The Geometry of Quantification

Climbing the Number Tree and Other Stories of Generalized Quantifiers

Dag Westerståhl

Stockholm University

Celebration event in honour of Johan van Benthem
ILLC and UvA
Amsterdam
September 26–27, 2014

Johan on generalized quantifiers and natural language 1983–89

Some publications (among many):

- Determiners and logic (*L&P*, 1983)
- Questions about quantifiers (*JSL*, 1984)
- Semantic automata (in a GRASS volume, 1986)

These and many others collected in the volume

- *Essays in Logical Semantics* (D. Reidel, Dordrecht, 1986)

Also,

- Polyadic quantifiers (*L&P*, 1989)

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1983–89: Johan takes the logical aspects of this work much further.

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To convince you of the opposite, I will look at several applications.

Quantifiers of type $\langle 1 \rangle$ and $\langle 1, 1 \rangle$

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$$Q_M(A) \Leftrightarrow (M, A) \in Q$$

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Examples:

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$(\exists_{\geq 3})_M(A) \Leftrightarrow |A| \geq 3$

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$\langle 1,1 \rangle$ $\mathit{some}_M(A, B) \Leftrightarrow A \cap B \neq \emptyset$

$\mathit{most}_M(A, B) \Leftrightarrow |A \cap B| > |A - B|$

$\mathit{the\ three}_M(A, B) \Leftrightarrow |A| = 3$ and $A \subseteq B$

$I_M(A, B) \Leftrightarrow |A| = |B|$ (Härtig quantifier)

Quantifiers as binary relations between numbers

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Q and Q^{rel} are the same binary relation.

The number tree

So (under these assumptions) quantifiers are subsets of $\mathbb{N}^2$:

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The number tree (triangle): rotate 45 degrees!

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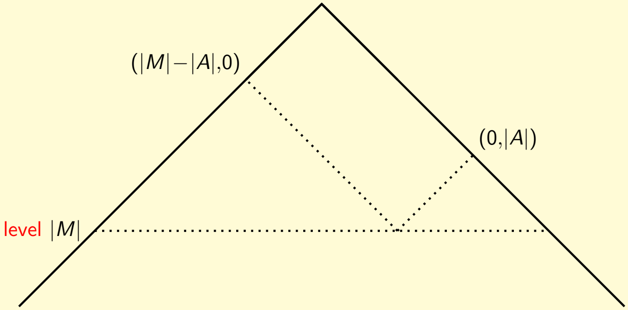
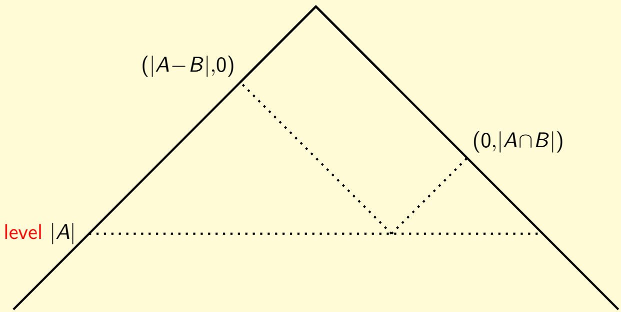
Type ⟨1⟩ quantifiers in the tree

$Q_M(A)$

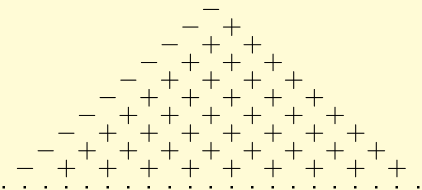


Type $\langle 1, 1\rangle$ quantifiers in the tree

$Q_M(A, B)$



Examples



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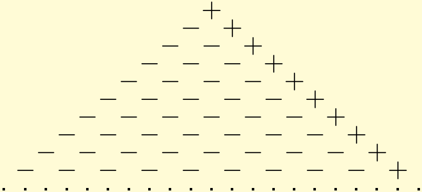
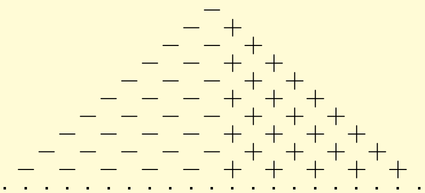
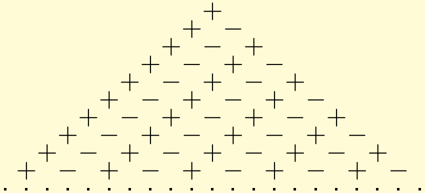


all (or ∀)

More examples

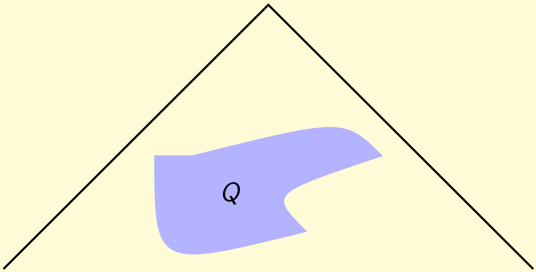


most (or Q^R)

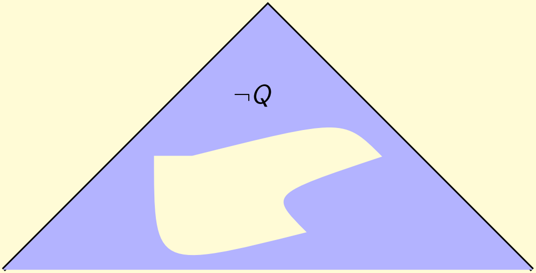


an-even-number-of (or Q_{even})

Outer negation: Q



Outer negation: $\neg Q$



Inner negation: $Q\neg(A, B) \Leftrightarrow Q(A, A - B)$

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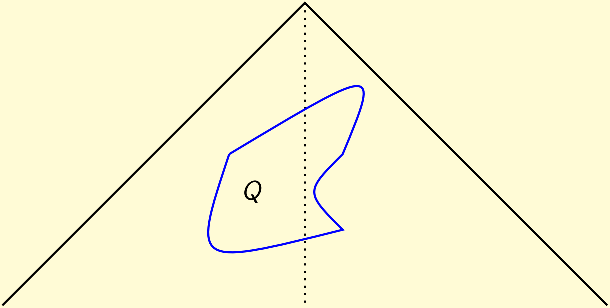

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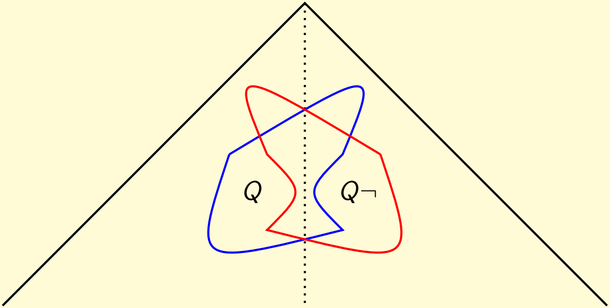

Application: 'midpoint' quantifiers: $Q = Q\neg$ (Keenan)

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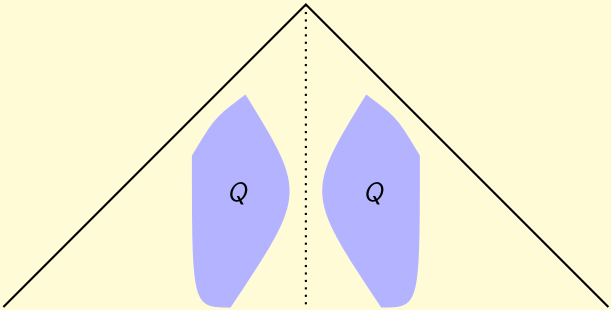

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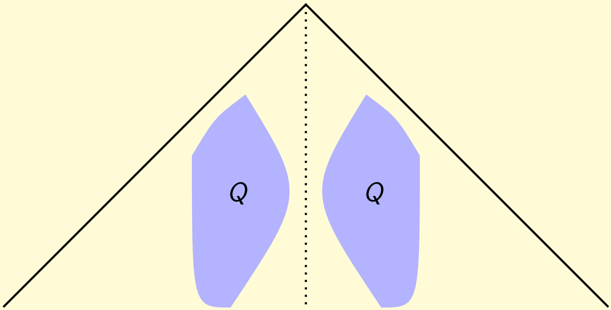

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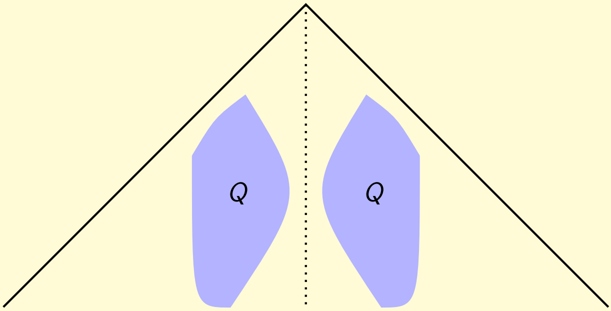

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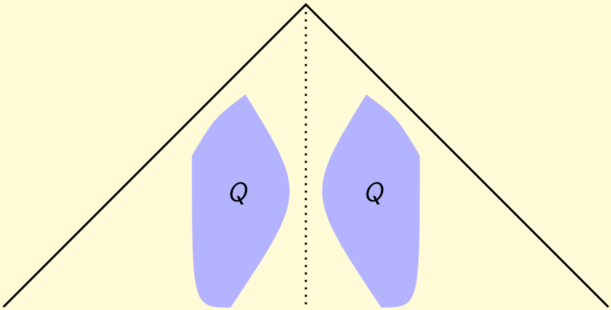

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Even with CONSERV, EXT, ISOM, and finite models, $2^{\aleph_0}$ many!

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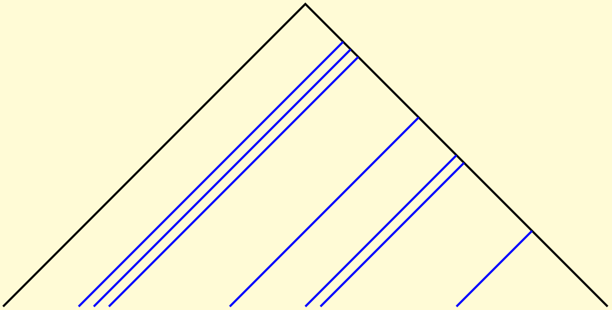

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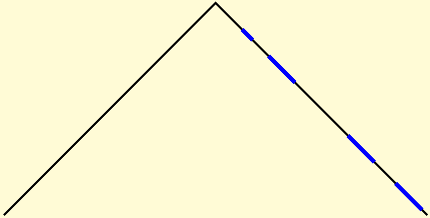

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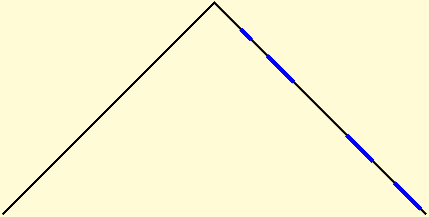

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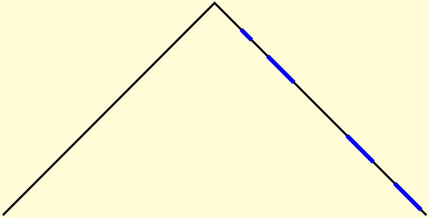

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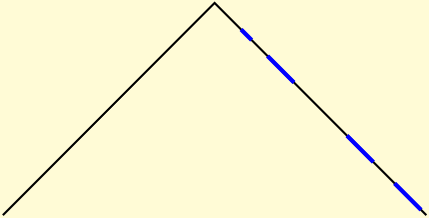
Asymmetry? $(Q(A, B) \Rightarrow \neg Q(B, A))$

Anti-symmetry: $Q(A, B) \wedge Q(B, A) \Rightarrow A = B$

Q is anti-symmetric iff $Q(m, n) \Rightarrow m = 0$



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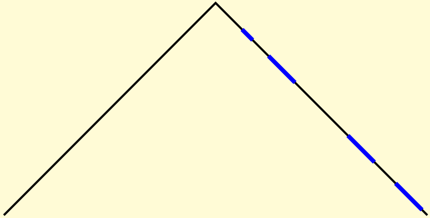

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Asymmetry? $(Q(A, B) \Rightarrow \neg Q(B, A))$

Q asymmetric iff $Q(m, n) \Rightarrow \forall k \, \neg Q(k, n)$

So Q asymmetric $\Rightarrow Q = \emptyset$: no non-trivial asymmetric quantifiers exist.

Transitivity: $Q(A, B) \wedge Q(B, C) \Rightarrow Q(A, C)$

Fact (W-hl 1984)

Q is transitive iff there are $X = \{k_1, k_2, \ldots\}$ and $Y = \{n_1, n_2, \ldots\}$ such that $X < Y$ and $Q(m, n) \Leftrightarrow m+n \in X \vee (n = 0 \wedge m \in Y)$.

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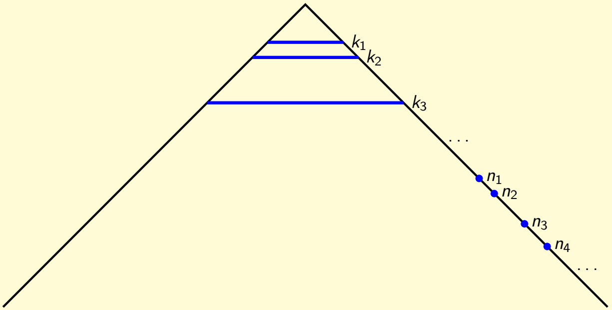

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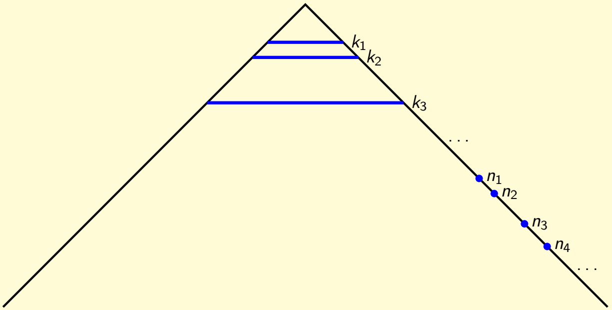

For example, it is immediate that anti-symmetry implies transitivity.

Right monotonicity

MON↑: $Q(A, B)\ \&\ B \subseteq B' \Rightarrow Q(A, B')$

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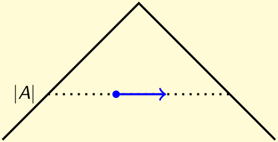

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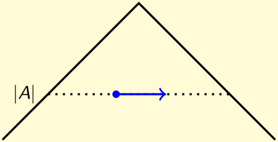

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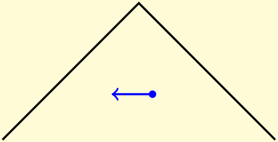

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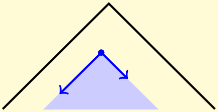

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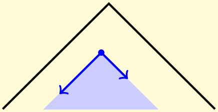

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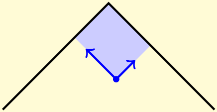

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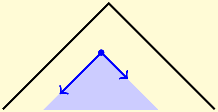

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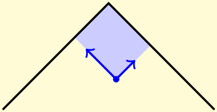

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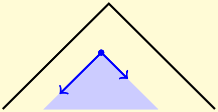

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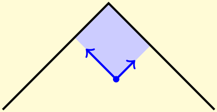

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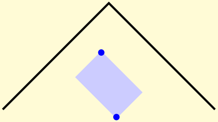

And four more

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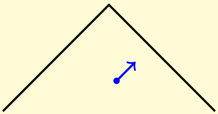

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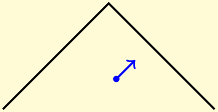

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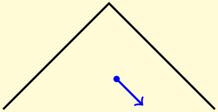

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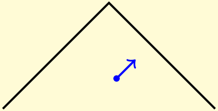
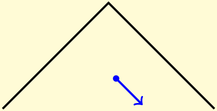
$\uparrow_{\mathsf{NE}}\mathrm{MON}$: 

$\uparrow_{\mathsf{SE}}\mathrm{MON}$: 

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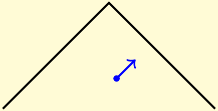

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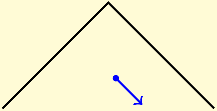

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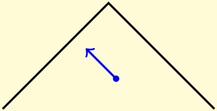

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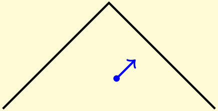

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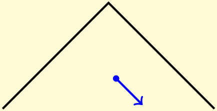

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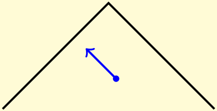

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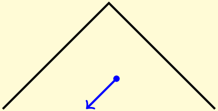

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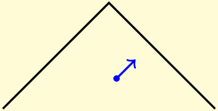

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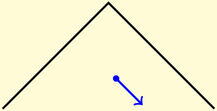

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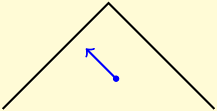

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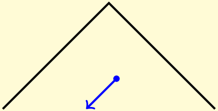

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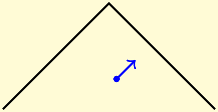

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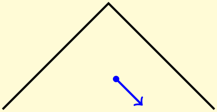

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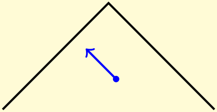

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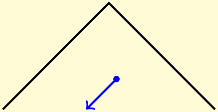

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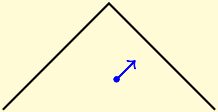

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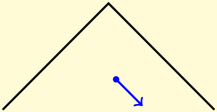

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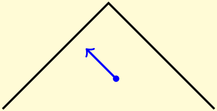

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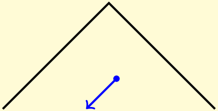

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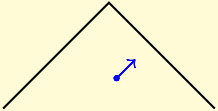
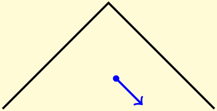
$\uparrow_{\mathsf{NE}}\mathrm{MON}$: 

$\uparrow_{\mathsf{SE}}\mathrm{MON}$: 

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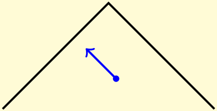
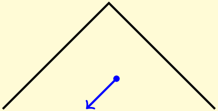
$\uparrow_{\mathsf{NW}}\mathrm{MON}$: 

$\uparrow_{\mathsf{SW}}\mathrm{MON}$: 

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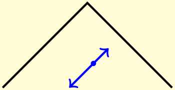
- symmetry $= \uparrow_{\mathsf{NE}}\mathrm{MON} + \uparrow_{\mathsf{SW}}\mathrm{MON}$ 

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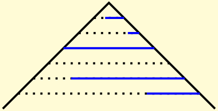

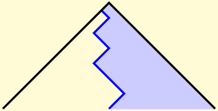

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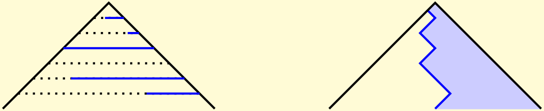

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The count complexity of Q is the smallest number of elements in a set A with n elements one needs to check in order to verify that $Q(A, B)$ holds + the corresponding number for falsification (this number is always $\geq n + 1$).

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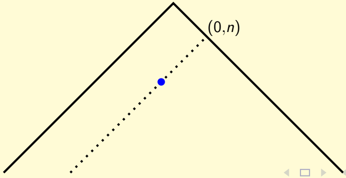

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Certain apparently non-trivial facts become immediate in the number tree.

For example, say that Q is quasi-reflexive if $Q(A,B) \Rightarrow Q(A,A)$, or

$$Q(m,n) \Rightarrow Q(m+n,0)$$

E.g. *some*, *at least two-thirds of*, *every* are quasi-reflexive.

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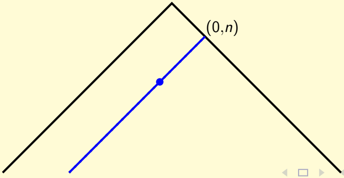

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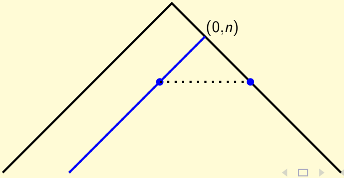

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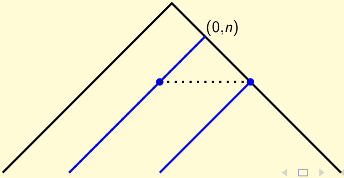

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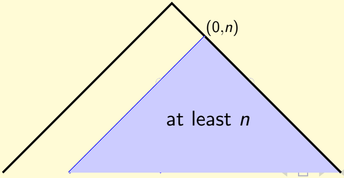

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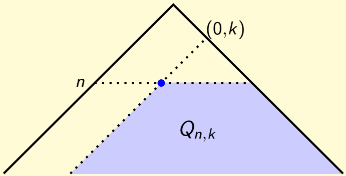

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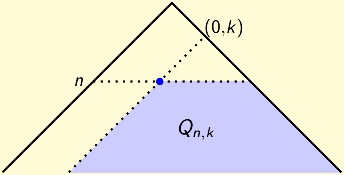

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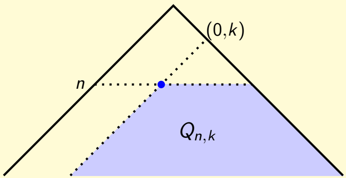

But after a finite number of steps from such a point you hit the left axis. Hence (in the tree) Q is a finite disjunction of the $Q_{n,k}$:

Fact

Q is $\uparrow$MON$\uparrow$ iff it is a finite disjunction of quantifiers of the form at least k of the n or more $(k \leq n)$.

Facts provable by 'reasoning' in the tree 2

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(3a) $Q(A \cup B, C)$
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Reasoning in the number tree one can show

Proposition (Peters and W-hl 2006)

The only non-trivial LAA quantifiers are all, no, *and* $Q(A, B) \Leftrightarrow A = \emptyset$.

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Not very hard to prove, but hard to come up with without the tree.

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Many Det denotations satisfy this. But (as Johan pointed out) a quick look at the number tree reveals that *some but not all* is a counter-example:

```
                  −
                 − −
                − + −
               − + + −
              − + + + −
             − + + + + −
            − + + + + + −
           − + + + + + + −
          − + + + + + + + −
         − + + + + + + + + −
        . . . . . . . . . . . . . . . . . . . . .
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Likewise Väänänen and W-hl (2002)—somewhat embarrassingly—suggested a smoothness universal (U2), which again is correct for many $\uparrow\text{Mon}$ Det denotations, such as *at least n*, *more than n/m:ths of*, *every*.

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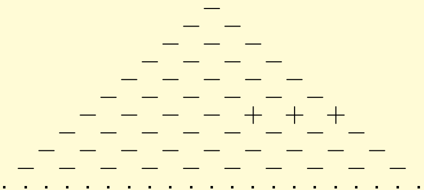

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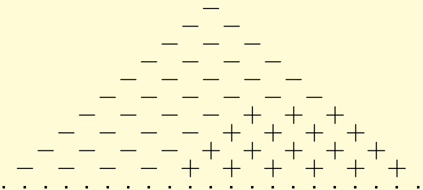

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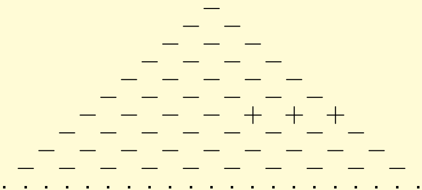

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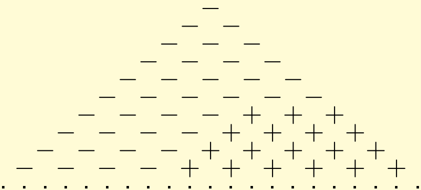

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In all of these cases, the number tree was instrumental.

First-order definability in the number tree

A simple use of EF-games shows:

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A quantifier of type $\langle 1 \rangle$ *or* ($\textsc{Conserv}$ *and* $\textsc{Ext}$ *and of type*) $\langle 1, 1 \rangle$ *is first-order definable iff for some* $r \geq 0$, *it has the pattern shown below.*

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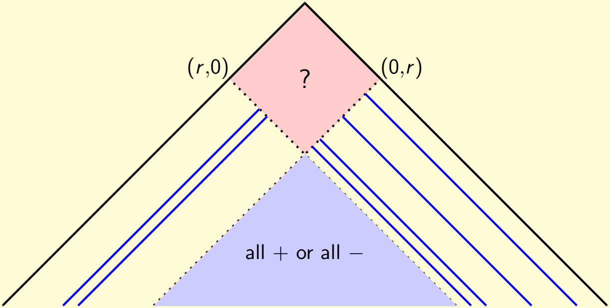

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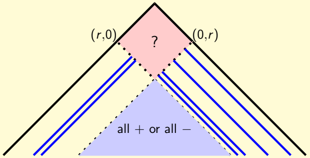

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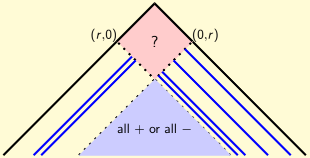

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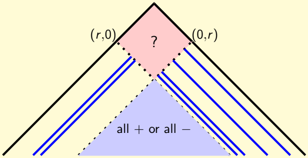

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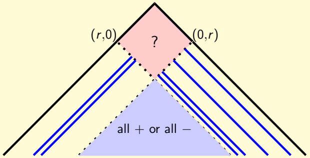

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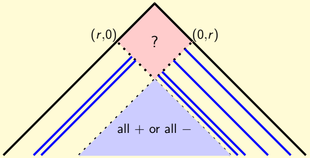

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Fact

All first-order definable quantifiers are Boolean combinations of ↑MON *quantifiers.* (*Looking a little closer, one sees that they are in fact disjunctions of* LEFT CONT *quantifiers.*)

Definability from monotone type $\langle 1 \rangle$ quantifiers

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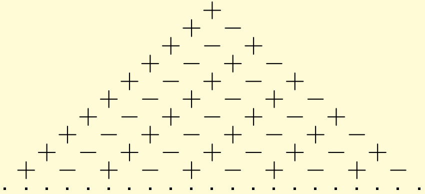
Theorem (Väänänen 1997)

A type $\langle 1 \rangle$ quantifier is definable from monotone type $\langle 1 \rangle$ quantifiers if and only if it has bounded oscillation.

Definability from monotone quantifiers, cont.

For example,

- Q_{even} is not definable from monotone type $\langle 1 \rangle$ quantifiers:



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```
                   +
                  + −
                 + − +
                + − + −
               + − + − +
              + − + − + −
             + − + − + − +
            + − + − + − + −
           + − + − + − + − +
          + − + − + − + − + −
         . . . . . . . . . . . . . . . . . . . . .
```

- Whereas $Q_M(A) \Leftrightarrow |M - A| = 2$ and $|A|$ is even, though not *FO*-definable, is definable from monotone type $\langle 1 \rangle$ quantifiers:

```
                   −
                  − −
                 + − −
                − − − −
               − − + − −
              − − − − − −
             − − − − + − −
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           − − − − − − + − −
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         . . . . . . . . . . . . . . . . . . . . .
```

Definability from monotone type $\langle 1 \rangle$ quantifiers, cont.

However, even though Q and Q^{rel} are the same binary relation in the number tree, the expressivity of monotone ($\mathrm{MON}\uparrow$ or $\mathrm{MON}\downarrow$) quantifiers of the latter form is much greater.

Definability from monotone type $\langle 1 \rangle$ quantifiers, cont.

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Question: Can the number tree help more generally to check definability properties of CONSERV and EXT quantifiers?

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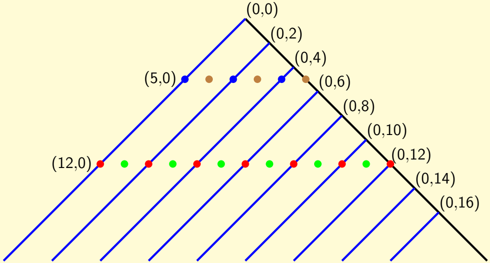

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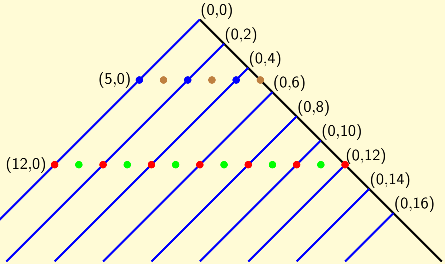

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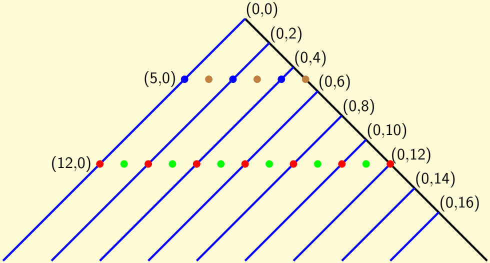

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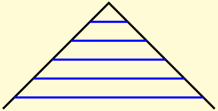

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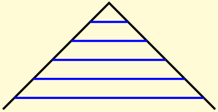

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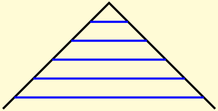

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If there is such a P, the number tree will presumably help finding it. . .

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THANK YOU