Chapter 2

The Rationality of Round Interpretation

2.1 Introduction

This chapter is about why round numbers are seen as round; that is, as an approximation that can be used to refer to other numbers close to them. Much has been said about round numbers already, but other work has mostly focused on explaining the distribution of round numbers (which I will not be getting into at all) and why a speaker would want to use round numbers.
Instead, we will look at things from the perspective of someone hearing a round number being used. The point will be to show that in addition to what other good reasons there may be, round meaning can also in large part be explained just by the mathematics of the situation and people making rational decisions when interpreting things. After that, we apply the analysis to vagueness.
Despite this difference in approach, I should mention that the idea for this analysis comes from the following remark in (Krifka 2007):

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(17) a. 0---------------------------------60----------------------...--120-...
     b. 0---------------30----------------60----------------90----...--120-...
     c. 0------15-------30-------45-------60-------75-------90----...--120-...
     d. 0-5-10-15-20-25-30-35-40-45-50-55-60-65-70-75-80-85-90-95-...--120-...
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> Let the a-priori probability on hearing *forty-five minutes* that one of the scales (17.c) or (17.d) be used be the same, say s. Then on hearing *forty-five minutes* the probability that the more fine-grained scale (17.d) is used is 5rs, and the probability that the more coarse-grained scale (17.c) is used is double the value of that, 10rs. Hence the hearer will assume the more coarse-grained scale.

This is almost a throwaway remark in the piece in question, but it suggests an underlying principle worth far more attention.

Now the central question I will look into in the next sections is: why is it rational for a hearer to interpret a round number as a rounding? I'll investigate this by looking into several questions and the mathematics behind them. The first question is a matter of conditional probability. Some game theory will follow later.

2.2 Conditional Probability

The first question is: *Given that a round number was used, what is the chance that it was meant roundly?* In Bayesian statistics there is a straightforward answer to this question: the probability of A given B is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If A means it was meant roundly and B that a round number was used, then the formula is as above, so we are looking for the chance of both happening divided by the (prior) chance a round number gets used. Keeping in mind that our A is in B and therefore $P(A \cap B) = P(A)$, we obtain

$$P(\textit{meant roundly}|\textit{round number is used}) = \frac{P(\textit{meant roundly})}{P(\textit{round number is used})}$$

Let us look into these chances using an example.

The example we're going to use is as follows: First we take a round number, say, 30. Now there are a bunch of numbers close-by enough that you might round them to 30. We will use the simplifying assumption that only integers are relevant sufficiently close ones have a chance of being rounded to 30. (See Section 2.5 for notes on how to drop both of these assumptions.) Suppose these sufficiently close ones are 25-34, or 10 numbers in total.
Now one of these numbers is randomly selected (with equally distributed probability) and the speaker wants to talk about that number. Finally, the speaker may or may not decide to round that number. Since we are interested in the hearer's side of things, we are going to just assign a value x to the chance that the speaker will choose to round to 30. For this example let us suppose $x = 50\%$. (This is perhaps on the high side, but not much depends on this; the point is to show how much larger than x the final conditional probability is. Also, Section 2.3 will show that a much smaller x can in fact suffice.) Let us see what happens given this situation.

	30	25-34 but not 30
Speaker rounds	$0,5 \cdot \frac{1}{10}$	$0,5 \cdot \frac{9}{10}$
Speaker does not round	$0,5 \cdot \frac{1}{10}$	$0,5 \cdot \frac{9}{10}$

This table outlines the probabilities of the four (a priori) possible situations. In the left column are the situations where the randomly selected number was exactly 30, in the right the ones where it was close but not 30 itself. Similarly, in the top row are the situations where the speaker chooses to round, while in the bottom are the ones where he does not.[1]
Now to get from these numbers to the conditional probability we want, the main thing to do is to apply the condition we were using. That condition was *Given that a round number was used.* Of course, if the number is not actually 30 and the speaker does not round to 30, then he will not say 30. Thus the lower-right corner is irrelevant for us. That is a lot of the total chance we're throwing out, so we can already see where this is going. But let us take a look.

	30	25-34 but not 30
Speaker rounds	$0,05$	$0,45$
Speaker does not round	$0,05$	$0,45$

$$\begin{aligned} P(\textit{Speaker rounded}) &= P(\textit{rounded}; 30) + P(\textit{rounded}; \textit{not 30}) \\ &= 0,05 + 0,45 = 0,5 \\ P(\textit{"30" is used}) &= P(\textit{rounded}) + P(\textit{didn't round}; \textit{30}) \\ &= 0,5 + 0,05 = 0,55 \end{aligned}$$

The other steps are straightforward. To get the chance the speaker rounded, take the chance he rounded and it was 30 and the chance he rounded and it was not and add them together. These are the ones in the top row, and the result is 50% again. For the chance a round number was used, we add to that the chance that the number was 30 and he did not round it, so we get 0,55.
Now we simple divide these, as per the formula. This gives

$$P(\textit{Speaker rounded}|\textit{"30" is used}) = \frac{P(\textit{both})}{P(\textit{"30" is used})} = \frac{0,5}{0,55} = \frac{10}{11} > 90\%$$

Thus, while the chance of the speaker rounding was just 50%, the chance that 30 was meant as round and should be interpreted like that is over 90%.

For the general picture, we replace our 50% chance by x, use an arbitrary round number R, and let k be the number of numbers that could be rounded to it (i.e.

[1] Keep in mind that when the actual number is exactly 30, "rounding" it still makes a difference: 30 meant sharply is not the same as 30 meant in a loose way that encompasses nearby numbers. Note also that the hearer cannot simply hear the difference between the two; indeed, figuring out how the hearer best deals with that is the point here.

10 in the above example). As mentioned before, the exact values of x and k will prove not to be too important.[2]

	Actually R	Merely close to R
Speaker rounded	$x\frac{1}{k}$	$x\frac{k-1}{k}$
Speaker didn't round	$(1-x)\frac{1}{k}$	$x\frac{k-1}{k}$

$$
\begin{aligned}
P(\textit{Speaker rounded}) &= P(\textit{rounded}; 30) + P(\textit{rounded}; \textit{not 30}) \\
&= x\frac{1}{k} + x\frac{k-1}{k} = x \\
P(\textit{"R" is used}) &= P(\textit{rounded}) + P(\textit{didn't round;}\mathrm{R}) \\
&= x + (1-x)\frac{1}{k} = \frac{k-1}{k}x + \frac{1}{k}
\end{aligned}
$$

Given these probabilities, the chance the speaker meant the number R as round is as follows:

$$
P(\textit{Speaker rounded}|\textit{"R" is used}) = \frac{x}{\frac{k-1}{k}x + \frac{1}{k}} = \frac{kx}{(k-1)x+1} = \frac{k}{k-1+\frac{1}{x}}
$$

With k on the large side, this is going to be close to 1. The only problem is if x is low, but for that to get problematic it has to get low enough to be inversely proportional to k.

Thus, just by the mathematics of it understanding numbers as round is the correct choice far more often than one might expect. It would seem to be the **rational** interpretation –and indeed we will be able able to say this with more confidence after section 2.3.

And it would be wrong to think that this will stay limited to hearers only. If round numbers are likely to be interpreted as such, a speaker is likely to anticipate and modify a round number if he actually means it non-roundly. But that makes round interpretation even more rational, since participants can expect this anticipation. This creates a self-reinforcing loop that makes round numbers get interpreted more and more as simply having a round meaning; in appropriate contexts, at any rate.

2.3 Game Theory

For the next part, we are going to look more closely into the rationality angle. The previous question was necessarily a bit indirect; but Game Theory is based on concepts like strategies and making the rational choice between them. Thus, it allows us to specifically ask *When is it rational to assume a round number was*

[2]See Appendix 2.5 for a treatment on how to generalize away from the discrete scale and even probability distribution.

meant roundly?, and to get an exact answer in the form of a value x has to exceed (where, as before, x is the chance of the speaker rounding). Furthermore, we will also be able to find out the exact importance of contextual factors.
To answer this question, Game Theory works by assigning so-called utility values to understanding and misunderstanding each other. Each outcome gets a value: the higher it is the better for everyone involved. These are just numbers, like the example values below. Each of the two hearer strategies then has an expected utility depending on the other player, and round interpretation simply is rational if the expected utility is higher than for non-round interpretation.

For this example, suppose the speaker has asked the hearer to show up for an appointment at 2 o'clock. This could be meant sharply, or could be meant to allow about five minutes either way. Obviously it would be preferable for the hearer to correctly understand the speaker's intent, so these outcomes get a higher value than the rest. We also assume that a greater need for precision gives rise to some inconvenience for one or both parties, so the correctly interpreted strict appointment has a slightly lower score.
Furthermore, showing up sharply on a loosely meant appointment is obviously not as bad as taking a sharply meant appointment loosely, so the values are fixed accordingly.[3,4]

	Round interpretation	*Non-round int*
Round intention	3	1
Non-round intention	0	2

Now as before we are interested in the hearer's point of view and simply let x be the chance that the speaker will round a given number. The better strategy is picked by maximizing expected utility, so round interpretation is rational if and only if

$$\begin{aligned} & P(\textit{Round intention}) \cdot 3 + P(\textit{Non-round intention}) \cdot 0 \\ > \; & P(\textit{Round intention}) \cdot 1 + P(\textit{Non-round intention}) \cdot 2 \end{aligned}$$

Filling in x, this becomes

$$3x + 0(1-x) > 1x + 2(1-x)$$

which simplifies to $2x > 2(1-x)$ which is if and only if $x > \frac{1}{2}$. This result does not actually look all that good, but there is something very important being

[3] There will also be some convenience in the fact that $3 - 1 = 2 - 0$, but this is not part of the story.

[4] Note that while the choice of payoffs here is convenient, it does *not* itself offer an advantage to round interpretation, as should become clear from the calculations as well as the generalized case later one.

overlooked here.
The thing we are overlooking is not unlike the condition we posed earlier. Essentially, if the speaker uses a non-round number, there is no way it can be misinterpreted as round. So the real strategies the hearer chooses from are not round and non-round interpretation; they are to interpret roundly if a round number is used or to never interpret roundly. This changes the analysis considerably.

	Round int [if a round number]	*Non-round int*
Round intention	3	1
Non-round intention	1, 8	2

In the lower-left corner instead of 0 we get 0-if-it's-round-and-two-if-it-isn't. That comes out to $0 \cdot \frac{1}{10} + 2 \cdot \frac{9}{10} = 1,8$.[5,6] This makes round interpretation look a lot better, yielding all the advantage and only a fraction of the disadvantage. As the calculation below shows, x need only be $\frac{1}{11}$ for round interpretation to be

[5]Assuming we are being precise to the minute, resulting in what amount to a $k = 10$ as before.

[6]Readers trying to interpret in terms of signaling games should note that the type t has two independent parameters here: one is the preferred time (even distribution over ten options), the other is the importance of showing up on the minute, which also governs the payoffs. The latter has a probability of x of corresponding to the upper row and $(1 - x)$ of corresponding to the lower one.
Now formalize as follows:

$$\begin{aligned} t_{1i} &: \textit{preferred time is 14.00} \\ t_{2i} &: \textit{preferred time not 14.00} \\ S_1 &: t_{1i}, t_{2i} \rightarrow \textit{"two o-clock"} \\ S_2 &: t_{1i} \rightarrow \textit{"two o-clock"} \\ &\quad\; t_{2i} \rightarrow \textit{specific other time} \\ H_1 &: \textit{"two o-clock"} \rightarrow \textit{interpret as round} \\ &\quad\; \textit{specific other time} \rightarrow \textit{interpret as precise} \\ H_2 &: \textit{any} \rightarrow \textit{interpret as precise} \end{aligned}$$

That it is rational for the sender to pick S_1 iff showing up on the minute is unimportant is left to the reader. Given this relationship the second parameter and the sender's strategy are both governed by x, and the rest of the analysis follows.

rational.

$$\begin{aligned}
3x + 1,8(1-x) &> x + 2(1-x) \\
3x + 1,8 - 1,8x &> x + 2 - 2x \\
1,2x + 1,8 &> 2 - x \\
2,2x + 1,8 &> 2 \\
2,2x &> 0,2 \\
x &> \frac{0,2}{2,2} = \frac{1}{11}
\end{aligned}$$

The general picture again is similar. In the general case we use not specific numbers but the following arbitrary game:

	Round interpretation	*Non-round int*
Round intention	a	b
Non-round intention	c	d

Any good example will of course have $a > b$ and $d > c$, but the numbers are otherwise open to be chosen freely. Of course, as before the factor k marginalizes the difference between c and d, so that this arbitrary game is transformed into the following actual game:

	Round int [if a round number]	*Non-round int*
Round intention	a	b
Non-round intention	$d - \frac{d-c}{k}$	d

The condition for round interpretation to be rational thus becomes

$$\begin{aligned}
ax + \left(d - \frac{d-c}{k}\right)(1-x) &> bx + d(1-x) \\
(a-b)x &> \frac{d-c}{k}(1-x) \\
(a-b)kx &> (d-c)(1x) \\
((a-b)k + (d-c))x &> d-c \\
x &> \frac{d-c}{(a-b)k + (d-c)}
\end{aligned}$$

Thus because of the generally largish k at the bottom, x can safely be quite small. Usually the breaking point is where it gets inversely proportional to k. If $(d-c) = (a-b)$ (that is, if the cost for misunderstanding is the same either way) then x need only be as little as $\frac{1}{k+1}$ for round interpretation to be the rational choice.

Now context can matter a lot, and that will work its way into what a, b, c and d really are, but clearly the factor k strongly pushes things towards round interpretation.

2.4 Discussion

This chapter shows that even a weak inclination to round can be enough to explain why rounding is [rationally] assumed: even if the chance the speaker chooses to round is low, round interpretation is still likely to be rational, and then people adapt and it gets more and more standard until it is a standard meaning. Roundness is a rational and natural outcome.

It does not purport to −and cannot− explain why speakers should have even a small inclination to round to begin with, but in this it should be favorably combinable with existing arguments focusing on the speaker side or on inherent benefits to rounding (eg arguments from irrelevance, high cost of precision, uncertainty on the part of the speaker, manipulation or mental restrictions). Such other arguments need no longer account for a preference for rounding, just for a sufficiently significant probability.

It also does not go into why such inclinations are limited to "round" numbers. In my opinion that matter is better dealt with through other methods of investigation, eg (Dehaene and Mehler 1992, Jansen and Pollmann 2001).

2.4.1 Generalization to Vagueness

Generalizing the results about round numbers to vagueness is often surprisingly straightforward. While vagueness doesn't have much to do with numbers as such, vague terms often do have an underlying scale that's numerical –or an underlying situation that is easily numerizable, so that the same arguments apply.

This is most clearly seen with absolute adjectives (using the term absolute adjective as used in (Kennedy 2007)). Take for example the word "bald". Loose use of the strictest sense of the word could be interpreted as rounding the number of hairs to zero. But then, given the number of hairs on a normal person's head, the k −the number of hairs that can be rounded to zero− for this situation can easily be in the hundreds or even thousands. The required prior chance of rounding x is thus so low that it can be accounted for even with just the various kinds of uncertainty. In this analysis that is obviously not a stable situation, so the word will quickly get used more and more loosely.

Importantly, this process does not stop. As soon as the meaning has changed (and stabilized), it is again subject to the same analysis. There is a slight difference in that more than one case counts as strictly bald now, but this can be accommodated by replacing k with a factor dividing the number of cases of the looser meaning by that of the new 'strict' meaning. k will be smaller and x may or may not change as well, but even looser interpretation is likely to be rational several more times, and further and further loosening will occur so long as this is so.

So just how loosely will it get used and where does the repeated loosening stop? That question gets hard to answer. Even if we and the people involved are pursuing a rational answer, just how loosely people should use and interpret the word soon depends on all kinds of factors nobody really knows; matters like how loosely everyone else is, should be, has been and should have been using it. Given that people might not use words equally loosely there will be much uncertainty and legitimate disagreement about such things, and this becomes more and more relevant as the process of loosening goes on. Eventually, the word becomes vague.[7] (Some people may prefer the following line of reasoning instead: if precise loose use is rational, there is also support for vague loose use, especially if people aren't actually capable of the former but can manage the latter. In this way we get a reduction of other vagueness to the vagueness inherent in loose use. When loosening stops, then, it is not so much because the term has become vague but because it has become vague enough/too vague, with further loosening making no difference: [current] vague terms are fixpoints of the loosening operator.)

What we have here then is a possible explanation for a lot of vagueness. Loose interpretation is often rational, this makes loose use become the norm over time, and therefore things eventually get vague.
There are a number of reasons to hypothesize that this is indeed the origin of much vagueness. The context-dependence of most vague terms can be explained in terms of the context-dependence of loose use. It also correctly predicts that vagueness occurs mostly for cases where there is an associated measurable property on a continuous or extremely fine scale, as these are the cases the argument is most naturally and easily applied to.[8] A number of vague terms do indeed have an associated "literal" or "absolute" meaning, e.g. "bald", "flat", "full".[9]
Furthermore, if we think absolute adjectives like "flat" and "full" as having prototypes, then the suggestion in prototype theory that the prototypes are by and large clear and universal across while the boundaries between concepts are not is consistent with an account where modern concepts are the result of repeated loosening of concepts that originally coincided with these prototypes far more

[7]There is also another possible reason, which I will not expand on here. If the loosenings of two related words start to overlap, the extensions may stop expanding there, since it remains more rational to use the "closer" word. Still, for the reasons above one would not expect the boundaries this results in to be sharp.

[8]Loose use can involve situations where no clear measurable property is involved –e.g. "I need a Kleenex." (where in fact any tissue would suffice) (Wilson and Sperber 2002)– but in such cases it cannot easily be argued that *repeated* loose use occurs often enough to achieve vagueness.

[9]In some cases, words that don't may have such a meaning at one point only for it to be evolved away or taken over by another word. See also the section on "very". Also, some vague terms may have evolved from other vague terms with the vagueness itself still coming about in the proposed way.
I wouldn't go so far as to propose that this process underlies *all* vagueness, though.

strictly. One example of such a suggestion is made in (Wierzbicka 1990) and supported in (Tribushinina 2008, p58-78)

When we are investigating a word like "bald", one might object that even if it is commonly used to refer to more than just an endpoint, the endpoint still remains and can be referred to with modifiers like "completely" and "absolutely". There would seem to be a difference between the loose use of absolute adjectives and the vagueness of other adjectives such as "tall". However, the section below outlines a big problem with such a view, further suggesting that repeated loosening can in fact produce vagueness.

On *very*, and the futility of remaximizing

It is well-known that many kinds of expressions can be vague, including adjectives, nouns, quantifiers and modifiers. This also includes the word "very", which may in fact be an even better example of this theory than "bald". I suggested just now that modifiers like "completely" and "absolutely" can refer to the endpoint of words like "bald", but is this really the case? In modern times nobody associates the word "very" with any specific endpoint. It is simply a strengthener. But in earlier centuries, they did. There is a paragraph about this in Elena Tribushinina's work(Tribushinina 2008) which is worth quoting at length.

> It is also worth noting that *extremely* is probably undergoing a semantic change from a maximizer to a booster. A similar development has taken place for *quite* and *very*. In the times of Chaucer, *quite* was only used in the sense of 'entirely' (e.g. *quite right*). The weaker sense of 'fairly' (as in *quite tall*) is attested from mid 19^{th} century (Paradis 1997: 74). Similarly, *very* originally meant 'true, genuine, really' (cf. Ger. *wahr*, Du. *waar*), and turned into a booster in the Middle English period (Cuzzolin & Lehmann 2004; Lorenz 2002; Mendez-Naya 2003; Peters 1994; Stoffel 1901).[10]

So as we can see here "very" originally meant something along the lines of "truly" or "completely", until it succumbed to the kind of pressures we have been talking about, which are also affecting "extremely", "totally", "completely" and pretty much every maximizer you can think of.[11] The phenomenon is well documented[12],

[10] The papers she cites are (Cuzzolin and Lehmann 2004), (Lorenz 2002), (Méndez-Naya 2003), (Peters 1994) and (Stoffel 1901), respectively.

[11] Indeed, many people have been annoyed at the way even "literally" gets (ab)used these days. From a discussion on the internet:

> A: I literally ROFL'd.
> B: You literally rolled over the floor laughing? Ouch.

People who understand both "literally" and "ROFL" can be hard to come by.

[12] See also (Ito and Tagliamonte 2003).

and is entirely natural and perhaps because of these arguments also fairly predictable.
And of course, if even "very" can turn out to have come about in this way, so can any other word.

2.4.2 Schelling Points and Evolutionary Game Theory; a problem?

During the course of writing this chapter it has come to my attention that Christopher Potts has done a related game-theoretical analysis on a related phenomenon. (Potts 2008) While his subject matter is different, one of his predictions contradicts an important one of my own. Before I mention how I account for this, a brief introduction of it is in order.
In (Potts 2008), Potts seeks to derive Kennedy's Interpretive Economy principle (Kennedy 2007), or rather, a substitute with the same practical consequences (in particular, solving Kennedy's puzzle) as that principle, from basic assumptions about cognitive prominence and evolutionary stability. This of course has little to do with general vagueness, much less round numbers, but his analysis would still be problematic for my own ideas discussed above.
Potts's argument rests on the notion that amongst the possible ways to interpret an adjective related to a scalar endpoint, the most strict one stands out as a so-called Schelling point, making it initially (at least marginally) more likely to be selected than other ways. The extent to which this is so is what he refers to as the strength of the "Schelling assumption". Insofar as the Schelling assumption is fairly weak, I will not argue against it here.
He then combines the Schelling assumption with evolutionary game theory, arguing that even a slight preference will result in strict interpretation becoming standard. This is a fairly straightforward application of evolutionary game theory, and I will mostly not argue against it either.
However, it does go against my own notions: in Section 2.4.1 in particular I argued that the evolution is likely to go the other way around, with vague words possibly being a result of repeated loosening of previously much sharper words. So how do I account for this? Naturally, the answer lies in doing what I have been doing in this chapter.
Potts's most important analysis starts from the following basic game:

	$[\![\textbf{full}]\!].$	$[\![\textbf{full}]\!]_d$
$[\![\textbf{full}]\!].$	10	9.9
$[\![\textbf{full}]\!]_d$	9.9	10

In this example, $[\![\textbf{full}]\!].$ represents the maximum (ie sharp) interpretation of "full" while $[\![\textbf{full}]\!]_d$ represents a looser interpretation. In order to let this conform more to the examples I have been using myself I will flip the table here, as follows:

	$[\![\textbf{full}]\!]_d$	$[\![\textbf{full}]\!]_.$
$[\![\textbf{full}]\!]_d$	10	9.9
$[\![\textbf{full}]\!]_.$	9.9	10

Now using evolutionary mechanics Potts shows that when a coordination game like this is repeated, even a very weak Schelling assumption will make the population evolve towards overwhelmingly favoring the Schelling point –in this case strict interpretation.

There is nothing inherently wrong with this analysis, except that it ignores the point I have been making in this chapter. Stay with this example, a loose usage of the word "full" can be used in more situations than strict use. Following the analyses of this chapter, we should assign a discrete scale or use the continuous analysis in Appendix 2.5 to find the appropriate number k for the amount/ratio of situations sufficiently close to be loosely referred to as "full".[13]

Assuming either an even distribution or one taken included as part of k as per Appendix 2.5, we should then follow Section 2.3 and replace the 9.9 in the lower-left by $9.9 \cdot \frac{1}{k} + 10 \cdot \frac{k-1}{k} = 10 - \frac{0.1}{k}$, thus replacing the basic game above by the following:[14]

	$[\![\textbf{full}]\!]_d$	$[\![\textbf{full}]\!]_.$
$[\![\textbf{full}]\!]_d$	10	9.9
$[\![\textbf{full}]\!]_.$	$10 - \frac{0.1}{k}$	10

By the math in the earlier Section 2.3, it follows that loose interpretation is rational if $x > \frac{1}{k+1}$. In this example the population distribution provides this x, and if loose interpretation is rational at the initial time t_0 it will only get more so, so the condition for loose interpreation to be the end result of evolution becomes $P^{t_0}([\![\textbf{full}]\!]_d) > \frac{1}{k+1}$. Therefore a weak Schelling assumption (where it suffices for $P^{t_0}([\![\textbf{full}]\!]_.)$ to be just barely higher than 50%) is nowhere near enough. To win, strict use would have to start out at more than $\frac{k}{k+1}$.

Given everything I've argued here, a factor benefiting strict use needs to be strong, not merely minimal, to be of much use against the k factor.

[13] The value of k in this depends on what specific d is being used, but since the stricter reading consists of a single point it depends even more on how fine the scale is. Indeed, increasingly fine scales can render k arbitrarily high.

[14]or in Potts's notation,

	$[\![\textbf{full}]\!]_.$	$[\![\textbf{full}]\!]_d$
$[\![\textbf{full}]\!]_.$	10	$10 - \frac{0.1}{k}$
$[\![\textbf{full}]\!]_d$	9.9	10

2.5 Appendix: Continuous Scale and k on Probability

It has been convenient to use the simplifying assumption of a discrete scale, but it is straightforward enough and interesting to drop this notion, especially in light of the discussion in section 2.4.1.
Starting from the general case scenario in Section 2.2, let R be some round number and as before let x be the prior chance that a sufficiently close number will be rounded to it. Let C be a set of real numbers sufficiently close to R to be rounded to it in this fashion. In order to avoid dividing by zero later on, we also let $A \subset C$ be a set of numbers so close to R as to be considered identical, or at least indistinguishable.[15]
Now let $B = C - A$, assume that the actual number is picked randomly with probability distributed evenly over C, and assume that $|\cdot|$ is an appropriate measure on $\mathbb{R}$.[16] Then we can "divide out"/ignore the probability part to obtain the following familiar-looking table:

	Actually R	Merely close to R
Speaker rounded	$x\|A\|$	$x\|B\|$
Speaker didn't round	$(1-x)\|A\|$	$(1-x)\|B\|$

I have not yet mentioned how k should be defined here, but by looking at the table it should surprise no one that the definition is simply $k = \frac{|C|}{|A|} = \frac{|A|+|B|}{|A|}$[17]. This leads to the following:

$$\begin{aligned} P(\textit{Speaker rounded}|\text{"}R\text{" is used}) &= \frac{x|A| + x|B|}{x|A| + x|B| + (1-x)|A|} = \frac{|A| + |B|}{|B| + |A|/x} \\ &= \frac{(|A| + |B|)/|A|}{\left(\frac{|A|+|B|}{|A|} - \frac{|A|}{|A|}\right) + 1/x} = \frac{k}{k - 1 + \frac{1}{x}} \end{aligned}$$

which is of course the same result as in the discrete case.
Taking the probability distribution out in this way may seem suspect, and in any case it is interesting to consider the impact of non-even distributions. The resulting formula threatens to get convoluted, but this is easily avoided through cheating: redefine k as

$$k = \frac{P(A \cup B)}{P(A)}$$

[15] Of course in more general situations A may also simply be whatever R refers to sharply, so long as that has non-zero measure.

[16] In the more general case, pick an appropriate measure on at least C.

[17] In the general case, the equality obtains because A and B are disjoint and we picked an appropriate measure function.

Then it is clear that we can just combine area and distribution into probability to get the following table:

	Actually R	Merely close to R
Speaker rounded	$xP(A)$	$xP(B)$
Speaker didn't round	$(1-x)P(A)$	$(1-x)P(B)$

Thus the results are exactly as before[18,19] except that now the effect of a change in probability distribution is a straightforward impact on k: for instance, the k in the above example could end up much fairly small if the distribution were a bell curve around R, with details depending on σ and the size of A.

[18]In this case the equality $P(A \cup B) = P(A) + P(B)$ follows from the laws of probability.

[19]Reobtaining the exact results from the sections involving game theory is not too difficult –and left to the reader.