\Gamma = \{ST_x(\chi) \mid \mathfrak{M}^+, w' \models \chi \text{ and } \textsc{prop}(\chi) \subseteq \textsc{prop}(\varphi) \wedge \textsc{prop}(\psi)\}$. We need to show that Γ is realized in $\mathfrak{N}^+$. By ω-saturatedness, it suffices to show that every finite subset of Γ is realized in $\mathfrak{N}^+$. Let $ST_x(\chi_1), \ldots, ST_x(\chi_n) \in \Gamma$. Then $\exists x.(ST_x(\chi_1) \wedge \cdots \wedge ST_x(\chi_n))$ is true in $\mathfrak{M}^+$ and therefore also in $\mathfrak{M}$ (recall that $\mathfrak{M}^+$ is an elementary extension of $\mathfrak{M}$). Since $\mathfrak{M}$ is generated by w, there are $\Diamond_1, \ldots, \Diamond_m \in \textsc{mod}$ such that $\mathfrak{M}, w \models \Diamond_1 \cdots \Diamond_m(\chi_1 \wedge \cdots \wedge \chi_n)$. Hence, since wZv, we have that $\mathfrak{N}, v \models \Diamond_1 \cdots \Diamond_m(\chi_1 \wedge \cdots \wedge \chi_n)$. Since $\mathfrak{N}^+$ is an elementary extension of $\mathfrak{N}$, it follows that $\mathfrak{N}^+, v \models \Diamond_1 \cdots \Diamond_m(\chi_1 \wedge \cdots \wedge \chi_n)$. We conclude that there is a point v' such that $\mathfrak{N}^+, v' \models \chi_1 \wedge \cdots \wedge \chi_n$. ⊣

Let $\mathfrak{F}$ and $\mathfrak{G}$ be the underlying frames of $\mathfrak{M}^+$ and $\mathfrak{N}^+$. Then, in particular, Z is a total frame bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$. Hence, by Proposition 2.5.2, there is a bisimulation product $\mathfrak{H} \in \mathsf{K}$ of $\mathfrak{F}$ and $\mathfrak{G}$ of which the domain is Z. By the definition of bisimulation products, the natural projections $f : \mathfrak{H} \to \mathfrak{F}$ and $g : \mathfrak{H} \to \mathfrak{G}$ are surjective bounded morphisms. For any proposition letter $p \in \textsc{prop}(\varphi)$, let $V(p) = \{u \mid \mathfrak{M}^+, f(u) \models p\}$, and for any proposition letter $p \in \textsc{prop}(\psi)$, let $V(p) = \{u \mid \mathfrak{N}^+, g(u) \models p\}$. The properties of Z guarantee that this V is well-defined for $p \in \textsc{prop}(\varphi) \cap \textsc{prop}(\psi)$. Finally, by a standard argument, the graph of f is a bisimulation between $(\mathfrak{H}, V)$ and $\mathfrak{M}^+$ with respect to $\textsc{prop}(\varphi)$, and the graph of g is a bisimulation between $(\mathfrak{H}, V)$ and $\mathfrak{N}^+$ with respect to $\textsc{prop}(\psi)$. It follows that $(\mathfrak{H}, V), \langle w, v\rangle \models \varphi \wedge \neg\psi$. This contradicts our initial assumption that $\mathsf{K} \models \varphi \to \psi$. □

This result cannot easily be strengthened. An example of an elementary frame class that is not closed under generated subframes but not under bisimulation products, on which the basic modal language lacks interpolation is the class defined by $\Diamond\Box p \to \Box\Diamond p$.[4]

An example of an elementary frame class closed under bisimulation products but not closed under generated subframes on which the basic modal language lacks interpolation is the class defined by $\forall x.(\forall y \exists z.R_1 yz \to R_1 xx) \wedge \forall x.(\exists y \forall z.(R_1 yz \to \bot) \to R_2 xx)$. It follows from Theorem 2.5.5 below that this first-order sentence is preserved under taking bisimulation products. Again, an easy bisimulation argument shows that there is no interpolant for the valid implication $p \wedge \neg\Diamond_1 p \to (q \to \Diamond_2 q)$. Note that this implication has an interpolant with global modality,

[4]To see that the basic modal language lacks interpolation on this frame class, consider the following implication.

$$\Big(\Box(s \to \Box(\neg p \to r)) \wedge \Box(t \to \Box(\neg p \to \neg r))\Big) \quad \to \quad \Big(\Diamond(s \wedge \Box(p \to q)) \to \Box(t \to \Diamond(p \wedge q))\Big)$$

This formula is valid on the given frame class, but a simple bisimulation argument shows that there is no interpolant. Note that, intuitively, an interpolant would have to express the fact that for every successor x satisfying s and for every successor y satisfying t, x and y have a common successor satisfying p.

namely $\mathsf{E}\Box_1\bot$. Indeed, a relatively straightforward adaptation of the proof of Theorem 2.5.3 shows that the modal language with global modality, $\mathcal{M}(\mathsf{E})$, has interpolation on any elementary frame class closed under bisimulation products.

The Beth property

Let $\models^{glo}$ denote the global entailment relation on models, i.e., $\Sigma \models^{glo} \varphi$ means that for all models $\mathfrak{M}$, if $\mathfrak{M}$ globally satisfies all formulas in Σ then $\mathfrak{M}$ globally satisfies φ. Global entailment relative to a class of frames, denoted by $\models^{glo}_{\mathsf{K}}$, is defined similarly. For a set of formulas $\Sigma(p)$ containing the proposition letter p (and possibly other proposition letters), we say that $\Sigma(p)$ *implicitly defines* p, relative to a frame class K, if $\Sigma(p) \cup \Sigma(p') \models^{glo}_{\mathsf{K}} p \leftrightarrow p'$. Here, p' is a proposition letter not occurring in Σ, and $\Sigma(p')$ is the result of replacing all occurrences of p by p' in $\Sigma(p)$. The basic modal language $\mathcal{M}$ is said to have the Beth property relative to a frame class K if whenever a set of modal formulas $\Sigma(p)$ implicitly defines a proposition letter p relative to K, then there is a modal formula ϑ in which p does not occur, such that $\Sigma \models^{glo}_{\mathsf{K}} p \leftrightarrow \vartheta$. The relevant formula ϑ is called an *explicit definition* of p, relative to Σ and K.

The Beth property is an important property. Intuitively, if a logic has it, this can be seen as evidence that its syntax and semantics match well. Tarski refers to the Beth property as completeness in the theory of definitions (as opposed to the theory of deductions).

By a standard argument, we obtain as a corollary of the above interpolation results the Beth property for the basic modal language, relative to every elementary frame class closed under bisimulation products and generated subframes.

2.5.4. Theorem. *If K is a elementary frame class closed under generated subframes and bisimulation products, then the basic modal language has the Beth property relative to K.*

Proof: For ease of presentation we restrict attention to the uni-modal case. The proof generalizes easily to languages containing more modalities.

Let $\Sigma(p)$ be any set of modal formulas containing the proposition letter p (and possibly other proposition letters and nominals), and suppose $\Sigma(p)$ implicitly defines the proposition letter p, relative to K. Let p' be a new proposition letter, and let $\Sigma(p')$ be the result of replacing all occurrences of p in Σ by p'. Then, by the definition of implicit definability, $\Sigma(p) \cup \Sigma(p') \models^{glo}_{\mathsf{K}} p \leftrightarrow p'$. Let $\Gamma(p) = \{\Box^n\varphi \mid \varphi \in \Sigma(p), n \in \omega\}$, and define $\Gamma(p')$ similarly.

Claim 1: $\Gamma(p) \cup \Gamma(p') \models_{\mathsf{K}} p \leftrightarrow p'$.

Proof of claim: Suppose $\mathfrak{M}, w \models \Gamma(p) \cup \Gamma(p')$ for some model $\mathfrak{M}$ based on a frame in K. Let $\mathfrak{M}_w$ be the submodel of $\mathfrak{M}$ generated by w. By closure under generated subframes, the underlying frame of $\mathfrak{M}_w$ is also in K. By construction, $\mathfrak{M}_w$ globally satisfies $\Sigma(p)$ and $\Sigma(p')$. It follows that $\mathfrak{M}_w$ globally satisfies $p \leftrightarrow p'$, hence, $\mathfrak{M}_w, w \models p \leftrightarrow p'$, hence $\mathfrak{M}, w \models p \leftrightarrow p'$. $\dashv$

By compactness, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0(p) \cup \Gamma_0(p') \models_{\mathsf{K}} p \leftrightarrow p'$. It follows that $\models_{\mathsf{K}} (p \land \bigwedge \Gamma_0(p)) \to (\bigwedge \Gamma_0(p') \to p')$. Let ϑ be an interpolant for this implication. Then the following facts hold.

1. The proposition letters p and p' do not occur in ϑ.
2. $\models_{\mathsf{K}} (p \land \bigwedge \Gamma_0(p)) \to \vartheta$.
3. $\models_{\mathsf{K}} \vartheta \to (\bigwedge \Gamma_0(p') \to p')$, and hence, by uniform substitution, $\models_{\mathsf{K}} \vartheta \to (\bigwedge \Gamma_0(p) \to p)$.

We conclude that $\Gamma_0(p) \models_{\mathsf{K}} p \leftrightarrow \vartheta$, and hence $\Sigma(p) \models^{glo}_{\mathsf{K}} p \leftrightarrow \vartheta$. $\square$

Here is a simple example of an elementary frame class on which the basic modal language *lacks* the Beth property. Let K be the class of frames satisfying $\exists x \forall yz.(Ryz \leftrightarrow y = x)$, and let $\Sigma = \{p \to \Box q, \neg p \to \Box \neg q\}$. Clearly, in models that are based on a frame in K and that globally satisfy Σ, q holds at a state iff p holds at the root, and hence, Σ implicitly defines q in terms of p, relative to K. However, a simple bisimulation argument shows that there is no explicit definition of q in terms of p, relative to Σ and K, in the basic modal language.

Preservation results for bisimulation products

One might ask for a syntactic characterization of the elementary frame properties that are preserved under taking bisimulation products. Such a preservation theorem can indeed be given. In what follows, we will characterize the first-order formulas that are preserved under bisimulation products, in the form of a preservation theorem. Recall the definition of p-formulas on page 18.

In the following proof we will refer to frames as models (models of the first-order frame correspondence language $\mathcal{L}^1_{fr}$, to be precise). This seems the more natural choice in the present context, since the theorem concerns first-order formulas.

2.5.5. THEOREM. *A first-order sentence φ is preserved under bisimulation products iff φ is equivalent to a conjunction of sentences of the form $\forall \vec{x}(\psi \to \chi)$, where ψ is a p-formula and χ is either an atomic formula or $\bot$.*

A similar characterization can be given for the first-order sentences that are preserved under bisimulation products and generated subframes. Call a strict p-sentence one that contains no unbounded universal quantifiers. In other words: bounded universal quantifiers, unbounded existential quantifiers, positive atoms.

2.5.6. THEOREM. *A first-order sentence is preserved under bisimulation products and generated subframes iff it is equivalent to a conjunction of formulas of the form $\forall \vec{x}(\varphi \to \psi)$ where φ is a strict p-formula and ψ is atomic or $\bot$.*

2.6 Decidability and complexity

Many decision problems can be formulated in the context of modal logic. We will mention a few. The *model checking* problem: given $\mathfrak{M}, w$ and φ, check if $\mathfrak{M}, w \models \varphi$.

2.6.1. THEOREM ([15]). *The model checking problem for modal formulas can be solved in polynomial time.*

The *frame checking* problem: given $\mathfrak{F}$ and φ, check if $\mathfrak{F} \models \varphi$.

2.6.2. THEOREM. *The frame checking problem for modal formulas is co-*NP*-complete.*

The *modal equivalence* problem: given $\mathfrak{M}, w$ and $\mathfrak{N}, v$, check if there is a modal formula that distinguishes w from v.

2.6.3. THEOREM ([23]). *The modal equivalence problem can be solved in polynomial time.*

The *frame satisfiability* problem: given a formula φ, check if there is a frame on which φ is valid.

2.6.4. THEOREM. *The frame satisfiability problem for modal formulas is highly undecidable, in fact not analytical.*

Proof: By Theorem 1.2.3, the satisfiability problem for monadic second order formulas in one binary relation is non-analytical. Thomason [27] reduced this problem to the following problem:

> Given uni-modal formulas φ, ψ of the basic modal language, such that ψ is closed (i.e., contains no proposition letters). Does φ entail ψ on frames (i.e., is ψ valid on every frame on which φ is valid)?

This problem can again be reduced to the frame satisfiability problem: it suffices to note that, for a modal operator $\Diamond$ not occurring in φ and ψ, φ entails ψ on frames iff $\varphi \wedge \Diamond\neg\psi$ has no frame (here, we use the fact that ψ is a closed formula). $\square$

Incidentally, the frame satisfiability problem for uni-modal formulas is trivially decidable in co-non-deterministic polynomial time, due to the fact that every frame satisfiable uni-modal formula has a singleton frame.

The *elementarity* problem: given a formula φ, does φ define an elementary frame class?

2.6.5. THEOREM. *The problem whether a given modal formula defines an elementary frame class is highly undecidable, in fact not analytical.*

Proof: Let φ be a modal formula, and let $\Diamond$ be a modal operator not occurring in φ. Then φ is frame satisfiable iff $\varphi \wedge (\Box\Diamond p \to \Diamond\Box p)$ is not elementary. It follows by Theorem 2.6.4 that the elementarity problem is not analytical. $\Box$

Finally, the decision problem that will receive most attention in this thesis is the *satisfiability problem*. For a given frame class K, the problem is to test if a modal formula is satisfiable on K or not. For different classes K, and for different extensions of the basic modal language, we will address the question if this problem is decidable, and if so, what is its complexity.

Let us say that a frame class K has the finite model property if whenever a modal formula is satisfiable on a frame in K, then it is satisfiable on a finite frame in K. If K has the finite model property, and if membership of a frame with respect to K can be tested effectively, then the modal formulas that are satisfiable on K can be enumerated: simply enumerate all triples $(\mathfrak{M}, w, \varphi)$, where $\mathfrak{M}$ is a finite model, w is a world of $\mathfrak{M}$ and φ is a modal formula, and check for each such triple if $\mathfrak{M}, w \models \varphi$ and if the underlying frame of $\mathfrak{M}$ is in K.

Dually, if $\mathbf{K}_{\mathcal{M}}\{\varphi\}$ is complete with respect to K, for some φ, then we can use this in order to enumerate the formulas that are not satisfiable with respect to K: simply enumerate all negations of formulas derivable in $\mathbf{K}_{\mathcal{M}}\{\varphi\}$.

If both the satisfiable and the non-satisfiable formulas can be enumerated, then the satisfiability problem is decidable: the decision procedure simply performs both enumerations in parallel, and stops as soon as the input formula occurs in one of the two enumerations. Since every formula is either satisfiable or non-satisfiable, the algorithm will stop after a finite amount of time. Note that while decidability might be shown in this way, no concrete bounds on the amount of time, or space, needed to solve the problem can be derived.

A useful method for proving the decidability and the finite model property is using *filtrations*. Let $\mathfrak{M}$ be a model based on a frame $\mathfrak{F} = (W, R)$ and let Σ be a set of formulas closed under subformulas. Define an equivalence relation $\sim_\Sigma$ on W such that for every $w, v \in W$:

$$w \sim_\Sigma v \ \text{ iff } \ \text{for every } \psi \in \Sigma,\ \mathfrak{M}, w \models \psi \text{ iff } \mathfrak{M}, v \models \psi$$

Denote by $[w]$ the $\sim_\Sigma$-equivalence class containing w and let $W/_{\sim_\Sigma}$ be the set of all $\sim_\Sigma$-equivalence classes of W. Define a valuation V_Σ on $W/_{\sim_\Sigma}$ such that $V_\Sigma(p) = \{[w] \mid w \in V(p)\}$. The model $\mathfrak{M}/_{\sim_\Sigma} = (W/_{\sim_\Sigma}, R_\Sigma, V_\Sigma)$ is called a *filtration* of $\mathfrak{M}$ through Σ if R_Σ is a binary relation on $W/_{\sim_\Sigma}$ such that for any $\psi \in \Sigma$ and $w \in W$, $\mathfrak{M}, w \models \psi$ iff $\mathfrak{M}/_{\sim_\Sigma}, [w] \models \psi$. This notion can be generalized to multi-modal languages as well.

2.6.6. Definition (Filtrations). *A frame class* K admits filtration *if for every modal formula* φ *there is a finite set of formulas* Σ_φ *containing all subformulas of* φ*, such that whenever* $\mathfrak{M}, w \models \varphi$ *and* $\mathfrak{M}$ *based on a frame in* K*, there is a filtration of* $\mathfrak{M}$ *over* Σ_φ *whose underlying frame is in* K.

We say that K admits polynomial filtration *if it admits filtration and the size of* Σ_φ *is polynomial in the length of* φ*. We say that* K admits simple *filtration if it admits filtration and for every formula* φ*,* Σ_φ *is the set of subformulas of* φ*.*

Since $|W/_{\sim_\Sigma}| \leq 2^{|\Sigma|}$, if K admits filtration then it has the finite model property.

Since the number of subformulas of φ is polynomial in the length of φ, every frame class that admits simple filtration admits polynomial filtration.

2.6.7. THEOREM. *Let* K *be any elementary frame class. If* K *admits polynomial filtration then satisfiability of modal formulas with respect to* K *can be decided in* NEXPTIME*.*

Proof: This can be considered a folklore result.

If K admits polynomial filtration, then every satisfiable formula φ has a model whose size can be bounded by an exponential in the length of φ. It therefore suffices to guess such a model and check if it satisfies φ and if the underlying frame is in K. Both of these checks can be performed in polynomial time (note that the model checking problem for a fixed first order formula can be solved in polynomial time). □

Frame classes defined by shallow formulas give us a nice example for the use of the filtration method.

2.6.8. THEOREM. *Every frame class defined by a finite set of shallow modal formulas admits polynomial filtration, hence has the finite model property and has a satisfiability problem that can be solved in* NEXPTIME*.*

Proof: Lewis [21] proved a restricted version of this result, for frame classes defined by modal formulas with modal depth at most 1. The same proof can be used to prove our more general result, with a small modification. Let K be a frame class defined by a finite set Γ of shallow modal formulas. For any modal formula φ, define Σ_φ to be the union of the set of subformulas of φ with the set of closed subformulas of formulas in Γ (recall that a formula is closed if it contains no proposition letters). Proceeding as in [21] using Σ_φ as the filtration set for φ, one can construct for every model $\mathfrak{M}$ based on a frame in K a filtration $\mathfrak{M}'$ with respect to Σ_φ, such that the underlying frame of $\mathfrak{M}'$ is in K, and φ is satisfied at some world in $\mathfrak{M}'$. □

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Part II

Topological Semantics for Modal Logic

Chapter 1

Topological semantics and completeness

The necessary topological definitions have been taken from [Engelking 1977] mostly, while for modal logic we used [Blackburn et al. 2001] as the dominating reference.

1.1 Preliminaries

We will define topological semantics for modal logic and present the soundness result in this section.
First of all, let us specify what kind of structure is a topological space.

Definition 1.1.1 A Topological Space $\mathcal{T} = (X, \tau)$ *is a set* X *with a collection of its subsets* τ *satisfying the following:*

T_1 $\emptyset \in \tau$ *and* $X \in \tau$,

T_2 *If* $O_1 \in \tau$ *and* $O_2 \in \tau$ *then* $O_1 \cap O_2 \in \tau$,

T_3 *If* $O_i \in \tau$ *for all* $i \in I$, *with* I *some index set, then* $\bigcup\limits_{i \in I} O_i \in \tau$.

The elements of τ are called *opens*. So, the empty set and the universe are opens, opens are closed under finite intersections and under arbitrary unions. If $x \in O \in \tau$ then we say that O is an open neighbourhood of x. Informally, open neighbourhoods of the given point tell us which points are "close" to the chosen one. It is known that the set $O \subseteq X$ is open iff every point in O has an open neighbourhood included in O. The universe is the (biggest) open neighbourhood of all of its points. The complements of opens are called *closed* and with the de Morgan rules in mind, it does not take long to see that arbitrary intersections, as well as finite unions of closed sets are closed, as are the universe and the empty set. There are many equivalent means of defining topological structure on a set (fixing the family of closed sets is one of them) and we will see some further on in this paper, but what we said so far is enough to define the topological semantics for the modal language with the single box operator.

Definition 1.1.2 *The basic modal language consists of countable stock of proposition letters $p, p_1, p_2, ..$ the constant truth $\top$, boolean connectives $\wedge, \neg$ and the modal operator $\Box$. Modal formulas are denoted by Greek letters and are built in the following way:*

$$\phi \ ::= \ p_i \mid \top \mid \phi \wedge \psi \mid \neg\phi \mid \Box\phi.$$

Now we are ready to define topological models for the basic modal language. The valuation on the topological space will be the function assigning a subset of the topological space to each propositional letter.

Definition 1.1.3 A topological model $\mathcal{M}$ *is a triple* (X, τ, ν) *where* $\mathcal{T} = (X, \tau)$ *is a topological space and the valuation* ν *sends propositional letters to subsets of* X*. The notation* $\mathcal{M}, x \models \phi$ *(or simply* $x \models \phi$*) will read "the point* x *of the model* $\mathcal{M}$ *makes the formula* ϕ *true"; if* A *is a subset of* X*,* $A \models \phi$ *will mean that* $x \models \phi$ *for all* x *in* A*. The definition of truth proceeds like this:*

$$\begin{array}{rll}
x \models p_i & \text{iff} & x \in \nu(p_i), \\
x \models \top & & \text{always}, \\
x \models \phi \wedge \psi & \text{iff} & x \models \phi \quad \text{and} \quad x \models \psi, \\
x \models \neg\phi & \text{iff} & x \not\models \phi, \\
x \models \Box\phi & \text{iff} & \exists O \in \tau : \ x \in O \models \phi.
\end{array}$$

The definitions of model validity and the space validity are standard. In words the above definition says that if we associate formulas with the set of points making this formula true, then the abstract operations $\neg$ and $\wedge$ become set-theoretic complement and intersection operations respectively. The last part of the truth definition is not that explicit - it says that formula $\Box\phi$ is true at x, just when ϕ is true at all the nearby points - everywhere in some open neighbourhood of x, that is. Taking a closer look, $\nu(\Box\phi)$, defined as the set of points making the formula $\Box\phi$ true, consists of all the points having an open neighbourhood $O \subseteq \nu(\phi)$. This is nothing else but the interior part of $\nu(\phi)$ - a topologist will say. More formally, here comes another definition from general topology:

Definition 1.1.4 *For* A *subset of the topological space, the* interior *of* A*, denoted* $\mathbf{I}(A)$ *is the union of all open sets included in* A*, or, equivalently, the biggest open subset of* A*.*

To make the picture complete, we recall the following proposition from [Engelking 1977]:

Proposition 1.1.5 *The point x belongs to $\mathbf{I}(A)$ iff there is an open neighbourhood O of x such that $O \subseteq A$.*

In the light of the latter definition and proposition, it should be clear that $\nu(\Box\phi) = \mathbf{I}(\nu(\phi))$. So the abstract $\Box$ operator becomes the interior operator over the subsets of the topological space in our interpretation. We would like to mention the topological equivalents for other logical connectives definable in our language. It is rather obvious that the boolean connectives obtain their traditional set-theoretic meaning, but what is the topological interpretation of the modal $\Diamond$? A simple derivation shows that $\nu(\Diamond\phi) = -\mathbf{I}-(\nu(\phi))$ where $-$ stands for set-theoretic complement operation. Recalling that $-\mathbf{I}(-A)$ with A subset of the topological space is nothing else then the *closure set* of A, we conclude $\nu(\Diamond\phi) = \mathbf{C}(A)$. Here $\mathbf{C}$ denotes the topological closure operator, assigning to each subset A of the topological space the smallest closed set containing this subset - the intersection of all closed sets which extend A, in other words.

Now, we know more about the $\mathbf{I}$ and $\mathbf{C}$ from general topology - they satisfy certain conditions. In fact, given the set equipped with the operator working on its subsets, imposing the interior- or closure- conditions on this operator is another way of defining the topological structure on the set. Let us have a look at these conditions and see what they oblige our $\Box$ to be like.

Proposition 1.1.6 *Let (X, τ) be any topological space,* $\mathbf{I}$ *and* $\mathbf{C}$ *the interior and closure operators defined by topology, then the following holds for any subsets of X:*

(I1)	$\mathbf{I}(X) = X$	*(C1)*	$\mathbf{C}(\emptyset) = \emptyset$
(I2)	$\mathbf{I}(A) \subseteq A$	*(C2)*	$A \subseteq \mathbf{C}(A)$
(I3)	$\mathbf{I}(A \cap B) = \mathbf{I}(A) \cap \mathbf{I}(B)$	*(C3)*	$\mathbf{C}(A \cup B) = \mathbf{C}(A) \cup \mathbf{C}(B)$
(I4)	$\mathbf{I}(\mathbf{I}(A)) = \mathbf{I}(A)$	*(C4)*	$\mathbf{C}(\mathbf{C}(A)) = \mathbf{C}(A)$

It is not difficult to notice that the conditions (I1),..,(I4) when read with $\Box$ instead of $\mathbf{I}$ look like the axioms for the modal system $S4$! Dually, conditions (C1),..,(C4) look like $S4$ axioms written with $\Diamond$. In fact, as we will see shortly, any topological model makes the modal logic $S4$ valid, turning $S4$ into the smallest normal modal logic of topological spaces. Let us put all these in precise mathematical form.

Definition 1.1.7 *The modal logic $S4$ is the smallest set of modal formulas which contains all the classical tautologies, the following axioms:*

(N) $\Box\top$

(T) $\Box p \to p$

(R) $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$

(4) $\Box p \to \Box\Box p$ *and is closed under modus ponens, substitution and monotonicity (from* $\phi \to \psi$ *derive* $\Box\phi \to \Box\psi$*).*

The above axioms are not exactly the standard way of defining $S4$, but equivalent to it and the best for our present needs they are. We claim that every topological space validates all the modal formulas in $S4$ thus proving the soundness of $S4$ with respect to the topological semantics.

Theorem 1.1.8 *The theorems of $S4$ are valid on every topological space.*

Proof: That the axioms (N), (T), (R) and (4) are valid on any topological space is an easy observation after comparing to the conditions (I1), (I2), (I3), (I4). That the derivation rules preserve the validity is a trivial exercise. $\dashv$

McKinsey and Tarski prove even more in [McKinsey and Tarski 1944], namely that $S4$ is the complete logic of topological spaces. We will approach this matter in the next section via Kripke completeness for $S4$.

1.2 Frames and Completeness

In this section we will build the connections between topological and Kripke semantics. We will consider $S4$-frames and see that they come in one-to-one correspondence with the special class of topological spaces, called Alexandroff spaces. This correspondence often facilitates some completeness proofs, showing that for the modal logics above $S4$ the topological semantics is a generalization of the frame semantics. Concluding this section, we will consider the modal logics $S4.1$ and $S4.2$ and see what class of topological spaces they characterize.
We have seen in the previous section that every topological model is a model for $S4$. This is not the case in Kripke semantics. The frame is validating $S4$ iff the relation on the frame is reflexive and transitive.

Definition 1.2.1 *The structure $\mathcal{F} = (W, R)$ is called a qo-set (quasi-ordered set) if W is a set and R is a reflexive and transitive relation on W.*

Such frames are also enough to refute any non-theorem of $S4$, as it is well-known.

Proposition 1.2.2 *The modal logic $S4$ is sound and strongly complete with respect to the class of all qo-sets.*

To illustrate the connection with the topological semantics, let us examine upward closed subsets of a qo-frame $\mathcal{F} = (W, R)$. The subset $A \subseteq W$ is upward closed, if together with any point w it contains all of its successors: $w \in A$ & $Rwv \Rightarrow v \in A$. It is not difficult to check that an arbitrary union or intersection of upward closed sets is again upward closed. This is sufficient for upward closed sets to form a topological structure of open sets on W. Of course, the topology obtained in this way is of a special kind - opens become closed under *arbitrary*, rather then *finite*, intersection. This sort of topological spaces are known from general topology:

Definition 1.2.3 *A topological space is called Alexandroff space (A-space for short) if either of the following equivalent conditions hold:*
1. Arbitrary intersections of opens are open.
2. Every point has a least open neighbourhood.

The second condition in the above definition opens the way from A-spaces to qo-sets. To define a reflexive-transitive relation on an A-space, we simply say that Rxy holds iff y is included in the least open neighbourhood of x. Reflexivity of such a relation is immediate, for transitivity note that if y is a member of the least open neighbourhood O_x of x, then O_x becomes an open neighbourhood of y as well and thus includes in itself all the members of the least open neighbourhood of y. It is worth mentioning that if we perform the A-space-qo-set transformation both ways starting from either one, we will get the initial structure back.

Well then, we have seen how any qo-set becomes an A-space, but why treat upward closed sets as opens, why not closed sets? They will happily satisfy the conditions imposed on the collection of closed sets for topology, will they not? The reason lies behind our definition of the interpretation of modal formulas. Indeed, the structures of a qo-set and the corresponding A-space are very much the same - just seen through different mathematical glasses, but what about the modal models built on them? Are they the same in any sense? The following theorem sheds light on these questions.

Theorem 1.2.4 *Let $\mathcal{F} = (W, R)$ be a qo-set and $\mathcal{T} = (W, \tau)$ corresponding A-space. For any valuation ν on W, for any point $w \in W$ and for any modal formula ϕ, we have:*

$$\mathcal{F}, \nu, w \models \phi \text{ iff } \mathcal{T}, \nu, w \models \phi.$$

Proof: The proof goes by induction on the length of ϕ. The propositional case and the case for the boolean connectives are trivial. Recalling that the R-successors of any point constitute precisely the least open neighbourhood of this point; and that in A-space $\Box\phi$ is true at a point iff ϕ is true in the least open neighbourhood of this point, we observe that $\mathcal{F}, \nu, w \models \Box\phi$ iff $\mathcal{T}, \nu, w \models \Box\phi$, which finishes the proof. $\dashv$

With the help of the latter theorem we can now prove the topological completeness of $S4$.

Theorem 1.2.5 *$S4$ is strongly complete with respect to the class of all topological spaces.*

Proof: Take any $S4$-consistent set of formulas Σ. We know that Σ can be satisfied in a model of which the underlying frame is a qo-set. Take the A-space corresponding to this qo-set, keeping the valuation. This will be a topological model where Σ is satisfied. $\dashv$

All we have done so far in this section shows that for $S4$ and its extensions topological semantics is more general then Kripke semantics. If the modal logic containing $S4$ is (strongly) complete with respect to some class of frames (qo-sets, in fact), then this logic is (strongly) complete with respect to the corresponding class of A-spaces. Certainly the most interesting topological spaces fail to have Alexandroff structure (although note that all finite topological spaces are clearly A-spaces), but we can still exploit A-spaces for completeness results. Witness, for example, the following picture: take any $S4$-consistent set of formulas Γ; surely Γ defines some class K of topological spaces which make all the formulas in Γ valid. If, in addition, $S4 + \Gamma$ is Kripke (strongly) complete, then it will be topologically (strongly) complete with respect to K, or, indeed, with respect to any subclass of K containing enough A-spaces to refute non-theorems of $S4 + \Gamma$ - thanks to theorem 1.2.4. This is quite a general and vague statement, but we are going to illustrate it in the following section.

Before proceeding to the next section though, we would like to address one interesting question concerning the topological semantics. Namely, is there a Kripke-incomplete extension of $S4$, which is topologically complete? In other words, is the topological semantics really more general than the frame semantics, from the completeness point of view? The answer is affirmative. In [Gerson 1975] the extension of $S4$ is presented which is complete with respect to the neighbourhood semantics, but incomplete with respect to Kripke semantics. We can prove that for $S4$ and its extensions, the neighbourhood semantics is equivalent to the topological semantics. It follows that the example given by Gerson will also work for proving that the topological semantics is strictly more general then the Kripke semantics. By the time of writing it was not known to us whether or not these facts have already appeared in literature.

1.3 Some more topological completeness

If the topological semantics is more general than Kripke semantics, we could just take any of the extensions of $S4$ known to be Kripke complete and automatically get their topological completeness, as outlined at the end of the previous section. In case we have the extension of $S4$ axiomatizable with Sahlqvist formulas, we would even get strong completeness! More challenging seems the question which topological property will then the logic in question define. We start with investigating the modal system $S4.2$.

Definition 1.3.1 *The modal logic $S4.2$ is the extension of $S4$ with the axiom (.2) $\Diamond\Box\phi \to \Box\Diamond\phi$.*

It is known that the frames for $S4.2$ are strongly directed, i.e. any two worlds having a common predecessor must share a successor, too. The formula (.2) being in a Sahlqvist form, gives strong Kripke completeness, and therefore strong topological completeness - with respect to the class of topological spaces characterized by (.2). To describe this class in topological terms, we will need the following definition:

Definition 1.3.2 *The topological space is* extremally disconnected *if any of the following two equivalent conditions hold:*

1. The closure of any open is open.
2. The closures of any two disjoint opens are disjoint.

Extremally disconnected spaces were defined in [Stone 1937] and have been around in general topology ever since. As it appears, (.2) characterizes the class of extremally disconnected topological spaces. This statement seems to be known to scholars working in the field. However we could not find the exact reference who first established it. Here is the proof of this fact:

Theorem 1.3.3 *$S4.2$ defines the class of extremally disconnected topological spaces.*

Proof: That $S4$ is valid on any topological space is already known, let us check that the formula (.2) is valid on a topological space $\mathcal{T} = (X, \tau)$ iff $\mathcal{T}$ is extremally disconnected.
$\boxed{\Rightarrow}$ Suppose $\mathcal{T}$ is not extremally disconnected, then there are O_1 and O_2 open disjoint sets, such that $\mathbf{C}(O_1) \cap \mathbf{C}(O_2) \neq \emptyset$. Define the valuation ν in such a way that $\nu(p) = O_1$. As O_1 is open, we have that $\nu(\Box p) = \mathbf{I}(\nu(p)) = \mathbf{I}(O_1) = O_1$. We also have $O_2 \models \neg p$ and by openness of O_2 this implies $O_2 \models \Box\neg p$. Take $x \in \mathbf{C}(O_1) \cap \mathbf{C}(O_2)$. We claim that $x \not\models \Diamond\Box p \to \Box\Diamond p$. Indeed, for any open neighbourhood O_x of x, we know from $x \in \mathbf{C}(O_1)$ that $O_x \cap O_1 \neq \emptyset \Rightarrow O_x \cap \nu(\Box p) \neq \emptyset \Rightarrow x \models \Diamond\Box p$. Likewise, from $x \in \mathbf{C}(O_2)$ we get $O_x \cap O_2 \neq \emptyset \Rightarrow O_x \cap \nu(\Box\neg p) \neq \emptyset \Rightarrow x \models \Diamond\Box\neg p$ and this means $x \not\models \Box\Diamond p$.
$\boxed{\Leftarrow}$ Say $\mathcal{T}$ is extremally disconnected and ν is any valuation on it. We ought to prove $\nu(\Diamond\Box\phi) \subseteq \nu(\Box\Diamond\phi)$ for any ϕ. In other words, we need to show $\nu(\Diamond\Box\phi) \cap -\nu(\Box\Diamond\phi) = \emptyset$, with $-$ standing for set-theoretic complement. Now, $\nu(\Diamond\Box\phi) = \mathbf{CI}(\nu(\phi))$ and $-\nu(\Box\Diamond\phi) = -\mathbf{IC}(\nu(\phi)) = \mathbf{CI}(-\nu(\phi))$. Clearly, $\mathbf{I}(-\nu(\phi))$ and $\mathbf{I}(\nu(\phi))$ are disjoint opens and their closures must be disjoint by extremal disconnectedness of $\mathcal{T}$, hence $\mathcal{T} \models \Diamond\Box\phi \to \Box\Diamond\phi$, for any ϕ. $\dashv$

As an easy consequence we obtain that extremally disconnected A-spaces are in one-to-one correspondence with strongly directed Kripke frames!
Summarizing all about $S4.2$ we have accomplished here, we get the following:

Theorem 1.3.4 *$S4.2$ is sound and strongly complete with respect to the class of extremally disconnected topological spaces.*

Proof: Soundness is the consequence of the previous theorem, for completeness we use the Kripke completeness of $S4.2$. Indeed, any Kripke frame validating $S4.2$ gives rise to an A-space which validates $S4.2$ by theorem 1.2.4 and is, thus, extremally disconnected by the previous theorem. Then any consistent set of modal formulas Σ can be satisfied in the A-space corresponding to the frame underlying the $S4.2$-Kripke-model for Σ. $\dashv$

We were quite lucky with the formula (.2) - it defined a well-known, topologically valuable class. Our next example shows that finding the topologically transparent property characterizing the modally defined class of spaces is not always possible. There are plethora of consistent extensions of $S4$ around (uncountably many, even) and no wonder some of them define rather bizzare topological properties. Nevertheless, if not entirely transparent and immediate to the intuition, some of the modal formulas still define interesting classes of spaces. A good example is the McKinsey formula $\Box\Diamond\phi \rightarrow \Diamond\Box\phi$. This formula first appeared in [McKinsey 1945] where the author baptized the corresponding extension of $S4$ as $S4.1$.

Definition 1.3.5 *The modal logic $S4.1$ is the extension of $S4$ with the axiom (.1) $\Box\Diamond\phi \rightarrow \Diamond\Box\phi$.*

It is known that in the presence of transitivity the McKinsey formula means atomicity for frames, i.e. every point sees a reflexive maximum: $\forall x \exists y (Rxy \,\&\, \forall z (Ryz \rightarrow y{=}z))$. The topological counterpart of this property can be defined in various equivalent ways, giving different intuitive grasp of what (.1) really says spatially. The following definition is inspired by the notion used in [Esakia 1979] for the analysis of the system $S4.1$.

Definition 1.3.6 *Let $\mathcal{T} = (X, \tau)$ be a topological space and $A \subseteq X$ its any subset. Define the* frontier *set for A to be the set $\mathbf{C}(A) \cap \mathbf{C}(-A)$. Call the operator $\mathbf{Fr}$ defined by the equation $\mathbf{Fr}(A) = \mathbf{C}(A) \cap \mathbf{C}(-A)$ the frontier operator.*
Call the topological space atomic if the frontier of any subset has an empty interior, i.e. $\mathbf{IFr}(A) = \emptyset$ for any A.

To say but a bit more, in atomic spaces no open set can be included in the frontier of some other set. We will demonstrate that (.1) characterizes the class of atomic spaces.

Theorem 1.3.7 *$S4.1$ is the logic of atomic spaces.*

Proof: We have to prove that for any $\mathcal{T} = (X, \tau)$, $\mathcal{T} \models \Box\Diamond\phi \to \Diamond\Box\phi$ iff $\mathcal{T}$ atomic.
$\boxed{\Rightarrow}$ Say $\mathcal{T}$ is not atomic. Then for some $A \subseteq X$ and for some non-empty open $O \in \tau$ we have $O = \mathbf{IFr}(A)$. Define the valuation ν on $\mathcal{T}$ in such a way that $\nu(p) = A$ and note that $\mathbf{Fr}(A)$ is exactly the set of points making $\Diamond p \land \Diamond\neg p$ true. Then for any point $x \in O$ it is clear that $x \models \Box(\Diamond p \land \Diamond\neg p)$ which is equivalent to say that $x \models \neg(\Box\Diamond p \to \Diamond\Box p)$ so we have refuted the McKinsey formula.
$\boxed{\Leftarrow}$ Assume $\mathcal{T} \not\models \Box\Diamond\phi \to \Diamond\Box\phi$. There must be a valuation ν under which some point x falsifies the McKinsey formula $x \not\models \Box\Diamond\phi \to \Diamond\Box\phi$ implying $x \models \Box\Diamond\phi$ and $x \models \neg\Diamond\Box\phi$. The latter is equivalent to $x \models \Box\Diamond\neg\phi$ which together with $x \models \Box\Diamond\phi$ means that there are open neighbourhoods O_1 and O_2 of x such that $O_1 \models \Diamond\phi$ and $O_2 \models \Diamond\neg\phi$. Then $x \in O_1 \cap O_2 \models \Diamond\phi \land \Diamond\neg\phi$ and $\nu(\phi)$ appears to be the set having the frontier with nonempty open subset $O_1 \cap O_2$, which denies $\mathcal{T}$ from being an atomic space. $\dashv$

For topological strong completeness witness the following:

Theorem 1.3.8 *$S4.1$ is sound and strongly complete with respect to all atomic spaces.*

Proof: Soundness follows from the previous theorem, for strong completeness we use the analogous result with respect to Kripke frames for $S4.1$ proved in [Lemmon and Scott 1966]. $\dashv$

While (.2) defined the essential topological property, spaces characterized by (.1) seem to receive minor attention in the topological literature. As we already mentioned, this is likely to happen with the most topological properties defined by modal formulas in the basic language. This is only one side of the coin though - taking modal formulas and tackling with the corresponding topological property to make some spatial sense of it. Another approach would be to take a topologically sensible class and see which modal formulas may describe it. Or indeed, when is a class of topological spaces *definable* at all? What are the conditions to impose on the class of topological spaces to be sure that this class can be characterized by a modal formula? We address this issue in the next part of this work.

Chapter 2

Definable Spaces

This part deals with the question of space definability. When is a class of topological spaces modally definable? In the case of Kripke semantics the Goldblatt-Thomason theorem has an answer. Something similar for topological spaces would have been useful. Let us recall this theorem:

Theorem (Goldblatt-Thomason) *A class of frames, closed under taking ultrafilter extensions, is modally definable, if and only if it is closed under the formation of bounded morphic images, generated subframes, and disjoint unions, and reflects ultrafilter extensions.*

In the first part we have defined topological models, translated frame structure into the topological one, but we did not mention which topological operations preserve space validity. The analogs for bounded morphism, generated subframe and disjoint union will be of a major importance in space definability; the first section of this chapter is devoted to developing this kind of topological tools. In the second section we will prepare some algebraic apparatus and work out algebraic duality to approach the definability; all of the latter will allow us to prove the topological Goldblatt-Thomason theorem in the third section, with the help of the new notion of Alexandroff extension, the topological equivalent of ultrafilter extensions for frames.

2.1 Validity preserving operations

We have built the bridge between Kripke frames and Alexandroff spaces in the first chapter. Certain operations on frames preserve the validity of modal formulas. Such operations - bounded morphisms, generated subframes and disjoint union, become crucial when dealing with frame definability. No wonder we need similar notions for the topological semantics and that is what we are about to bring in this section.

While we need to rethink in topological terms the notions of bounded morphism and generated subframe, the formation of disjoint union appears to have the straightforward topological equivalent. Here is the definition:

Definition 2.1.1 *For a family of disjoint topological spaces* $\mathcal{T}_i = (X_i, \tau_i)$, $i \in I$, *their* topological sum *is the topological space* $\mathcal{T} = (X, \tau)$ *with* $X = \bigcup_{i \in I} X_i$ *and* $\tau = \{O \in X | \forall i \in I : O \cap X_i \in \tau_i\}$.

It is easy to observe that when restricted to A-spaces, this operation will yield the structure corresponding to the disjoint union of respective frames. This means we are on the right track, and the following theorem, proving that the topological sum of topological spaces preserves the validity of modal formulas, justifies our choice.

Theorem 2.1.2 *Let* $(\mathcal{T}_i)_{i \in I}$ *be a family of disjoint topological spaces and let* ϕ *be a modal formula such that* $\mathcal{T}_i \models \phi$ *holds for each* $i \in I$. *Then for* $\mathcal{T} = \bigcup_{i \in I} \mathcal{T}_i$ *we have* $\mathcal{T} \models \phi$.

Proof: Suppose $\mathcal{T} \not\models \phi$ for the sake of contradiction. This means there is a valuation ν and a point $x \in X$ such that $\mathcal{T}, \nu, x \not\models \phi$. Obviously, $x \in X_i$ for some $i \in I$. Define the valuation ν_i as described: $\nu_i(p) = \nu(p) \cap X_i$. An easy induction argument shows that:

$$\mathcal{T}_i, \nu_i, x_i \models \psi \text{ iff } \mathcal{T}, \nu, x_i \models \psi,$$

for all $x_i \in X_i$ and all modal formulas ψ. This gives the desired contradiction, because we get $\mathcal{T}_i, \nu_i, x \not\models \phi$. $\dashv$
NB In fact an easy argument shows that the converse holds as well - each member of a family of spaces validates some modal formula, if their topological sum validates this formula.

This already allows us to show that certain topologically interesting properties (or rather, their corresponding classes) are not definable by (sets of) modal formula(s) in the basic modal language. Our examples are connectedness and compactness.

Definition 2.1.3 *For a topological space* $\mathcal{T} = (X, \tau)$ *we say that:*
(i) $\mathcal{T}$ *is* connected *if* X *can not be presented as the union of two disjoint open subsets.*
(ii) $\mathcal{T}$ *is* compact *if any family* $(F_i)_{i \in I}$ *of closed subsets of* X *with the finite intersection property has a non-empty intersection.*

Connected and compact topological spaces play the starring role in general topology. Unfortunately, neither of these properties can be defined in our language. Witness the following:

Theorem 2.1.4 *The class of connected topological spaces and the class of compact topological spaces are not modally definable.*

Proof: The following example serves for both reasons: consider countably many disjoint sets $X_i = \{i\}$ with the corresponding topologies $\tau_i = \{\emptyset, X_i\}$, $i \in I = \{1, 2, ..\}$. These are one-point sets with the only possible topology. All of them are compact (any finite topological space clearly is), and connected. If connectedness (compactness) were definable by a modal formula ϕ, all $\mathcal{T}_i = (X_i, \tau_i)$ would validate ϕ and, by the preservation of modal validity under the formation of topological sum, so would $\mathcal{T} = \bigcup_{i \in I} \mathcal{T}_i$. This would imply that $\mathcal{T}$ is connected (compact), which it is not. To observe why, note that by the definition of topology on the topological sum of spaces, $\mathcal{T}$ is just a countable set equipped with the discrete topology. In other words, the topology is such, that every subset is both open and closed. In particular, every cofinite set is closed. The collection of cofinite sets form a family of closed sets with the finite intersection property, but their intersection is empty. This shows $\mathcal{T}$ is not compact. Any cofinite set different from the universe, together with its complement, form a disjoint couple of opens giving the universe in union. Thus connectedness of $\mathcal{T}$ holds wrong as well. $\dashv$

This is rather unfortunate. We will see how the enriching of the modal language with global diamond will help us to define connectedness in the last chapter of this work, but compactness (being essentially higher-order property) will still remain wild in this sense. We leave this issue till later and go on to see the topological equivalents for other basic frame operations. Next comes the notion of generated subframe in our agenda. Recall that a generated subframe is based on a subset of a frame which together with any of its members contains all of its successors. But this is exactly what we called "upward closed sets" in the first chapter and we saw that the topological notion of open subset is the equivalent for that. At least this was the case with A-spaces - upward closed sets constituted the topology when forming the A-space from a qo-set. The following theorem shows that open subspaces of topological spaces inherit the validity of modal formulas thus proving that open subspaces are the right topological substitute for the notion of generated subframe.

Theorem 2.1.5 *Let U be the open subset of the topological space $\mathcal{T} = (X, \tau)$ and ϕ be a modal formula. If $\mathcal{T} \models \phi$ and $\mathcal{T}_U = (U, \tau_U)$ is the topological space with $\tau_U = \{O \subseteq U | O \in \tau\}$, then $\mathcal{T}_U \models \phi$.*

Proof: For any valuation ν on U let ν' be any extension of ν to all of X, agreeing with ν on U. We will prove by induction on the length of modal formulas that for any point $u \in U$, any modal formula ψ:

$$\mathcal{T}, \nu', u \models \psi \text{ iff } U, \nu, u \models \psi,$$

which readily implies the claim of the theorem.
Since the propositional letters and boolean connectives need not any special care, consider the case $\psi = \Box\psi_1$.
$\boxed{\Rightarrow}$ $\mathcal{T}, \nu', u \models \Box\psi_1$ means there is an open neighbourhood O of u, such that $\mathcal{T}, \nu', O \models \psi_1$. By the definiton of topology on U and the induction hypothesis, we get $U, \nu, O \cap U \models \psi_1$ with $O \cap U$ the open neighbourhood of u in τ_U. This means $U, \nu, u \models \Box\psi_1$.
$\boxed{\Leftarrow}$ $U, \nu, u \models \Box\psi_1$ means there is an open neighbourhood O_U of u in U such that $U, \nu, O_U \models \psi_1$. From the definition of τ_U it follows that $O_U \in \tau$. By induction hypothesis then $\mathcal{T}, \nu', O_U \models \psi_1$ with O_U open neighbourhood of u in $\mathcal{T}$, which yields $\mathcal{T}, \nu', u \models \Box\psi_1$. $\dashv$

The class of disconnected topological spaces is that of complementing the class of connected topological spaces. We have shown the latter to be modally undefinable. The former appears to be undefinable as well. Just consider the two-point space with discrete topology - it is disconnected, but the one-point open subspace of it is clearly connected. As open subspaces preserve validity, disconnectedness can not be modally defined. Although recall that we defined modally the special subclass of disconnected spaces - namely the class of extremally disconnected topological structures - in the first chapter.

So open subspaces are the topological twins of generated subframes. The difference should be mentioned though - in frames, if A is any subset of a frame, there always exists the least upward closed set containing A - the A-generated subframe. This is the case with Alexandroff spaces as well, but not generally. In the arbitrary topological space we can always take *some* open subspace surrounding the given subset A, but one can not always find a least open of this kind.

With this correspondence in hand, it is not hard to find out what the topological notion of bounded morphism should be. Look what a bounded morphism does with upward closed sets - the forth condition in the definition of a bounded morphism is just a monotonicity requirement saying that bounded-morphic preimages of the upward closed sets are upward closed. In topological terms this would mean that preimages of opens are open. Such maps are called *continuous* in general topology:

Definition 2.1.6 *The map $f : \mathcal{T}_1 \to \mathcal{T}_2$ between topological spaces $\mathcal{T}_1 = (X_1, \tau_1)$ and $\mathcal{T}_2 = (X_2, \tau_2)$ is said to be* continuous *if whenever $O_2 \subseteq X_2$ is open in $\mathcal{T}_2$, the set $f^{-1}(O_2)$ is open in $\mathcal{T}_1$, i.e.:*

$$O_2 \in \tau_2 \text{ implies } f^{-1}(O_2) \in \tau_1.$$

The back condition in the definition of bounded morphism, when slightly rephrased, is claiming that the images of upward closed sets are upward closed. This corresponds to the definition of *open* maps in topology.

Definition 2.1.7 *The map $f : \mathcal{T}_1 \to \mathcal{T}_2$ between topological spaces $\mathcal{T}_1 = (X_1, \tau_1)$ and $\mathcal{T}_2 = (X_2, \tau_2)$ is called* open *if it sends open sets to open sets, i.e.:*

$$O_1 \in \tau \text{ implies } f(O_1) \in \tau_2.$$

A map is called an interior map*, if it is both open and continuous.*

The latter definition and our earlier remarks suggest interior maps play the same role in topological semantics as bounded morphisms do in Kripke semantics. The following theorem proves this suggestion correct.

Theorem 2.1.8 *Let $\mathcal{T}_1 = (X_1, \tau_1)$ be a topological space and f an interior map from $\mathcal{T}_1$ onto $\mathcal{T}_2 = (X_2, \tau_2)$. For any modal formula ϕ, $\mathcal{T}_1 \models \phi$ implies $\mathcal{T}_2 \models \phi$.*

Proof: Again, for any valuation ν_2 on $\mathcal{T}_2$ define the valuation ν_1 as follows: $\nu_1(p) \equiv f^{-1}(\nu(p))$. We claim for any modal formula ψ and any $x \in X_1$ the following holds:

$$\mathcal{T}_1, \nu_1, x \models \psi \text{ iff } \mathcal{T}_2, \nu_2, f(x) \models \psi.$$

This will then yield the desired validity preservation result, exploiting the surjectiveness of f. The proof proceeds by induction, with propositional and boolean cases straightforward. Assume $\psi = \Box\psi_1$:
$\boxed{\Rightarrow}$ From $\mathcal{T}_1, \nu_1, x \models \Box\psi_1$ it follows that $\mathcal{T}_1, \nu_1, U_x \models \psi_1$ for some open neighbourhood U_x of x. It is by openness of f that we get $f(U_x)$ is an open neighbourhood of $f(x)$. By induction hypothesis, $\mathcal{T}_2, \nu_2, f(U_x) \models \psi_1$ and we arrive at $\mathcal{T}_2, \nu_2, f(x) \models \Box\psi_1$.
$\boxed{\Leftarrow}$ Same as above, with f^{-1} instead of f. The continuity of f ensures $f^{-1}(O_2)$ to be open in $\mathcal{T}_1$ provided O_2 is open in $\mathcal{T}_2$. $\dashv$

We are almost done with finding the topological equivalents for basic validity-preserving operations on frames. The only one left is the formation of ultrafilter extension and we will postpone the definition of the analogous topological construction till the last section of this chapter. Before that, let us put the constructions we have already defined above to work. In particular, our immediate aim is to show that the separation axioms are not definable by the modal formula. The definitions of higher separation axioms employ some involved topological terminology, which we would like to avoid here. The axioms T_0 and T_1 are defined in the last part of the work. For our present purposes it is sufficient to know (cf. [Engelking 1977]) that the real line with its usual topology obeys the separation axioms $T_0, T_{\frac{1}{2}}, T_1, T_2, T_3, T_{3\frac{1}{2}}, T_4$. Unlucky as it is, none of these separation axioms can be defined by the modal formula, not being preserved under interior maps. Here is the evidence:

Theorem 2.1.9 *The separation axioms T_i with $i \leq 4$ are not definable in the basic modal language.*

Proof: Consider the interior map from real line with standard topology to $X = \{1, 2\}$ equipped with antidiscrete topology, sending rationals to 1 and irrationals to 2. It is easy to verify that the reals obey all separation axioms, while X obeys none. As interior maps preserve the validity, none of the separation axioms could be defined by a formula in the basic modal language. $\dashv$

Note that this example will also exhibit non-compactness to be modally undefinable.
Again, we can gain a little by enriching the language we are working with, and we address this issue in the third part of the present work, where we will

see how nominals can help to define some lower separation axioms.
We would like to make a few remarks on the constructions developed in this section before going on to the next one.

Other topological constructions: In some cases well-known topological constructions give rise to natural interior maps. The validity preservation automatically transfers to these constructions. As an example we briefly discuss the topological product of spaces and the quotient of the topological space.
The product of topological spaces can be defined in two ways. We would like to avoid bringing in the respective definitions, but in both cases the natural projections are interior maps. This means that if the product of topological spaces validates some modal formula, then the components must validate this formula as well. For the quotient of a topological space note that any equivalence relation defined on the universe of the space determines a quotient space with the natural topology on it (so that the natural quotient map becomes continuous). Under the (necessary and sufficient) condition that for any open set O the union of all the equivalence classes intersecting with O is open, the natural quotient map becomes an interior map. This would mean that whenever the above condition is satisfied for a space and an equivalence relation on it, and the space validates the modal formula ϕ, the quotient space validates ϕ as well. For the exact definitions of these topological constructions we refer to [Engelking 1977].

Topological invariance: Say P is a topological property which our modal language is powerful enough to express. It is worth mentioning that the results of this section imply P is invariant for the formation of topological sums, open subspaces and interior images. This obvious consequence of the earlier theorems bears topologically interesting insight. Consider, for example, atomicity. It is straightforward from the modal point of view that being an atomic space is a topological property which is invariant under topological sums, open subspaces and interior images. This is easy to establish by direct topological proof, but still not entirely obvious. Describing invariance of properties under various topological transformations is the major task of general topology. For modally definable properties and three basic validity-preserving topological operations the invariance is automatic, witness the earlier theorems of this section. The same holds for extremal disconnectedness. We will say a bit more about extremal disconnectedness in the last chapter and get back to the topological invariance in the concluding remark of the thesis.

In the meantime, we are going to proceed to the next section of this chapter. We choose to approach the desired definability result via algebraic machinery and the next section is devoted to developing such techniques. Let us conclude this section with the remark that when restricted to A-spaces, the topological notions of topological sum, open subspace and interior map coincide with the frame-theoretic notions of disjoint union, generated subframe and bounded morphism respectively, applied on corresponding qo-sets.

2.2 Interior algebras

First introduced by McKinsey and Tarski, Interior Algebras are BAOs with the topological operator. Actually, the pioneering completeness proof for modal logic has been carried out with the help of Interior Algebras and their topological interpretations. Here we repeat some of the basic definitions and prepare the firm grounds to base our definability result upon. An Abstract Interior Algebra is just a boolean algebra with $S4$-operator, as formally stated below:

Definition 2.2.1 *An Interior Algebra is a pair $(B, \Box)$ where B is a boolean algebra, $\Box$ is an operator, assigning to each element a of B element $\Box a \in B$, such that the following holds:*

(I_1) $\Box(a \wedge b) = \Box a \wedge \Box b$,

(I_2) $\Box a \leq a$,

(I_3) $\Box\Box a = \Box a$,

(I_4) $\Box\top = \top$.

An obvious example of such an algebra would be the full set algebra over any topological space, with **I** operator. Indeed, any set algebra over the topological space, closed under **I**, will do. Following the route of the celebrated Stone Representation Theorem, we can actually embed any abstract Interior Algebra into the power set algebra over some topological space. The carrier of the topology, as in the case of boolean algebras, is the set of all ultrafilters of the algebra. The topology on this set can be defined in various ways, giving rise to different structures. Our aim favours the definition where the representation space has an Alexandroff topology. As we will see shortly, this can be done in a natural way. Namely, we declare the set of ultrafilters over an abstract Interior Algebra to be open, if it consists of all the ultrafilters extending some open filter.

Definition 2.2.2 *We call a filter F of an interior algebra $(B, \Box)$ an* open filter *if for any $a \in B$ we have that $a \in F$ implies $\Box a \in F$.*
We call an element a of an interior algebra open, *if $a = \Box a$.*

Before checking this definition of the topology to be correct, it should be mentioned ahead that the representation space for an interior algebra $\mathcal{A}$ we are about to bring turns out to be the A-space obtained from the ultrafilter frame of $\mathcal{A}$ by declaring upward-closed sets open. This means our next theorem, although topologically flavoured, is simply building up the ultrafilter frame - in topological terms.

Theorem 2.2.3 *Any Interior Algebra $\mathcal{A} = (A, \Box)$ is isomorphic to subalgebra of all subsets of some A-space.*

Proof: Consider the set $A^* \equiv Uf(A)$ of all ultrafilters of A. For arbitrary filter F of the algebra by F^* we will denote the set of all ultrafilters extending F. In case F is generated by an element a of the algebra, $F = \{b \in A | b \geq a\}$, it is clear that F^* consists of all ultrafilters having a as an element. In this case we may also use the notation $a^* \equiv \{u \in A^* | a \in u\}$.
Let O be any open filter, consider the set O^* of all ultrafilters which extend O. An easy argument shows that the collection of all O^* will qualify for the base of topology. Denote the resulting topological space $\mathcal{A}^* = (A^*, \tau^*)$ and call it the *Alexandroff extension* of A.
To show that τ^* is an Alexandroff topology, it suffices to prove that any ultrafilter has the smallest open neighbourhood. Let u be an arbitrarily chosen ultrafilter of A, consider the set of all open elements from this ultrafilter $O_u^1 \equiv \{o \in u | o = \Box o\}$, which is closed under finite meets according to (I_1) and thus generates the filter $O_u \equiv \{a \in A | a \geq o \in O_u^1\}$ which obviously is an open filter. From the fact that u is an extension of O_u it follows that O_u^* is an open neighbourhood of u, it is also the smallest such, as all the open filters contained in u are also contained in O_u.
Now consider the sets $a^* = \{u \in A^* | a \in u\}$ for each $a \in A$. By Stone's

representation theorem, family of subsets $(a^*)_{a\in A}$ of A^* is a boolean algebra isomorphic to A. Now, showing that $\mathcal{A}$ is a subalgebra of $(\wp A^*, I^*)$, where I^* stands for the interior operator defined by τ^*, boils down to showing that $(\Box a)^* = I^*a^*$ for each $a \in A$:
First observe that $(\Box a)^*$ is an open set in τ^*. Indeed, as we already mentioned, $(\Box a)^*$ coincides with the set of all ultrafilters extending the filter generated by $\Box a$. This filter is open because $\Box a \leq b$ implies $\Box a \leq \Box b$ by (I_1) and (I_3). The derivation $u \in (\Box a)^* \Rightarrow \Box a \in u \Rightarrow a \in u \Rightarrow u \in a^*$ shows that $(\Box a)^* \subseteq a^*$. We thus proved that $(\Box a)^*$ is an open set included in a^*. To complete the proof we will show that $(\Box a)^*$ is the biggest open set included in a^*.
Indeed, take any open set $O \subseteq a^*$, from the definition of τ^* we retrieve $O = \bigcup\limits_{i\in I} O_i^*$ with O_i^* an open filter for each $i \in I$. It is straightforward that $O_i^* \subseteq a^*$ for each $i \in I$. This means that an ultrafilter extending O_i must contain a and by ultrafilter theorem this is precisely to say $a \in O_i$. Thus $\Box a \in O_i$ by openness of O_i for each $i \in I$, whence $O_i^* \subseteq (\Box a)^*$, for each $i \in I$. But then $O = \bigcup\limits_{i\in I} O_i^* \subseteq (\Box a)^*$. Since O was the arbitrary open set included in a^*, we conclude that $(\Box a)^*$ is the greatest open contained in a^*. So $(\Box a)^* = I^*a^*$. $\dashv$

To get a grasp of what the above construction really does, consider the following.

Definition 2.2.4 *For an Interior Algebra $\mathcal{A} = (A, \Box)$ define the relation R^* on A^* as follows:*

$$R^*uv \quad \text{iff} \quad v \in O_u.$$

Where O_u denotes the smallest open neighbourhood of u in τ^*.

We will show now that R^* coincides with the relation R_+ of the ultrafilter frame for $\mathcal{A}$, thus linking the Alexadroff space we have just built with the ultrafilter frame of the respective algebra.

Theorem 2.2.5 *Let $\mathcal{A} = (A, \Box)$ be any Interior Algebra. The frame (A^*, R^*) coincides with the ultrafilter frame $\mathcal{A}_+$ of $\mathcal{A}$.*

Proof: Let us first turn $\mathcal{A}$ into BAO by means of defining the operator $\Diamond \equiv -\Box-$ on the boolean algebra A. This is merely another way of looking at $\mathcal{A}$. Now of two arbitrary ultrafilters u and v the following holds:

$$R^*uv \text{ iff } \Diamond a \in u \text{ for all } a \in v.$$

This is to say that the relation R^* coincides with the relation R_+ of the ultrafilter frame of $\mathcal{A}$. The proof is as follows:
$\boxed{\Rightarrow}$ Assume R^*uv and for some $a \in v$, $\Diamond a \notin u$. Since u is an ultrafilter $-\Diamond a \in u \Rightarrow --\Box -a \in u \Rightarrow \Box -a \in u$. $\Box -a$ is an open element, recall form the proof of the previous theorem that v is some extension of the open filter of all open elements from u, therefore $\Box -a \in v$ and as far as $\Box -a \leq -a$ and v is an ultrafilter, $-a \in v$ which contradicts the assumption $a \in v$.
$\boxed{\Leftarrow}$ Now let $\Diamond a \in u$ hold for all $a \in v$ and assume R^*uv fails; then v must lack at least one open element from u, or, putting it more precisely, there exists $o \in u$ such that $o = \Box o$ and $o \notin v$. By the properties of ultrafilters, then, $-o \in v$ and by our assumption $\Diamond -o \in u$. An easy verification shows that $\Diamond -o = -\Box --o = -\Box o = -o$, so $-o \in u$ and again, we find a contradiction with $o \in u$. $\dashv$

To make the picture complete, we just need some duality results concerned with homomorphisms and products. In the light of the last theorem we could just mention that the homomorphism between two interior algebras gives rise to the bounded morphism between respective ultrafilter frames and thus, the interior map between their Alexandroff extensions (the correspondence established in the first chapter). We supply the proof of this fact in the terms of theorem 2.2.3 for interested reader. First let us define how to lift the homomorphism between algebras to the map between corresponding Alexandroff extensions.

Definition 2.2.6 *Let $h : \mathcal{A}_1 \to \mathcal{A}_2$ be homomorphism of Interior Algebras. Define $h^* : \mathcal{A}_2^* \to \mathcal{A}_1^*$ as follows:*

$$h^*(u_2) = \{a_1 \in A_1 | h(a_1) \in u_2\}.$$

A simple verification shows that h^* is defined correctly (it maps ultrafilters to ultrafilters). Topologically, it also appears an interior map. We prove this fact below.

Theorem 2.2.7 *If $h : \mathcal{A}_1 \to \mathcal{A}_2$ is a homomorphism of Interior Algebras, then $h^* : \mathcal{A}_2^* \to \mathcal{A}_1^*$ is an interior map between topological spaces. Additionally, if h is injective (surjective) then h^* is surjective (injective).*

Proof: h^* can be easily checked to be sending ultrafilters to ultrafilters, surjectively if h is injective and injectively if h is surjective. The only non-trivial part to prove remains h^* being interior (that is, open and continuous). This suffices to be checked just on the elements of the respective bases (provided we first establish continuity and then openness), as it is known from general topology (cf. [Engelking 1977], section 1.4). Let us proceed towards this aim. To prove the continuity of h^* consider $O_1^* \in \tau_1^*$ element of the base generated by the open filter O_1 and take $h^{*-1}O_1^*$ which consists of all such ultrafilters of A_2 h^*-images of which contain O_1 as a subset, or, in other words, all ultrafilters of A_2 which contain the set $h(O_1)$. Although $h(O_1)$ may not be a filter, it is closed under finite meets because O_1 was closed under finite meets and h respects the boolean operations. Hence the set of all ultrafilters containing $h(O_1)$ will coincide with the set of all ultrafilters containing the smallest filter generated by $h(O_1)$. This filter is an open filter, because O_1 was an open filter and h respects the $\Box$. So we obtain $h^{*-1}O_1^* = (h(O_1))^* \in \tau_2^*$.
For openness of h^* we take an element of the base of τ_2 denoted O_2^* and consider the set $h^*(O_2^*)$. A typical representative of this set has the form $h^*(u_2)$ where u_2 is such that $O_2 \subseteq u_2$. Since $h^*(u_2) = \{a_1 \in A_1 | h(a_1) \in u_2\}$, it is immediate that $h^{-1}(O_2) \subseteq h^*(u_2)$. For h is a homomorphism and O_2 is an open filter, it follows that $h^{-1}(O_2)$ is an open filter of A_1 and this means nothing but $h^*(O_2^*) = (h^{-1}(O_2))^* \in \tau_1^*$.

Concluding this section, we show how to deal with the products of the set interior algebras.

Proposition 2.2.8 *Let $\mathcal{T}_i$ be the collection of topological spaces indexed with the set I. Denote by $(\mathcal{T}_i)_*$ the power set algebra over $\mathcal{T}_i$ with respective interior operator. $(\mathcal{T}_i)_*$ are clearly interior algebras, and the following holds:*

$$\prod_{i\in I} (\mathcal{T}_i)_* \cong (\bigcup_{i\in I} \mathcal{T}_i)_* \ .$$

With $\cong$ standing for IA*-isomorphism and $\bigcup$ for the topological sum of topological spaces.*

Proof: Define the map $f : \prod_{i\in I} (\mathcal{T}_i)_* \to (\bigcup_{i\in I} \mathcal{T}_i)_*$ as follows:
For $(a_i)_{i\in I} \in \prod_{i\in I} (\mathcal{T}_i)_*$ note that each $a_i \in (\mathcal{T}_i)_*$ corresponds to the subset

$A_i \subseteq \mathcal{T}_i$. Then define $f((a_i)_{i\in I}) = (\bigcup_{i\in I} A_i)_*$. In other words, take the topological sum of all A_i and send $f((a_i)_{i\in I})$ to corresponding element of Interior Algebra of all subsets of $\bigcup_{i\in I} \mathcal{T}_i$. It is a subject of a routine check that f is an isomorphism of interior algebras. To clarify why f commutes with the interior operator, note that on the product of algebras interior operator is defined componentwise; on the topological sum of topological spaces interior operator also works componentwise. $\dashv$

2.3 Topological Goldblatt-Thomason theorem

We have exhibited the topological equivalents for basic frame operations of disjoint union, generated subframe and bounded morphic image. The fourth crucial operation on frames is formation of the ultrafilter extension. We still lack the topological equivalent for that. Since A-spaces are in one-to-one correspondence with $S4$-frames, such an equivalent should yield the ultrafilter extension of corresponding frame when applied to an A-space. But then we already have one implicit candidate that will do this job! Just take the interior algebra of all subsets of topological space and consider the Alexandroff extension of it - when starting from an A-space, this procedure will result in the ultrafilter extension for this frame. This entire section is the justification of this choice, with application to topological version of Goldblatt-Thomason theorem.

Definition 2.3.1 *For a given topological space $\mathcal{T} = (X, \tau)$ define its* Alexandroff extension *to be the A-space $\mathcal{T}^* \equiv (\wp(X)^*, \tau^*)$ of the interior algebra of all subsets of X.*

In other words, just take the upward-closed set topology on the ultrafilter frame for the power set algebra of $\mathcal{T}$ and define the standard valuation on it as follows:

Definition 2.3.2 *If $\mathcal{M} = (\mathcal{T}, \nu)$ is a topological model, define the standard valuation ν^* on the $\mathcal{T}^*$ as follows:*

$$u \in \nu^*(p) \text{ iff } \nu(p) \in u.$$

This definition will help us to show that Alexandroff extension anti-preserves the modal validity. We could use the relational, rather than topological, structure on $\mathcal{T}^*$ in the proof, reproducing the analogous result about ultrafilter extension of a frame, but the presented proof is shorter and fits better in the framework of this chapter.

Theorem 2.3.3 *Let $\mathcal{T} = (X, \tau)$ be a topological space, $\mathcal{M} = (\mathcal{T}, \nu)$ a topological model based on it, $\mathcal{T}^*$ the Alexandroff extension of $\mathcal{T}$ and ν^* the standard valuation, then the following holds for any modal formula ϕ:*

$$u \in \nu^*(\phi) \text{ iff } \nu(\phi) \in u.$$

Proof: We proceed by induction on the length of ϕ. The propositional case is taken care of by the definition of ν^*, the cases of the boolean connectives are rather obvious, so we only address the modality case. Consider a formula of $\Box\phi$ form:
$\boxed{\Rightarrow}$ By definition $u \in \nu^*(\Box\phi)$ means that u has an open neighbourhood (restrict to the element of the base without loss of generality) O_u^* such that $v \models \phi$ holds for all $v \in O_u^*$. In other words, $O_u \subseteq v$ implies $v \models \phi$. By the induction hypothesis this can be rephrased as $O_u \subseteq v$ implies $\nu(\phi) \in v$ for all v. This indicates that $\nu(\phi)$ is an element of O_u, for if $\nu(\phi) \notin O_u$ were the case, by ultrafilter theorem we could find an ultrafilter extending O_u and still leaving $\nu(\phi)$ aside. So $\nu(\phi) \in O_u$. Hence, O_u being an open filter yields $I(\nu(\phi)) \in O_u$ and since u extends O_u, we arrive at $I(\nu(\phi)) \in u$.
$\boxed{\Leftarrow}$ Suppose $\nu(\Box\phi) \in u$, this means $I(\nu(\phi)) \in u$. Consider any ultrafilter v from the smallest open neighbourhood of u. Any such ultrafilter contains all the open elements from u. Thus $I(\nu(\phi)) \in v$ because $I(\nu(\phi))$ is an open element of u. Then $I(\nu(\phi)) \subseteq \nu(\phi)$ implies $\nu(\phi) \in v$. By the induction hypothesis $v \in \nu^*(\phi)$. As v was arbitrarily chosen from the smallest open neighbourhood of u, we get $u \in \nu^*(\Box\phi)$. $\dashv$

We are finally in the position of presenting the main result of this chapter: a topological version of Goldblatt-Thomason theorem about definable classes.

Theorem 2.3.4 (Topological Goldblatt-Thomason) *The class K of topological spaces which is closed under formation of Alexandroff extensions is modally definable iff it is closed under taking opens subspaces, interior images, topological sums and it reflects Alexandroff extensions.*

Proof: We only deal with the right to left direction, as the other direction appears trivial given the earlier theorems 2.1.2, 2.1.5, 2.1.8 and 2.3.3.

Assume that K is any class of spaces satisfying the conditions of the theorem and let $\mathcal{T} = (X, \tau)$ be a space validating the modal theory of K, then the Interior Algebra of all subsets of X, denoted $\mathcal{T}_*$, is validating the equational theory of algebras corresponding to the spaces from K. Thus by Birkhoff's theorem, $\mathcal{T}_*$ is isomorphic to **HSP** of such algebras. So there are algebras $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{P}$ and mappings h and s such that:

(1) $h : \mathcal{H} \to \mathcal{T}_*$ is onto IA-homomorphism.

(2) $s : \mathcal{H} \to S$ is injective IA-homomorphism.

(3) $\mathcal{S} = \prod_{i \in I} (\mathcal{P}_i)_*$ where $\mathcal{P}_i \in K$ and $(\mathcal{P}_i)_*$ is the IA of all subsets of $\mathcal{P}_i$ for each $i \in I$.

From (3), definition of Alexandroff extension and the proposition 2.2.8 we obtain that $\mathcal{S} = (\bigcup_{i \in I} \mathcal{P}_i)_*$ and denoting $\mathcal{P} = \bigcup_{i \in I} \mathcal{P}_i$ we get $\mathcal{P} \in K$ by the closure under the formation of topological sum. So $\mathcal{S} = \mathcal{P}_*$.
Now we exploit (2) lifting up s to Alexandroff extensions, and as far as s is injective IA-homomorphism, we find that $s^* : (\mathcal{P}_*)^* \to H^*$ is an onto interior map. Note that $(\mathcal{P}_*)^*$ is nothing else but the Alexandroff extension of $\mathcal{P}$ and thus belongs to K by the conditions we imposed on this class; but then so is H^* being the interior image of the space from K. So $H^* \in K$.
Finally we treat (1), again lifting up h to Alexandroff extensions and obtaining $h^* : (\mathcal{T}_*)^* \to H^*$ where h^* in now an injective interior map, so $h^*((\mathcal{T}_*)^*)$ is an open subset of H^* and, by closure conditions, $h^*((\mathcal{T}_*)^*) \in K$. Note that $h^* : (\mathcal{T}_*)^* \to h^*((\mathcal{T}_*)^*)$ is bijective interior map (homeomorphism). This implies $(\mathcal{T}_*)^* \in K$. We just have to mention that $(\mathcal{T}_*)^*$ is simply an Alexandroff extension of $\mathcal{T}$, so, by K reflecting Alexandroff extensions, $(\mathcal{T}_*)^* \in K$ readily implies desired $T \in K$. $\dashv$

As an immediate consequence, we get that the class of Alexandroff spaces, obviousely closed under the formation of Alexandroff extensions, is not modally definable - it clearly does not *reflect* Alexandroff extensions!

Chapter 3

Richer Languages

3.1 Enriching the language

As promised in the earlier chapters, we will show in this section how extending the modal language helps us to define some of otherwise undefinable topological properties/classes. We will consider adding the global modality, the difference operator and nominals. The case with global modality has already been considered in [Shehtman 1999] where connectedness is expressed as well. Let us consider all these options in turn.

Global Modality: Add to the language with topological box operator the global modality. We could even view the global box (written A) as a special kind of topological box with the fixed *antidiscrete*[1] topology interpretation! The global modality combined with the topological one gives rise to the logic $S4 * S5$ and the connectedness becomes expressible by the modal formula $A(\Diamond\phi \to \Box\phi) \to (A\phi \vee A\neg\phi)$, as shown in [Shehtman 1999]. We would like to add that the example displayed in the theorem 2.1.9 still shows non of the separation axioms can be expressed even in this richer language - interior maps preserve validity here too.

What happens with definability when we add the global diamond to our language? In the case of Kripke semantics we know that two of the closure conditions in the Goldblatt-Thomason theorem drop out. Namely the closure under disjoint unions and generated subframes, for those no longer preserve the validity of modal formulas in the enriched language. This of course implies that the topological sum and the open subspace do not preserve the modal validity in the language with additional global modalities. However, interior images and Alexandroff extensions still do. We sketch the proofs below.

Theorem 3.1.1 *Let $\mathcal{T}_1 = (X_1, \tau_1)$ be a topological space and f an interior*

[1]Antidiscrete topology (also called trivial topology in the literature) consists solely of empty set and the universe.

map from $\mathcal{T}_1$ onto $\mathcal{T}_2 = (X_2, \tau_2)$. For any modal formula ϕ in the language with additional global modality, $\mathcal{T}_1 \models \phi$ implies $\mathcal{T}_2 \models \phi$.

Proof sketch: Note that the mere fact that f is a surjective mapping ensures that f is still an interior map when we equip X_1 and X_2 with the antidiscrete topologies. Use the theorem 2.1.8 thinking of A as a topological box with the fixed antidiscrete interpretation. $\dashv$

Theorem 3.1.2 *Let $\mathcal{T} = (X, \tau)$ be a topological space, $\mathcal{M} = (\mathcal{T}, \nu)$ a topological model based on it, $\mathcal{T}^*$ the Alexandroff extension of $\mathcal{T}$ and ν^* the standard valuation, then the following holds for any modal formula ϕ in the modal language with additional global modality:*

$$u \in \nu^*(\phi) \ \text{ iff } \ \nu(\phi) \in u.$$

Proof sketch: Proving by induction on the length of formulas, witness the following chain of equivalent statements: $u \in \nu^*(A\phi) \Leftrightarrow \forall v(v \in \nu^*(\phi)) \Leftrightarrow \forall v(\nu(\phi) \in v) \Leftrightarrow \nu(\phi) = X \Leftrightarrow \nu(A\phi) = X \in u$. $\dashv$.

These prove the easy direction of the definability theorem to proved in the next part of this reader.
The difference operator: Enrich the basic modal language with the new diamond D saying something is true somewhere else. The global diamond is definable in this language by the formula $E\phi \leftrightarrow \phi \vee D\phi$. It follows readily that connectedness is expressed as well. We can do even better - the separation axioms T_0 and T_1 become modally definable. Let us first give the topological definition of these axioms.

Definition 3.1.3 *We say that a topological space $\mathcal{T}$ obeys the T_0 separation axiom if for any two distinct points x and y, one of them has an open neighbourhood not containing the other.*
We say that a topological space $\mathcal{T}$ obeys the T_1 separation axiom if for any two distinct points x and y, each of them has an open neighbourhood not containing the other.

The following is how the separation axioms T_0 and T_1 can be defined in the modal language enriched with the difference operator.

Theorem 3.1.4 *Consider the basic modal language enriched with the difference operator D and with diamond interpreted as closure operator. The following two formulas define the T_0 and T_1 separation axioms respectively:*

$$t_0 = Up \wedge DUq \to \Box\neg q \vee D(q \wedge \Box\neg p),$$

$$t_1 = Up \to A(p \leftrightarrow \Diamond p).$$

Where $U\phi$ is defined as $\phi \wedge \neg D\phi$ and $A\phi$ is defined $\phi \wedge \neg D\neg\phi$.

Proof: First recall that $U\phi$ is true at a world x in a model iff ϕ is true at x and only there.
$\boxed{T_0 \Rightarrow t_0}$ Take a T_0 topological space $\mathcal{T} = (X, \tau)$ and any valuation ν on it. If some point x_1 makes the antecedent of t_0 true, this proves $\nu(\phi) = \{x_1\}$ and $\nu(\psi) = \{x_2\}$ for some $x_2 \neq x_1$. At least one of the points x_1 or x_2 has an open neighbourhood not containing the other one, which is exactly what the consequent of t_0 needs to be true.
$\boxed{t_0 \Rightarrow T_0}$ If $\mathcal{T} \models t_0$ then it does so for any valuation and any substitution instance of t_0. For any two points $x_1, x_2 \in X$ consider the valuation $\nu(p_1) = \{x_1\}$, $\nu(p_2) = \{x_2\}$ and employ $x_1 \models Up_1 \wedge DUp_2 \to \Box\neg p_2 \vee D(p_2 \wedge \Box\neg p_1)$. Antecedent is true (we designed the valuation for it) and the consequent must be true as well, which means either $x_1 \models \Box\neg p_2$ or $x_1 \models D(p_2 \wedge \Box\neg p_1)$. If $x_1 \models \Box\neg p_2$ is true, then x_1 obtains the open neighbourhood with $\neg p_2$ true throughout. Thus none of the points in this neighbourhood can be x_2. If $x_1 \models D(p_2 \wedge \Box\neg p_1)$ is the case, for some $x_3 \in X$ we have $x_3 \models p_2 \wedge \Box\neg p_1$. Since p_2 is only true at x_2, from $x_3 \models p_2$ we get $x_3 = x_2$. Thus $x_2 \models \Box\neg p_1$, which means x_2 has an open neighbourhood making $\neg p_1$ true. Hence x_2 has an open neighbourhood disjoint from x_1.
$\boxed{T_1 \Rightarrow t_1}$ Recall that T_1 spaces are precisely those where singletons are closed, or equivalently, where the closure of every point is this point itself. If $\mathcal{T}$ is such a space and ν any valuation on it, consider any point x making the antecedent of t_1 true. This means $\nu(\phi) = \{x\}$ and by T_1 also $\nu(\Diamond\phi) = \{x\}$ implying that the consequent is true as well.
$\boxed{t_1 \Rightarrow T_1}$ If $\mathcal{T} \models t_1$ and $x \in X$ is any point, take the valuation $\nu(p) = \{x\}$ and exploit $x \models Up \to A(p \leftrightarrow \Diamond p)$ bearing in mind that $x \models Up$ by the definition of ν. We get $\nu(\Diamond p) = \{x\} = \mathbf{C}\{x\}$ which is to say $\mathcal{T}$ is T_1-space given x was arbitrary. $\dashv$
See [de Rijke 1992] for a thorough survey of the modal languages with difference operator. Some aspects are discussed in [Blackburn et al. 2001] also, where a hybrid languages are considered as well.

Hybrid language: Consider the basic modal language equipped with countably many additional propositional letters denoted $i, j, ..$ and meant to be sent to exactly one point with the valuation, topologically interpreted $\Box$ and the satisfaction operators $@_i$ for each nominal with the interpretation

$$x \models @_i\phi \text{ iff } \nu(i) \models \phi.$$

Again, we can express T_0 and T_1 separation axioms as stated below:

Theorem 3.1.5 *Consider the following formulas in the hybrid modal language with @ operator:*

$$t_0 = @_i\neg j \to @_j\Box\neg i \vee @_i\Box\neg j,$$

$$t_1 = @_i\neg j \to @_j\Box\neg i \wedge @_i\Box\neg j.$$

(i) The topological space $\mathcal{T} = (X, \tau)$ is a T_0-space iff $\mathcal{T} \models t_0$.

(ii) The topological space $\mathcal{T} = (X, \tau)$ is a T_1-space iff $\mathcal{T} \models t_1$.

Proof: (i) Recall that the T_0 separation axiom requires one of the any two distinct points to have an open neighbourhood not containing the other.
$\boxed{t_0 \Rightarrow T_0}$ If $\mathcal{T} \models t_0$, then for any two distinct points $x, y \in X$ with valuation ν such that $\nu(i) = \{x\}$, $\nu(j) = \{y\}$, $\mathcal{T}, \nu, x \models @_i\neg j$. Then $\mathcal{T}, \nu, x \models @_j\Box\neg i \vee @_i\Box\neg j$, which means that either $\mathcal{T}, \nu, x \models \Box\neg j$ or $\mathcal{T}, \nu, y \models \Box\neg i$. This is precisely to say that either x has an open neighbourhood not containing y or vice versa.
$\boxed{T_0 \Rightarrow t_0}$ If $\mathcal{T}$ is not a T_0-space, then it has two distinct points x, y such that they are inseparable by an open set. Take any valuation which sends i to x and j to y, then $\mathcal{T}, \nu, x \models @_i\neg j$ and at the same time, as every open neighbourhood of x (y) contains y (x), we have $X, x \models \Diamond j$ and $X, y \models \Diamond i$. This means $X, x \not\models \Box\neg j$ and $X, y \not\models \Box\neg i$. In other words, $X, x \not\models @_j\Box\neg i \vee @_i\Box\neg j$ and $X \not\models t_0$.
(ii) Follows the same thread using "and" instead of "or". $\dashv$

Several questions naturally present themselves:

1. Can we extend the Goldblatt-Thomason Theorem to these enriched languages with the global modality, the Difference operator and the nominals?

2. Are higher Separation Axioms definable in the Hybrid Language?

We answer the first question – "Yes" and the second question – "No" in the next part of this reader.

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Part III

Expressivity and Definability for Extended Modal languages

Chapter 1

Introduction

Modal logic, as a language for talking about topological spaces, has been studied for at least 60 years. Originally, the motivations for this study were purely mathematical. More recently, computer science applications have lead to a revival of interest, giving rise to new logics of space, many of which are (extensions of) modal languages (e.g., [22, 2, 21, 4], to name a few).

The design of such logics is usually guided by considerations involving expressive power and computational complexity. Within the landscape of possible spatial languages, the basic modal language interpreted on topological spaces can be considered a minimal extreme. It has a low computational complexity, but also a limited expressive power.

In this part, we characterize the expressive power of the basic modal language, as a language for talking about topological spaces, by comparing it to the well established topological language $\mathcal{L}_t$ [15]. Among other things, we obtain the following results:

Theorem 3.4. Let $\phi(x)$ be any $\mathcal{L}_t$ formula with one free variable. Then $\phi(x)$ is equivalent to (the standard translation of) a modal formula iff $\phi(x)$ is invariant under topo-bisimulations.

Theorem 3.8. Let K be a class of topological spaces definable in $\mathcal{L}_t$. Then K is definable in the basic modal language iff K is closed under topological sums, open subspaces and images of interior maps, while the complement of K is closed under Alexandroff extensions.

These can be seen as topological generalizations of the Van Benthem theorem and the Goldblatt-Thomason theorem, respectively. We give similar characterizations for some extensions of the modal language, containing nominals, the global modality, the difference modality, and the $\downarrow$-binder (for a summary of our main results, see Section 5).

Characterizations such as these help explain why certain languages (in this case the basic modal language) are natural to consider. They can also guide us in finding languages that provide the appropriate level of expressivity for an application.

Chapter 2

Preliminaries

In this section we recall some basic notions from topology, topological model theory, and the topological semantics for modal logic.

2.1 Topological spaces

Definition 1 (Topological spaces). A *topological space* (X, τ) is a non-empty set X together with a collection $\tau \subseteq \wp(X)$ of subsets that contains $\emptyset$ and X and is closed under finite intersections and arbitrary unions. The members of τ are called *open sets* or simply *opens*. We often use the same letter to denote both the set and the topological space based on this set: $X = (X, \tau)$.

If $A \subseteq X$ is a subset of the space X, by $\mathbb{I}A$ (read: 'interior A') one denotes the greatest open contained in A (i.e. the union of all the opens contained in A). Thus $\mathbb{I}$ is an operator over the subsets of the space X. It is called the *interior operator*.

Complements of open sets are called *closed*. The *closure* operator, which is a dual of the interior operator, is defined as $\mathbb{C}A = -\mathbb{I}-A$ where '$-$' stands for the set-theoretic complementation. Observe that $\mathbb{C}A$ is the least closed set containing A.

A standard example of a topological space is the real line $\mathbb{R}$, where a set is considered to be open if it is a union of open intervals (a, b).

For technical reasons, at times it will be useful to consider topological bases—collections of sets that generate a topology.

Definition 2 (Topological bases). A *topological base* σ is a collection $\sigma \subseteq \wp(X)$ of subsets of a set X such that closing σ under arbitrary unions gives a topology on X (i.e., such that $(X, \{\bigcup \sigma' \mid \sigma' \subseteq \sigma\})$ is a topological space). The latter requirement is in fact equivalent to the conjunction of the following conditions:

1. $\emptyset \in \sigma$
2. $\bigcup \sigma = X$
3. For all $A, B \in \sigma$ and $x \in A \cap B$, there is a $C \in \sigma$ such that $x \in C$ and $C \subseteq A \cap B$.

For (X, σ) a topological base, we denote by $\widehat{X} = (X, \widehat{\sigma})$ the topological space it generates, i.e., the topological space obtained by closing σ under arbitrary unions. Furthermore, we say that σ *is a base for* $\widehat{\sigma}$.

For example, a base for the standard topology on the reals is the set of open intervals $\{(a, b) \mid a \leq b\}$.

2.2 The basic modal language

We recall syntax and the topological semantics for the basic modal language.

Definition 3 (The basic modal language)**.** The *basic modal language* $\mathcal{ML}$ consists of a set of propositional letters $\textsc{Prop} = \{p_1, p_2, \dots\}$, the boolean connectives $\wedge, \neg$, the constant truth $\top$ and a modal box $\Box$. Modal formulas are built according to the following recursive scheme:

$$\phi \ ::= \ \top \mid p_i \mid \phi \wedge \phi \mid \neg\phi \mid \Box\phi$$

Unless specifically indicated otherwise, we will always assume that the set of propositional letters is countably infinite.

Nowadays, the best-known semantics for $\mathcal{ML}$ is the Kripke semantics. In this paper, however, we study the topological semantics, according to which modal formulas denote regions in a topological space. The regions denoted by the propositional letters are specified in advance by means of a *valuation*, and $\wedge$, $\neg$ and $\Box$ are interpreted as intersection, complementation and the interior operator. Formally:

Definition 4 (Topological models)**.** A *topological model* $\mathfrak{M}$ is a tuple (X, ν) where $X = (X, \tau)$ is a topological space and the valuation $\nu : \textsc{Prop} \to \wp(X)$ sends propositional letters to subsets of X.

Each modal formula ϕ defines a set of points in a topological model (namely the set of points at which it is true). With a slight overloading of notation, we will sometimes denote this set by $\nu(\phi)$. It is not hard to see that $\nu(\Box\phi) = \mathbb{I}\nu(\phi)$.

Given a class K of spaces, the set of modal formulas $\{\phi \in \mathcal{ML} \mid \mathsf{K} \models \phi\}$ ("*the modal logic of* K") is denoted by $\mathrm{Log}(\mathsf{K})$. Conversely, given a set of modal formulas Γ, the class of spaces $\{X \mid X \models \Gamma\}$ is denoted by $\mathrm{Sp}(\Gamma)$. Thus, in this notation, a class K is modally definable iff $\mathrm{Sp}(\mathrm{Log}(\mathsf{K})) = \mathsf{K}$.

The following example illustrates the concept of modal definability.

Definition 5 (Hereditary Irresolvability)**.** A subset $A \subseteq X$ of a space X is said to be *dense* in X if $\mathbb{C}A = X$ (or, equivalently, if A intersects each non-empty open in X). A topological space X is called *irresolvable* if it cannot be decomposed into two disjoint dense subsets. It is *hereditarily irresolvable* (HI) if all its subspaces[1] are irresolvable.

Theorem 2.1. *The modal formula $\Box(\Box(p \to \Box p) \to p) \to \Box p$ (Grz) defines the class of hereditarily irresolvable spaces.*

Proof. Follows from results in [13] and [6]. For purposes of illustration, we will give a direct proof, inspired by [6].

We are to show that X is HI iff $X \models (Grz)$.

[1]Recall that a *subspace* of a space X is a non-empty subset $A \subseteq X$ endowed with the relative topology $\tau_A = \{O \cap A \mid O \in \tau\}$.

First note that $X \models$ (Grz) iff $X \models \Diamond\neg p \to \Diamond(\neg p \wedge \Box(p \to \Box p))$ iff $X \models \Diamond q \to \Diamond(q \wedge \neg\Diamond(\neg q \wedge \Diamond q))$ iff $\forall A \subseteq X.[\mathbb{C}A \subseteq \mathbb{C}(A - \mathbb{C}(\mathbb{C}A - A))]$.

Suppose X is not HI. Then there exists a non-empty subset $A \subseteq X$ and two disjoint sets $B, B' \subset A$ such that $A \subseteq \mathbb{C}B \cap \mathbb{C}B'$. We show that $\mathbb{C}B \not\subseteq \mathbb{C}(B - \mathbb{C}(\mathbb{C}B - B))$ so X does not make (Grz) valid. Indeed, since $A \subseteq \mathbb{C}B$ it is clear that $B' \subseteq \mathbb{C}B - B$, hence $B \subseteq A \subseteq \mathbb{C}B' \subseteq \mathbb{C}(\mathbb{C}B - B)$ and $\mathbb{C}B \not\subseteq \mathbb{C}(B - \mathbb{C}(\mathbb{C}B - B)) = \emptyset$.

Suppose $X \not\models (Grz)$. Then there exists a non-empty subset $A \subseteq X$ such that $\mathbb{C}A \not\subseteq \mathbb{C}(A - \mathbb{C}(\mathbb{C}A - A))$. Denote $Y = \mathbb{C}A$. We will show that Y is not HI thus proving that X is not HI (it is easily seen that a closed subspace of an HI space must itself be HI). Since Y is a closed subspace of X the operator $\mathbb{C}_Y$ coincides with $\mathbb{C}$ on subsets of Y. Thus $Y \not\subseteq \mathbb{C}_Y(A - \mathbb{C}_Y(Y - A)) = \mathbb{C}_Y\mathbb{I}_Y A$. It follows that A is dense in Y while $\mathbb{I}_Y A$ is not dense in Y. Then there exists a subset $U \subseteq Y$ that is open in the relative topology of Y such that $\emptyset = \mathbb{I}_Y A \cap U = \mathbb{I}_Y(U \cap A) = \mathbb{I}_Y((U \cap (Y - U)) \cup (U \cap A)) = \mathbb{I}_Y(U \cap (A \cup (Y - U))) = U \cap \mathbb{I}_Y(A \cup (Y - U)) = U - \mathbb{C}_Y(U - A)$. This implies that $U \subseteq \mathbb{C}_Y(U - A)$. But at the same time $U \subseteq \mathbb{C}_Y(U \cap A)$ since U is open in Y and A is dense in Y. As $U = (U - A) \cup (U \cap A)$ it follows that U is decomposed into two disjoint dense in U subsets $U - A$ and $U \cap A$, so U is resolvable. Thus Y is not HI and hence X is not HI either. $\dashv$

One of the central questions in this paper is which properties of topological spaces are definable in the basic modal language and its various extensions.

2.3 The topological correspondence language $\mathcal{L}_t$

In the relational semantics, the van Benthem theorem and the Goldblatt-Thomason theorem characterize the expressive power of the basic modal language by comparing it to the 'golden standard' of first-order logic. In the topological setting, it is less clear what the golden standard should be. Let us imagine for a moment a perfect candidate for a 'first-order correspondence language for topological semantics of modal logic'. Such a language should have the usual kit of nice properties of first-order languages like Compactness and the Löwenheim-Skolem theorem; it should be able to express topological properties in a natural way; moreover, it should be close enough to the usual mathematical language used for speaking about topologies so that we could determine easily whether a given topological property is expressible in it or not; and it should be suitable for translating modal formulas into it nicely.

The language $\mathcal{L}_t$ which we describe in this section satisfies all these requirements. Moreover, its model theory has been quite well studied and the corresponding machinery will serve us well in the following sections. With the exception of Theorems 2.2 and 2.7, all results on $\mathcal{L}_t$ discussed in this section, and much more, can be found in the classical monograph on topological model theory by Flum and Ziegler [15].

Before defining $\mathcal{L}_t$, we will first introduce the two-sorted first order language $\mathcal{L}^2$. In its usual definition, this language can contain predicate symbols of arbitrary arity. Here, however, since the models we intend to describe are the *topological models* introduced in the previous section, we will restrict attention to a specific signature, containing a unary predicate for each propositional letter $p \in \textsc{Prop}$.

Definition 6 (The quantified topological language $\mathcal{L}^2$)**.** $\mathcal{L}^2$ is a two-sorted first-order language: it has terms that are intended to range over elements, and terms that are intended to range over open sets. Formally, the alphabet is constituted by a countably infinite set of "point variables" $x, y, z, \dots$ a countably infinite set of "open variables" $U, V, W, \dots$, unary predicate symbols P_p corresponding to propositional letters $p \in \textsc{Prop}$ and a binary predicate symbol ε that relates point variables with open variables. The formulas of $\mathcal{L}^2$ are given by the following recursive definition:

$$\phi \ ::= \ \top \ \mid \ x = y \ \mid \ U = V \ \mid \ P_p(x) \ \mid \ x \varepsilon U \ \mid \ \neg\phi \ \mid \ \phi \wedge \phi \ \mid \ \exists x.\phi \ \mid \ \exists U.\phi$$

where x, y are point variables and U, V are open variables. The usual shorthand notations (e.g., $\forall$ for $\neg\exists\neg$) apply.

Due to the chosen signature, formulas of $\mathcal{L}^2$ can be naturally interpreted in topological models (relative to assignments that send point variables to elements of the domain and open variables to open sets). However, as we show in Appendix A, under this semantics, $\mathcal{L}^2$ is rather ill-behaved: it lacks the usual model theoretic features such as Compactness, the Löwenheim-Skolem theorem and the Łoś theorem. For this reason, we will first consider a more general semantics in terms of *basoid models.*

Definition 7 (Basoid models)**.** A *basoid model* is a tuple (X, σ, ν) where X is a non-empty set, $\sigma \subseteq \wp(X)$ is a topological base, and the valuation $\nu : \textsc{Prop} \to \wp(X)$ sends propositional letters to subsets of X.

Interpret $\mathcal{L}^2$ on a basoid model as follows: point variables range over X, open variables range over σ, the valuation ν determines the meaning of the unary predicates P_p, while ε is interpreted as the set-theoretic membership relation.

Under this interpretation, $\mathcal{L}^2$ displays all the usual features of a first-order language, including Compactness, the Löwenheim-Skolem property and the Łoś theorem [15].[2]As we mentioned already, these properties are lost if we further restrict attention to topological models.

Theorem 2.2. *$\mathcal{L}^2$ interpreted on topological models lacks Compactness, Löwenheim-Skolem and Interpolation, and is Π^1_1-hard for validity.*

[2]Essentially, this is due to the fact that, within the class of all two-sorted first-order structures, the basoid models can be defined up to isomorphism by conjunction of the following sentences of $\mathcal{L}^2$ (cf. Definition 2, see also [16, p. 14]):

$$\begin{aligned}
\text{Ext} &\equiv \forall U, V.(U = V \leftrightarrow \forall x.(x\varepsilon U \leftrightarrow x\varepsilon V)) \\
\text{Union} &\equiv \forall x.\exists U.(x\varepsilon U) \\
\text{Empty} &\equiv \exists U.\forall x.(\neg x\varepsilon U) \\
\text{Bas} &\equiv \forall x.\forall U, V.(x\varepsilon U \wedge x\varepsilon V \to \exists W.(x\varepsilon W \wedge \forall z.(z\varepsilon W \to z\varepsilon U \wedge z\varepsilon V)))
\end{aligned}$$

The proof can be found in Appendix A.

Thus, in order to work with topological models *and* keep the nice first-order properties we need to somehow 'tame' $\mathcal{L}^2$. This is where $\mathcal{L}_t$ enters the picture, a well behaved fragment of $\mathcal{L}^2$. Let us call an $\mathcal{L}^2$ formula α *positive (negative) in an open variable U* if all free occurrences of U are under an even (odd) number of negation signs.

Definition 8 (The language $\mathcal{L}_t$)**.** $\mathcal{L}_t$ contains all atomic $\mathcal{L}^2$-formulas and is closed under conjunction, negation, quantification over the point variables and the following restricted form of quantification over open variables:

- if α is positive in the open variable U, and x is a point variable, then $\forall U.(x\varepsilon U \to \alpha)$ is a formula of $\mathcal{L}_t$,
- if α is negative in the open variable U, and x is a point variable, then $\exists U.(x\varepsilon U \wedge \alpha)$ is a formula of $\mathcal{L}_t$.

(recall that $\phi \to \psi$ is simply an abbreviation for $\neg(\phi \wedge \neg\psi)$).

The reason $\mathcal{L}_t$ is particularly well-suited for describing topological models lies in the following observation: $\mathcal{L}_t$-formulas cannot distinguish between a basoid model and the topological model it generates. More precisely, for any basoid model $\mathfrak{M} = (X, \sigma, \nu)$, let $\widehat{\mathfrak{M}} = (X, \widehat{\sigma}, \nu)$, where $\widehat{\sigma}$ is the topology generated by the topological base σ.

Theorem 2.3 ($\mathcal{L}_t$ is the base-invariant fragment of $\mathcal{L}^2$)**.** *For any $\mathcal{L}_t$-formula $\alpha(x_1, \dots, x_n, U_1, \dots, U_m)$, basoid model $\mathfrak{M} = (X, \sigma, \nu)$, and for all $d_1, \dots, d_n \in X$ and $O_1, \dots, O_m \in \sigma$,*

$$\mathfrak{M} \models \alpha\,[d_1, \dots, d_n, O_1, \dots, O_m] \quad \textit{iff} \quad \widehat{\mathfrak{M}} \models \alpha\,[d_1, \dots, d_n, O_1, \dots, O_m]\ .$$

Moreover, every $\mathcal{L}^2$-formula $\phi(x_1, \dots, x_n, U_1, \dots, U_m)$ satisfying this invariance property is equivalent on topological models to an $\mathcal{L}_t$-formula with the same free variables.

It follows that $\mathcal{L}_t$ satisfies appropriate analogues of Compactness, the Löwenheim-Skolem property, and the Łoś theorem *relative to the class of topological models*. Let us start with the Löwenheim-Skolem property. Call a topological model $\mathfrak{M} = (X, \tau, \nu)$ *countable* if X is countable and τ has a countable base.

Theorem 2.4 (Löwenheim-Skolem for $\mathcal{L}_t$)**.** *Let Γ be any set of $\mathcal{L}_t$-formulas (in a countable signature). If Γ has an infinite topological model, then it has a countable topological model.*

Next, we will discuss an analogue of Łoś's theorem for $\mathcal{L}_t$. First we need to define ultraproducts of topological models.

Definition 9 (Ultraproducts of basoid models)**.** Let $(\mathfrak{M}_i)_{i\in I}$ be an indexed family of basoid models, where $\mathfrak{M}_i = (X_i, \sigma_i, \nu_i)$, and let $\mathfrak{D}$ be an ultrafilter over the index set I. Define an equivalence relation $\sim_{\mathfrak{D}}$ on $\prod_{i\in I} X_i$ as follows:

$$x \sim_{\mathfrak{D}} y \quad \text{iff} \quad \{i \mid x_i = y_i\} \in \mathfrak{D}$$

We define the *ultraproduct* $\prod_{\mathfrak{D}} \mathfrak{M}_i$ to be (X, σ, ν), where $X = (\prod_{i\in I} X_i)/_{\sim_{\mathfrak{D}}}$, $\sigma = \{(\prod_{i\in I} O_i)/_{\sim_{\mathfrak{D}}} \mid \text{each } O_i \in \sigma_i\}$, and $\nu(p) = (\prod_{i\in I} \nu_i(p))/_{\sim_{\mathfrak{D}}}$.

If $\mathfrak{M}_i = \mathfrak{M}_j$ for all $i, j \in I$, then $\prod_{\mathfrak{D}} \mathfrak{M}_i$ is called an *ultrapower.*

It is not hard to see that, under this definition, every ultraproduct of basoid models is again a basoid model. The same does *not* hold for topological models. Hence, rather than the basoid ultrapower $\prod_{\mathfrak{D}} \mathfrak{M}_i$, we will use the topological model it generates, i.e., $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}_i}$. We will call the latter the *topological ultraproduct* (or, *topological ultrapower*, if all factor models coincide). Note that, by Theorem 2.3, the topological ultraproduct $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}_i}$ cannot be distinguished from the basoid ultraproduct $\prod_{\mathfrak{D}} \mathfrak{M}_i$ in $\mathcal{L}_t$.

Theorem 2.5 (Łoś theorem for $\mathcal{L}_t$)**.** *Let α be any $\mathcal{L}_t$-sentence, $(\mathfrak{M}_i)_{i\in I}$ an indexed set of topological models, and $\mathfrak{D}$ an ultrafilter over I. Then*

$$\widehat{\prod_{\mathfrak{D}} \mathfrak{M}_i} \models \alpha \quad \textit{iff} \quad \{i \in I \mid \mathfrak{M}_i \models \alpha\} \in \mathfrak{D}$$

In particular, if $\mathfrak{N}$ is a topological ultrapower of $\mathfrak{M}$, then for all $\mathcal{L}_t$-formulas ϕ and assignments g, $\mathfrak{M} \models \phi\ [g]$ iff $\mathfrak{N} \models \phi\ [f \cdot g]$, where $f : \mathfrak{M} \to \mathfrak{N}$ is the natural diagonal embedding.

A typical use of ultraproducts is for proving compactness.

Theorem 2.6 (Compactness for $\mathcal{L}_t$)**.** *Let Γ be any set of $\mathcal{L}_t$-formulas. If every finite subset of Γ is satisfiable in a topological model, then Γ itself is satisfiable in a topological ultraproduct of these models.*

Another common use of ultraproducts is for obtaining *saturated models.* One can generalize this construction to topological models, provided that the notion of saturation is defined carefully enough. The following definition of saturatedness is probably not the most general, but will suffice for the purposes of this paper.

Definition 10 ($\mathcal{L}_t$-saturatedness)**.** By an $\mathcal{L}_t$*-type* we will mean a set of $\mathcal{L}_t$-formulas $\Gamma(x)$ having exactly one free point variable x and no free open variables. An open set O in a topological model is called *point-saturated* if, whenever all finite subtypes of an $\mathcal{L}_t$-type $\Gamma(x)$ are realized somewhere in O, then $\Gamma(x)$ itself is realized somewhere in O. A topological model $\mathfrak{M} = (X, \tau, \nu)$ is said to be $\mathcal{L}_t$*-saturated* if the following conditions hold:

1. The entire space X is point-saturated.

2. The collection of all point-saturated open sets forms a base for the topology. Equivalently, for each point d with open neighborhood O, there is a point-saturated open subneighborhood $O' \subseteq O$ of d.

3. Every point d has an open neighborhood O_d such that, for all $\mathcal{L}_t$-formulas $\phi(x)$, if $\phi(x)$ holds throughout *some* open neighborhood of d then $\phi(x)$ holds throughout O_d.

Theorem 2.7. *Every topological model $\mathfrak{M}$ has an $\mathcal{L}_t$-saturated topological ultrapower. This holds regardless of the cardinality of the language.*

Proof. Let $\mathfrak{M}$ be any topological model. It follows from classical model theoretic results that $\mathfrak{M}$ has a basoid ultrapower $\prod_{\mathfrak{D}} \mathfrak{M} = (X, \sigma, \nu)$ that is countably saturated (in the classical sense, for the language $\mathcal{L}^2$) [11, Theorem 6.1.4 and 6.1.8]. We claim that $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}}$ is $\mathcal{L}_t$-saturated.

In what follows, with *basic open sets* we will mean open sets from the basoid model $\prod_{\mathfrak{D}} \mathfrak{M}$.

1. Suppose every finite subset of an $\mathcal{L}_t$-type $\Gamma(x)$ is satisfied by some point in $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}}$. In other words, for every finite $\Gamma'(x) \subseteq \Gamma$, $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}} \models \exists x. \bigwedge \Gamma'(x)$. Note that the latter formula belongs to $\mathcal{L}_t$. It follows by base invariance (Theorem 2.3) that every finite subset of $\Gamma(x)$ is satisfied by some point in $\prod_{\mathfrak{D}} \mathfrak{M}$. Hence, by the countable saturatedness of this basoid model, there is a point d satisfying all formulas of $\Gamma(x)$. Applying the base invariance again, we conclude that d still satisfies all formulas of $\Gamma(x)$ in $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}}$.

2. Let d be any point and O any open neighborhood of d. By definition, O is a union of basic open sets from $\prod_{\mathfrak{D}} \mathfrak{M}$. It follows that d must have a basic open subneighborhood O'. Of course, O' is still an open neighborhood of d in $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}}$. By the same argument as before, we know that O' is point-saturated—just consider the type $\Sigma(x) = \{x \varepsilon O'\} \cup \Gamma(x)$.

3. Let d be any point and let Σ be the collection of all $\mathcal{L}_t$-formulas $\phi(x)$ that hold throughout some open neighborhood of d. Recall that each open neighborhood of d contains a basic open subneighborhood of d. It follows that each $\phi(x) \in \Sigma$ holds throughout some basic open neighborhood of d.

   Next, we will proceed using the language $\mathcal{L}^2$, and the fact that $\prod_{\mathfrak{D}} \mathfrak{M}$ is countably saturated as a model for this language. Consider the following set of $\mathcal{L}^2$-formulas (where $\mathbf{d}$ is used as a parameter referring to d, and U is a free open variable):

   $$\Gamma(U) = \{\mathbf{d} \varepsilon U\} \cup \{\forall y. (y \varepsilon U \to \phi(y)) \mid \phi(y) \in \Sigma\}$$

   Every finite subset of $\Gamma(U)$ holds throughout some basic open neighborhood of d (in $\prod_{\mathfrak{D}} \mathfrak{M}$). This follows from the definition of Σ, the base invariance of $\mathcal{L}_t$, and the fact that every open neighborhood of d contains a basic open neighborhood.

It follows by the countable saturatedness of $\prod_{\mathfrak{D}} \mathfrak{M}$ with respect to $\mathcal{L}^2$ that there is a basic open set O_d satisfying all formulas in $\Gamma(U)$. In particular, O_d is an open neighborhood of d and (applying base invariance once more) all formulas in Σ hold throughout O_d in $\widehat{\prod_{\mathfrak{D}} \mathfrak{M}}$. $\dashv$

We can conclude that $\mathcal{L}_t$ is model theoretically quite well behaved. Computationally, $\mathcal{L}_t$ is unfortunately less well behaved.

Theorem 2.8. *The $\mathcal{L}_t$-theory of all topological spaces is undecidable, even in the absence of unary predicates. The same holds for T_0-spaces, for T_1-spaces, and for T_2-spaces. The $\mathcal{L}_t$-theory of topological models based on T_3-spaces, on the other hand, is decidable.*

The next natural question is which topologically interesting properties we can express in this language. Table 1 lists some examples of properties that can be expressed in $\mathcal{L}_t$ (where $x \not\varepsilon U$ is used as shorthand for $\neg(x \varepsilon U)$, $\forall U_x.\alpha$ as shorthand for $\forall U.(x \varepsilon U \rightarrow \alpha)$, and $\exists U_x.\alpha$ as shorthand for $\exists U.(x \varepsilon U \wedge \alpha)$). Recall that the *separation axioms* $T_0 - T_5$ are properties of spaces that allow separating distinct points and/or disjoint closed sets [12].

Table 1: Some examples of properties that can be expressed in $\mathcal{L}_t$

T_0	$\forall xy.(x \neq y \rightarrow \exists U_x.(y \not\varepsilon U) \vee \exists V_y.(x \not\varepsilon V))$
T_1	$\forall xy.(x \neq y \rightarrow \exists U_x.(y \not\varepsilon U))$
T_2	$\forall xy.(x \neq y \rightarrow \exists U_x.\exists V_y.\forall z.(z \not\varepsilon U \vee z \not\varepsilon V))$
Regular	$\forall x.\forall U_x.\exists V_x.\forall y.(y \varepsilon U \vee \exists V'_y.\forall z.(z \varepsilon V' \rightarrow z \not\varepsilon V))$
T_3	$T_2 \wedge$ Regular
Discrete	$\forall x.\exists U_x.\forall y.(y \varepsilon U \rightarrow y = x)$
Alexandroff	$\forall x.\exists U_x.\forall V_x.\forall y.(y \varepsilon V \rightarrow y \varepsilon U)$

A typical example of a property not expressible in $\mathcal{L}_t$ is *connectedness*.

Definition 11 (Connectedness)**.** A topological space (X, τ) is said to be *connected* if $\emptyset$ and X are the only sets that are both open and closed.

Theorem 2.9 ([15, page 8])**.** *Connectedness is not expressible in $\mathcal{L}_t$.*

Note that connectedness *is* expressible in $\mathcal{L}^2$, namely by the sentence $\forall U, U'.(\forall x.(x \varepsilon U \leftrightarrow x \not\varepsilon U') \rightarrow (\forall x.(x \varepsilon U) \vee \forall x.(x \not\varepsilon U)))$.

We have the following translation from the basic modal language to $\mathcal{L}_t$.[3]

[3]In fact, a slight variation of this translation shows that modal formulas can be mapped to $\mathcal{L}_t$-formulas containing at most two point variables and one open variable.

Definition 12. [Standard translation] The *standard translation ST* from the basic modal language $\mathcal{ML}$ into $\mathcal{L}_t$ is defined inductively:

$$\begin{array}{lcl} ST_x(\top) & = & \top \\ ST_x(p) & = & P_p(x) \\ ST_x(\neg\phi) & = & \neg ST_x(\phi) \\ ST_x(\phi \wedge \psi) & = & ST_x(\phi) \wedge ST_x(\psi) \\ ST_x(\Box\phi) & = & \exists U.(x\varepsilon U \wedge \forall y.(y\varepsilon U \to ST_y(\phi))) \end{array}$$

where x, y are distinct point variables and U is an open variable.

Theorem 2.10. *For $\mathfrak{M}$ a topological model and $\varphi \in \mathcal{ML}$ a modal formula, $\mathfrak{M}, a \models \varphi$ iff $\mathfrak{M} \models ST_x(\varphi)[a]$*

Proof. By induction on the complexity of φ. $\dashv$

In other words, modal formulas can be seen as $\mathcal{L}_t$-formulas in one free variable, and sets of modal formulas can be seen as $\mathcal{L}_t$*-types* in the sense of Definition 10. This shows that all the above results on $\mathcal{L}_t$ also apply to modal formulas. For example,

Theorem 2.11 (Löwenheim-Skolem theorem for $\mathcal{ML}$)**.** *Let $\Sigma \subseteq \mathcal{ML}$ be a set of modal formulas (in a countable signature). If Σ is satisfied in a topological model, then it is satisfied in a countable topological model.*

Chapter 3

First Steps

The expressive power of the basic modal language *on relational structures* is relatively well understood. The Van Benthem theorem characterizes the modally definable *properties of points in Kripke models*, in terms of bisimulations, while the Goldblatt-Thomason theorem characterizes modal definability of *classes of Kripke frames*, in terms of closure under operations such as disjoint union.

In this section, we will prove topological analogs of these results. First, we present a topological version of Van Benthem's theorem, using the notion of *topo-bisimulations* [1]. Next, we identify four operations on topological spaces that preserve validity of modal formulas. Finally, we apply these closure conditions in order to determine which $\mathcal{L}_t$-definable classes are modally definable, and vice versa.

3.1 Topological bisimulations

In this section we characterize the modal fragment of $\mathcal{L}_t$ in terms of topo-bisimulations.

Definition 13. Consider topological models $\mathfrak{M} = (X, \nu)$ and $\mathfrak{M}' = (X', \nu')$. A non-empty relation $Z \subseteq X \times X'$ is a *topo-bisimulation between* $\mathfrak{M}$ *and* $\mathfrak{M}'$ if the following conditions are met for all $x \in X$ and $x' \in X'$:

Zig If xZx' and $x \in O \in \tau$ then there exists $O' \in \tau'$ such that $x' \in O'$ and for all $y' \in O'$ there exists a $y \in O$ such that yZy'.

Zag If xZx' and $x' \in O' \in \tau'$ then there exists $O \in \tau$ such that $x \in O$ and for all $y \in O$ there is a $y' \in O'$ such that yZy'.

Atom If xZx' then $x \in \nu(p)$ iff $x' \in \nu'(p)$ for all $p \in \textsc{Prop}$.

Elements $x \in X$ and $x' \in X'$ are said to be *bisimilar*, denoted by $(\mathfrak{M}, x) \underline{\leftrightarrow} (\mathfrak{M}', x')$, if there exists a bisimulation Z between $\mathfrak{M}$ and $\mathfrak{M}'$ such that xZx'.

This definition can be formulated more naturally if we use some standard mathematical notation. For a binary relation $Z \subseteq X \times X'$ and a set $A \subseteq X$, let us denote by $Z[A]$ the *image* $\{x' \in X' \mid \exists x \in A.(xZx')\}$, and let us define the *preimage* $Z^{-1}[A']$ of a set $A' \subseteq X'$ analogously.

Proposition 3.1. *The* ***Zig*** *and* ***Zag*** *conditions in Definition 13 are equivalent to the following:*

Zig′ *For all* $O \in \tau$, $Z[O] \in \tau'$.

Zag′ *For all $O' \in \tau$, $Z^{-1}[O'] \in \tau$.*

Proof. We will only show the equivalence for **Zig′**, the proof for **Zag′** is analogous. In one direction, suppose that Z satisfies **Zig**, and take an open $O \in \tau$. The **Zig** condition ensures that, for each $x' \in Z[O]$, we can find an open neighborhood $O' \in \tau'$ with $x' \in O'$, such that $O' \subseteq Z[O]$. It follows that $Z[O]$, being the union of these neighborhoods, is open in τ'. For the other direction, suppose $Z[O] \in \tau'$ holds for all $O \in \tau$. Consider an arbitrary $x \in O \in \tau$ and $x' \in X'$ such that xZx'. Then $Z[O]$ qualifies for an open neighborhood O' of x' satisfying the condition **Zig** since $x' \in Z[O] \in \tau'$. $\dashv$

In what follows, we will freely use this equivalent formulation whenever it is convenient. Topo-bisimulations are closely linked with the notion of modal equivalence.

Definition 14. We say that two pointed topological models $(\mathfrak{M}, x)$ and $(\mathfrak{M}', x')$ are *modally equivalent* and write $(\mathfrak{M}, x) \leftrightsquigarrow (\mathfrak{M}', x')$ if for all formulas $\phi \in \mathcal{ML}$, $(\mathfrak{M}, x) \models \phi$ iff $(\mathfrak{M}', x') \models \phi$.

Theorem 3.2 ([1])**.** *For arbitrary topological pointed models $(\mathfrak{M}, x)$ and $(\mathfrak{M}', x')$, if $(\mathfrak{M}, x) \underline{\leftrightarrow} (\mathfrak{M}', x')$ then $(\mathfrak{M}, x) \leftrightsquigarrow (\mathfrak{M}', x')$.*

Proof. The proof proceeds via straightforward induction on the complexity of modal formulas. We only treat the case $\phi = \Box\psi$.

Suppose $(\mathfrak{M}, x) \models \Box\psi$. Then there exists an open neighborhood O of x such that $O \models \psi$. By **Zig** we obtain that $Z[O]$ is an open neighborhood of x' and, by induction hypothesis, $Z[O] \models \psi$. Therefore $(\mathfrak{M}', x') \models \Box\psi$. The other direction is proved similarly. $\dashv$

The converse does not hold in general, but it holds on a restricted class of $\mathcal{L}_t$-saturated topological models.

Theorem 3.3. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be $\mathcal{L}_t$-saturated topological models, and suppose that $(\mathfrak{M}, x) \leftrightsquigarrow (\mathfrak{M}', x')$. Then $(\mathfrak{M}, x) \underline{\leftrightarrow} (\mathfrak{M}', x')$.*

Proof. Let $\mathfrak{M} = (X, \tau, \nu)$ and $\mathfrak{M}' = (X', \tau', \nu')$, and let $Z \subseteq X \times X'$ be the modal indistinguishability relation (i.e., xZx' iff $(\mathfrak{M}, x) \leftrightsquigarrow (\mathfrak{M}', x')$). We will show that Z is a topo-bisimulation, and hence, $(\mathfrak{M}, x) \underline{\leftrightarrow} (\mathfrak{M}', x')$. That the **Atom** condition holds follows immediately from the construction of Z. In the remainder of this proof, we will show that **Zag** holds. The case for **Zig** is analogous.

Consider any a, a' such that aZa', and let $O' \in \tau'$ be an open neighborhood of a'. Since $\mathfrak{M}'$ is $\mathcal{L}_t$-saturated, we may assume that O' is point-saturated (if not, just take a point-saturated subneighborhood of a'). We need to find an open neighborhood O of a such that for each $b \in O$ there exists a $b' \in O'$ with bZb'.

By $\mathcal{L}_t$-saturatedness of $\mathfrak{M}$, we know that a has an open neighborhood O_a such that, for every modal formula ϕ, if $a \models \Box\phi$ then ϕ holds throughout O_a. Dually, this means that

(*) For any $b \in O_a$ and modal formula ϕ, if $b \models \phi$ then $a \models \Diamond\phi$.

To show that O_a meets the requirements of the **Zag** condition, consider any $b \in O_a$. We will find a $b' \in O'$ such that bZb'. Let Σ_b be the set of modal formulas true at b. Every finite subset of Σ_b is satisfied somewhere in O'. For, consider any finite $\Sigma' \subseteq \Sigma_b$. Then by (*), $\mathfrak{M}, a \models \Diamond \bigwedge \Sigma'$, and hence $\mathfrak{M}', a' \models \Diamond \bigwedge \Sigma'$. Therefore $\bigwedge \Sigma'$ must be satisfied somewhere in O'. Recall that O' is point-saturated. We conclude that there is a point $b' \in O'$ satisfying Σ_b. It follows that $(\mathfrak{M}, b) \leftrightsquigarrow (\mathfrak{M}', b')$, and hence bZb'. $\dashv$

Combining this with Theorem 2.7, we obtain

Theorem 3.4. *An $\mathcal{L}_t$-formula $\alpha(x)$ is invariant under topo-bisimulations iff it is equivalent to the standard translation of a modal formula.*

Proof. Easily adapted from the proof of the van Benthem Characterization Theorem for relational semantics (see e.g. [8, Theorem 2.68] for details). $\dashv$

3.2 Alexandroff extensions

We saw in Part II (section 2.1) the three validity-preserving operations on topological spaces — taking open subspaces, topological sums and interior images. We also introduced a new construction of Alexandroff extensions that *anti*-prserves validity. Here we recall this definition identify a connection between Alexandroff extensions and topological ultraproducts.

Definition 15 (Alexandroff extensions)**.** A filter $\mathcal{F} \subseteq \wp(X)$ over a topological space (X, τ) is called *open* if for all $A \in \mathcal{F}$, also $\mathbb{I}A \in \mathcal{F}$. The *Alexandroff extension* of a space (X, τ) is the space $X^* = (\mathtt{Uf}X, \tau^*)$, where $\mathtt{Uf}X$ is the set of ultrafilters over X, and τ^* is the topology over $\mathtt{Uf}X$ generated by the sets of the form $\{\mathfrak{u} \in \mathtt{Uf}X \mid \mathcal{F} \subseteq \mathfrak{u}\}$ for $\mathcal{F}$ an open filter over X.

Theorem 3.5. *For any space X, X^* is Alexandroff.*

Proof. For any point $\mathfrak{u} \in X^*$ consider a filter $\mathcal{F}$ generated by all open sets that belong to $\mathfrak{u}$. Then the set $\{\mathfrak{v} \in X^* \mid \mathcal{F} \subseteq \mathfrak{v}\}$ is a least open neighborhood of $\mathfrak{u}$. It follows that $\mathfrak{v}$ is in the least open neighborhood of $\mathfrak{u}$ iff for each $\mathbb{I}A \in \mathfrak{u}$ we have $\mathbb{I}A \in \mathfrak{v}$ iff $\mathbb{C}A \in \mathfrak{u}$ for each $A \in \mathfrak{v}$. $\dashv$

Note that the map $\pi : X \to X^*$ that sends $a \in X$ to the corresponding principal ultrafilter π_a need not be open, or even continuous [7, Example 5.13]. Indeed the image $\pi(X)$, as a subspace of X^*, might not be homeomorphic to X—as soon as X is T_1 the subspace $\pi(X)$ is discrete. Nevertheless, it is worth mentioning that the topology τ^* preserves the information about the original topology τ in a curious way. It is an easy exercise for the reader familiar with ultrafilter convergence (see, e.g., [12, pp. 91-93]) that $\mathfrak{u} \in X^*$ belongs to the least open neighborhood of the principal ultrafilter π_a in X^* iff $\mathfrak{u} \to a$ (i.e. $\mathfrak{u}$ converges to $a \in X$ according to τ).

Basic open sets of the Alexandroff extension X^* have a nice characterisation that follows immediately from their definition. For any topological space X and subset $A \subseteq X$, let $A^* = \{\mathfrak{u} \in X^* \mid A \in \mathfrak{u}\}$. It easily seen that:

- $\{a\}^* = \{\pi_a\}$;
- $(A \cap B)^* = A^* \cap B^*$, $(A \cup B)^* = A^* \cup B^*$;
- A^* is open iff A is open.

Now, the basic open sets of X^* are precisely the sets of the form $\bigcap_{A \in \mathcal{F}} A^*$ for $\mathcal{F}$ an open filter on X.

We saw in Part II (Theorem 2.3.3) that the formation of the Alexandroff extension *anti-preserves* modal validity.

Theorem 3.6. *Let X be a topological space and X^* its Alexandroff extension. For all modal formulas ϕ, if $X^* \models \phi$ then $X \models \phi$.*

The following key theorem (which can be seen as a topological analogue of [8, Theorem 3.17]) connects Alexandroff extensions to topological ultrapowers.

Theorem 3.7. *For every topological space $X = (X, \tau)$ there exists a topological ultrapower $\widehat{\prod_{\mathfrak{D}} X}$ and a surjective interior map $f : \widehat{\prod_{\mathfrak{D}} X} \to X^*$. In a picture:*

$$\begin{tikzcd} \widehat{\prod_{\mathfrak{D}} X} \arrow[d, dash] \arrow[dr, "f"] & \\ X & X^* \end{tikzcd}$$

Proof. Let us consider an $\mathcal{L}^2$-based language containing a unary predicate P_A for each $A \subseteq X$, interpreted naturally on X, i.e., $(P_A)^X = A$. In what follows we will treat X as a topological model for this (possibly uncountable) language. By Theorem 2.7, X has an $\mathcal{L}_t$-saturated topological ultrapower $\widehat{\prod_{\mathfrak{D}} X}$. Denote $Y \equiv \widehat{\prod_{\mathfrak{D}} X}$. The following $\mathcal{L}_t$-sentences are clearly true in X, and hence, by Theorem 2.5, also in Y:

(1) $Y \models \exists x.P_A(x)$ for each non-empty $A \subseteq X$,

(2) $Y \models \forall x.(P_A(x) \wedge P_B(x) \leftrightarrow P_{A \cap B}(x))$ for each $A, B \subseteq X$,

(3) $Y \models \forall x.(\neg P_A(x) \leftrightarrow P_{-A}(x))$ for each $A \subseteq X$,

(4) $Y \models \forall x.(P_{\mathbb{I}A}(x) \leftrightarrow \exists U.(x \varepsilon U \wedge \forall y.(y \varepsilon U \to P_A(y))))$ for each $A \subseteq X$,

(5) $Y \models \forall x.(P_{\mathbb{C}A}(x) \leftrightarrow \forall U.(x \varepsilon U \to \exists y.(y \varepsilon U \wedge P_A(y))))$ for each $A \subseteq X$.

We define the desired interior map $f : Y \to X^*$ in the following way:

$$f(a) = \{A \subseteq X \mid a \in (P_A)^Y\}$$

In the remainder of this proof, we will demonstrate that f is indeed a surjective interior map from Y to X^*. First we show that f is a well-defined onto map.

- For any $a \in Y$, $f(a)$ is an *ultrafilter* over X.

  Recall that an ultrafilter over X is any set $\mathfrak{u}$ of subsets of X satisfying (i) $A \cap B \in \mathfrak{u}$ iff both $A \in \mathfrak{u}$ and $B \in \mathfrak{u}$, and (ii) $A \in \mathfrak{u}$ iff $(X \setminus A) \notin \mathfrak{u}$. By (2) and (3) above, $f(a)$ indeed satisfies these properties.

- f is *surjective* (i.e., every ultrafilter over X is $f(a)$ for some $a \in Y$).

  Take $\mathfrak{u} \in X^*$, and let $\Gamma_{\mathfrak{u}}(x) = \{P_A(x) \mid A \in \mathfrak{u}\}$. It follows from (1) and (2) that every finite subset of $\Gamma_{\mathfrak{u}}(x)$ is satisfied by some point in Y. Since Y is point-saturated, there exists $a \in Y$ satisfying $\Gamma_{\mathfrak{u}}(x)$, hence $f(a) = \mathfrak{u}$.

Next we show that f is open and continuous. Note that by Proposition 3.1 it suffices to prove that the graph of f is a topo-bisimulation.

Take arbitrary $a \in Y$ and let O_a be as described in Definition 10. Let $O_{\mathfrak{u}}$ be a least open neighborhood of $\mathfrak{u} = f(a)$. We proceed by verifying the conditions **Zig** and **Zag** for the pair $(a, \mathfrak{u})$.

- **Zig**. Take arbitrary O' such that $a \in O'$. By $\mathcal{L}_t$-saturatedness of Y, there exists a *point-saturated* $O \subseteq O'$ such that $a \in O$.

  Take arbitrary $\mathfrak{v} \in O_{\mathfrak{u}}$. We will find a $b \in O$ such that $\mathfrak{v} = f(b)$. Let

  $$\Gamma_{\mathfrak{v}}(x) = \{P_A(x) \mid A \in \mathfrak{v}\}$$

  Every finite subset of $\Gamma_{\mathfrak{v}}(x)$ is satisfied somewhere in O. Indeed, if $P_{A_1}, \dots, P_{A_n} \in \Gamma_{\mathfrak{v}}$, denote $B \equiv \bigcap_i A_i$. Then $B \in \mathfrak{v}$ and hence $\mathbb{C}B \in \mathfrak{u}$. Therefore $Y \models P_{\mathbb{C}B}(a)$. It follows by (5) that P_B holds somewhere in O. By the point-saturatedness of O we may conclude that some $b \in O$ satisfies all of $\Gamma_{\mathfrak{v}}(x)$, and hence $f(b) = \mathfrak{v}$.

- **Zag**. It suffices to show that for any $b \in O_a$ we have $f(b) \in O_{\mathfrak{u}}$. Suppose the contrary. Then we have $b \in O_a$ and $f(b) \notin O_{\mathfrak{u}}$. The latter means that there exists a set $A \subseteq X$ such that $A \in f(b)$ but $\mathbb{C}A \notin \mathfrak{u}$. From $A \in f(b)$ we obtain $Y \models P_A(b)$. While $\mathbb{C}A \notin \mathfrak{u}$ iff $-\mathbb{C}A \in \mathfrak{u}$ iff $\mathbb{I} - A \in \mathfrak{u}$ iff $Y \models P_{\mathbb{I}-A}(a)$ iff $Y \models \exists U.[a\varepsilon U \wedge \forall y.(y\varepsilon U \to P_{-A}(y))]$ iff $P_{-A}(x)$ is true throughout some open neighborhood of a iff $P_{-A}(x)$ is true throughout O_a, which contradicts $Y \models P_A(b)$ since $b \in O_a$.

$\dashv$

3.3 Modal definability vs $\mathcal{L}_t$-definability

In this section we are seeking *necessary and sufficient* conditions, in the spirit of the Goldblatt-Thomason theorem, for a class of topological spaces to be modally definable. We have already found some necessary conditions: we have seen that every modally definable class of topological spaces is closed under the formation of *topological sums*, *open subspaces* and *interior images* and reflects *Alexandroff extensions.* Our aim is to prove a converse, in other words, to characterize modal definability in terms of these closure properties.

Theorem 3.8. *Let* K *be any* $\mathcal{L}_t$*-definable class of topological spaces. Then* K *is modally definable iff it is closed under taking open subspaces, interior images, topological sums and it reflects Alexandroff extensions.*

Proof. We will only prove the difficult right-to-left direction. The left-to-right direction already follows from theorems 2.1.2, 2.1.5 and 2.1.8 of Part II and Theorem 3.6.

Let K be any class satisfying the given closure conditions. Take the set $Log(\mathsf{K})$ of modal formulas valid on K. We will show that, whenever $X \models Log(\mathsf{K})$, then $X \in \mathsf{K}$. In other words, $Log(\mathsf{K})$ *defines* K.

Suppose $X \models Log(\mathsf{K})$ for some topological space X. Introduce a propositional letter p_A for each subset $A \subseteq X$, and let ν be the natural valuation on X for this (possibly uncountable) language, i.e. $\nu(p_A) = A$ for all $A \subseteq X$. Let Δ be the set of all modal formulas of the following forms (where A, B range over subsets of X):

$$\begin{array}{lcl} p_{A\cap B} & \leftrightarrow & p_A \wedge p_B \\ p_{-A} & \leftrightarrow & \neg p_A \\ p_{\mathbb{I}A} & \leftrightarrow & \Box p_A \\ p_{\mathbb{C}A} & \leftrightarrow & \Diamond p_A \end{array}$$

By definition, Δ is valid on $\mathfrak{M} = (X, \nu)$. Note that the standard translations of the formulas in Δ correspond exactly to the formulas listed in conditions (2)–(5) of Theorem 3.7 (in the corresponding $\mathcal{L}_t$-language, which has a one-place predicate $P_A(x)$ for each $A \subseteq X$). What is missing is the condition (1). The following claim addresses this.

> *Claim:* For each $a \in X$ there is a model $\mathfrak{N}_a = (Y_a, \mu_a)$ with $Y_a \in \mathsf{K}$, such that $\mathfrak{N}_a \models \Delta$ and some point in $\mathfrak{N}_a$ satisfies p_a.
>
> Proof. Take any $a \in X$, and let $\Delta_a = \{\Box\varphi \mid \varphi \in \Delta\} \cup \{p_{\{a\}}\}$. As a first step, we will show that there is a topological model $\mathfrak{K}$ based on a space in K, such that some point a' of $\mathfrak{K}$ satisfies Δ_a. By the compactness of $\mathcal{L}_t$ (Theorem 2.6), it suffices to show that every finite conjunction δ of formulas in Δ_a is satisfiable on K. Since δ is satisfied at a in $\mathfrak{M}$ and $\mathfrak{M} \models Log(\mathsf{K})$, $\neg\delta$ cannot belong to $Log(\mathsf{K})$. Hence δ is satisfiable on K.
>
> By Theorem 2.7 we may assume $\mathfrak{K}$ is $\mathcal{L}_t$-saturated. Let $O_{a'}$ be an open neighborhood of a' as described in Definition 10, and let $\mathfrak{N}_a$ be the submodel of $\mathfrak{K}$ based on $O_{a'}$. Then $\mathfrak{N}_a$ satisfies all requirements of the claim. $\dashv$

Note how, in the above argument, we used the fact that K is $\mathcal{L}_t$-definable (for the compactness argument, and for the saturation), and that it is closed under taking open subspaces. Next, we will use the fact that K is closed under taking *topological sums*.

Let $Y = \biguplus_{a \in X} Y_a$, and let $\mathfrak{N} = (Y, \mu)$, where μ is obtained from the μ_a's in the obvious way. By closure under taking topological sums, $Y \in \mathsf{K}$. Moreover, by Theorem 2.1.2 of Part II, $\mathfrak{N} \models \Delta$. Finally, each p_A, for non-empty $A \subseteq X$, holds at some point in $\mathfrak{N}$ (more precisely, at some point in $\mathfrak{N}_a$ for any $a \in A$). It follows (using the standard translation) that the conditions (1)–(5) from the proof of Theorem 3.7 hold for $\mathfrak{N}$.

We can now proceed as in the proof of Theorem 3.7, and construct an interior map from an ultrapower of $\mathfrak{N}$ onto the Alexandroff extension X^* of X. Since K is closed under topological ultrapowers (Theorem 2.5) and images of interior maps, and reflecs Alexandroff extensions, we conclude that $X \in \mathsf{K}$. $\dashv$

Inspection of the proof shows that Theorem 3.8 applies not only to $\mathcal{L}_t$-definable classes but to any class of spaces closed under ultraproducts. In fact, by Lemma 3.9 below, closure under ultra*powers* suffices. Some further improvements are still possible. Most importantly, using algebraic techniques, we will show in the next section that *closure under Alexandroff extensions* already suffices. For the complete picture, see Corollary 3.15.

The opposite question. Theorem 3.8 characterizes, among all $\mathcal{L}_t$-definable classes of topological spaces, those that are modally definable. It makes sense to ask the opposite question: *which modally definable classes of spaces are $\mathcal{L}_t$-definable?* In classical modal logic the answer was provided by van Benthem in [3] (see also [19]). We follow the route paved in these papers. First we prove a topological analogue of an observation due to Goldblatt:

Lemma 3.9. *An ultraproduct of topological spaces is homeomorphic to an open subspace of the ultrapower (over the same ultrafilter) of their topological sum.*

Proof. Suppose $(X_i)_{i \in I}$ is a family of topological spaces and $\mathfrak{D}$ is an ultrafilter over I. Denote by $X = \biguplus_{i \in I} X_i$ the topological sum of X_i and by $Y = \widehat{\prod_{\mathfrak{D}} X_i}$ their topological ultraproduct. Take arbitrary $a : I \to \biguplus_{i \in I} X_i$ such that $a(i) \in X_i$. Then a can be viewed both as an element of $\prod_{i \in I} X_i$ and as an element of $\prod_{i \in I} X$. This defines a natural embedding from Y into $\widehat{\prod_{\mathfrak{D}} X}$ which is clearly injective. That this embedding is open is easily seen (recall that this suffices to be checked on the elements of the base). To show that it is also continuous, suppose $[a]_{\mathfrak{D}} \in \prod_{\mathfrak{D}} X$ is such that $A = \{i \mid a(i) \in X_i\} \in \mathfrak{D}$ (so $[a]_{\mathfrak{D}}$ comes from Y). Then any basic ultrabox neighborhood $\prod_{\mathfrak{D}} O_i$ of $[a]_{\mathfrak{D}}$ is such that $B = \{i \mid a(i) \in O_i \subseteq X\} \in \mathfrak{D}$. We clearly have $A \cap B \in \mathfrak{D}$, so $\prod_{\mathfrak{D}}(O_i \cap X_i)$ is another open neighborhood of $[a]_{\mathfrak{D}}$, now also in Y. The required continuity follows. Since we have established that Y can be embedded into $\widehat{\prod_{\mathfrak{D}} X}$ by an interior map, it follows that Y is homeomorphic to an open subspace of $\widehat{\prod_{\mathfrak{D}} X}$. $\dashv$

We are one step away from finding a nice criterion for a modally definable class to be $\mathcal{L}_t$-definable. It follows from Garavaglia's theorem [16] (the topological analogue of the Keisler-Shelah Theorem) that a class K of spaces is $\mathcal{L}_t$-definable iff K is closed under isomorphisms and ultraproducts and the complement of K is closed under ultrapowers.

Theorem 3.10. *A modally definable class* K *of spaces is* $\mathcal{L}_t$*-definable iff it is closed under ultrapowers.*

Proof. If K is $\mathcal{L}_t$-definable, then it is clearly closed under ultrapowers. For the converse direction take a modally definable class K that is closed under ultrapowers. Then K is closed under topological sums and open subspaces. It follows from Lemma 3.9 that K is closed under ultraproducts. It is easily seen that any modally definable class is closed under $\mathcal{L}_t$-isomorphisms. It follows from Theorems 2.5 and 2.10 that the complement of K is closed under ultrapowers. Hence K is $\mathcal{L}_t$-definable. $\dashv$

Since modally definable classes are closed under interior images and ultrapowers are interior images of box products via the canonical quotient map, we obtain

Corollary 3.11. *A modally definable class of spaces that is closed under box powers is* $\mathcal{L}_t$*-definable.*

Separating examples. To close this section we give examples separating $\mathcal{L}_t$-definability from modal definability. We have exhibited earlier $\mathcal{L}_t$-sentences defining the separation axioms $T_0 - T_2$. We have also shown in Theorem 2.1.9 of Part II that $T_0 - T_2$ are not definable in the basic modal language. Thus we have examples of $\mathcal{L}_t$-definable classes of spaces that are not modally definable.

To show that there are modally definable classes of spaces that are not $\mathcal{L}_t$-definable requires more work. Recall that the class of Hereditarily Irresolvable (HI) spaces is modally definable (Theorem 2.1). This class is not $\mathcal{L}_t$-definable. We will use the following lemma:

Lemma 3.12. *Any class* K *of spaces that is both modally definable and* $\mathcal{L}_t$*-definable is closed under Alexandroff extensions.*

Proof. Suppose $X \in \mathsf{K}$. By Theorem 3.7 there exists a topological ultrapower Y of X and an onto interior map $f : Y \to X^*$. Being an $\mathcal{L}_t$-definable class, K is closed under topological ultrapowers. Hence $Y \in \mathsf{K}$. Being a modally definable class, K is closed under interior images. Therefore $X^* \in \mathsf{K}$, as required. $\dashv$

Theorem 3.13. *The class of HI spaces is not* $\mathcal{L}_t$*-definable.*

Proof. By Theorem 2.1 and the above lemma, to prove that the class of HI spaces is not $\mathcal{L}_t$-definable it suffices to show that this class is not closed under Alexandroff extensions. In [7, Example 5.12] a space X is exhibited that is HI, but its Alexandroff extension is not HI. We reproduce this example for reader's convenience.

Let $X = (\mathbb{N}, \tau)$ be a topological space with carrier $\mathbb{N} = \{1, 2, \dots\}$ and topology $\tau = \{[1, n) \mid n \in \mathbb{N}\} \cup \mathbb{N}$. This is the Alexandroff topology corresponding to the order $\geq$. To show that X is HI, observe first that for an arbitrary subset $A \subseteq \mathbb{N}$ we have $\mathbb{C}A = [\min A, \infty)$. Further, if $A, A' \subseteq B$ are such that $A \cap A' = \emptyset$ it is easily seen that either $\min A > \min B$ or $\min A' > \min B$. Hence either $B \not\subseteq \mathbb{C}A$ or $B \not\subseteq \mathbb{C}A'$. This shows that no subset of X can be decomposed into two disjoint dense in it sets, so X is HI.

Consider the Alexandroff extension X^*. Let $\mathfrak{F} \subseteq X^*$ denote the set of all the free ultrafilters over X. Fix two distinct free ultrafilters $\mathfrak{u}, \mathfrak{v} \in \mathfrak{F}$. We will show that both $\mathfrak{u}$ and $\mathfrak{v}$ belong to the least open neighborhood of any $\mathfrak{w} \in \mathfrak{F}$. To see this it suffices to check that for any non-empty $A \subseteq X$ we have $\mathbb{C}A \in \mathfrak{w}$. But if A is non-empty, then $\mathbb{C}A = [\min A, \infty)$ is cofinite and thus belongs to the free ultrafilter $\mathfrak{w}$. It follows that $\{\mathfrak{u}\}$ and $\{\mathfrak{v}\}$ are two disjoint dense in $\mathfrak{F}$ subsets. Hence X^* is not HI. $\dashv$

3.4 Interlude: an algebraic perspective

In Part II (Theorem 2.3.4) we have proved the following theorem using algebraic techniques and duality theory:

Theorem 3.14. *The class* K *of topological spaces which is closed under the formation of Alexandroff extensions is modally definable iff it is closed under taking open subspaces, interior images, topological sums and it reflects Alexandroff extensions.*

Another characterization of the modal definability for topological spaces that applies to *any* class of spaces is contained in [7, Theorem 5.10] and is also based on the duality outlined in this section. For the co-algebraic perspective on modal definability that encompasses both the relational and the topological cases, as well as more general semantical frameworks, we refer to [24].

Combining Theorem 3.14 with Theorem 3.7, we obtain our most general version of the definability theorem for the basic modal language:

Corollary 3.15. *Let* K *be a class of topological spaces satisfying at least one of the following conditions:*

(i) K *is* $\mathcal{L}_t$*-definable;*

(ii) K *is closed under box powers;*

(iii) K *is closed under ultrapowers;*

(iv) K *is closed under Alexandroff extensions.*

then K *is modally definable iff it is closed under taking open subspaces, interior images, topological sums and it reflects Alexandroff extensions.*

Proof. The easier 'only if' part follows from theorems 2.1.2, 2.1.5 and 2.1.8 of Part II and theorem 3.6.

To prove the 'if' part suppose that K is closed under taking open subspaces, interior images, topological sums and it reflects Alexandroff extensions. Let us prove that under these conditions, if K satisfies any of the conditions (i)-(iii) above, then it also satisfies the condition (iv).

First we show that each of (i) and (ii) implies (iii). Indeed, if K is $\mathcal{L}_t$-definable, then it is closed under ultrapowers; also, if K is closed under box powers, since ultrapowers are interior images of box powers under the canonical quotient map and K is closed under interior images, we obtain that K is closed under ultrapowers.

Next we show that (iii) implies (iv). Indeed, it follows from Theorem 3.7 and the closure under interior images that if K is closed under taking ultrapowers, then K is closed under Alexandroff extensions.

Thus, in any of the cases (i)-(iv), K is closed under Alexandroff extensions. Now apply Theorem 3.14. $\dashv$

The analogue of Theorem 3.7 for relational semantics has a neat algebraic proof [18]. A similar proof for the topological case is lacking and we leave this as a challenge for the interested reader.

Chapter 4

Extended Modal Languages

In order to increase the topological expressive power of the basic modal language, various extensions have been proposed. For instance, Shehtman [28] showed that *connectedness* becomes definable when the basic modal language is enriched with the *global modality.* Similarly, T_0, T_1 and *density-in-itself* become definable when we enrich the basic modal language either with nominals or with the difference modality. In this section, we show *exactly how much definable power* we gain by these additions, by giving analogues of Theorem 3.8 for these extended languages. Our findings are summarized in Table 2 and 3 on page 102 and 103.

We believe Theorem 3.4 could also be generalized to the languages studied in this section, using appropriate analogues of topo-bisimulations. However, we have decided not to pursue this here suspecting the lack of many new insights.

4.1 The global modality

In the basic modal language with $\Diamond$ and $\Box$, one can only make statements about points that are arbitrarily close to the current point of evaluation. It appears impossible to say, for instance, that *there is a point satisfying* p (i.e., to express non-emptyness of p). The *global modality*, denoted by E, gives us the ability to make such global statements. For example, Ep expresses non-emptyness of the set p, and $A(p \to q)$ expresses that p is contained in q.

Formally, $\mathcal{M}(E)$ extends the basic modal language with an extra operator E that has the following semantics:

$$\mathfrak{M}, w \models E\phi \quad \text{iff} \quad \exists v \in X.\ (\mathfrak{M}, v \models \phi)$$

The dual of E is denoted by A, i.e., $A\varphi$ is short for $\neg E \neg \varphi$. The standard translation can be extended in a straightforward way, by letting $ST_x(E\phi) = \exists x.(ST_x(\phi))$. In other words, $\mathcal{M}(E)$ is still a fragment of $\mathcal{L}_t$.

Shehtman [28] showed that connectedness can be defined using the global modality:

Proposition 4.1. *$A(\Box p \vee \Box \neg p) \to Ap \vee A\neg p$ defines connectedness.*

As connectedness is not definable in the basic modal language (Theorem 2.1.4 of Part II), this shows that $\mathcal{M}(E)$ is more expressive than the basic modal language. As a consequence of this increased expressive power, certain operations on spaces do not preserve validity anymore.

Proposition 4.2. *Taking open subspaces, or taking topological sums, in general does not preserve validity of $\mathcal{M}(E)$-formulas.*

Proof. It suffices to show that connectedness is not preserved by these two operations. The real interval $(0,1)$, with the usual topology, is connected, but its open subspace $(0,\frac{1}{2})\cup(\frac{1}{2},1)$ is not. Likewise, for any connected space X, the topological sum $X \biguplus X$ is no longer connected. $\dashv$

Taking interior images, on the other hand, *does* preserve validity of $\mathcal{M}(E)$-formulas, and taking Alexandroff extensions anti-preserves it. In fact, these two operations characterize definability in $\mathcal{M}(E)$, as the following analogue of Theorem 3.8 shows.

Theorem 4.3. *Let K be any $\mathcal{L}_t$-definable class of topological spaces. Then K is definable in the basic modal language with global modality iff it is closed under interior images and it reflects Alexandroff extensions.*

Proof. The 'only if' direction is a straightforward adaptation of Theorem 2.1.8 of Part II and Theorem 3.6. The proof of the 'if' direction is essentially a simplification of the proof of Theorem 3.8: suppose X is a topological space validating the $\mathcal{M}(E)$-theory of K, and let Δ be the following set of formulas, for all $B, C \subseteq X$:

$$\begin{array}{l}
Ep_B \text{ for non-empty } B \\
A(p_{B\cap C} \leftrightarrow p_B \wedge p_C) \\
A(p_{-B} \leftrightarrow \neg p_B) \\
A(p_{\mathbb{I}B} \leftrightarrow \Box p_B) \\
A(p_{\mathbb{C}B} \leftrightarrow \Diamond p_B)
\end{array}$$

Note that these formulas exactly correspond to conditions (1)–(5) from the proof of Theorem 3.7. As in the proof of Theorem 3.8, we can find a topological model $\mathfrak{N} = (Y, \mu)$ with $Y \in \mathsf{K}$, such that $\mathfrak{N} \models \Delta$. Finally, we proceed as in the proof of Theorem 3.7, and construct an interior map from an ultrapower of $\mathfrak{N}$ onto the Alexandroff extension X^* of X. Since K is closed under topological ultrapowers (Theorem 2.5) and images of interior maps, and reflects Alexandroff extensions, we conclude that $X \in \mathsf{K}$. $\dashv$

4.2 Nominals

Another natural extension of the basic modal language is with *nominals.* Nominals are propositional variables that denote singleton sets, i.e., they name points. In point-set topology one often finds definitions that involve both open sets and individual points. In the language $\mathcal{L}_t$, one can refer to the points in the space by means of point variables. The basic modal language lacks such means, and nominals can be seen as a way to solve this problem. Here are some examples of properties that can be defined using nominals:

T_0	$@_i\Diamond j \wedge @_j\Diamond i \rightarrow @_i j$
T_1	$\Diamond i \rightarrow i$
Density-in-itself	$\Diamond\neg i$

These properties are not definable in the basic modal language (Theorem 2.1.9 of Part II). T_2-separation, on the other hand, remains undefinable even with nominals (Theorem 4.9).

Modal languages containing nominals are often called *hybrid languages*. In this section we investigate the topological expressive power of two hybrid languages, namely $\mathcal{H}(@)$ and $\mathcal{H}(E)$. Formally, fix a countably infinite set of nominals $\textsc{Nom} = \{i_1, i_2, \ldots\}$, disjoint from the set $\textsc{Prop}$ of proposition letters. Then the formulas of $\mathcal{H}(@)$ are given by the following recursive definition:

$$\mathcal{H}(@) \qquad \phi ::= \top \mid p \mid i \mid \phi \wedge \phi \mid \neg\phi \mid \Box\phi \mid @_i\phi$$

where $p \in \textsc{Prop}$ and $i \in \textsc{Nom}$. $\mathcal{H}(E)$ further extends $\mathcal{H}(@)$ with the global modality, which was described in the previous section. Thus, the formulas of $\mathcal{H}(E)$ are given by the following recursive definition:

$$\mathcal{H}(E) \qquad \phi ::= \top \mid p \mid i \mid \phi \wedge \phi \mid \neg\phi \mid \Box\phi \mid @_i\phi \mid E\phi$$

As in the previous section, we will use $A\phi$ as an abbreviation for $\neg E \neg \phi$. We have introduced $@_i$ as a primitive operator, but it will become clear after introducing the semantics that $@_i$ can be defined in terms of the operator E.

Definition 16. A *hybrid topological model* $\mathfrak{M}$ is a topological space (X, τ) and a valuation $\nu : \textsc{Prop} \cup \textsc{Nom} \to \wp(X)$ which sends propositional letters to subsets of X and nominals to singleton sets of X.

The semantics for $\mathcal{H}(@)$ and $\mathcal{H}(E)$ is the same as for the basic modal language for the propositional letters, nominals, Boolean connectives, and the modality $\Box$. The semantics of $@$ and E is as follows:

$$\begin{array}{lll} \mathfrak{M}, w \models @_i\phi & \text{iff} & \mathfrak{M}, v \models \phi \text{ for } \nu(i) = \{v\} \\ \mathfrak{M}, w \models E\phi & \text{iff} & \exists v \in X.(\mathfrak{M}, v \models \phi) \end{array}$$

Validity and definability are defined as for the basic modal language, but considering only valuations that assign singleton sets to the nominals.

Proposition 4.4. *Taking topological sums or interior images in general does not preserve validity of $\mathcal{H}(@)$-formulas.*

Proof. The one-point space $X = \{0\}$ with the trivial topology validates $@_i\Diamond j$, but this formula is not valid on $X \biguplus X$. Thus topological sums do not preserve validity for $\mathcal{H}(@)$.

To see that $\mathcal{H}(@)$-validity is not preserved by interior maps, consider natural numbers with the topology induced by the ordering, i.e. the space $(\mathbb{N}, \tau)$ where $\tau = \{[a, \infty) \mid a \in \mathbb{N}\} \cup \{\emptyset\}$. The formula $\varphi = @_i\Box(\Diamond i \to i)$ (which defines antisymmetry in the relational case) is easily seen to be valid in it. Then consider a topological space $X = \{0, 1\}$ with the trivial topology $\tau' = \{\emptyset, X\}$ and a map f that sends even numbers to 0 and odd numbers to 1. This is an interior map, however, φ is not even satisfiable on X. $\dashv$

On the other hand, the validity of $\mathcal{H}(@)$-formulas is preserved under taking open subspaces.

Lemma 4.5. *The validity of $\mathcal{H}(@)$ formulas is preserved under taking open subspaces.*

The proof is identical to that of Theorem 2.1.5 of Part II. Recall from Part II, Chapter 2 that connectedness is not preserved under taking open subspaces. As a corollary, we obtain that connectedness is not definable in $\mathcal{H}(@)$.

Also, validity of $\mathcal{H}(E)$-formulas *is* reflected by Alexandroff extensions. We can in fact improve on this a bit, using the notion of a *topological ultrafilter morphic image*.

Definition 17. Let X and Y be topological spaces. Y is called a *topological ultrafilter morphic image* (or simply an *u-morphic image*) of X if there is a surjective interior map $f : X \to Y^*$ such that $|f^{-1}(\pi_y)| = 1$ for every principal ultrafilter $\pi_y \in Y^*$ (one can say figuratively 'f is injective on principal ultrafilters').

Clearly, every space is a u-morphic image of its Alexandroff extension.

Lemma 4.6. *The validity of $\mathcal{H}(E)$ formulas is preserved under taking u-morphic images.*

Proof. Let X and Y be topological spaces and $f : X \to Y^*$ an interior map that is injective on principal ultrafilters. Suppose further that $Y \not\models \phi$. We will show that $X \not\models \phi$.

Since $Y \not\models \phi$, there is a valuation ν on Y such that $\nu(\phi) \neq Y$. Consider the valuation on Y^* defined by $\nu^*(p) = \{\mathfrak{u} \in Y^* \mid \nu(p) \in \mathfrak{u}\}$, where p can be a propositional letter or a nominal. It is not hard to see that ν^* assigns to each nominal a singleton set consisting of a principal ultrafilter. Next, we define the valuation ν' on X by $\nu'(p) = f^{-1}(\nu^*(p))$. Since f is injective on principal ultrafilters, ν' again assigns singleton sets to the nominals. Finally, a straightforward induction argument reveals that for all $a \in X$ and $\psi \in \mathcal{H}(E)$,

$$(X, \nu'), a \models \psi \quad \Leftrightarrow \quad (Y^*, \nu^*), f(a) \models \psi \quad \Leftrightarrow \quad \nu(\psi) \in f(a)$$

As $\nu(\neg\phi) \neq \emptyset$ there exists an ultrafilter $\mathfrak{u} \in Y^*$ which contains $\nu(\neg\phi)$. Since f is onto, there exists $a \in X$ such that $f(a) = \mathfrak{u}$. It follows that $(X, \nu'), a \models \neg\phi$ and therefore $X \not\models \phi$, as required. $\dashv$

The following two results characterize definability in $\mathcal{H}(E)$ and $\mathcal{H}(@)$ in terms of closure under taking u-morphic images. The proofs are inspired by relational results presented in [10].

Theorem 4.7. *Let K be any $\mathcal{L}_t$-definable class of topological spaces. Then K is definable in $\mathcal{H}(E)$ iff K is closed under u-morphic images.*

Proof. Lemma 4.6 constitutes the proof of the left-to-right direction. We will prove the right-to-left direction. Let $Log(\mathsf{K})$ be the set of $\mathcal{H}(E)$-formulas valid on K. We will show that every space $X \models Log(\mathsf{K})$ belongs to K, and hence $Log(\mathsf{K})$ *defines* K.

Suppose $X \models Log(\mathsf{K})$. We introduce a propositional letter p_A for every subset $A \subseteq X$, as well as a nominal i_a for every $a \in X$. These propositional letters and nominals are interpreted on X by the natural valuation. Let Δ be the following set of formulas, where B and C range over all subsets of X and a ranges over all points of X:

$$\begin{array}{l} A(i_a \leftrightarrow p_{\{a\}}) \\ A(p_{-B} \leftrightarrow \neg p_B) \\ A(p_{B\cap C} \leftrightarrow p_B \wedge p_C) \\ A(p_{\mathbb{I}B} \leftrightarrow \Box p_B) \\ A(p_{\mathbb{C}B} \leftrightarrow \Diamond p_B) \end{array}$$

As in the proof of Theorem 4.3, we can find an $\mathcal{L}_t$-saturated (hybrid) topological model, based on a space $Y \in \mathsf{K}$, that makes Δ globally true. Note that conditions (1)–(5) from the proof of Theorem 3.7 hold for Y (the truth of $A(i_a \leftrightarrow p_{\{a\}})$ ensures that the predicates $P_{\{a\}}$ have non-empty interpretation). It follows, by the same argument as in the proof of Theorem 3.7, that the map $f : Y \to X^*$ defined by

$$f(a) = \{A \subseteq X \mid Y \models P_A(a)\}$$

is a surjective interior map. We will now show that f is injective on principal ultrafilters. Suppose there exist $w, v \in Y$ and $f(w) = f(v) = \pi_a$ where $a \in X$ and π_a is the principal ultrafilter containing $\{a\}$. By definition of f we get $Y \models P_{\{a\}}(w) \wedge P_{\{a\}}(v)$. By global truth of Δ we obtain $Y, w \models i_a$ and $Y, v \models i_a$, hence $w = v$.

It follows that X is an u-morphic image of Y. As K is closed under u-morphic images, we conclude that $X \in \mathsf{K}$ as required. $\dashv$

Theorem 4.8. *Let K be any $\mathcal{L}_t$-definable class of topological spaces. Then K is definable in $\mathcal{H}(@)$ iff it is closed under topological ultrafilter morphic images and under taking open subspaces.*

Proof. The 'only if' part is taken care of by Lemmata 4.5 and 4.6. The proof of the 'if' part proceeds as in Theorem 3.8, with some modifications.

The first difference is that the set of formulas Δ is augmented with formulas of the form $@_{i_a} p_A$, for all points a that belong to a non-empty set $A \subseteq X$.

A compactness argument similar to the one used in the proof of Theorem 3.8 shows that $\{@_{i_a}\Box\phi \mid \phi \in \Delta, a \in X\}$ is true in some $\mathcal{L}_t$-saturated topological model $\mathfrak{N} = (Y, \mu)$ with $Y \in \mathsf{K}$. For each $b \in Y$ named by a nominal, choose an open neighborhood O_b as described in Definition 10. Let O be the union of all these open neighborhoods. Note that by closure under open subspaces we obtain $O \in \mathsf{K}$. It is not hard to see that the submodel $\mathfrak{K}$

of $\mathfrak{N}$ based on the open subspace O globally satisfies Δ, and hence satisfies the conditions (1)–(5) described in the proof of Theorem 3.7.

Thus there exists an interior map f from O onto X^*. That f is injective on principal ultrafilters can be proved as in Theorem 4.7. Thus X is an u-morphic image of $O \in \mathsf{K}$. Since K is closed under u-morphic images, we obtain $X \in \mathsf{K}$ as required. $\dashv$

As an application, we will show that $\mathcal{H}(@)$ and $\mathcal{H}(E)$ are not expressive enough to be able to define the T_2 separation property. Recall the definition of irresolvability (Definition 5). We call a space X *α-resolvable* for a cardinal number α if X contains α-many pairwise disjoint dense subsets. In [14], an $2^{2^{\aleph_0}}$-resolvable T_2-space was constructed. We use this space to prove that

Theorem 4.9. *The class of T_2 topological spaces in not definable in $\mathcal{H}(@)$ and $\mathcal{H}(E)$.*

Proof. We employ an argument similar to, but more complicated than, the one used in Theorem 2.1.9 of Part II. Our strategy is as follows: we construct spaces X and Y such that: Y is a T_2 space, X is an u-morphic image of Y, and X is not a T_2 space. Then we apply Theorem 4.7.

Take $X = (\mathbb{N}, \tau)$ where τ is the co-finite topology. That is

$$\tau = \{\emptyset\} \cup \{A \subseteq \mathbb{N} \ \mid \ \mathbb{N} \backslash A \text{ is finite}\}$$

Then X is T_1 since every singleton is closed, but not T_2 as any two non-empty opens necessarily meet. Denote by $\mathfrak{F}$ the set of all the free ultrafilters over $\mathbb{N}$. Then the following holds:

> *Claim 1:* The topology τ^* of the Alexandroff extension X^* of X is described as follows:
>
> $$O \in \tau^* \quad \Leftrightarrow \quad \mathfrak{F} \subseteq O$$
>
> *Proof:* Suppose $O \in \tau^*$. If $O = X^*$ the claim follows. Otherwise O contains a basic open set $\mathcal{G}^*$ which consists of all the ultrafilters extending a proper open filter $\mathcal{G}$. Note that if $A \in \mathcal{G}$ is not cofinite, then $\mathbb{I}A = \emptyset \notin \mathcal{G}$. Therefore, $\mathcal{G}$ consists of cofinite sets only. Since each free ultrafilter contains *all* cofinite sets, we obtain $\mathfrak{F} \subseteq \mathcal{G}^* \subseteq O$.
>
> Now for the other direction. Suppose $\mathfrak{F} \subseteq O$. First note that $\mathfrak{F}$, being the extension of the open filter of all cofinite subsets of X, is a basic open in τ^*. Further, if $x \in X$, then the open filter $O_x = \{A \mid x \in A, \ A \text{ cofinite}\}$ is such that $O_x^* = \pi_x \cup F_x$ where π_x denotes the principal filter of x and $F_x \subseteq \mathfrak{F}$. It follows that
>
> $$O = \mathfrak{F} \cup \bigcup_{\pi_x \in O} (\pi_x \cup F_x)$$
>
> Since each $\pi_x \cup F_x = O_x^* \in \tau^*$ we obtain that $O \in \tau^*$. The claim is proved.

Next we will construct the space Y. Let $Z = (Z, \tau_1)$ be a $2^{2^{\aleph_0}}$-resolvable topological space which satisfies T_2 (according to [14] such a space exists). We will denote $2^{2^{\aleph_0}}$ many dense disjoint subsets of Z by Z_ι where $\iota \in \mathfrak{F}$. Here $\mathfrak{F}$ is again the set of all free ultrafilters over $\mathbb{N}$. Since the cardinality of $\mathfrak{F}$ is known to be $2^{2^{\aleph_0}}$ [12, Corollary 3.6.12], such indexing is possible. Let $\bar{Z} = Z - \bigcup_{\iota \in \mathfrak{F}} Z_\iota$. Thus

$$Z = \bar{Z} \cup \bigcup_{\iota \in \mathfrak{F}} Z_\iota$$

Put $Y = (\mathbb{N} \cup Z, \tau')$ where τ' is as follows:

$$\tau' = \{\emptyset\} \cup \{O \subseteq Y \mid O \cap Z \in \tau_1, O \cap Z \neq \emptyset\}$$

In words—the topology of Z as a subspace of Y is τ_1 and the neighborhoods of the points from $\mathbb{N}$ are the sets of the form $\{x\} \cup O$ where $x \in \mathbb{N}$ and $\emptyset \neq O \in \tau_1$.

Claim 2: Y is a T_2 space.

Proof: Indeed, any two points that belong to Z can be separated by two opens from τ_1, since (Z, τ_1) is a T_2 space. Any two points $x, y \in \mathbb{N}$ can be separated by open sets of the form $\{x\} \cup O_x$ and $\{y\} \cup O_y$ where $O_x, O_y \in \tau_1$ are non-empty open sets from Z such that $O_x \cap O_y = \emptyset$. Finally, two points x, y such that $x \in \mathbb{N}$ and $y \in Z$ can be separated by the sets $\{x\} \cup O_x$ and O_y where again O_x and O_y are disjoint non-empty open subsets of Z.

Now we construct the mapping $f : Y \to X^*$. Pick any $\zeta \in \mathfrak{F}$ and define $f : \mathbb{N} \cup Z \to X$ as follows:

$$f(x) = \begin{cases} \pi_x & \text{if } x \in \mathbb{N} \\ \iota & \text{if } x \in Z_\iota \\ \zeta & \text{if } x \in \bar{Z} \end{cases}$$

Claim 3: The map f is a surjective interior map.

Proof: That f is surjective follows from the construction.

Let us show that f is continuous. Take $O \in \tau^*$. By Claim 1 we have $\mathfrak{F} \subseteq O$. It follows from the definition of f that $f^{-1}O$ is of the form $Z \cup A$ where $A \subseteq \mathbb{N}$. From the definition of τ' we obtain $f^{-1}(O) \in \tau'$.

To show that f is an open map, take an arbitrary open set $O \in \tau'$. It follows from the definition of τ' that $O \cap Z \in \tau_1$ and $O \cap Z \neq \emptyset$. Then, as each Z_ι is dense in Z, it follows that $O \cap Z_\iota \neq \emptyset$ for all $\iota \in \mathfrak{F}$. Hence, $f(O)$ contains $\mathfrak{F}$ and is open in X^* according to Claim 1.

Note that f is injective on principal ultrafilters, by construction. Therefore X is an u-morphic image of Y. Since Y is T_2 and X is not, it follows that the class of T_2 spaces is not closed under u-morphic images. Recall that the class of T_2 spaces is $\mathcal{L}_t$-definable. It follows by Theorem 4.7 and Theorem 4.8 that the class of T_2 spaces is not definable in $\mathcal{H}(E)$ and $\mathcal{H}(@)$. ⊣

4.3 The difference modality

In this section, we consider $\mathcal{M}(D)$, the extension of the basic modal language with the *difference modality* D. Recall that the global modality allows us to express that a formula holds *somewhere*. The *difference modality* D allows us to express that a formula holds *somewhere else*. For example, $p \wedge \neg Dp$ expresses that p is true at the current point and nowhere else. Formally,

$$\mathfrak{M}, w \models D\varphi \quad \text{iff} \quad \exists v \neq w.(\mathfrak{M}, v \models \varphi)$$

The global modality is definable in terms of the difference modality: $E\phi$ is equivalent to $\phi \vee D\phi$. It follows that $\mathcal{M}(D)$ is at least as expressive as $\mathcal{M}(E)$. Furthermore, one can express in $\mathcal{M}(D)$ that a propositional letter p is true at a unique point (i.e., behaves as a nominal): this is expressed by the formula $E(p \wedge \neg Dp)$. Combining these two observations, it is not hard to show that every class of topological spaces definable in $\mathcal{H}(E)$ is also definable in $\mathcal{M}(D)$. The opposite also holds [17, 25]:

Theorem 4.10. *$\mathcal{M}(D)$ can define exactly the same classes of topological spaces as $\mathcal{H}(E)$.*

Corollary 4.11. *An $\mathcal{L}_t$-definable class of topological spaces is definable in $\mathcal{M}(D)$ iff it is closed under u-morphic images.*

Recall that the separation axioms T_0 and T_1, as well as *density-in-itself*, are definable in the language $\mathcal{H}(E)$. They are definable in $\mathcal{M}(D)$ as follows, where $U\phi$ is short for $\phi \wedge \neg D\phi$:

$$\begin{array}{ll} T_0: & Up \wedge DUq \to \Box\neg q \vee D(q \wedge \Box\neg p) \\ T_1: & Up \to A(p \leftrightarrow \Diamond p) \\ \text{Density-in-itself}: & p \to \Diamond Dp \end{array}$$

For more on topological semantics of $\mathcal{M}(D)$ we refer to a recent study [23].

4.4 The ↓-binder

The last extension we will consider is the one with explicit point variables, and with the ↓-binder. The point variables are similar to nominals, but their interpretation is not fixed in the model. Instead, they can be bound to the current point of evaluation using the ↓-binder. For instance, $\downarrow x.\Box x$ expresses that the current point is an isolated point.

$\mathcal{H}(@, \downarrow)$ and $\mathcal{H}(E, \downarrow)$ are the extensions of $\mathcal{H}(@)$ and $\mathcal{H}(E)$, respectively, with state variables and the ↓-binder. Formally, let $\text{VAR} = \{x_1, x_2, \ldots\}$ be a countably infinite set of point variables, disjoint from PROP and NOM. The formulas of $\mathcal{H}(@,\downarrow)$ and $\mathcal{H}(E,\downarrow)$ are given by the following recursive definitions (where $p \in$ PROP, $i \in$ NOM, and $x \in$ VAR):

$$\begin{array}{lll}\mathcal{H}(@,\downarrow) & \phi ::= p \mid i \mid x \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi \mid @_i\phi \mid \downarrow x.\phi \\ \mathcal{H}(E,\downarrow) & \phi ::= p \mid i \mid x \mid \neg\phi \mid \phi \wedge \psi \mid \Box\phi \mid @_i\phi \mid E\phi \mid \downarrow x.\phi\end{array}$$

These formulas are interpreted, as usual, in topological models. However, the interpretation is now given relative to an *assignment* g of points to point variables (just as in $\mathcal{L}_t$). The semantics of the state variables and $\downarrow$-binder is as follows:

$$\begin{array}{lll}\mathfrak{M}, w, g \models x & \text{iff} & g(x) = w \\ \mathfrak{M}, w, g \models \downarrow x.\phi & \text{iff} & \mathfrak{M}, w, g^{[x\mapsto w]} \models \phi\end{array}$$

where $g^{[x\mapsto w]}$ is the assignment that sends x to w and that agrees with g on all other variables. We will restrict attention to *sentences*, i.e., formulas in which all occurences of point variables are bound. The interpretation of these formulas is independent of the assignment.

It turns out that $\mathcal{H}(E,\downarrow)$ is essentially a notational variant for a known fragment of $\mathcal{L}_t$, called $\mathcal{L}_I$. This is the fragment of $\mathcal{L}_t$ where quantification over opens is only allowed in the form, for U not occurring in α:

$$\begin{array}{ll}\exists U.(x\varepsilon U \wedge \forall y.(y\varepsilon U \to \alpha)), & \text{abbreviated as } [I_y\alpha](x)\text{, and, dually,} \\ \forall U.(x\varepsilon U \to \exists y.(y\varepsilon U \wedge \alpha)), & \text{abbreviated as } [C_y\alpha](x).\end{array}$$

Comparing the above with the Definition 12 reveals that the formulas of the basic modal language translate inside $\mathcal{L}_I$ by the standard translation. So $\mathcal{ML}$ can be thought of as a fragment of $\mathcal{L}_I$. Apparently, adding nominals, $\downarrow$ and E to the language is just enough to get the whole of $\mathcal{L}_I$.

Theorem 4.12. *$\mathcal{H}(E,\downarrow)$ has the same expressive power as $\mathcal{L}_I$.*

Proof. The standard translation from modal logic to $\mathcal{L}_t$ can be naturally extended to $\mathcal{H}(E)$, treating nominals as first-order constants. The extra clauses are then

$$\begin{array}{lll}ST_x(t) & = & x = t \quad \text{for } t \in \text{NOM} \cup \text{VAR} \\ ST_x(@_i\varphi) & = & \exists x.(x = c_i \wedge ST_x(\varphi)) \\ ST_x(E\varphi) & = & \exists x.ST_x(\varphi) \\ ST_x(\downarrow y.\varphi) & = & \exists y.(y = x \wedge ST_x(\varphi))\end{array}$$

It is easily seen that this extended translation maps $\mathcal{H}(E,\downarrow)$-sentences to $\mathcal{L}_I$-formulas in one free variable. Conversely, the translation HT_x below maps $\mathcal{L}_I$-formulas $\alpha(x)$ to $\mathcal{H}(E,\downarrow)$-sentences:

$$
\begin{array}{lcl}
HT(s=t) & = & @_s t \\
HT(Pt) & = & @_t p \\
HT(\neg\alpha) & = & \neg HT(\alpha) \\
HT(\alpha\wedge\beta) & = & HT(\alpha)\wedge HT(\beta) \\
HT(\exists x.\alpha) & = & E\downarrow x.HT(\alpha) \\
HT([I_y\alpha](t)) & = & @_t\Box\downarrow y.HT(\alpha) \\
HT([C_y\alpha](t)) & = & @_t\Diamond\downarrow y.HT(\alpha) \\
HT_x(\alpha(x)) & = & \downarrow x.HT(\alpha)
\end{array}
$$

It is not hard to see that both translations preserve truth, in the sense of Theorem 2.10. $\dashv$

This connection allows us to transfer a number of known results. For instance, $\mathcal{L}_I$ has a nice axiomatization, it is know to have interpolation, and the $\mathcal{L}_I$-theory of the class of T_1-spaces is decidable (see [26]). Hence, these results transfer to $\mathcal{H}(E,\downarrow)$. It is also known that $\mathcal{L}_I$ is strictly less expressive than $\mathcal{L}_t$. In particular, there is no $\mathcal{L}_I$-sentence that holds precisely on those topological models that are based on a T_2-space. Hence, the same holds for $\mathcal{H}(E,\downarrow)$.[4] Note that this does not imply *undefinability* of T_2 in $\mathcal{H}(E,\downarrow)$. Nevertheless, we conjecture that T_2 is not definable in $\mathcal{H}(E,\downarrow)$.

The precise expressive power of $\mathcal{L}_I$ on topological models can be characterized in terms of *potential homeomorphisms*.

Definition 18. A *potential homeomorphism* between topological models $\mathfrak{M}=(M,\tau,\nu)$ and $\mathfrak{N}=(N,\sigma,\mu)$ is a family F of partial bijections $f: M\to N$ satisfying the following conditions for each $f\in F$:

1. f preserves truth of proposition letters and nominals (in both directions).
2. - For each $m\in M$ there is a $g\in F$ extending f, such that $m\in dom(g)$.
   - For each $n\in N$, there is a $g\in F$ extending f, such that $n\in rng(g)$.
3. - For each $(m,n)\in f$ and open neighborhood $U\ni m$, there is an open neighborhood $V\ni n$ such that for all $n'\in V$ there is a $g\in F$ extending f and an $m'\in U$ such that $(m',n')\in g$.
   - Likewise in the opposite direction.

The following characterization follows from results in [26].

Theorem 4.13. *An $\mathcal{L}_t$-formula $\phi(x_1,\ldots,x_n)$ is equivalent to an $\mathcal{L}_I$-formula in the same free variables iff it is invariant for potential homeomorphisms.*

Substituting $\mathcal{H}(E,\downarrow)$ for $\mathcal{L}_I$, this gives us a Van Benthem-style characterization of $\mathcal{H}(E,\downarrow)$ as a fragment of $\mathcal{L}_t$. We leave it as an open problem to find a similar characterization of the expressive power of $\mathcal{H}(@,\downarrow)$. We also

[4] In fact, Makowsky and Ziegler [26] showed that, in the absence of proposition letters and nominals, every two dense-in-itself T_1-spaces have the same $\mathcal{L}_t$-theory.

leave it as an open problem to characterize the classes of topological spaces definable in these languages.

Note that the union of the graphs of the partial bijections that constitute a potential homeomorphism gives rise to a total topo-bisimulation between the models in question. Thus a formula that is invariant for topo-bisimulations is also invariant for potential homeomorphisms. This is a semantical side of the fact that the basic modal language is a fragment of $\mathcal{L}_I$. In fact, we could have taken the language $\mathcal{L}_I$ as our first-order correspondence language from the very beginning. A feeling that $\mathcal{L}_I$ might be the 'right' candidate for the topological correspondence language might be strengthened by the fact that in its relational interpretation (i.e., on Kripke structures), $\mathcal{H}(E, \downarrow)$ has the full expressive power of the first-order correspondence language. We stand, however, by our choice of $\mathcal{L}_t$ since: (a) it provides stronger definability results (there are more $\mathcal{L}_t$-definable classes than $\mathcal{L}_I$-definable ones); (b) $\mathcal{L}_t$ is closer to both the usual first-order signature and the usual set-theoretic language used to formalize concepts in general topology.

Chapter 5

Discussion

We have studied the expressive power of various (extended) modal languages interpreted on topological spaces. Tables 2 and 3 summarize and illustrate our main findings, concerning definability of classes of spaces. We also obtained a Van Benthem-style characterization of the basic modal language in terms of topo-bisimulations, thereby solving an open problem from [9].

Some of the key innovative elements in our story are (*i*) identifying the appropriate topological analogues of familiar operations on Kripke frames such as taking bounded morphic images, or, ultrafilter extensions (*ii*) identifying $\mathcal{L}_t$ as being the appropriate correspondence language on topological models (indeed, our result confirm once again that, as has been claimed before, $\mathcal{L}_t$ functions as the same sort of "landmark" in the landscape of topological languages as first-order logic is in the landscape of classical logics), and (*iii*) formulating the right notion of saturation for $\mathcal{L}_t$ (which many of our technical proofs depend on).

Our results on the hybrid language $\mathcal{H}(E, \downarrow)$ are remarkable. For example, they show that, while $\mathcal{H}(E, \downarrow)$ is expressively equivalent to the first-order correspondence language on relational structures, it is strictly less expressive than $\mathcal{L}_t$ on topological models. This seems one more instance of the more sensitive power of topological modeling.

Given that Alexandroffness is definable in $\mathcal{L}_t$, many of our results can be seen as generalizing known results for modal languages on (transitive reflexive) relational structures, and it is quite well possible that results on the topological semantics will yield new consequences for the relational semantics.

Table 2: Definability in extended modal languages

	characterization of definability for $\mathcal{L}_t$-definable classes	
$\mathcal{ML}$	*closed under topological sums, open subspaces and interior images, reflecting Alexandroff extensions*	Theorem 3.8
$\mathcal{M}(E)$	*closed under interior images, reflecting Alexandroff extensions*	Theorem 4.3
$\mathcal{H}(@)$	*closed under open subspaces and u-morphic images*	Theorem 4.8
$\mathcal{H}(E)$	*closed under u-morphic images*	Theorem 4.7
$\mathcal{M}(D)$	*closed under u-morphic images*	Corollary 4.11

Table 3: Properties of topological spaces definable in various languages

	$\mathcal{L}_t$	$\mathcal{M}$	$\mathcal{M}(E)$	$\mathcal{H}(@)$	$\mathcal{H}(E)$ (or $\mathcal{M}(D)$)
T_0	$\forall xy.(x \neq y \rightarrow \exists U_x.(y \notin U) \vee \exists V_y.(x \notin V))$	no	no	$@_i \Diamond j \wedge @_j \Diamond i \rightarrow @_i j$	idem
T_1	$\forall xy.(x \neq y \rightarrow \exists U_x.(y \notin U))$	no	no	$\Diamond i \rightarrow i$	idem
T_2	$\forall xy.(x \neq y \rightarrow \exists U_x.\exists V_y.\forall z.(z \notin U \vee z \notin V))$	no	no	no	no
Density-in-itself	$\forall x \forall U_x(\exists y \neq x(y \in U))$	no	no	$\Diamond \neg i$	idem
Connectedness	no	no	$\mathsf{A}(\Box p \vee \Box \neg p) \rightarrow \mathsf{A} p \vee \mathsf{A} \neg p$	no	as in $\mathcal{M}(E)$
Hereditary irresolvability	no	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow \Box p$	idem	idem	idem

We finish by mentioning interesting directions for future research.

- **Correspondence theory for alternative semantics.** There are at least two other semantic paradigms where the approach taken in this paper might prove useful. We discuss them briefly.

  Diamond as derived set operator. For any subset S of a topological space, the derived set $\mathbf{d}S$ is the set of limit points of S, i.e., all points x of which each open neighborhood contains an element of S distinct from x itself. The closure operator can be defined in terms of the derived set operator: $\mathbb{C}S = S \cup \mathbf{d}S$. The converse does not hold, as $\mathbf{d}$ is strictly more expressive than $\mathbb{C}$ [28, 5]. Indeed, if we interpret the $\blacklozenge$ as the derived set operator, then the modal formula $\blacklozenge\top$ defines density-in-itself. With the derived set operator we can also partially mimic nominals: $p \wedge \neg\blacklozenge p$ expresses that, within small enough neighborhoods, p acts as a nominal for the current point. Conversely, with nominals we can partially mimic the $\mathbf{d}$-operator: $@_i(\blacklozenge\phi \leftrightarrow \Diamond(\phi \wedge \neg i))$ is valid. The precise connection between $\mathbf{d}$ and nominals remains to be investigated. The standard translation should be modified in the following way to account for the new semantics:

  $$ST_x(\blacklozenge\phi) \;:=\; \forall U.(x\varepsilon U \rightarrow \exists y.(y\varepsilon U \wedge x \neq y \wedge ST_y(\phi)))$$

  This is still a $\mathcal{L}_t$-formula. Whether or not the expressive power of the corresponding fragment of $\mathcal{L}_t$ can be characterized in a way we have presented here remains to be seen. The interested reader is referred to [13, 28, 5] for more details on this topological semantics.

  Neighborhood semantics. This is a generalization of the topological semantics for modal logic that allows to tackle non-normal modal logics. The corresponding structures are *neighborhood frames* (W, n) where W is a non-empty set and $n \subseteq W \times \wp(W)$ is a binary relation between points of W and subsets of W. The correspondence theory for *Monotonic* modal logic has been explored in [20] and analogues of the Goldblatt-Thomason theorem and the Van Benthem theorem have been proved for the neighborhood semantics. Quite general definability results are also put forth in [24] using the co-algebraic approach. We believe that a modification of the approach presented in this paper will further strengthen the investigation of the precise expressive power of non-normal modal logics over neighborhood frames. We outline one possible route in this direction that is similar to, but more general than, the one pursued in [20].

  Extend the language $\mathcal{L}^2$ by another intersorted binary relation symbol η. To ensure that ε behaves like the membership relation we postulate

  $$\forall U, V.(U = V \leftrightarrow \forall x.(x\varepsilon U \leftrightarrow x\varepsilon V))$$

  The models for this new language $\mathcal{L}^2_\eta$ are of the form (X, σ, ν) with X a set and $\sigma \subseteq \wp X$. The relation ε is interpreted as set-theoretic membership, while the interpretation of η defines a relation between elements of X and elements of σ so that (X, η^X) becomes a neighborhood frame. Conversely, each neighborhood frame (W, n) gives rise to a structure $(W, \{A \subseteq W \mid \exists w \in W.(wnA)\})$.

The standard translation can also be modified to suit the new semantics:

$$ST_x(\Box\phi) \quad := \quad \exists U.(x\eta U \wedge \forall y.(y\varepsilon U \rightarrow ST_y(\phi)))$$

Note that the whole story now becomes simpler than in the case of topological semantics, since there is no restriction on η. Thus the full apparatus of the model theory for first-order logic is at hand when considering the expressivity and characterization of modal logic over neighborhood frames, in terms of $\mathcal{L}^2_\eta$. At least this is the case for the modal logic **E** determined by the class of *all* neighborhood frames. The situation might change if some other non-normal modal logic is taken as a base. We briefly discuss two representative examples.

Consider the modal logic determined by the class of neighborhood frames that are closed under intersection. The closure under intersection is easily seen to be $\mathcal{L}^2_\eta$-definable by the formula:

$$\forall x.\forall U, V.\, [x\eta U \wedge y\eta V \rightarrow \exists W.(y\eta W \wedge \forall z.(z\varepsilon U \wedge z\varepsilon V \leftrightarrow z\varepsilon W))]$$

Consequently, we expect the situation in this and similar, $\mathcal{L}^2_\eta$-definable cases to be rather straightforward.

However, consider the modal logic **M**, determined by the class of all *monotone* neighborhood frames. Recall that (W, n) is monotone, if wnA and $A \subseteq B$ imply wnB. This condition is not expressible in $\mathcal{L}^2_\eta$, so in this case part of the story we witnessed in this paper might reappear. That is to say, one needs to find a well-behaved fragment of $\mathcal{L}^2_\eta$ that is invariant for monotone frames. One possibility is to restrict the quantification over open variables by admitting only formulas of the kind $\exists U.(x\varepsilon U \rightarrow \phi)$ with U occuring positively in ϕ. We leave it to further research to decide whether fully developing this approach is worthwhile in this and other interesting cases.

- **Axiomatizations**. In this paper, we have investigated *expressive power* of extended modal languages interpreted on topological spaces. However, in order for these logics to be of practical use, their *proof theory* will have to be studied as well. In the case of the basic modal language, topological completeness has already been studied for a long time [27], but for more expressive modal languages, this is a new area of research. Some first results for $\mathcal{M}(D)$ and $\mathcal{H}(E)$ and can be found in [23].

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A $\mathcal{L}^2$ over topological models

In this section, we prove Theorem 2.2, here stated once again for reference:

Theorem A.1. *$\mathcal{L}^2$ interpreted on topological models lacks Compactness, Löwenheim-Skolem and Interpolation, and is Π^1_1-hard for validity.*

This was already known for the more general case where $\mathcal{L}^2$-formulas can contain k-ary relation symbols with $k \geq 2$. The topological models we work with in this paper contain only unary predicates, but we will show that the bad properties of $\mathcal{L}^2$ already occur in this more restricted setting.

Proof. These facts can all be derived from the observation that $\mathcal{L}^2$ can define $(\mathbb{N}, \leq)$ up to isomorphism.

- Definability of $(\mathbb{N}, \leq)$.

  Let $x \leq y$ stand for the $\mathcal{L}^2$-formula $\forall U.(x \varepsilon U \rightarrow y \varepsilon U)$, which defines the well known *specialisation order* ($x \leq y$ iff $x \in \mathbb{C}\{y\}$). For each topological space (X, τ), $\leq$ defines a quasi-order on X. Conversely, every quasi-order on a set X is the specialisation order of some topology on X (in fact, of an Alexandroff topology on X).

  A special feature of $\leq$ is that every open set U is an *up-set* with respect to $\leq$ (i.e., whenever $x \varepsilon U$ and $x \leq y$ then also $y \varepsilon U$). Likewise, closed sets are *down-sets* with respect to $\leq$. If a space is Alexandroff, the converse holds as well: a set is open if and only if it is an up-set with respect to $\leq$, and it is closed if and only if it is a down-set.

  Now, let χ_N be the conjunction of the following formulas (where we use $x < y$ as shorthand for $x \leq y \wedge x \neq y$):

  $\leq$ is a linear order
  $\forall xy.(x \leq y \;\wedge\; y \leq x \;\rightarrow\; x = y)$
  $\forall xy.(x \leq y \;\vee\; y \leq x)$

  There is a least element
  $\exists x.\forall y.(x \leq y)$

  Each element has an immediate successor in the ordering
  $\forall x.\exists y.(x < y \;\wedge\; \forall z.(x < z \;\rightarrow\; y \leq z))$

  The space is Alexandroff (the down-sets are the closed sets)
  $\forall x.\exists U_x.\forall V_x.\forall y.(y \varepsilon V \rightarrow y \varepsilon U)$

  Each down-set other than $X, \emptyset$ has a least and a greatest element
  $$\forall U.(\exists x.(x \not\varepsilon U) \wedge \exists x.(x \varepsilon U) \rightarrow \exists z_l, z_g.(z_l \not\varepsilon U \wedge z_g \not\varepsilon U \wedge \forall y.([y < z_l \vee z_g < y] \rightarrow y \varepsilon U)))$$

  It is not hard to see that, if we take the open sets to be the up-sets, then $(\mathbb{N}, \leq)$ is a model for χ_N. In other words, χ_N is satisfiable. Now, suppose χ_N is true in some topological space (X, τ). We claim that $(X, \leq)$ is isomorphic to the natural numbers with their usual ordering.

To prove this, it suffices to show that, for any $w \in X$, the set $\{v \mid v \leq w\}$ is finite (this property, together with the fact that $\leq$ is a linear order and each element has an immediate successor, characterizes the natural numbers up to isomorphism). In other words, we need to demonstrate that no infinite ascending or descending chains exist below an arbitrary point of X.

Suppose that for some $w \in X$ the set $(w] = \{v \in X \mid v \leq w\}$ contains an infinite ascending chain $A = \{a_1, a_2, \dots\}$ with $a_i < a_{i+1}$ for each $i \in \mathbb{N}$. Consider the down-set $(A] = \{w \in X \mid \exists a_i \in A.(w \leq a_i)\}$ generated by the set A. Since $(A] \subseteq (w]$, we know that $(A] \neq X$, and hence there is a greatest element $g \in (A]$. By the definition of $(A]$, we have $g \leq a_i$ for some $i \in \mathbb{N}$. By definition of A, we also have $a_i < a_{i+1}$ and so $g < a_{i+1}$, contradicting the maximality of g in $(A]$. Hence no infinite ascending chains exist below w.

Next, suppose that for some $w \in X$ the set $(w] = \{v \in X \mid v \leq w\}$ contains an infinite descending chain $D = \{d_1, d_2, \dots\}$, with $d_{i+1} < d_i$ for each $i \in \mathbb{N}$. Then the set $X \setminus [D) = \{w \in X \mid \forall d_i \in D.(w < d_i)\}$ is a non-empty down-set (for non-emptyness, note that the least element of X cannot belong to D, and hence belongs to $X \setminus [D)$). But then, there must be a greatest element $g \in X \setminus [D)$. Let g' be the immediate successor of g. Note that by maximality of g we must have $d_i \leq g'$ for some $i \in \mathbb{N}$. By definition of D, $d_{i+1} < d_i$ and we obtain $d_{i+1} \leq g$, hence $g \in [D)$, a contradiction. Thus no infinite descending chains exist below w.

- Failure of Compactness

  Consider the following set of $\mathcal{L}^2$-sentences with one unary predicate P:

  $$\Gamma \equiv \{\chi_{_N}, \exists x.P(x)\} \cup \{\varphi_n \mid n \in \mathbb{N}\}$$

  where $\varphi_k \equiv \forall x.(P(x) \to \exists y_1, \dots, y_k.(y_1 < y_2 < \dots < y_k < x)$ express that every point in P has at least k predecessors. Every finite subset of Γ is satisfiable but Γ itself is not.

  In fact, it is possible to show failure of compactness even without using any unary predicates.

- Failure of upward and downward Löwenheim-Skolem

  Since $\chi_{_N}$ characterizes $(\mathbb{N}, \leq)$ up to isomorphism, clearly, it has only countable models. Thus, the upward Löwenheim-Skolem theorem fails for $\mathcal{L}^2$. The downwards Löwenheim-Skolem theorem fails as well: we can easily express in $\mathcal{L}^2_=$ that the specialisation order $\leq$ is a dense linear ordering without endpoints. Further, we can express (on Alexandroff spaces) that each non-empty up-set has an infimum:

  $$\text{Inf} \quad \forall U.\big(\exists x.(x \varepsilon U) \to \exists y.\forall z.\,((y < z \to z \varepsilon U) \land (z \varepsilon U \to y \leq z))\big)$$

  Combining these formulas, we can enforce a *complete dense linear order without endpoints.* An example of an infinite model satisfying this is

$\mathbb{R}$ with its usual ordering. Any *countable* model, on the other hand, would have to be isomorphic to $\mathbb{Q}$, as a countable dense linear order without endpoints, which contradicts the conjunct Inf (e.g., the up-set $\{w \in \mathbb{Q} \mid w^2 > 2\}$ has no infimum).

- Failure of Interpolation

  Let P, Q, R be distinct unary predicates. Let $\phi_{even}(P)$ be the $\mathcal{L}^2$-sentences expressing that, on the natural numbers, P is true exactly of the even numbers, and $\phi_{even}(Q)$ likewise (it is not hard to see that there are such formulas). Then the following implication is valid:

  $$\chi_{N} \wedge \phi_{even}(P) \wedge \exists x.(Px \wedge Rx) \quad \rightarrow \quad (\phi_{even}(Q) \rightarrow \exists x.(Qx \wedge Rx))$$

  Any interpolant for this implication has to express that R is true of some even number, without the help of additional predicates. Using an Ehrenfeucht-Fraïsse-style argument, one can show that this is impossible (note that we are essentially in first-order logic: quantification over open sets does provide any help, as the only open sets are the up-sets).

- Σ^1_1-hard satisfiability problem.

  Using χ_{N}, we can reduce the problem of deciding whether an existential second order (ESO) formulas is true on $(\mathbb{N}, \leq)$ —a well known Σ^1_1-complete problem— to the satisfiability problem of $\mathcal{L}^2$. For simplity we will discuss here only the case for ESO sentences of the form $\exists R.\phi(R, \leq)$, where R is a single binary relation. The argument generalizes to more relations, and relations of other arities.

  Let an ESO sentence $\exists R.\phi$ be given. Let N, P_1 and P_2 be distinct unary predicates. Intuitively, the elements of the model satisfying N will stand for natural numbers, while the other elements only play a technical role for coding up the binary relation R. Let $x <^{+} y$ be short for $x < y \wedge Ny \wedge \forall z.(x < z \wedge Nz \rightarrow y \leq z)$, expressing that y is the least N-element greater than x. By induction, we define an $\mathcal{L}^2$-formula ϕ^* as follows:

  $$\begin{array}{lcl} (x = y)^* & = & Nx \wedge Ny \wedge x = y \\ (x \leq y)^* & = & Nx \wedge Ny \wedge x \leq y \\ (Rxy)^* & = & \exists x'y'z.(z{<}x'{<}^{+}x \ \wedge \ z{<}y'{<}^{+}y \ \wedge \ P_1x' \ \wedge \ P_2y') \\ (\phi \wedge \psi)^* & = & \phi^* \wedge \psi^* \\ (\neg\phi)^* & = & \neg\phi^* \\ (\exists x.\phi)^* & = & \exists x.(Nx \wedge \phi^*) \end{array}$$

  We claim that $(\mathbb{N}, \leq) \models \exists R.\phi$ iff $\phi^* \wedge \chi^{\circ}_{N}$ is satisfiable, where χ°_{N} is the relativisation of χ_{N} to N (i.e., the formula obtained from χ_{N} by relativising all quantifiers by N, thus expressing that the subspace defined by N with its specialisation order is isomorphic to $(\mathbb{N}, \leq)$).

  The difficult direction is *left-to-right*. We give a rough sketch. Suppose that $(\mathbb{N}, \leq) \models \exists R.\phi$. Let $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$ be a witnessing binary relation.

Now, we define our model for $\phi^* \wedge \chi^\circ_{_N}$ as follows: the subspace defined by N is simply the Alexandroff topology generated by $(\mathbb{N}, \leq)$. For each pair $(m, n) \in \mathcal{R}$, we create three distinct $\neg N$-elements, $(m, n)_0$, $(m, n)_1$ and $(m, n)_2$. Then we make sure that m is the least N-successor of $(m, n)_1$ and P_1 holds at $(m, n)_1$, n is the least N-successor of $(m, n)_2$ and P_2 holds at $(m, n)_2$, $(m, n)_0 < (m, n)_1$ and $(m, n)_0 < (m, n)_2$. In this way, we ensure that, for any pair of natural numbers m, n, $(m, n) \in \mathcal{R}$ iff the $\mathcal{L}^2$-formula $(Rxy)^*$ is true of (m, n) in the constructed model. Once this observation is made, the claim becomes easy to prove.

$\dashv$

B General Topology

In this appendix we try to gather the definitions from general topology used throughout the thesis. Most of the material exposed here is based on the source [12].

B.0.1 Topological spaces

A **topological space** $X = \langle X, \tau \rangle$ is a nonempty set X together with the collection of its **open** subsets $\tau \subseteq \wp(X)$ closed under finite meets and arbitrary joins. If $x \in X$ and $x \in O \in \tau$ then O is said to be an open **neighbourhood** of x. For a subset $A \subseteq X$ the topology **induced** by τ can be defined as $\tau_A = \{O \cap A \mid O \in \tau\}$. Then $A = \langle A, \tau_A \rangle$ is called a **subspace** of X.

The simplest example of a topological space is the one point space $X = \{x\}$ with the only possible topology $\{\emptyset, X\}$. In general on any nonempty set X one can define two topologies—the **trivial** topology $\{\emptyset, X\}$ and the **discrete** topology $\wp(X)$. A bit more interesting is the topology of the **Sierpinski space** $\langle \{a, b\}, \{\emptyset, a, \{a, b\}\} \rangle$. The real line $\mathbb{R}$ is another example of a topological space in which open sets are arbitrary unions of open intervals (x, y).

In general arbitrary intersections of open sets need not be open. Spaces in which the intersection of opens is always an open set are called **Alexandroff** spaces. In particular, any finite topological space is Alexandroff.

A topology τ' on a set X is aid to be **finer** than the topology τ on the same set if $\tau \subseteq \tau'$. It is also said that τ' **extends** τ or that τ is **coarser** than τ'.

B.0.2 Closure, interior and derived set operators

For a subset $A \subseteq X$ its **interior** $\mathbb{I}A$ is defined as the largest open set contained in A. More precisely:

$$\mathbb{I}A = \bigcup \{O \mid O \subseteq A,\ O \in \tau\}$$

Dually, the **closure** $\mathbb{C}A$ of a set $A \subseteq X$ is defined as the smallest closed set containing A, where by a **closed** set we mean a set that has the open complement. Alternative ways of defining a topological space is to pose certain (Kuratowski) axioms which the operators of closure and interior should satisfy. In particular, $X = \langle X, \mathbb{C} \rangle$ is said to be a topological space, if the following hold for arbitrary subsets of X:

(C1) $\mathbb{C}\emptyset = \emptyset$,
(C2) $A \subseteq \mathbb{C}A$,
(C3) $\mathbb{C}(A \cup B) = \mathbb{C}A \cup \mathbb{C}B$,
(C4) $\mathbb{C}\mathbb{C}A = \mathbb{C}A$.

Dually, $X = \langle X, \mathbb{I} \rangle$ is said to be a topological space, if the following hold for arbitrary subsets of X:

(I1) $\mathbb{I}X = X$,
(I2) $\mathbb{I}A \subseteq A$,
(I3) $\mathbb{I}(A \cap B) = \mathbb{I}A \cap \mathbb{I}B$,
(I4) $\mathbb{I}\mathbb{I}A = \mathbb{I}A$.

If the closure operator $\mathbb{C}$ satisfying the above conditions is given, then the interior operator can be defined as $\mathbb{I}A \equiv (\mathbb{C}A^c)^c$. Saying that A is open iff $A = \mathbb{I}A$ defines the collection $\tau = \{A \subseteq X \mid A = \mathbb{I}A\}$ which qualifies for topology and induces the very same interior operators $\mathbb{I}$ and $\mathbb{C}$. Dually, A is closed iff $A = \mathbb{C}A$. Both closure and interior operators are **monotone**, that is if $A \subseteq B$ then $\mathbb{C}A \subseteq \mathbb{C}B$ and $\mathbb{I}A \subseteq \mathbb{I}B$. Another monotone operator is a **derived set** operator . For a subset $A \subseteq X$ of a topological space A is the collection of all limit points of A, where $x \in X$ is said to be a **limit point** of A if for every open neighbourhood $O \in \tau$ of x the set $O \cap A$ contains a point y such that $x \neq y$. Formally

$$A = \{x \in X \mid \forall O \in \tau (x \in O \ \rightarrow \ (O - \{x\}) \cap A \neq \emptyset)\}$$

The derived set operator satisfies the following axioms in an arbitrary topological space:

(D1) $\emptyset = \emptyset$,
(D2) $(A \cup B) = A \cup B$,
(D3) $A \subseteq A \cup A$.

The connection between $\mathbb{C}$ and is expressed in the following equation valid in an arbitrary topological space

$$\mathbb{C}A = A \cup A$$

B.0.3 Resolvable and irresolvable spaces

A subset $A \subseteq X$ is **dense-in-itself** if $A \subseteq A$. A space X that is dense-in-itself is also sometimes called **crowded**. More generally, a set $A \subseteq X$ is said to be **dense** in a set $B \subseteq X$ if $A \subseteq B$. A subset $A \subseteq X$ is called **everywhere dense** if $\mathbb{C}A = X$. A space X that can be decomposed into two disjoint dense subsets is called **resolvable**. Spaces that are not resolvable are called **irresolvable**. We say that a space X is **hereditarily irresolvable** (or **HI** for short), if its every subspace is irresolvable. A subset $A \subseteq X$ of a topological space is called **nowhere dense** if $\mathbb{I}\mathbb{C}A = \emptyset$.

B.0.4 Scattered spaces and the Cantor-Bendixson Theorem

A point $x \in X$ is called an **isolated** point of A if x is not a limit point of A. In other words, x is an isolated point of A if there exists an open neighbourhood $O \in \tau$ of x such that $O \cap A \subseteq \{x\}$. A point $x \in X$ is an isolated point (of X) if $\{x\} \in \tau$. A subset $A \subseteq X$ of a space X is called **discrete** if every $x \in A$ is an isolated point of A. A space X is called **scattered** if every subspace of X has an isolated point. A space is called **weakly scattered** if the set of all isolated points is everywhere dense. A subset $A \subseteq X$ is called **perfect** if A is closed and dense-in-itself.

Theorem B.1 (Cantor-Bendixson). *An arbitrary topological space X can be decomposed into the union of two disjoint subspaces P and S such that P is perfect and S is scattered.*

B.0.5 Separation axioms

A space X is said to satisfy the T_0 separation axiom if for any two distinct points in X there exists an open set which contains one of them, but not the other. A space satisfies T_1 **separation axiom** if for any two distinct points $x, y \in X$ there exists an open neighbourhood O of x such that $y \notin O$. The equivalent characterization of T_1 spaces is that every singleton set is closed. A space is called **Hausdorff**if any two distinct points have disjoint open neighbourhoods. A space X is said to satisfy T_D **separation axiom** if every singleton can be obtained as the intersection of an open a closed subsets of X. It is known that every Hausdorff space is T_1, every T_1 space is T_D and every T_D space is T_0

B.0.6 Connectedness and disconnectedness

A topological space X is called **connected** if the only closed and open subsets of X are $\emptyset$ and X. A space is called **well-connected** if for any two open subsets $O_1, O_2 \in \tau$, $O_1 \cup O_2 = X$ implies $O_1 = X$ or $O_2 = X$. X is said to be **extremally disconnected** if the closure of any open subset is open. A space is **perfectly disconnected** if it is T_0 and for any two disjoint subsets $A \cap B = \emptyset$ we have $A \cap B = \emptyset$.

B.0.7 Maps between spaces

A map $f : X \to Y$ between the topological space X and Y is called **continuous** if $f^{-1}(O) \subseteq X$ is open whenever $O \subseteq Y$ is open (pre-image of any open is open). A map $f : X \to Y$ is called **open** if $f(O) \subseteq Y$ is open whenever $O \subseteq X$ is open (image of any open is open). A map $f : X \to Y$ is an **homeomorphism** if it is a bijective open continuous map, that is if f is injective, onto and it is continuous together with its inverse.