

Truth in Metamathematics

Albert Visser

Theoretical Philosophy, Department of Philosophy,
Faculty of the Humanities, Utrecht University

Palmyr VI,
Friday, December 14, 2007

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

**The Undefinability
of Truth**

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Tarski's Theorem

Conventional Wisdom: Suppose the theory U has enough coding. You cannot define a truth-predicate for U in U itself. You need a metalanguage to do that.

Example: we cannot define a truth predicate for Peano Arithmetic, PA, in PA itself, but we can define a truth predicate for PA in Zermelo-Fraenkel set theory, ZF. *Note that the last statement is modulo translation/interpretation.*

Tarski 1944 uses *language* ambiguously between: theory and model.

Thesis: An object-meta pair is a certain kind of arrow in a category, to be specific: a certain kind of interpretation between theories.

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Can Meta be Object?

There are interpretations K of Peano Arithmetic into itself such that there is a truthpredicate \tilde{K} for the interpretation, i.o.w.:

$$\blacktriangleright \text{PA} \vdash \tilde{K}(\underline{\#A}) \leftrightarrow A^K.$$

A sufficient condition for this phenomenon is that our theory U is (globally) reflexive, i.e., the theory $\mathcal{U}_U := \mathcal{S}_2^1 + \{\text{con}_n(U) \mid n \in \omega\}$ is interpretable in U .

We can even get an equivalence if we demand more from what it is to have a truthpredicate.

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results



Universiteit Utrecht

Questions

1. Develop a framework in which we can think in a natural way about the object-meta relation in connection with translation.
2. Under what conditions do we have Tarski's Theorem of the Undefinability of Truth?
3. If Tarski's Theorem fails, are there still traces of it? (Just like the First Incompleteness Theorem is a trace of the Liar.)
4. Are there further metamathematical applications?

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Translations

A *relative translation* $\tau : \Sigma \rightarrow \Theta$ is a pair $\langle \delta, F \rangle$.

- ▶ δ is Θ -formula
- ▶ F associates to R of Σ of arity n a Θ -formula $F(R)$ with variables among v_0, \dots, v_{n-1} .

Induced extension mapping:

- ▶ $(R(y_0, \dots, y_{n-1}))^\tau := F(R)(y_0, \dots, y_{n-1})$;
- ▶ $(\cdot)^\tau$ commutes with propositional connectives;
- ▶ $(\forall y A)^\tau := \forall y (\delta(y) \rightarrow A^\tau)$;
- ▶ $(\exists y A)^\tau := \exists y (\delta(y) \wedge A^\tau)$.

Composition:

- ▶ $\delta_{\tau\nu} := (\delta_\nu \wedge (\delta_\tau)^\nu)$,
- ▶ $R_{\tau\nu} = (R_\tau)^\nu$.

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Interpretations

A *relative interpretation*: $K : U \rightarrow V$ is a triple $K = \langle U, \tau, V \rangle$, such that, for all U -sentences A , we have $U \vdash A \Rightarrow V \vdash A^\tau$.

$K = K' : U \rightarrow V$ if:

- ▶ $V \vdash \delta_K V \leftrightarrow \delta_{K'} V$,
- ▶ $V, \vec{v} : \delta \vdash P^K \vec{v} \leftrightarrow P^{K'} \vec{v}$.

We obtain a category of interpretations: INT. There are other notions of equality of interpretations that give us different categories.

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

**The Canonical
Metatheory**

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

The Meta-Construction

We start with a theory U . We construct a minimal metatheory U^{meta} for U as follows.

- ▶ The language of U^{meta} is the language of U , plus the language of arithmetic, plus two domain constants for U -objects and for numbers, plus a truthpredicate.
- ▶ The theory is the disjoint union U and Robinson's Arithmetic Q , plus Tarski's Convention T: $\vdash \text{true}(\#A) \leftrightarrow A$, for U -sentences A .
- ▶ om_U is the obvious embedding of U in U^{meta} .

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

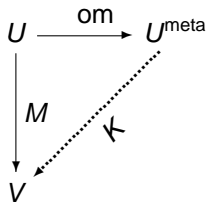
Further Results



Universiteit Utrecht

Restrictedness

We say that an interpretation $M : U \rightarrow V$ is *restricted* iff we can find a K such that the following diagram commutes.



Restrictedness is our explication of *object-meta pair*.

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

A Good Property

Suppose $K : U \rightarrow V$. We can lift K to an interpretation

$$K^{\text{meta}} : U^{\text{meta}} \rightarrow V^{\text{meta}}$$

as follows. Everything translates in the obvious way. We assign 'true(v_0^K)' to 'true'. Here $(\cdot)^K$ is the arithmetization of $A \mapsto A^K$.

We have:

$$\begin{array}{ccc} U & \xrightarrow{\text{om}} & U^{\text{meta}} \\ \downarrow K & & \vdots K^{\text{meta}} \\ V & \xrightarrow{\text{om}} & V^{\text{meta}} \end{array}$$

Warning: $(\cdot)^{\text{meta}}$ is not a functor.

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Verification

We have in V^{meta} :

$$\begin{aligned} \vdash \text{true}^{K^{\text{meta}}}(\#A) &\leftrightarrow \text{true}((\#A)^{\kappa}) \\ &\leftrightarrow \text{true}(\#(A^K)) \\ &\leftrightarrow A^K \\ &\leftrightarrow A^{K^{\text{meta}}} \end{aligned}$$

The Undefinability
of Truth

Interpretations

**The Canonical
Metatheory**

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



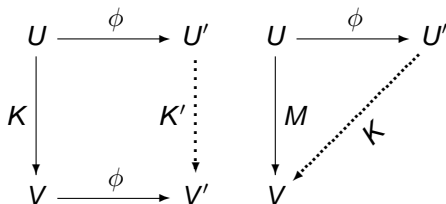
e-Schemes

We call a function ϕ from theories to interpretations a *uniform e-scheme* iff:

- ▶ $\text{dom}(\phi_U) = U$.
- ▶ For all $K : U \rightarrow V$, there is $K' : K' \circ \phi_U = \phi_V \circ K$.

We define:

- ▶ $M \models ?\phi :\Leftrightarrow \exists K M = K \circ \phi_{\text{dom}(M)}$.



The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

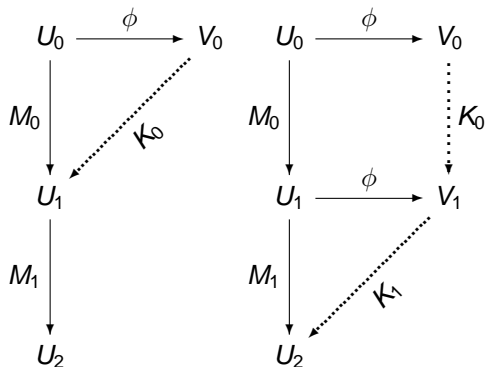
Further Results



Universiteit Utrecht

Composition

Suppose $M_0 : U_0 \rightarrow U_1$ and $M_1 : U_1 \rightarrow U_2$. Then, $M_1 \circ M_0 \models \phi$ if $M_0 \models \phi$ or $M_1 \models \phi$.



The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Tarski's Undefinability Result

Theorem

id_U is not restricted.

Theorem

Suppose $N : Z \rightarrow U$, $K : U \rightarrow V$, $M : V \rightarrow W$:

1. If K is restricted, then so are $M \circ K$. and $K \circ N$.
2. If $M \circ K$ is restricted and K is surjective, then M is restricted.
So surjective K cannot be restricted.

Theorem

No split monomorphism is restricted. No split epimorphism is restricted.

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

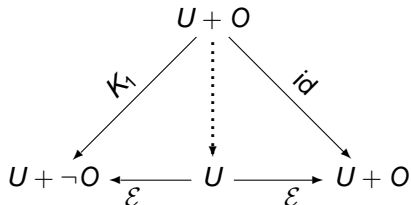
**The Orey
Phenomenon**

Further Results



The Orey Phenomenon

Suppose $K : U \rightarrow U$ is restricted. Let: $U \vdash O \leftrightarrow \neg \text{true}^{K'}(\#O)$, where K' witnesses restrictedness. We find: $U \vdash O \leftrightarrow \neg O^K$. So there are K_0 and K_1 such that: $K_0 : U + \neg O \rightarrow U + O$, and $K_1 : U + O \rightarrow U + \neg O$.



So, $(U + O) \rightarrow U$. Similarly: $(U + \neg O) \rightarrow U$. These interpretations are split monomorphisms and, hence, faithful.

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results



Universiteit Utrecht

Overview

The Undefinability of Truth

Interpretations

The Canonical Metatheory

e-Schemes

Consequences

The Orey Phenomenon

Further Results

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht

Further Results

ZF is mutually interpretable with an extension of PA. However no extension of PA is synonymous or even bi-interpretable with ZF. So, in one sense, set theory is more or less the same as arithmetic and, in another sense, set theory is 'really different' from arithmetic.

For a somewhat different choice of the canonical metatheory, we do have: there is a restricted arrow $U \rightarrow U$ iff U is reflexive. This also corresponds to U interprets U with first-order comprehension, for sequential theories.

For yet another choice of the canonical metatheory, we do have: there is a restricted arrow $U \rightarrow U$ iff U is +-reflexive. This also corresponds to U interprets U with predicative comprehension, for sequential theories.

The Undefinability
of Truth

Interpretations

The Canonical
Metatheory

e-Schemes

Consequences

The Orey
Phenomenon

Further Results



Universiteit Utrecht