Chapter 3.

The Square Universe.

Outline.

Giving a detailed treatment of two-dimensional modal logics, this chapter forms the pivot of the thesis.

In the introduction we give an overview of the literature on two-dimensional modal logics; we also provide a uniform, technical framework for the topic. Section 2 deals with a rather simple formalism, two-dimensional cylindric modal logic. In section 3 we give an extensive account of the modal counterpart $CC\text{5}$ of relation algebras. In section 4 we add a temporal component to the formalism $CC\text{5}$, obtaining a two-dimensional temporal logic. The last but one section shows what results in algebraic logic our modal approach yields. We finish in section 6 with conclusions, and questions for further research.
3.1 Introduction.

3.1.1. Why.

There are several reasons for developing a framework of modal logic in which the possible worlds are pairs of elements of the model instead of the points themselves, and in the literature one can find similar ideas arising in this direction, from different backgrounds, sometimes in quite different disciplines.

First, in tense logic there is a research line inspired by linguistic motivations. In ordinary tense logic as developed by Prior (cf. [100]), the truth of a formula is only dependent of the form of the formula itself and the time when its truth is evaluated. However, temporal discourse has both a referential and a deictic side: the truth of a proposition may change not only with the point of reference, but also with the point of utterance by the speaker. In the seventies, the development of formal linguistics and the strive to provide a logical foundation for it, led people like Gabbay, Guenthner, Kamp, Segerberg and Åqvist to investigate two-dimensional temporal logics taking care of these phenomena, cf. for example [30, 33, 62, 119, 142]. So these papers all have a temporal aspect in common, e.g. models always having some ordering relation. For an overview we refer to Gabbay [35], in section 4 we will give some more details on two-dimensional temporal logic.

In order to provide a modal framework where the absolute (logical) and the relative (e.g. physical) necessity can be distinguished, both Humberstone [55] and van Fraassen [29] suggest a two-dimensional approach. Studying Humberstone’s system, Kuhn gives a technical account in [67] of the two-dimensional modal logic of the domino relation \((u, v)R(x, y) \iff v = x\).

More abstract is the approach of V. Shekhtman, whose motivation to study many-dimensional modal logics seems to come mainly from pure logic. In [120], his aim is to axiomatize the ‘Cartesian product’ of two modal logics. For example, the intended frames of the Cartesian product of \(S4\) and \(S5\) consists of a set of possible worlds of the form \(U_0 \times U_1\), with a reflexive transitive relation on \(U_0\) and an equivalence relation \(U_1\).

An earlier article in this direction is Davis [26].

Related frameworks with a two-dimensional flavour are branching-time temporal logics (cf. Zanardo [140]), combinations of modal and temporal logics (cf. Thomason [128]), and in computer science, formalizations of the behaviour of distributed systems (cf. Spaan [124]. In these three areas, the worlds in the intended frames can be seen as pairs consisting of a timepoint, together with respectively a branch of time, a state of affairs and a computation sequence.

In chapter 5 we will give a detailed account of how temporal logics of intervals fit in the two-dimensional picture.

Many of the systems described above may be perceived as following a general trend in modal logic, namely of bridging the gap between the old-fashioned intensional framework, which is simple, elegant and has nice computational properties, and classical first order logic which is expressive and perhaps still more familiar. Examples of other extensions
of the classical modal formalism are: Blackburn [19] and Gargov and Goranko [39] add ‘nominals’ or ‘names’ to the language, these being atomic propositions holding at unique possible worlds. Orłowska [92], Humberstone [56] and Goranko [44] enrich the system with new operators, e.g. one for having the complement of the accessibility relation of the original ⊗ as its accessibility relation. The $D$-operator discussed in the previous chapter can be seen as another example.

Let us have a closer look at this relation between modal logics and classical logic. It is well-known (cf. van Benthem [14]), that (on the model level) modal logic corresponds to a fragment of first order logic, and that there exists standard translations from modal logics to first order formulas. The co-domain of these translations is formed by a set of first order formulas having one free variable, in a language with a fixed set of accessibility predicates and arbitrarily many monadic predicate variables. One-dimensional extensions of the simple modal logic aim at reaching a larger set of first order formulas, but still these formulas have only monadic predicates (besides the accessibility predicates) and one free variable. In the above-mentioned two-dimensional extensions, these parameters are shifted too: the co-domain of the translation\(^1\) may now contain formulas in two free variables and with dyadic predicate variables.

These characteristics of two-dimensional modal logics are shared by the algebraic theory of binary relations (cf. Németi [89] for an introductory overview), which forms our last source motivating the study of two-dimensional modal logics. The analogies between algebras of binary relations and two-dimensional modal logics are striking but (as far as we know) they have never been made this explicit. We hope to do so in this chapter — in fact the idea can be put into one slogan:

Algebras of binary relations are the modal algebras of two-dimensional modal logics.

This idea forms our guideline in choosing the particular examples of two-dimensional logics that we will study.

### 3.1.2. How.

In this subsection we will acquaint the reader with the technicalities of our approach to many-dimensional modal logics. We will first give the general setting, then we mention two examples known from the literature. So, to start with the general idea, we need:

**Definition 3.1.1.**

Consider the similarity type $S_2$ with a modal constant $\delta$, the following monadic operators: $\Diamond, \Diamond', \Diamond'', \Diamond^*, \Diamond^\ast$, and the dyadic operator $\circ$.

By Appendix A.4.5 we have a definition of a semantics for $S_2$ and its subtypes. The intended semantics for $S_2$ however has a two-dimensional character, the set of possible

\(^1\)Note that for two-dimensional logics, this translation (cf. section 3.3.3 for details) is not the same as the standard correspondence map as defined for the similarity type.
worlds consisting of a Cartesian square and the interpretation map having a fixed and uniform definition for this kind of universes.

Definition 3.1.12.
A two-dimensional frame or a square is a frame \( \mathfrak{A} = (W, I) \) where \( W = U \times U \) for some set \( U \), and the definition of \( I \) is given as follows:

\[
\begin{align*}
I(\delta) &= \{ (u, v) \mid u = v \} \\
I(\varnothing) &= \{ ((u, v), (x, y)) \mid v = y \} \\
I(\Phi) &= \{ ((u, v), (x, y)) \mid u = x \} \\
I(\Phi') &= \{ ((u, v), (x, y)) \mid u \neq x, v = y \} \\
I(\Phi') &= \{ ((u, v), (x, y)) \mid u = x, v \neq y \} \\
I(\Phi) &= \{ ((u, v), (x, y)) \mid u = y \} \\
I(\Phi) &= \{ ((u, v), (x, y)) \mid v = x \} \\
I(\Phi) &= \{ ((u, v), (x, y)) \mid u = y, v = x \} \\
I(\Phi) &= \{ ((u, v), (x, y)) \mid u = x = y \} \\
I(\Phi) &= \{ ((u, v), (x, y)) \mid v = x = y \} \\
I(\Phi) &= \{ ((u, v), (x, y), (w, x), (y, z)) \mid u = w, v = z, x = y \}.
\end{align*}
\]

A two-dimensional or square model is a model based on a two-dimensional frame.

We are interested in the following questions for subtypes \( S \) of \( S_2 \):

1. Can we distinguish the squares among the (abstract) \( S \)-frames? And if so, in which language \( (M_S \) or \( L_S \) and how exactly?
2. Can we give a derivation system generating the \( S \)-formulas valid in the class of two-dimensional frames?
3. Which fragment of first order logic does the set of \( S \)-formulas capture?

A nice thing about two-dimensional modal logic is that structures can be represented geometrically, in a very intuitive way. Let \( \mathfrak{M} \) be a square model, and \((x, y)\) a world in \( \mathfrak{M} \), then

\[\mathfrak{M}, x, y \models \Phi \phi \iff \text{there is a } z \text{ in } U \text{ with } \mathfrak{M}, z, y \models \phi\]
\[\mathfrak{M}, x, y \models \Phi \phi \iff \text{there is a } z \text{ in } U \text{ with } \mathfrak{M}, x, z \models \phi\]
\[\mathfrak{M}, x, y \models \delta \iff x = y\]

viz.

\[\Phi \phi \quad \phi \]
\[\Phi \phi \]
\[\delta\]

\[^2\text{We do not consider the case where the universe is of the form } U_0 \times U_1 \text{ with possibly different base sets } U_0 \text{ and } U_1. \text{ For this wider case an analogous theory can be developed.}\]
We do not show the pictures of the irreflexive versions $\otimes', \odot'$ of $\otimes, \odot$; for the picture of the dyadic operator $\circ$, we refer to section 3.3.2.

In the literature some subsystems of $D_2$ have been studied explicitly: we mention the similarity type $SEG$, with operators $\{\otimes, \odot, \otimes, \odot\}$. It was first studied by Segerberg in [119], where he gave a finite axiomatization of the $SEG$-formulas valid in the class of two-dimensional frames. He also showed that this validity problem is decidable, and he introduced the intuitive symbols $\ominus, \odot$, etc. Kuhn treated the $\{\otimes\}$-fragment $KU$ in a paper [68], providing a complete derivation system with infinitely many derivation rules and a non-$\xi$ rule. In Venema [135] it was shown, that adding the inverse operator $\ominus$ to $KU$, one can give a finite and orthodox axiomatization.

Finally, a remark about conventions: as the interpretation of the operators in $D_2$ is uniformly defined for all squares, we usually neglect mentioning the interpretation when referring to a two-dimensional frame. Also, par abus de notation, we will not write $\mathfrak{M} = (U \times U, V)$ but $\mathfrak{M} = (U, V)$ to denote a two-dimensional model. The purpose of this is that now models can be seen as structures for classical first order logic with dyadic predicates: every modal propositional variable, corresponding to a dyadic predicate in the first order logic, is indeed interpreted as a binary relation over $U$. 
3.2 Two-dimensional cylindric modal logic.

3.2.1. Two-dimensional cylindric modal logic.

In this section we will focus our attention on a particular two-dimensional modal logic:

**Definition 3.2.1.**
Consider the similarity type of cylindric modal logic of dimension 2 \( CML_2 = \{ \Theta, \Phi, \delta \} \), for which we introduce some auxiliary terminology. A \( CML_2 \)-frame is called a 2-frame and usually represented as a quadruple \( F = (W, H, V, D) \) where \( H, V \) and \( D \) are the accessibility relations of respectively \( \Theta, \Phi \) and \( \delta \). A 2-model is a model based on a 2-frame. Two-dimensional frames (squares) have been defined in 3.1.2. The class of \( CML_2 \)-squares is denoted by \( C_2 \).

For a 2-formula \( \phi \), its mirror image \( \phi^m \) is obtained by replacing all occurrences in \( \phi \) of \( \Theta \) by \( \Phi \) and vice versa.

The symbols \( H, V \) and \( D \) are mnemonics for the horizontal resp. vertical accessibility relation and the diagonal, cf. the pictures in the introduction. Note that in two-dimensional models, the projection operators \( \Theta, \Phi \), and the domino operators \( \Theta, \Phi \) can be defined as abbreviations in \( CML_2 \). This is not the case for the irreflexive versions \( \Theta', \Phi' \) of \( \Theta, \Phi \), but then, strangely enough, we do have a definition of the D-operator (cf. section 2.4) in our language:

**Definition 3.2.2.**
We use the following abbreviations:

- \( \Phi \phi = \Theta \Phi \phi \), \( Z_H \phi = \delta \Theta (\delta \wedge \Phi \phi) \)
- \( \Theta \phi = \Theta (\delta \wedge \phi) \), \( Z_V \phi = \Theta \Phi (\delta \wedge \Theta \phi) \)
- \( \Phi \phi = \Phi (\delta \wedge \phi) \), \( D_2 \phi = Z_H \phi \vee Z_V \phi \).

**Proposition 3.2.3.**
Let \( \mathfrak{M} = (U, V) \) be a two-dimensional model, \( u, v \in U \). Then

(i) \( \mathfrak{M}, u, v \models \Theta \phi \iff \) there are \( x, y \) in \( U \) with \( \mathfrak{M}, x, y \models \phi \),
(ii) \( \mathfrak{M}, u, v \models \Phi \phi \iff \mathfrak{M}, v, v \models \phi \),
(iii) \( \mathfrak{M}, u, v \models \Theta \phi \iff \mathfrak{M}, u, u \models \phi \),
(iv) \( \mathfrak{M}, u, v \models D_2 \phi \iff \) there is a \( (u', v') \neq (u, v) \) with \( \mathfrak{M}, u', v' \models \phi \).

**Proof.**
We only treat (iv). First we show

\[
\mathfrak{M}, u, v \models Z_H \phi \\
\iff \mathfrak{M}, u, v \models \Theta (\delta \wedge \Phi \phi) \\
\iff \mathfrak{M}, u, u \models (\delta \wedge \Phi \phi) \\
\iff \text{there is a } u' \text{ with } \mathfrak{M}, u', u \models (\delta \wedge \Phi \phi) \\
\iff \text{there is a } u' \neq u \text{ with } \mathfrak{M}, u', u \models \Phi \phi \\
\iff \text{there are } u', w \text{ with } u' \neq u \text{ and } \mathfrak{M}, u', w \models \phi
\]
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So, \( u, v \models Z_y \phi \) iff there is a point with a different first coördinate where \( \phi \) holds. Likewise, \( u, v \models Z_y \phi \) iff there is a point with a different second coördinate where \( \phi \) holds. As two pairs of points are different iff (at least) one of their coördinates is different, this implies that \( D_2 \) indeed has the inequality relation as its accessibility relation, in two-dimensional frames. 

We chose to study the fragment \( CML_2 \) of \( S_2 \) in order to have the important two-dimensional cylindric algebras (cf. Henkin-Monk-Tarski [53]) as (a subclass of) our modal algebras; this connection however will not be studied before section 3.5.

As was already mentioned in the introduction, we are interested in the connection between 2-frames and two-dimensional frames. It is not very hard to give a characterization of two-dimensional frames in the first order language \( L_{CML_2} \), and as the analogous task for an extension of \( CML \) will be undertaken in the next section, we omit it here.

The axiomatization problem is much more interesting, and the remainder of this section will be devoted to it. We propose the following axioms:

Definition 3.2.4.
Consider the following \( CML_2 \)-formulas:

\[
\begin{align*}
(\text{CH1}) & \quad p \rightarrow \Diamond p \\
(\text{CH2}) & \quad p \rightarrow \Box \Diamond p \\
(\text{CH3}) & \quad \Diamond \Diamond p \rightarrow \Diamond p \\
(\text{CH4}) & \quad \Diamond \Diamond p \rightarrow \Diamond \Diamond p \\
(\text{CH5}) & \quad \Diamond \delta \\
(\text{CH6}) & \quad \Diamond (\delta \land p) \rightarrow \Box (\delta \rightarrow p) \\
(\text{CH7}) & \quad (\delta \land \Diamond (\neg p \land \Diamond p)) \rightarrow \Diamond (\neg \delta \land \Diamond p)
\end{align*}
\]

The mirror image of \( (CHi) \) is denoted by \( (CVi) \), \( (CHVi) \) denotes \( (CHi \land CVi) \).

All these formulas are in Sahlqvist form. It is convenient to have names for their Sahlqvist correspondents:

Definition 3.2.5.
Define the following \( L_{CML_2} \)-formulas:

\[
\begin{align*}
\text{(NH1)} & \quad Hxx \\
\text{(NH2)} & \quad \forall y (Hxy \rightarrow Hyx) \\
\text{(NH3)} & \quad \forall y \forall z ((Hxy \land Hyz) \rightarrow Hxz) \\
\text{(NH4)} & \quad \forall y (Hxy \land Vuy) \rightarrow \exists v (Vxv \land Hvy) \\
\text{(NH5)} & \quad \exists y (Hxy \land Dy) \\
\text{(NH6)} & \quad \forall y' ((Hxy \land Hxy' \land Dy \land Dy') \rightarrow y = y') \\
\text{(NH7)} & \quad \forall y (Dx \land Hxx \land Vuy \land u \neq y) \rightarrow \exists v (Vxv \land \neg Dv \land Hvy)).
\end{align*}
\]

The mirror image of \( (NHi) \) is denoted by \( (NVi) \), \( NHVi = NHi \land NVi \).

So, \( NH1, NH2 \) and \( NH3 \) express that \( H \) is respectively reflexive, symmetric and transitive; together they state that \( H \) is an equivalence relation. \( NH5 \) and \( NH6 \) then mean that in every \( H \)-equivalence class there is exactly one element on the diagonal \( D \) (\( NH5 \) for existence and \( NH6 \) for unicity). The meaning of \( NH4 \) and \( NH7 \) is best made clear by the following pictures:
For these formulas, the correspondence part of Sahlqvist’s theorem means the following:

**Theorem 3.2.6.**

Let $\mathfrak{s}$ be a 2-frame, then for $i = 1, \ldots, 7$, $Z \in \{H, V\}$:

$$\mathfrak{s}, w \models CZi \iff \mathfrak{s} \models NZi[x \mapsto w].$$

**Proof.**

The claim is immediate by the fact that $(CZi)^* = (NZi)$ and theorem 2.2.2. We give some details of the computation of the Sahlqvist equivalent $(CH7)^*$ of $CH7$, as defined in 2.3.13. Clearly we have $\mathfrak{s}, w \models CH7$ iff

$$\mathfrak{s}, w \models \forall P \quad [(Dx_0 \land \exists x_1(Hx_0x_1 \land \neg Px_1 \land \exists x_2(Vx_1x_2 \land Px_2))$$

$$\quad \quad \quad \quad \quad \quad \quad \rightarrow \exists x_1(Vx_0x_1 \land \neg Dz_1 \land \exists x_2(Hz_1z_2 \land Pz_2))],$$

which is equivalent to

$$\mathfrak{s}, w \models \forall P \forall x_1 \forall x_2 \quad [(Dx_0 \land Hx_0x_1 \land \neg Px_1 \land Vx_1x_2 \land Px_2$$

$$\quad \quad \quad \quad \quad \quad \quad \rightarrow \exists x_1(Vx_0x_1 \land \neg Dz_1 \land \exists x_2(Hz_1z_2 \land Pz_2))].$$

Then the previous statements are equivalent to

$$\mathfrak{s}, w \models \forall P \forall x_1 \forall x_2 \quad [(Dx_0 \land Hx_0x_1 \land Vx_1x_2 \land \forall y(x_2 = y \rightarrow Py))$$

$$\quad \quad \quad \quad \quad \quad \quad \rightarrow (Px_1 \lor \exists z_1(Vx_0z_1 \land \neg Dz_1 \land \exists x_2(Hz_1z_2 \land Pz_2))).$$

So by definition 2.3.13 we obtain $(CH7)^*$ by substituting $v = x_2$ for $Pv$ everywhere in the above formula (the ‘minimal’ substitution making the antecedent true). This gives

$$\mathfrak{s}, w \models \forall x_1 \forall x_2 \quad [(Dx_0 \land Hx_0x_1 \land Vx_1x_2)$$

$$\quad \quad \quad \quad \rightarrow (x_1 = x_2 \lor \exists z_1(Vx_0z_1 \land \neg Dz_1 \land \exists x_2(Hz_1z_2 \land z_2 = x_2))],$$

or, equivalently

$$\mathfrak{s}, w \models \forall x_1 \forall x_2 \quad [(Dx_0 \land Hx_0x_1 \land Vx_1x_2 \land x_1 \neq x_2)$$

$$\quad \quad \quad \quad \rightarrow \exists x_1(Vx_0z_1 \land \neg Dz_1 \land Hz_1x_2)],$$
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which is NH7.

In fact, \((CH7)\) is the modal counterpart of (a simplified version of) Henkin's equation which plays an important rôle in the theory of cylindric algebras. For details we refer to subsection 3.5.2.

### 3.2.2. Two-dimensional cylindric completeness.

In this section we prove a completeness result for \(CML_2\): we will give a strongly sound and complete axiom system for the set of 2-formulas that are valid in the class \(C_2\) of squares.

**Definition 3.2.7.**

Let \(A_2\) be the basic axiom system \(K_{CML_2}\) for \(CML_2\) extended with the axioms \(CHV1, \ldots, CHV7\).

**Theorem 3.2.8.**

\(A_2\) is strongly sound and complete with respect to \(C_2\).

The proof of theorem 3.2.8 will consist of two parts: First we show that \(A_2\) is strongly sound and complete for the class of so-called hypercylindric frames, and then we show that these hypercylindric frames and the two-dimensional ones have the same modal theory.

**Definition 3.2.9.**

A 2-frame \(F\) is hypercylindric if \(F \models CHV1 \land \ldots \land CHV7\). The class of hypercylindric 2-frames is denoted by \(HCF_2\).

**Theorem 3.2.10.**

\(A_2\) is strongly sound and complete with respect to \(HCF_2\).

**Proof.**

An immediate consequence of the completeness part 2.2.2(ii) of Sahlqvist's theorem.

Note that for \(CML_2\) the notion of a zigzag morphism boils down to the following: let \(F\) and \(F'\) be two 2-frames, then a map \(f : W \rightarrow W'\) is a zigzag morphism if it satisfies the following properties:

1. \(f\) is a homomorphism
2. \(Du\) if \(D'f'\)
3. If \(H'fuv'\) then there is a \(v \in W\) such that \(Huv\) and \(fuv = v'\)
4. If \(V'fuv'\) then there is a \(v \in W\) such that \(Vuv\) and \(fuv = v'\)

We will call a map \(f\) satisfying the first two conditions a potential zigzag morphism.

The following lemma states that every hypercylindric frame is a zigzag morphic image of a disjoint union of two-dimensional frames. This immediately implies that \(\Theta(C_2) = \Theta(HCF_2)\).
Theorem 3.2.11.

\[ \text{HCF}_2 = \text{H}_f \text{P}_f \text{C}_2. \]

Proof.
Clearly every square is hypercylindrical, so \( \text{H}_f \text{P}_f \text{C}_2 \subseteq \text{HCF}_2. \)
For the other direction, observe that \( H[V] \) is an equivalence relation, in fact the accessibility relation of the S5-diamond \( \Phi \). Call a frame connected if this relation \( H[V] \) is total, nice if it is connected and hypercylindrical. It is an easy observation that every hypercylindrical frame is a disjoint union of nice frames, so it suffices to show that

every nice frame is a zigzagomorphic image of a square.

So, let \( \mathcal{F} = (W, H, V, D) \) be a nice frame. We will define a chain of potential zigzag morphisms \( (f_\xi)_{\xi < \lambda} \) (where \( \lambda \) is the maximum of \( |W| \) and \( \omega \)), such that the union \( f_\lambda \) of this chain is the desired zigzag. Every map \( f_\xi \) should be seen as an approximation of \( f_\lambda \).
Look at the set of potential defects \( P = \lambda \times \lambda \times W \times \{ \Phi, \Diamond \} \). Call the quadruple \( (\beta, \gamma, v, \Diamond) \in P \) a defect of a homomorphism \( f: \xi \times \xi \rightarrow W \) (where \( \xi < \lambda \)), if it defies one of the zigzag conditions (3) or (4), e.g. for (4): \( \Diamond = \Phi \), \( (\beta, \gamma) \in \xi \times \xi \) and \( Vf(\beta, \gamma)v \)
while there is no \( \eta' \in \xi \) such that \( f(\beta, \gamma') = v \); \( f \) is called perfect if it has no defects.
Assume that \( P \) is well-ordered, then we may speak of the first defect \( D(f) \) of an imperfect potential zigzag morphism \( f: \xi \times \xi \rightarrow W \). By the following lemma such a map has an extension \( f' \) lacking the defect \( D(f) \):

Claim.
Let \( f: \xi \times \xi \rightarrow W \) be a potential zigzag morphism, \( (\beta, \gamma, v, \Diamond) \) a defect of \( f \). Then there is a potential zigzag morphism \( f' \supset f \), \( f': (\xi + 1) \times (\xi + 1) \rightarrow W \) such that \( (\beta, \gamma, v, \Diamond) \) is not a defect of \( f' \).

Proof.
Without loss of generality we assume that \( \beta = \gamma = 0 \) and \( \Diamond = \Phi \).
We first set

\[
\begin{align*}
    f'(\zeta, \eta) &= f(\zeta, \eta) \text{ for } \zeta, \eta < \xi, \\
    f'(0, \xi) &= v,
\end{align*}
\]

viz. the left picture on the next page (where we denote \( f'(\zeta, \theta) \) by \( f'_\theta \)).
Next we are concerned with the \( f'(\eta, \xi), \eta < \xi \). By assumption we have \( v \neq f(0, \eta) \), and as \( f \) is a potential zigzag morphism we get a situation as showed in the right picture. By \( \mathcal{F} \models NV7(f'(\eta, \eta)) \), \( \mathcal{F} \) has a \( v_\eta \notin D \) with \( Hv_\eta v \) and \( Vv_\eta f'(\eta, \eta) \). We define

\[ f'(\eta, \xi) = v_\eta. \]

and set \( f'(\xi, \xi) \) as the unique diagonal \( H \)-successor of any/all of the \( f'(\eta, \xi) \).
It is straightforward to verify that with this definition the part of \( f' \) defined up till now satisfies both conditions (1) and (2).
For the definition of \( f^*(\xi, \eta) \) (\( \eta < \xi \)), we use the same trick as above to ensure \( f^*(\xi, \eta) \notin D \): as \( f^*(\xi, \xi) \) is in \( D \) and \( f^*(\eta, \xi) \) is not, they cannot be identical. So \( f^*(\eta, \xi) \) can be defined as any non-diagonal \( H \)-successor of \( f^*(\eta, \eta) \) which is a \( V \)-successor of \( f^*(\xi, \xi) \) (such a \( f^*(\xi, \eta) \) exists by \( NH7 \)).

We now define the chain of maps as follows:

\[
\begin{align*}
    f_0 &= \{((0,0), u)\} \quad \text{for some } u \text{ on the diagonal of } F.
    f_{\xi+1} &= \{ f_\xi \} \quad \text{if } f_\xi \text{ is perfect.}
    f_\theta &= \bigcup_{\xi < \theta} f_\xi \quad \text{otherwise}
\end{align*}
\]

It is now straightforward to verify that \( f_\lambda \) has the desired properties: first of all it is a potential zigzag morphism as all the maps in the chain are. Suppose that \( f_\lambda \) is not a zigzag morphism, then there are quadruples in \( P \) witnessing this shortcoming. Let \( \pi = (\beta, \gamma, v, \bowtie) \) be the first of these in the well-ordering of \( P \), suppose its ordinal number is \( \eta \). Take \( \theta = \text{max}(\beta+1, \gamma+1) \), then \( \pi \) is a defect of \( f_\theta \). It need not be its first one, but there can be at most \( \eta \) problems before \( \pi \) that are more urgent. So \( \pi \) must be the first defect of \( f_{\theta+\eta} \), whence it can not be a defect of \( f_{\theta+\eta+1} \). But this gives a contradiction, since \( f_{\theta+\eta+1} \subseteq f_\lambda \). So \( f_\lambda \) is a zigzag.

Finally, the proof that \( f_\lambda \) is surjective is straightforward by the connectedness of \( G \).

**Proof of theorem 3.2.8.**

Immediate by 3.2.10 and 3.2.11.
3.3 A Modal Logic of Binary Relations.

In this rather large section we treat a second two-dimensional logic, in more detail. Instead of giving an overview, we let the titles of the subsections speak for themselves.

3.3.1. (Representable) Relation Algebras.

In the algebraic theory of binary relations (cf. Németi [89] for an overview), one studies operations on the set of binary relations.

**Definition 3.3.1.**

Let $U$ be some unspecified set. $Re(U)$ is defined as the set of all binary relations on $U$, i.e. $Re(U) = \{ R \mid R \subseteq U \times U \}$.

The composition $R \mid S$ of two relations $R$ and $S$ is defined by

$$R \mid S = \{ (s, t) \in U \times U \mid \exists u ((s, u) \in R \land (u, t) \in S) \},$$

the converse of a relation $R$ is

$$R^{-1} = \{ (s, t) \in U \times U \mid (t, s) \in R \}$$

and finally the diagonal is the relation

$$Id = \{ (s, t) \in U \times U \mid s = t \}.$$  

Although Tarski [126] was not the first one to suggest an algebraic treatment to the subject (cf. Maddux [80] for a historical introduction), his approach set the standard:

**Definition 3.3.2.**

A relation type algebra is defined as a Boolean Algebra with the following operators: a binary $\cup$, a unary $^c$ and a constant 1'.

The class FRA of full relation algebras consists of those relation type algebras that are isomorphic to an algebra of the form

$$Re(U) = (Re(U), U, ^c, |, -1, Id).$$

The class RRA of representable relation algebras is defined as the variety generated by FRA, i.e. $RRRA = V(FRA)$.  

The question naturally arises as to study the representable relation algebras. By applying some elementary universal algebra (cf. Burris-Sankappanavar [24]), we obtain $RRRA = HSP(FRA)$ and by Birkhoff's theorem we have $\mathfrak{a}$ is in $RRRA$ iff all the equations holding in FRA are true in $\mathfrak{a}$. Tarski proved that every representable relation algebra is a subalgebra of a direct product of full relation algebras, i.e. $RRRA = SP(FRA)$. Some reflection then shows that every RRA can be embedded in an algebra of the form
(\(P(E), U, c, |, ^{-1}, Id\)) where \(E\) is an equivalence relation over some set \(U\).

In order to enumerate the equations holding in the variety RRA, Tarski proposed the following axiomatization:

**Definition 3.3.3.**
A relation algebra is a relation type algebra \(\mathfrak{A} = (A, +, -, \cdot, ^{-}, 1')\) in which the following axioms are valid:

- **RA0** Axioms stating that \((A, +, -)\) is a Boolean Algebra
- **RA1** \((x + y); z = x; x + y; z\)
- **RA2** \((x + y); = x' + y'\)
- **RA4** \((x; y); z = x; (y; x)\)
- **RA5** \(x; 1' = x\)
- **RA6** \((x') = x\)
- **RA7** \((x; y); = y'; x'\)
- **RA8** \(x; - (x; y) \leq -y\).

The class of relation algebras is denoted by RA.

For an introduction to the theory of relation algebras and their arithmetic, we refer to Jónsson [58, 59], or to Tarski-Givant [127], where the formalism \(L^\infty\) can be seen as an alphabetic variant for the arithmetic of relation algebras that are generated by one element.

It soon turned out that the RA-axioms do not exhaustively generate all valid principles governing binary relations. There are RAs that are not representable, as was first shown by Lyndon [73]; perhaps the simplest, finite example was provided by McKenzie and can be found in [127]. The question whether finitely many equations might be added to the RA-axioms was answered negatively by Monk in [81], while in [83] he showed that it is not even sufficient to add infinitely many axioms in only a finite number of relation variable symbols. Recent further strengthenings of this negative result have been found by Haiman [47] and by Andréka [5].

On the other hand, explicit infinite axiomatizations are known: cf. Lyndon [73] or McKenzie [74]. Unfortunately, these axiomatizations are intuitively not very appealing. Wadge gave another way of recursively enumerating \(Equ(RRA)\) using a Gentzen-type deduction method, cf. Wadge [137] or Maddux [77]. Here variables referring to elements of the domain are introduced again in the proofs, thus violating the paradigm of algebraic logic not having such variables.

Natural sufficient but not necessary conditions for representability can be found in Maddux [75].

The study of relation algebras is not restricted to algebraic logic: in de Roever [106], they are used in a computer framework science framework, for proving program correctness. Recently, van Benthem [15] found interesting applications for relation algebras in a general theory of information processing. He shows various connections with linguistics and inference systems. These patterns are also present in Roorda [107], who treats a modal logic closely related to the formalism presented in the next section.
3.3.2. A Modal Logic of Binary Relations.

In this section we give the modal system in which the modal algebras have the type of relation algebras. Therefore this logic must have the following signature:

Definition 3.3.4.
CC8 is the modal similarity type \( \{ o, \otimes, \delta \} \) with \( o \) a dyadic, \( \otimes \) a monadic modal operator, and \( \delta \) a modal constant.

Just like in the case of CML2, we have two kinds of semantics: the intended models have a cartesian square as the set of possible worlds; they form a subclass of the class of CC8-models provided by the general definition of a semantics for a modal similarity type.

Definition 3.3.5.
A CC8-frame is a quadruple \( \mathfrak{F} = (W, C, R, I) \) with \( C \subseteq \, ^3W \), \( R \subseteq \, ^2W \) and \( I \subseteq W \). A CC8-frame is two-dimensional, or a square, if \( W \) is of the form \( W = U \times U \) for some set \( U \), and

\[
C = \{ ((u,v), (w,x), (y,z)) \in \, ^3(U \times U) \mid u = w \land x = y \land v = z \} \\
R = \{ ((u,v), (x,y)) \in \, ^2(U \times U) \mid u = y \land v = x \} \\
I = \{ (u,v) \in U \times U \mid u = v \}.
\]

The class of two-dimensional CC8-frames (squares) is denoted by SQ.

So, in a two-dimensional model \( \mathfrak{M} \) we have

\[
\begin{align*}
\mathfrak{M}, u, v & \models o & \iff & \ u = v \\
\mathfrak{M}, u, v & \models \otimes \phi & \iff & \mathfrak{M}, v, u \models \phi \\
\mathfrak{M}, u, v & \models \phi \circ \psi & \iff & \text{there is a } w \text{ with } \mathfrak{M}, u, w \models \phi \text{ and } \mathfrak{M}, w, v \models \psi,
\end{align*}
\]

viz.

As we have already seen, the constant \( \delta \) is true precisely on the diagonal, to verify if \( A \models \otimes \phi \) we look at the image \( A' \) of \( A \) after reflecting in the diagonal. The formula \( \phi \circ \psi \) holds at a point \( A \) if we can build a rectangle \( ABCD \) with the following properties:
B \models \phi, D \models \psi \text{ and } C \text{ lies on diagonal, where } B \text{ is the vertex on the same vertical line as } A \text{ and } D \text{ on the same horizontal as } A.

The connection with RRAs is given by the following

**Proposition 3.3.6.**

$\mathfrak{F}$ is a square iff $\text{cm}\mathfrak{F}$ is (isomorphic to) a full relation algebra. 

Note that the cylindric operators $\Theta$ and $\Phi$, their irreflexive companions $\Theta'$, $\Phi'$ and the $D$-operator can be seen as abbreviated operators of $CC\delta$:

**Definition 3.3.7.**

Define

$$
\Theta \phi = T \circ \phi
$$

$$
\Phi \phi = \phi \circ T
$$

$$
\Theta' \phi = -\delta \circ \phi
$$

$$
\Phi' \phi = \phi \circ -\delta
$$

$$
D' \phi = T \circ \phi \circ -\delta \lor -\delta \circ \phi \circ T.
$$

It is a straightforward exercise to verify that in the squares, these defined operators indeed have the right semantics. We only state

**Proposition 3.3.8.**

$$
SQ \models D\phi \leftrightarrow D'\phi.
$$

### 3.3.3. $CC\delta$ and Classical Logic.

In the introduction we have already mentioned that two-dimensional models for $CC\delta$ can also be seen as ordinary structures for a first order language with dyadic predicates, and that in fact, this identification was precisely the reason to study (the algebraic version of) systems like $CC\delta$. In this section we will see how far the expressive power of $CC\delta$ takes us in classical logic. In section 3.4 we give some examples of how this language can express properties of binary relations like transitivity or irreflexivity.

**Definition 3.3.9.**

Let $L$ be an ordinary signature of first order logic, $N$ a set of $L$-formulas, $k \leq \omega$ an ordinal and $X$ a set of variables in $L$. We set

$$
N^2 = \{ \phi \in N \mid \phi \text{ contains only dyadic predicates} \}
$$

$$
N(X) = \{ \phi \in N \mid \text{all free variables of } \phi \text{ are in } X \}
$$

$$
N_k = \{ \phi \in N \mid \text{all variables of } \phi \text{ are in } \{ x_0, \ldots, x_{k-1} \} \}.
$$

We will show that $CC\delta$ has the same expressive power as $L^2_3(x_0, x_1)$, a fragment of $L$ which we will call "the three variable fragment of first order logic", by a slight abus de langue. We hasten to remark that this claim is an immediate consequence of the fact that the corresponding relation algebraic system has the same property. This matter is also
treated in detail in Tarski-Givant [127]. We prove this correspondence directly in order to give a clear picture of what is going on.

Note that, as we have two kinds of semantics for \( CC\delta \), namely the squares and the wider class of more general \( CC\delta \)-models, we also have two kinds of correspondence maps. The one below is directed to the squares:

**Definition 3.3.10.**
Let \((\cdot)^{\circ}\) be the following translation from \( CC\delta \)-formulas to \( L_{3}^{2}(x_{0}, x_{1})\):

\[
\begin{align*}
p_{i}^{\circ} &= P_{i}x_{0}x_{1} \\
(\phi \land \psi)^{\circ} &= \phi^{\circ} \land \psi^{\circ} \\
(\neg \phi)^{\circ} &= \neg \phi^{\circ} \\
(\forall \phi)^{\circ} &= \phi^{\circ}(x_{0}/x_{1}, x_{1}/x_{0}) \\
(\phi \circ \psi)^{\circ} &= \exists x_{2}(\phi^{\circ}(x_{2}/x_{1}) \land \psi^{\circ}(x_{2}/x_{0})).
\end{align*}
\]

Note that in the above definition we tacitly assumed that the substitution of variables can be performed in \( L_{3}^{2} \). This is not very difficult but rather tedious to establish, so we refer to Gabbay [32] or Tarski-Givant [127]. The following proposition states that every \( CC\delta \)-formula has an equivalent in \( L_{3}^{2}(x_{0}, x_{1})\):

**Proposition 3.3.11.**
Let \( \mathcal{M} = (U, V) \) be a two-dimensional model. Then

\[
\mathcal{M}, u_{0}, u_{1} \models \phi \iff \mathcal{M} \models \phi^{\circ}[x_{i} \mapsto u_{i}].
\]

**Proof.**
By a trivial formula induction.

To show that conversely, every \( L_{3}^{2}(x_{0}, x_{1}) \)-formula has an equivalent in \( CC\delta \), is a bit harder. Maybe the easiest proof uses a second subset of \( L \) as an intermediate system:

**Definition 3.3.12.**
In this definition we assume \( \{i, j, k\} = \{0, 1, 2\} \). By induction we define the sets \( L^{+}(x_{i}, x_{j})\):

All atomic formulas in \( L^{2}(x_{i}, x_{j}) \) are (atomic) formulas of \( L^{+}(x_{i}, x_{j}) \). \( L^{+}(x_{i}, x_{j}) \) is closed under Boolean formula-building, and finally, if \( \phi \) is in \( L^{+}(x_{i}, x_{j}) \) and \( \psi \) is in \( L^{+}(x_{j}, x_{k}) \), then \( \exists x_{j}(\phi \land \psi) \) is in \( L^{+}(x_{i}, x_{k}) \).

\( L^{+}(x_{i}, x_{j}) \) is designed to be the exact first order counterpart of \( CC\delta \):

**Definition 3.3.13.**
Let \((\cdot)^{ij}\) be the following translation from \( L^{+}(x_{i}, x_{j}) \) to \( CC\delta \):

\[
\begin{align*}
(P_{i}x_{i}x_{j})^{ij} &= p_{i} \quad & (x_{i} = x_{j})^{ij} &= \delta \\
(P_{i}x_{j}x_{i})^{ij} &= \Phi p_{i} \quad & (x_{j} = x_{i})^{ij} &= \delta \\
(P_{i}x_{i}x_{i})^{ij} &= \Phi (p_{i} \land \delta) \quad & (x_{i} = x_{i})^{ij} &= T \\
(P_{i}x_{j}x_{j})^{ij} &= \Phi (p_{i} \land \delta) \quad & (x_{j} = x_{i})^{ij} &= T \\
(\neg \phi)^{ij} &= \neg \phi^{ij} \\
(\phi \land \psi)^{ij} &= \phi^{ij} \land \psi^{ij} \\
(\exists x_{k}(\phi(x_{i}, x_{j}) \land \psi(x_{k}, x_{j})))^{ij} &= \phi^{ik} \circ \psi^{kj}.
\end{align*}
\]
Proposition 3.3.14.
Let $\mathfrak{M}$ be a two-dimensional model, $\phi$ in $L^+(x_i, x_j)$. Then

$$\mathfrak{M} \models \phi[x_i \mapsto u_i, x_j \mapsto u_j] \iff \mathfrak{M}, u_i, u_j \models \phi^{ij}.$$  

Proof.
By a trivial formula-induction. □

So we are finished if we can prove that every $L_3^2(x_0, x_1)$-formula has an equivalent in $L^+(x_0, x_1)$. We need something stronger:

Proposition 3.3.15.
Every formula in $L_3^2$ is equivalent to a Boolean combination of $L^+$-formulas.

Proof.
By induction to the complexity of $L_3^2$-formulas. We only treat the induction step involving the existential quantifier.

Let $\phi \in L_3^2$ be of the form $\phi = \exists x_2 \psi$. As $\phi$ is in $L_3^2$, so is $\psi$. By induction hypothesis then, $\psi$ is equivalent to a Boolean combination of $L^+(x_i, x_j)$-formulas. Assume that this combination is in disjunctive normal form, and distribute the $\exists x_2$ over the disjuncts. This shows that in fact we may assume that $\psi$ is equivalent to a conjunction $\psi_{01} \land \psi_{02} \land \psi_{12}$ with $\psi_{ij} \in L^+(x_i, x_j)$. Clearly then $\phi$ is equivalent to $\psi_{01} \land \exists x_2(\psi_{02} \land \psi_{12})$ which is a Boolean combination of two $L^+(x_0, x_1)$-formulas. □

Proposition 3.3.16.
Let $\phi$ be in $L_3^2(x_0, x_1)$. Then $\phi$ has an equivalent $\phi^*$ in $\textit{CC6}$ such that for all two-dimensional models:

$$\mathfrak{M} \models \phi[x_i \mapsto u_i] \iff \mathfrak{M}, u_0, u_1 \models \phi^*.$$  

Proof.
Let $\phi^*$ be $\phi$'s $L^+(x_0, x_1)$-equivalent, which exists by the previous proposition. Then take $\phi^* = (\phi^*)^{01}$. The claim then follows by proposition 3.3.14. □

3.3.4. $\textit{CC6}$ and Relation Algebras.

In this subsection the exact relation between $\textit{CC6}$ and RAs will be discussed. By a result of Maddux [76], it is known that the algebraic equations defining RA have first order equivalents in the frame language. Our aim is to show that there are Sahlqvist formulas characterizing $\textit{AtRA}$. Particularly in this area, we are deeply indebted to Johan van Benthem for his guidance into correspondence theory. The observation that the RA-axioms are Sahlqvist forms was made and worked out during conversations with him, and is also reported on in van Benthem [16].

The first order language of the Sahlqvist correspondents is slightly different from our $L_{\textit{CC6}}$ having $C$, $R$ and $I$ as accessibility predicates, however. Because of the intended interpretation of $(\cdot)^{-1}$ as taking the unique converse of a relation, the relation symbol $R$ is replaced by a function symbol $f$. This inspires the following definition:
Definition 3.3.17.
An arrowframe is a quadruple \( \mathfrak{F} = (W, C, f, I) \) with \( C \subseteq W \), \( f \in \mathcal{W} \) and \( I \subseteq W \). Elements of \( W \) are called arrows. An arrowmodel is a \( CC\theta \)-model based on an arrowframe. \( L_A \) is the first order language for arrowframes, having relation symbols \( C \) (ternary), \( I \) (unary) and a function symbol \( f \) (unary).

Of course, the arrowframes form a subclass of the \( CC\theta \)-frames, viz. those \( CC\theta \) frames where \( R \) is functional. For the truth relation in arrowmodels we obtain
\[
\mathfrak{M}, w \models \phi \iff \mathfrak{M}, fw \models \phi.
\]

Note that in all squares \( R \) is functional.

From now on we will concentrate on arrowframes rather than on \( CC\theta \)-frames, and for example use the first order language \( L_A \) instead of \( L_{CC\theta} \). This change of language is not essential, because the class of arrowframes is quite easy characterizable by a Sahlqvist formula:

Proposition 3.3.18.
Let \( \mathfrak{F} = (W, C, R, I) \) be a \( CC\theta \)-frame, then
\[
\begin{align*}
\mathfrak{F} \models \ominus \phi & \iff \mathfrak{F} \models \forall x \forall y \forall z (Rxy \land Rxz \rightarrow y = z) \\
\mathfrak{F} \models \neg \ominus \phi & \iff \mathfrak{F} \models \forall x \exists y Rx y \\
\mathfrak{F} \models \ominus \phi & \iff R \text{ is functional.}
\end{align*}
\]

Proof.
These are standard Sahlqvist equivalences, cf. 2.2.2.

Arrowframes are known from the literature, cf. Maddux [76]. (The nice name ‘arrowframe’ is due to Johan van Benthem.) Informally, we will use the following pictures to represent arrows and their relations:

- \[ \begin{array}{c}
\Diamond \\
Iu
\end{array} \]
- \[ \begin{array}{c}
v \\
w
\end{array} \]
- \[ \begin{array}{c}
\rightarrow \\
u \\
fu
\end{array} \]
- \[ Cuvw \]

We now give the modal version of the RA-axioms:
3.3. A MODAL LOGIC OF BINARY RELATIONS.

Definition 3.3.19.
Set the following (pairs of) $CC\delta$- and $L_A$-formulas

\[(CC0)\quad \otimes p \leftrightarrow \neg \otimes \neg p\]
\[(CC1)\quad p \to \otimes \otimes p\]
\[(AR1)\quad ff x = x\]

\[(CC2)\quad p \circ (q \circ r) \to (p \circ q) \circ r\]
\[(CC3)\quad (p \circ q) \circ r \to p \circ (q \circ r)\]
\[(AR2)\quad \forall yzu((Cxyz \land Czuv) \to \exists w(Cxwv \land Cwuy))\]
\[(AR3)\quad \forall yuw((Cxwv \land Cwuy) \to \exists z(Cxyz \land Czuw))\]

\[(CC4)\quad p \to \delta \circ p\]
\[(CC5)\quad \delta \circ p \to p\]
\[(AR4)\quad \exists y(Iy \land Cxyz)\]
\[(AR5)\quad \forall yz(Cxyz \land Iy) \to x = z)\]

\[(CC6)\quad \otimes(p \circ q) \to (\otimes q \circ \otimes p)\]
\[(CC7)\quad (\otimes q \circ \otimes p) \to \otimes(p \circ q)\]
\[(AR6)\quad \forall yz(Cfxyz \to Cxfzfy)\]
\[(AR7)\quad \forall yz(Cxfzfy \to Cfxyz)\]
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\[(CC8) \quad \Diamond p \circ \neg(p \circ q) \land q \rightarrow \bot\]
\[(AR8) \quad \forall yz(Cx.fyz \rightarrow Cxyz)\]

\[(A CC8-frame or) an \text{ arrowframe is called relational if } (CC0 \text{ and }) CC1, \ldots, CC8 \text{ hold in it. The class of relational frames is denoted by AR.} \]

The relational frames are the atom structures of the relation algebras; we state the converse proposition, which is immediate by Appendix A.19:

**Theorem 3.3.20.**
\[RA = CmAR.\]

**Warning 3.3.21:** It is tempting to see these pictures as graphs, and reason accordingly. But as not all relational frames are squares, such reasoning could lead to wrong conclusions: one should be careful not to use intuitions about graphs that are not explicitly justified by the axioms. However, it might follow from theorem 5 of Maddux [77] that in subgraphs containing not more than four points, all graph-based intuitions are sound. (By lack of space, we can not go into details.)

As we had already announced, the CC-axioms are Sahlqvist formulas; so we have both a correspondence and a completeness result.

**Definition 3.3.22.**
Let \(AR\) be the minimal axiom system \(K_{CC8}\), extended with \(CC0 \ldots CC8\) as axioms.

**Theorem 3.3.23.**
For \(i = 1, \ldots, 8\): \(\exists \models CCi \iff \exists \models ARI.\)

**Theorem 3.3.24.**
\(AR\) is strongly sound and complete with respect to \(AR\).

**Proofs.**
Immediate by the Sahlqvist form of the axioms and theorem 2.2.2.
3.3. A MODAL LOGIC OF BINARY RELATIONS.

3.3.5. Characterizing squares.

In this section we set out to characterize the class SQ of two-dimensional arrow- resp. CC8-frames. The simplest way to do so is in the first order language $L_A$.

**Definition 3.3.25.**

Define the following $L_A$-formulas:

$(ARU) \quad \forall xyuv'wv'w'y'y'((C xv \land Cx'v' \land Cyw \land Cy'w') \rightarrow v = v')$

$(ARE) \quad \forall uv \exists vxy(C xv \land Cyw)$.

$\mathfrak{z} \models ARE$ means that for every arrow pair $u$ and $w$ there is a connecting arrow $v$, while $\mathfrak{z} \models ARU$ means that such a connecting arrow must be unique, viz.

![Diagram](image)

**Theorem 3.3.26.**

Let $\mathfrak{z}$ be an arrowframe. Then

$\mathfrak{z}$ in SQ $\iff \mathfrak{z} \models AR1, \ldots, AR8, ARU & ARE$.

**Proof.**

The direction from left to right is straightforward, so we only prove the other side: suppose $\mathfrak{z} = (W,C,f,I)$ is a relational arrow frame satisfying $ARU$ and $ARE$. We will show that $\mathfrak{z}$ is isomorphic to the square based on $I \subseteq W$. To this end we give an isomorphism $g : W \rightarrow I \times I$. Before doing so, we prove an extra fact about $F$, namely that the mirror images of $AR4$ and $AR5$ hold in it:

**Claim 1**: $\mathfrak{z} \models \forall xy(1y \land Cxxy)$ and $\mathfrak{z} \models \forall xyz(Cxzy \land Iy \rightarrow z = y)$.

**Proof**: It belongs to the standard arithmetic of relation algebras that $x;1^1 = x$ holds in an RA. This implies that if $\mathfrak{z}$ is in AR, we have $\mathfrak{z} \models p \circ \delta \leftrightarrow p$. Taking the Sahlqvist correspondents of the formulas $p \circ \delta \rightarrow p$ and $p \rightarrow p \circ \delta$, we immediately obtain claim 1.

We can now define a unique 'left point'- arrow $l_u$ and 'right point'-arrow $r_u$ of an arrow $u$:

**Claim 2**: $\mathfrak{z} \models \forall u \exists ! l(Cul \land Il)$ and $\mathfrak{z} \models \forall u \exists ! r(Cur \land Ir)$.

**Proof**: Existence is given for $l$ by $(AR4)$, and for $r$ by claim 1.

Uniqueness we only prove for $l$: suppose there are $l, l'$ with $Cul \land Cul'u \land Il \land Il'$. By $AR2$ there is an $m$ with $Cml'l'$. It is straightforward to verify that by $Il$ and $Il'$ this implies $l = m = l'$.
So we are justified in defining the ‘left point’ \( l_u \) and ‘right point’ \( r_u \) of an arrow \( u \) as the arrows satisfying the first resp. last condition of the second claim. We can now define the isomorphism \( g : W \rightarrow 2^I \) by

\[ gu = (l_u, r_u) \]

**Claim 3:** \( g \) is surjective.

**Proof:** Let \( l, r \) be in \( I \). By \( ARE \) there are \( v, x, y \) with \( Cxlv \) and \( Cyvr \). \( Cxlv \) and \( II \) imply \( x = v \) by \( AR5 \), \( Cyvr \) and \( Ir \) imply \( y = v \) by claim 1.

So we have \( Cuv \) and \( Cuw \), implying \( gu = (l, r) \).

**Claim 4:** \( g \) is injective.

**Proof:** Suppose \( gu = gu' \). Set \( l = l_u = l_{u'} \), \( r = r_u = r_{u'} \), then both \( u \) and \( u' \) connect \( l_u = l_{u'} \) and \( r_u = r_{u'} \), so by \( ARU \) we have \( u = u' \).

**Claim 5:** \( g \) is a homomorphism.

**Proof:** Let \( u \in I \). By \( Cuw \) we find \( u = l_u = r_u \), so \( gu = (u, u) \in I \).

For \( f \) we have to show \( fgu = fgw \). Now \( fgu = (r_u, l_u) \). For \( fgw \), we find \( l_{fu} = r_u \), as \( fl_u \in I \) and \( Cufu \) implies \( Cfu \). Similarly, \( r_{fu} = l_u \). So, indeed, \( fgw = fgu \).

For \( C \), suppose \( Cuw \). We have to show \( l_w = l_u, r_w = r_u \) and \( r_w = r_u \). We only treat \( r_w = r_u \); using \( AR2 \) and \( AR3 \) we can easily show that both \( x = r_u \) and \( x = l_u \) satisfy \( Cuux \) and \( Cuux \), so by \( ARU \) we have \( r_u = l_u \).

**Claim 6:** \( g \) is an anti-homomorphism.

**Proof:** For \( I \): if \( l_u = r_u \), we have both \( u \) and \( l_u \) connecting \( fu \) and \( u \), so \( u = l_u \) whence \( u \in I \).

The part for \( f \) is already proved in the previous claim.

For \( C \), suppose \( Cguguyv \), then \( l_w = l_u, r_w = r_v \) and \( r_u = l_v \). Define \( m = r_u (= l_v) \).

\( Cuum \) and \( Cvuv \) imply the existence of an \( x \) with \( Cxuv \). We will prove that \( x = w \). \( Cxuv \) implies \( l_x = l_u \) (e.g. by the previous claim, where we proved \( g \) to be a homomorphism) and \( r_x = r_u \). But then \( l_x = l_w \) and \( r_x = r_w \), so we obtain \( x = w \) by the injectivity of \( g \). \( Cwuv \) is then immediate.

Unfortunately, SQ is not characterizable in \( C\overline{C} \). Of course, this is obvious as \( FRA (= CmSQ) \) is not a variety: it is not closed under products or subalgebras. However, after adding the \( D \)-operator (cf. section 2.4), we can define SQ in the new language, by Sahlqvist formulas:

**Definition 3.3.27.**

\( CCD \) is the similarity type \( C\overline{C} \) augmented with the difference operator \( D \).

Note that the \( C\overline{C} \)-frames can be identified with the class of standard \( CCD \)-frames. (We refer to definition 2.4.1 and the remarks below it for notions concerning the \( D \)-operator, like the definition of \( O \) and \( E \), or the identification of \( C\overline{C} \)-frames with standard \( CCD \)-frames.)

**Definition 3.3.28.**

Define the following \( CCD \)-formulas:
(CCU) \( (O \circ p \circ q \circ r \land E(\neg p \circ \neg q \circ \neg r)) \to \top \)
(CCE) \( p \land Eq \to E(p \circ T \circ q) \).

Proposition 3.3.29.
Let \( F = (W, C, f, I) \) be a (D-standard) relational arrowframe. Then
\[
\not\models F \models CCU \iff \not\models F \models ARU \\
\not\models F \models CCE \iff \not\models F \models ARE.
\]

Proof.
The proposition can be proved by the Sahlqvist form of CCU and CCE, but in this case a direct proof is much more perspicuous; we only treat CCU \( \Rightarrow \) ARU:
Suppose \( \not\models F \models ARU \), then there are arrows \( u, w, v \neq v', x, x', y \) and \( y' \) as in the picture above theorem 3.3.26. Define a valuation \( V \) on \( F \) with \( V(p) = W - \{w\} \), \( V(r) = W - \{w\} \) and \( V(q) = \{v\} \). Recall that (in a standard frame) \( O\phi \) holds at a world iff this world is the only one where \( \phi \) holds. Now let \( z \) be an arrow with \( Czuy \) and \( Cxzv \); \( z' \) is defined likewise. It is straightforward to prove that under this valuation, \( z \models O\circ p \circ q \circ r \) and \( z' \models \neg p \circ \neg q \circ \neg r \). This latter fact implies \( z \models E(\neg p \circ \neg q \circ \neg r) \), showing \( \not\models F, z \models CCU \).

We now have our desired characterization of the squares: they are the D-standard relational arrowframes where CCU and CCE are valid:

Theorem 3.3.30.
\( SQ = AR_{CCE,CCU}^F \).

Proof
By 3.3.23 and 3.3.26.

We have already seen that on the class of squares, the D-operator is definable. Theorem 3.3.30 seems to contradict the fact that SQ is not characterizable in CCD — why not take the \( D' \)-versions of CCE and CCU? The point is that for the D-operator as a primitive we have stipulated that inequality be its accessibility relation, for the defined \( D' \)-operator this will not hold for all frames in AtRRA, only for the squares.

3.3.6. Axiomatizing Squares.

As we have found a characterization of the class of two-dimensional frames in terms of Sahlqvist CCD-formulas, we can obtain a completeness result which is an almost immediate consequence of the SD-theorem. Formally however, we have to extend the language once again, as in the formulation of this theorem one needs an versatile similarity type:

Definition 3.3.31.
Let \( CCD' \) be the similarity type CCD extended with two dyadic modal operators \( \circ_1 \) and \( \circ_2 \) besides \( \circ \).

The set \( \{\circ, \circ_1, \circ_2\} \) should be seen as a triple of versatile dyadic operators, cf. definition
2.7.1. Note that we can identify $CC\delta$-structures (with $CCD$-structures and thus) with versatile $CCD'$-structures: in versatile $CCD'$-frames, the accessibility relations of $\alpha_1$ and $\alpha_2$ are given by $R_{\alpha_1}uvw \leftrightarrow Cuvw \leftrightarrow R_{\alpha_2}uvw$.

Definition 3.3.32.
Let $ACCD'$ be the axiom system $K'_{CCD'}D'$ (cf. 2.7.2 and 2.4.2), extended with the axioms $CC0, \ldots, CC8, CCE$ and $CCU$ and the following:

(XD) $Dp \leftrightarrow (\lnot \delta \circ p \circ T) \lor (T \circ p \circ \lnot \delta)$.

Theorem 3.3.33.
$ACCD'$ is strongly sound and complete with respect to the class of squares.

Proof.
Immediate by the $SD$-theorem 2.7.7 and 3.3.30. (In fact, the axiom $XD$ is superfluous; note that it is sound.)

We will now show that we can in fact formulate a much simpler sound and complete axiom system which does not need to go beyond the borders of the old language $CC\delta$.

Definition 3.3.34.
Let $AR'$ be the extension of $AR$ with the irreflexivity rule for $D'$:

$\hspace{1cm} (IR_{D'}) \vdash (p \land \lnot D'p) \to \phi \to \top, \text{ if } p \not\in \phi. \hspace{1cm}$

In other words, $AR'$ has as its axioms: all propositional tautologies, distribution for $\circ$ and $\otimes$, and $CC0 \ldots CC8$. Its rules are $MP, UG, SUB$ and $IR_{D'}$.

We will prove that $ACCD'$ is a conservative extension of $AR'$, using induction to the length of $ACCD'$-derivations. Loading the induction hypothesis makes the proof more perspicuous; we need the following definition:

Definition 3.3.35.
Let $(\cdot)^o$ be the following translation from $CCD'$-formulas to $CC\delta$-formulas:

\[
\begin{align*}
p^o & = p \\
\delta^o & = \delta \\
(\lnot \phi)^o & = \lnot \phi^o \\
(\phi \circ \psi)^o & = \phi^o \circ \psi^o \\
\phi \land \psi)^o & = \phi^o \land \psi^o \\
\otimes \phi)^o & = \otimes \phi^o \\
(D\phi)^o & = D'\phi^o
\end{align*}
\]

Proposition 3.3.36.
For a $CCD'$-formula $\phi$:

$ACCD' \vdash \phi \iff AR' \vdash \phi^o$.

Proof.
The direction $\leftarrow$ is easy, as all $AR'$-axioms are $ACCD'$-axioms and the irreflexivity rule for $D'$ is easily seen to be a derived rule of $ACCD'$. (Here the axiom $XD$ comes handy.)
The proof for the other direction is by induction to the derivation of $\phi$ in $ACCD'$.

For the basis step, in the next section we will show that

\[
(\ast) \hspace{1cm} AR \vdash \alpha^o \text{ for all axioms } \alpha \text{ of } ACCD'.
\]
If $\phi$ is derived from earlier theorems by applying one of the orthodox derivation rules, $\phi^o$ is derived from the translations of these earlier ACCD$^t$-theorems by applying the same rule in $AR^+$. 

So the only case left is where the last step in the ACCD$^t$-derivation of $\phi$ used the $D'$-irreflexivity rule: we have $ACCD^t \vdash (p \land \neg Dp) \rightarrow \phi$, where $p$ does not occur in $\phi$. By the induction hypothesis, $AR^+ \vdash (p \land \neg D'p) \rightarrow \phi^o$, so an application of $(IR_{D'})$ gives $AR^+ \vdash \phi^o$. 


Theorem 3.3.37. SOUNDNESS AND COMPLETENESS. 

$$\Sigma \vdash_{AR^+} \phi \iff \Sigma \models_{SQ} \phi.$$ 

Proof. 

Soundness is immediate. For completeness, let $\Sigma \models_{SQ} \phi$, then $\Sigma \vdash_{ACCD^t} \phi$ by 3.3.33. By definition this means that there are $\sigma_1, \ldots, \sigma_n$ in $\Sigma$ such that $ACCD^t \vdash (\sigma_1 \land \ldots \land \sigma_n) \rightarrow \phi$. By 3.3.36 we obtain $AR^+ \vdash ((\sigma_1 \land \ldots \land \sigma_n) \rightarrow \phi)^o$, but as $\phi$ and the $\sigma_i$ are $CCD^t$-formulas, we have $((\sigma_1 \land \ldots \land \sigma_n) \rightarrow \phi)^o = (\sigma_1 \land \ldots \land \sigma_n) \rightarrow \phi$. So $(\sigma_1 \land \ldots \sigma_n) \rightarrow \phi$ itself is an $AR^+$-theorem. This gives $\Sigma \vdash_{AR^+} \phi$. 


3.3.7. Some Arrow-Arithmetic. 

The previous section had an open end: for some $CCD^t$-formulas we have to show that they are derivable in $AR^+$. 

Proposition 3.3.38. 

Let $(\cdot)^o$ be as in 3.3.35. For every $ACCD^t$-axiom $\alpha$, $AR^+ \vdash \alpha^o$. 

Proof. 

In fact, the $D'$-irreflexivity rule is needed for none of these derivations. We will frequently use the completeness theorem 3.2.24, giving semantic proofs about relational arrowframes instead of formal derivations. We let $O'$ and $E'$ denote the obvious abbreviations, i.e. $O'\phi = \phi \land \neg D'\phi$, $E'\phi = \phi \lor D'\phi$.

Claim 1: $AR \vdash V^o$ 

Proof. Recall that $V \equiv V0 \land V1 \land V2$ is the tense axiom associated with the operator triple $\{o, o_1, o_2\}$ of $CCD^t$. 

We only treat 

$$V_2 = p \land \neg (\circ_2 p) o_1 r \rightarrow \bot.$$ 

An evaluation shows that $V_2^o$ is 

$$p \land \circ o \circ \neg (\circ p \circ r) \rightarrow \bot,$$

which by $CC0$, $CC6$ and $CC1$ is equivalent to 

$$p \land \circ o \circ \neg (\circ r \circ p) \rightarrow \bot$$ 

and then by $CC1$ to 

$$p \land \circ \circ o \circ \neg (\circ r \circ p) \rightarrow \bot.$$
So by one application of \( SUB \) to the axiom \( CC8 \) it follows that this formula is an \( AR \)-theorem.

**Claim 2:** \( AR \vdash D1^o \).

**Proof.** Recall that \( D1 \) is the axiom \( p \rightarrow DDp \), so it is sufficient to show that \( AR \vdash p \rightarrow \neg D' \neg D'p \).

So suppose that \( \mathfrak{M} \) is a model based on a relational arrowframe, and that \( \mathfrak{M}, u \models p \land D' \neg D'p \). We will derive a contradiction from this.

By \( u \models D' \neg D'p \) and the definition of \( D' \), there are \( w_1, w_2 \) with \( Cu w_1 w_2 \) and either \( w_1 \notin I \), \( w_2 \models \neg D'p \circ T \), or \( w_2 \notin I \) and \( w_1 \models T \circ \neg D'p \).

Without loss of generality we assume the first, so there are \( v, x \) with \( Cw_2 vx \) and \( v \models \neg D'p \). By \( AR2 \) there is a \( y \) with \( Cw_1 y \) and \( Cuyx \), viz.

\[ Cuyx \implies Cyfx, \text{ so } u \models p \text{ gives } y \models p \circ T. \]

\( Cyw_1 y \) implies \( Cvfw_1 y \), and as \( fw_1 \notin I \) (by \( w_1 \notin I \)), we get \( v \models \neg \delta \circ (p \circ T) \), contradicting \( v \models \neg D'p \).

To show that \( AR \vdash D2^o \), \( AR \vdash D3^o \) (cf. 2.4.2 for definitions), we first prove an auxiliary result. Recall that \( \Phi \phi \) is the formula \( T \circ \phi \circ T \).

**Claim 3:** \( AR \vdash \Phi p \rightarrow p \lor D'p \).

**Proof.** All of the following formulas are \( AR \)-theorems:

\[
\begin{align*}
\Phi p & \rightarrow (\delta \lor \neg \delta) \circ p \circ T \\
\Phi p & \rightarrow (\neg \delta \circ p \circ T) \lor (\delta \circ p \circ T) \\
\Phi p & \rightarrow D'p \lor (\delta \circ p \circ (\delta \lor \neg \delta)) \\
\Phi p & \rightarrow D'p \lor ((\neg \delta \circ p \circ \neg \delta) \lor (\delta \circ p \circ \delta)) \\
\Phi p & \rightarrow D'p \lor (T \circ p \circ \neg \delta) \lor \circ p \\
\Phi p & \rightarrow D'p \lor D'p \lor \circ p.
\end{align*}
\]

**Claim 4:** \( AR \vdash D2^o \), \( AR \vdash D3^o \).

**Proof.** It is fairly easy to establish that the following formulas are \( AR \)-theorems:

\[
D' D'p \rightarrow \Phi p, \quad \circ p \rightarrow \Phi p, \quad p \circ q \rightarrow \Phi p, \quad q \circ p \rightarrow \Phi p.
\]

The derivation of \( D2^o \) and \( D3^o \) in \( AR \) is then easy to find, by claim 3.

**Claim 5:** \( AR \vdash CC8^o \).
Proof. We will show that $AR \models p \land E'q \rightarrow E'(p \circ \top \circ q)$.

Let $u$ be a world with $u \models p \land E'q$. By definition of $E'$, we have $u \models q$ or $u \models \top \circ q \circ \delta$ or $u \models \neg \delta \circ q \circ \top$, of which we only treat the last case. Analogous to the proof of claim 2, using the same terminology (and the same picture), we find $w_1 \models \neg \delta$, $v \models q$ and $u \models p$.

This gives $y \models p \circ \top \circ q$.

If $x \in I$, we have $u = y$, so $u \models p \circ \top \circ q$.

If $x \notin I$ we have $u \models (p \circ \top \circ q) \circ \neg \delta \Rightarrow u \models \top \circ (p \circ \top \circ q) \circ \neg \delta \Rightarrow u \models D'(p \circ \top \circ q)$.

So we obtain $u \models p \vee D'p$, QED.

Claim 6: $AR \models CCU^\circ$.

Proof.

We will show that

$$AR \models (O'\neg p \circ q \circ O'\neg r) \land E'(-p \circ q \circ \neg r) \rightarrow \bot.$$ 

Suppose otherwise, that in a relational arrowmodel, $x \models (O'\neg p \circ q \circ O'\neg r) \land E'(-p \circ q \circ \neg r)$. It is a tedious, but not too difficult exercise to show that this implies $x \models \neg p \circ q \circ \neg r$. So there are $u, y, v, w, u', y', v', w'$ such that $u \models O'\neg p$, $v \models q$, $w \models O'\neg r$, $u' \models \neg p$, $v' \models \neg q$, $w' \models \neg r$, where these arrows are situated as depicted as in the left figure:

![Diagram showing the relation between worlds](image)

First we show that $u = u'$: by $Cxuy$ and $Cuxy'$ there is a $z$ with $Cu'zxu$ and $Cy'zy$. Suppose that $z \notin I$, then $u' \models \neg p \Rightarrow u \models \neg p \circ \delta \Rightarrow u \models \top \circ \neg p \circ \delta \Rightarrow u \models D'\neg p$, which would contradict $u \models O'\neg p$. So $z$ is in $I$. But then $u = u'$ by $AR5$ and $y = y'$ by claim 1 in 2.3.26.

So we get a picture as in the right figure. (Note that now we have a reduced 'graph' of four 'points', cf. 3.3.21).

There is a $t$ with $Cu'vt$ and $Cwtu'$. In the same way as for $x$, we can show that $t \in I$.

But then $Cu'vt$ implies $v = v'$, contradicting $v \models q$ and $v' \models \neg q$.  

\[\blacksquare\]
3.4 A Two-Dimensional Temporal Logic.

In the introduction to this chapter we mentioned extended tense logics as one of the main examples of modal formalisms with a two-dimensional semantics. In this section we will develop such a two-dimensional temporal logic as a simple extension of CCB, compare it with some of the existing two-dimensional tense systems and prove some results concerning expressiveness and completeness.

The main idea behind our system CCA is very simple: as the ordering relation of temporal structures is a binary relation, in a two-dimensional modal logic we can introduce a modal constant referring to this relation.

The technical framework of this section is closely connected to that in chapter 5. Many results carry over, in both directions. We have tried to keep the overlap as small as possible. For more information on the technical side of this congruence we refer to [131].

**Definition 3.4.1.**

CCB is the similarity type CCB extended with a modal constant λ. A two-dimensional frame for CCA is a pair $\mathcal{F} = (T, \prec)$ with $\prec$ a binary relation on $T$. A two-dimensional model is a CCA-model of which the frame is two-dimensional.

The CCB-operators obtain their usual interpretation in two-dimensional models for CCA, for λ we have

$$\mathcal{M}, s, t \models \lambda \iff s \prec t.$$ 

As abbreviations we define the compass-operators by

\[
\begin{align*}
\diamond \phi &= \phi \circ \otimes \lambda \\
\otimes \phi &= \lambda \circ \phi \\
\otimes \phi &= \phi \circ \lambda.
\end{align*}
\]

Note that in a two-dimensional model $\mathcal{M}$, these compass-operators get their natural interpretation, e.g.

$$\mathcal{M}, s, t \models \diamond \phi \iff \begin{array}{l}
\quad \text{there is a } u \text{ with } s, u \models \phi \text{ and } u, t \models \lambda \\
\quad \text{there is a } u \text{ with } s, u \models \phi \text{ and } u \prec t \\
\quad \text{there is a point } (s, u) \text{ south of } (s, t) \text{ with } s, u \models \phi,
\end{array}$$

\[\lambda, \delta \text{ and } \otimes \lambda\quad \text{south.}\]
A nice consequence of having an explicit referent to the (ordering) relation in the object language, is that it becomes very easy to characterize properties of $\langle$. 

**Definition 3.4.2.**

Consider the following $\text{CC}_\lambda$-formulas:

- $(TR)$ $\lambda \circ \lambda \rightarrow \lambda$ (transitivity)
- $(IR)$ $\lambda \rightarrow \neg \delta$ (irreflexivity)
- $(TO)$ $\lambda \lor \delta \lor \otimes \lambda$ (totality)
- $(LN)$ $TR \land IR \land TO$ (linearity)
- $(DI)$ $\lambda \circ \lambda \rightarrow \lambda \circ (\lambda \land \neg (\lambda \circ \lambda)) \land (\lambda \land \neg (\lambda \circ \lambda)) \circ \lambda$ (discreteness)
- $(DE)$ $\lambda \rightarrow \lambda \circ \lambda$ (denseness)
- $(W)$ $\boxdot p \rightarrow \boxdot (p \land \square \boxdot \neg p)$ (well-orderings)
- $(FP)$ $\diamond \top$ (first point)
- $(LP)$ $\Diamond \top$ (last point)

**Proposition 3.4.3.**

Let $\mathcal{I} = (T, \langle)$ be a two-dimensional frame. Then

1. $\mathcal{I} \models TR \iff \langle \text{ is transitive.}$
2. $\mathcal{I} \models IR \iff \langle \text{ is irreflexive.}$
3. $\mathcal{I} \models TO \iff \langle \text{ is total.}$
4. $\mathcal{I} \models LN \iff \langle \text{ is linear.}$

Now suppose $\langle$ is linear. Then

5. $\mathcal{I} \models DI \iff \langle \text{ is discrete.}$
6. $\mathcal{I} \models DE \iff \langle \text{ is dense.}$
7. $\mathcal{I} \models W \iff \langle \text{ is well-ordered.}$
8. $\mathcal{I} \models FP \iff \text{ $T$ has a first point.}$
9. $\mathcal{I} \models LP \iff \text{ $T$ has a last point.}$

**Proof.**

As an example, we prove (v). Let $\mathcal{I}$ be linear.

First, assume that $\mathcal{I}$ is discrete, and that $\mathcal{M}$ is a model on $\mathcal{I}$ with $\mathcal{M}, s, t \models \lambda \circ \lambda$. Clearly then $t$ is a successor of $s$, but not the immediate one. So let $u$ be the immediate successor of $s$. By linearity of $\langle$ we have $s < u < t$, and as $u$ is the immediate successor of $s$: $s, u \models \lambda \land \neg (\lambda \circ \lambda)$. So $s, t \models (\lambda \land \neg (\lambda \circ \lambda)) \circ \lambda$.

We treat the other conjunct of the consequent in $DI$ likewise, here considering the immediate predecessor of $t$.

Now, assume that $\mathcal{I} \models DI$ and let $s < t$. We have to find an immediate successor for $s$. If $t$ is the immediate successor of $s$, we are finished. Otherwise, $s, t \models \lambda \circ \lambda$ (in every model on $\mathcal{I}$), so $s, t \models (\lambda \land \neg (\lambda \circ \lambda)) \circ \lambda$ by assumption. By the truth definition, there is a $u$ with $s, u \models \lambda \land \neg (\lambda \circ \lambda)$ and $u, t \models \lambda$. It is then straightforward to verify that this $u$ is the immediate successor of $s$.

Compared to the existing two-dimensional tense logics, we feel that $\text{CC}_\lambda$ has the advantage of being both quite expressive and perspicuous. In fact, concerning the first point, all of the systems known to us can be seen as subsystems of $\text{CC}_\lambda$. For example, the system
studied by Åqvist in [142] uses a set of operators that can be defined as the following subtype of CCA:

\[\{bf = \lambda, id = \delta, af = \otimes \lambda, \langle P \rangle = \Diamond, \langle F \rangle = \Diamond, \langle Q \rangle = \Box, \langle X \rangle = \boxdot\} .\]

As a second example, one of the systems discussed by Gabbay in [35] has two modal operators, \( F \) and \( P \), with \( F \) having the following semantics:

\[\mathfrak{M}, s, t \models F \phi \iff \text{Either } s = t \text{ and for some } t' > t, \mathfrak{M}, s, t' \models \phi \text{ or } s < t \text{ and } \mathfrak{M}, t, t \models \phi \text{ or } s > t \text{ and for some } s < u < t, \mathfrak{M}, u, u \models \phi .\]

It is a straightforward exercise to show that \( F \phi \) can be defined in CCA as

\[(\delta \rightarrow \Diamond \phi) \land (\lambda \rightarrow \Box \phi) \land (\otimes \lambda \rightarrow \Diamond \Diamond (\delta \land \phi)).\]

Of course, for practical purposes such operators may be necessary: Gabbay’s motivation for the introduction of \( F \) is that it exactly captures the future perfect tense in English. However, we feel that it is better to use a formalism where the basic operators have a more perspicuous semantics, provided that this clarity does not stand in the way of the system’s expressive power.

We will now pin down this expressive power of CCA precisely:

**Definition 3.4.4.**
Let \( L^< \) denote the set of first order formulas in an ordinary signature with one fixed dyadic predicate symbol \(<\) (which is, of course, to be interpreted as the ordering relation in two-dimensional models).

Recall from the previous section that \( L^2(x_0, x_1) \) is the set of first order formulas, in a language with binary predicates, one of which is \(<\), using only the variables \( x_0, x_1 \) and \( x_2 \), of which \( x_0 \) and \( x_1 \) are free. The results in section 3.3 immediately give:

**Proposition 3.4.5.**
Over the class of two-dimensional models, every CCA-formula has a \( L^2(x_0, x_1) \)-equivalent, and vice versa.

Suppose now that we restrict the valuations on two-dimensional models in such a way that atomic propositions correspond to monadic predicates instead of dyadic ones. This is not an unusual or unnatural restriction in two-dimensional temporal logics; in fact, both of the systems mentioned above satisfy this constraint.

**Definition 3.4.6.**
Let \( \mathfrak{T} = (T, <) \) be a two-dimensional frame. A CCA-valuation \( V : Q \mapsto P(T \times T) \) is called flat if

\[\text{for all } s, t, t' \in T, q \in Q, (s, t) \in V(q) \iff (s, t') \in V(q) .\]

A two-dimensional model \( \mathfrak{M} = (T, V) \) is flat if \( V \) is so.

So a model is flat if the truth of the atomic formulas only depends on the first coordinate
of the evaluation point. This means that in fact, flat models are not structures for dyadic predicates, but for monadic ones. For more information on the subject of flat versus ordinary two-dimensional tense logic we refer to Gabbay [35] (where these types of logic are called weak resp. strong). To connect the notion of flatness with first order logic, we need

**Definition 3.4.7.**

$L^{1<}$ is the set of first order formulas in a language with one binary predicate $<$ and arbitrary monadic predicates. MOD($L^{1<}$) denotes the class of structures for $L^{1<}$ (in the ordinary sense of first order model theory).

Clearly then MOD($L^{1<}$) can be identified with the class of flat $CC\lambda$-models. An almost immediate consequence of 3.4.5 is

**Proposition 3.4.8.**

Over the class of flat two-dimensional models, every $CC\lambda$-formula has an equivalent in $L^{1<}_{3}(x_0,x_1)$, and vice versa.

**Proof.**

By slightly adapting the definitions of the translations already given for ordinary $CC5$ in section 3.3.3.

The basic $CC\lambda$-proposition $p$ now can be translated as $Px_0$.

For the other direction, we may proceed as if an atom $Px_0$ were of the form $Px_0x_0$ and then continue as in 3.3.16.

A fortiori, every $CC\lambda$-formula has an equivalent in $L^{1<}(x_0,x_1)$. We will now prove the converse of this fact, establishing an expressive completeness result, in the style of Kamp's famous result stating that over the class of continuous linear orderings, every formula in $L^{1<}(x_0)$ has an equivalent in the one-dimensional formalism with the operators $S$ ('Since') and $U$ ('Until') (cf. Kamp [61], or Gabbay [35] for a more accessible proof).

**Theorem 3.4.9.**

Over the class of flat models based on a linear frame, every $L^{1<}(x_0,x_1)$-formula has an equivalent in $CC\lambda$, and vice versa.

**Proof.**

Let $\phi$ be a formula in $L^{1<}(x_0,x_1)$. By results of Gabbay [32], resp. Immerman and Kozen [57], $L^{1<}$ has Henkin-dimension three, resp. the three-variable property over the class of linear orderings, both implying that $\phi$ has an equivalent in $L^{1<}_{3}(x_0,x_1)$. By proposition 3.4.8 then, $\phi$ has a $CC\lambda$-equivalent.

Note that the restriction to flat $CC\lambda$/monadic predicates is essential here, as it is shown in Venema [136] that no finite system of two-dimensional temporal operators can be as expressive as $L^{1<}_{3}(x_0,x_1)$.

Our notion of expressive completeness is not the only one possible, and $CC\lambda$ is not the only two-dimensional expressively complete system. We refer to Gabbay [35] for more details.
We now turn to completeness matters, the last topic of this section. We will show that $CC\Lambda$ allows very simple axiomatizations, simple at least on top of the completeness theorem for $CC\mathfrak{B}$.

Definition 3.4.10.
Let $AL^+$, $ADE^+$ and $AD^+I$ be the axiom system $AR^+$ of $CC\mathfrak{B}$, extended with the following axioms:

$AL^+ : AR^+ + LN$

$ADE^+ : AR^+ + LN + DI$

$ADI^+ : AR^+ + LN + DE$.

Theorem 3.4.11.
$AL^+$, $ADI^+$ and $ADE^+$ are strongly sound and complete with respect to respectively the classes of linear orderings, discrete linear orderings and dense linear orderings.

Proof.
We only show completeness. For $AL^+$, let $\Delta$ be an $AL^+$-consistent set of formulas. Considering $\lambda$ as a propositional variable in $CC\mathfrak{B}$, we obtain by $AL^+ \supset AR^+$ and the completeness theorem 3.3.37 for $CC\mathfrak{B}$, that $\Delta$ is satisfiable in a two-dimensional model $\mathfrak{m} = (U, V)$. Define $\prec := V(\lambda)$, then by 3.4.2(iv) and the fact that $\mathfrak{m} \models LN$, $\prec$ is a linear ordering. Clearly then $\Delta$ is satisfiable in a linear two-dimensional model.

The proofs for $ADE^+$ and $ADI^+$ are analogous, using 3.4.2(iv) and 3.4.2(v).

3.5 Two-Dimensional Algebras.

3.5.1. Two-dimensional Cylindric Algebras.

$CML_2$ is not the only ‘simple’ subtype of $S_2$; the reason why we picked it out is that the Boolean $CML_2$-algebras are well-known in algebraic logic under the name cylindric type algebras (of dimension two). We change some of our notation, in order to keep in tune with the algebraic standard:

Definition 3.5.1.
In this section we write $c_0$, $c_1$ and $d_{01}$ for the operators $\Theta$, $\Phi$ and $\delta$, and $T_0$ (or: $\sim_0$), $T_1$ (or: $\sim_1$) resp. $E_{01}$ for the accessibility relations $H$, $V$ resp. $D$.

Definition 3.5.2.
Boolean $CML_2$-algebras are called cylindric type algebras; cylindric type algebras in the variety generated by $CmC_2$ are called Representable Cylindric Algebras of dimension 2; this variety itself is denoted by $RCA_2$.

For a discussion of these notions we refer to the next chapter where (representable) cylin-
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dric algebras of arbitrary dimension are treated; the standard textbook is Henkin-Monk-
Tarski [53].
The set $\text{Eq}(\text{RCA}_2)$ is finitely axiomatizable, and a finite, explicit set of equational ax-
ioms has been known for a long time, cf. [53], pp. 79–84. It is interesting to note that
the method used there to prove the representation theorem (originating with Andréka
and Németi), is virtually the same as the two-dimensional bulldozing technique Segerberg
used to prove his completeness result for the two-dimensional similarity type $\text{SEG}$, cf. Segerberg [119].

By Appendix A.19, our completeness result 3.2.8 has an immediate algebraic counterpart
in the form of a new finite axiom system for $\text{Eq}(\text{RCA}_2)$:

**Definition 3.5.3.**
Define the following $\text{CML}_2$-equations (where $\{i, j\} = \{0, 1\}$):

- $(CE_{1_i}) \quad x \leq c_ix$
- $(CE_{2_i}) \quad x \leq -c_i - c_ix$
- $(CE_{3_i}) \quad c_i c_j x \leq c_i x$
- $(CE_{4_i}) \quad c_i c_j x \leq c_j c_i x$
- $(CE_{5_i}) \quad c_i d_{01} = 1$
- $(CE_{6_i}) \quad c_i (d_{01} \cdot x) \leq -c_i (d_{01} \cdot -x)$
- $(CE_{7_i}) \quad d_{01} \cdot c_i (-x \cdot c_j x) \leq c_j (-d_{01} \cdot c_i x)$

$CE = \{CE_{l_{i,j}} \mid l = 1, \ldots, 7, \{i, j\} = \{0, 1\}\}$. 

**Theorem 3.5.4.**
$CE$ axiomatizes $\text{RCA}_2$.

**Proof.**
Immediate by A.19 and 3.2.8.

3.5.2. Simplifying Henkin's equation.

As we have mentioned before, it has been known for a long time that the class $\text{RCA}_2$ can
be finitely axiomatized. This standard axiomatization is slightly different from ours:

**Definition 3.5.5.**
Consider the following $\text{CML}_2$-equations:

- $(CE_{2_i}^{1_2}) \quad c_i(x \cdot c_j y) = c_i x \cdot c_j y$
- $(H_{ij}) \quad c_i(x \cdot -y \cdot c_j(x \cdot y) \leq c_j(-d_{01} \cdot c_0 x))$

A cylindric type algebra is a **Cylindric Algebra** if $CE_1, CE_{2_i}^{1_2}, CE_4, CE_5$ and $CE_6$
hold in it. The variety of Cylindric Algebras is denoted by $\text{CA}_2$.

Among algebraic logicians, $H$ is known as **Henkin's equation**. Traditionally, it is the
equations $CE_1, CE_{2_i}^{1_2}, CE_4, CE_5, CE_6$ and $H$ that are used to axiomatize $\text{RCA}_2$. It
is immediate that this system is equivalent to the set $CE$, as both axiomatize the same
variety RCA₂. We will prove the equivalence directly however, because this proof is a nice illustration of how easy Sahlgqvist's theorem can make life, enabling us to reason in the frames instead of giving algebraic derivations. First:

**Proposition 3.5.6.**
Let \( \mathfrak{A} \) be a CA₂-type algebra with \( \mathfrak{A} \models CE1 \). Then

\[
\mathfrak{A} \models CE2^{\frac{1}{2}} \iff \mathfrak{A} \models CE2 \land CE3.
\]

**Proof.**
By 2.2.6, the Sahlgqvist form of the equations gives us the advantage it is sufficient to prove that the *first order* Sahlgqvist correspondents to be equivalent. As we have \((CE2^{\frac{1}{2}})^* = (\dagger)\)

\[
\forall x \forall y (\langle T_i u x \land T_i x y \rangle \leftrightarrow \langle T_i u x \land T_i y u \rangle),
\]

we have to prove that for \( T_i \)-reflexive frames \( \mathcal{F} \):

\[
\mathcal{F} \models (\dagger) \iff T_i \text{ is transitive and symmetric}.
\]

But this is almost immediate by the definitions.

**Corollary 3.5.7.**
A cylindric type algebra \( \mathfrak{A} \) is a cylindric algebra iff \( \mathfrak{A} \models CE1, \ldots, CE6 \).

Now we will prove that in the variety of cylindric algebras, Henkin's equation is equivalent to \( CE7 \):

**Proposition 3.5.8.**
Let \( \mathfrak{A} \) be in CA₂. Then

\[
\mathfrak{A} \models H \iff \mathfrak{A} \models CE7.
\]

**Proof.**
By a similar argument as before, it suffices to prove that on the class of cylindric frames, the Sahlgqvist correspondents of \( H \) and \( CE7 \) are equivalent. Now \( CE7^* = \forall x. NHV7(x) \)
and \( H^* \) has the form

\[
(H^*) \quad \forall u \forall v \forall w ((u \sim_0 v \hateq_1 w \land v \neq w) \rightarrow \exists x (\neg D x \land u \hateq_1 x \land (x \sim_0 v \lor x \sim_0 w))).
\]

So we have to show that for a cylindric frame \( \mathcal{F} \)

\[
\mathcal{F} \models H^* \iff \mathcal{F} \models \forall x. NHV7(x).
\]

The following pictures show the meaning of \( H^* \) and \( NHV7 \) for cylindric frames:
3.5. TWO-DIMENSIONAL ALGEBRAS.

(⇒)
Assume that \( \mathfrak{F} \models NHV7 \). To prove that \( \mathfrak{F} \models H^s \), let \( u, v \) and \( w \) be worlds in \( \mathfrak{F} \) with \( u \sim_0 v \sim_1 w \) and \( v \neq w \). We have to find an \( x \) with \( x \notin D \), \( u \sim_1 x \) such that \( x \) is in the 0-equivalence class with \( v \) or with \( w \).

Distinguish the following cases:
Case 1: \( u \in D \). Then \( \mathfrak{F} \models NHV7(u) \) immediately gives us the desired \( x \), with \( x \sim_0 w \).
Case 2: \( u \notin D \). Then \( u \) itself is the desired \( x \), as \( u \sim_0 v \) and \( u \sim_1 u \).

(⇐)
For the other direction, we assume that \( \mathfrak{F} \models H^s \), we consider arbitrary worlds \( u, v \) and \( w \) in \( \mathfrak{F} \) with \( u \notin D \), \( u \sim_0 v \sim_1 w \) and \( v \neq w \), and set ourselves the task to find an \( x \) with \( x \notin D \) and \( u \sim_1 x \sim_0 w \), viz. figure 2.

By \( \mathfrak{F} \models H^s \), there is a \( y \notin D \) with \( u \sim_1 y \) and \( y \sim_0 v \) or \( y \sim_0 w \). Distinguish

Case 1: \( y \sim_0 w \).
This means we are finished immediately: take \( x = y \).

Case 2: \( y \sim_0 v \).
By \( \mathfrak{F} \models NHV4 \), there is a \( z \) in \( \mathfrak{F} \) with \( u \sim_1 z \sim_0 w \), viz. figure 3.
Distinguish

Case 2.1: \( z \not\in D \). Again we are finished: take \( x = z \).

Case 2.2: \( z \in D \). This implies \( z = u \) by \( \mathfrak{A} \models NHV6 \), so we have the situation as in figure 4.

We now have \( w \sim_0 z = u \sim_0 v \sim_0 y \), so \( y \sim_0 w \) after all, and we are back in case 1: take \( x = y \).

\[ \square \]

3.5.3. Relation Algebras.

The completeness theorem for \( C \mathcal{C} \mathcal{B} \) has as an immediate corollary a finite derivation system generating \( Equ(RRA) \):

Definition 3.5.9.

Define the following derived \( C \mathcal{C} \mathcal{B} \) term:

\[ d'(x) = 0; x; 1 + 1; x; 0' \].

\( \square \)

Clearly \( d'(x) \) is the algebraic version of the defined \( D' \)-operator.

Definition 3.5.10.

Let \( \Psi_2 \) be the smallest set of \( C \mathcal{C} \mathcal{B} \)-equations containing \( RA0, \ldots, RA8 \) which is closed under ordinary algebraic deduction and under the following closure operation:

\[ y \cdot d'(y) \leq \frac{t(x_0, \ldots, x_{n-1})}{t(x_0, \ldots, x_{n-1})} = 1 \]

provided \( y \) does not occur among the \( \bar{x} \).

Theorem 3.5.11.

\[ \Psi_2 = Equ(RRA) \]

Proof.

Immediate by the fact that the derivation system generating \( \Psi_2 \) is the algebraic version (cf. Appendix A.33) of \( AR^+ \), the completeness theorem 3.3.37 and 3.3.6.

\[ \square \]

3.6 Conclusions, Remarks and Questions.

3.6.1. General Conclusions.

In this chapter we have given a setup for a uniform, systematic study of two-dimensional modal logics (section 1). Three examples have been worked out in detail: \( CML_2 \) (section 2), \( C \mathcal{C} \mathcal{B} \) (section 3) and \( C \mathcal{C} \Lambda \) (section 4). \( CML_2 \) and \( C \mathcal{C} \mathcal{B} \) form the modal counterpart of two-dimensional cylindric resp. relation algebras. In these two formalisms we have put some modal machinery in action: for \( CML_2 \) we used Sahlqvist's theorem to find a
complete axiomatization of the formulas valid in the squares; to obtain an analogous result for "CC§" we needed the SD-theorem of the first chapter as well. In section 4 we added to "CC§" a constant referring to the ordering relation in a time structure and obtained a two-dimensional temporal logic, "CCλ". For this similarity type we found, besides a rather easy completeness proof, a result on expressive completeness, in the style of Kamp’s theorem. The modal approach to the algebraic framework turned out to be rather fruitful: in section 5 we saw how the modal completeness results for "CML2" and "CC§" yielded finite derivation systems for the equations valid in the classes of representable cylindric resp. relation algebras. We also proved that in cylindric algebras, Henkin’s equation could be simplified to "CE7" (cf. definition 3.5.3), a possible candidate as the shortest equation in Equ(RCA2) = Equ(CA2) (cf. Problem 10 in [?]).

3.6.2. Questions and Remarks.

Lots of topics have been left untouched. To mention a few:

(1) There are more two-dimensional modal operators possible besides the ones we have mentioned; for example, operators corresponding to the algebraic Q-operators defined by Jónssom [59]. The corresponding modal similarity type would contain all two-dimensional, first order definable, additive modal operators. Andréka-Németi [9] and Venema [136] study the expressive power of this similarity type; in both papers it is proved that over the class of squares, the system with Q-operators is still less expressive than first order logic L²(x0, x1).

Other examples are non-additive modal operators, like two-dimensional versions of S and U (‘Since’ and ‘Until’), or modal versions of operators that are not first order definable, like the Kleene star.

(2) We have confined ourselves to studying squares; it is also interesting to study rectangle models where the set of possible worlds W is of the form W = U0 x U1 with U0 and U1 possibly distinct. This is the approach taken by e.g. Shekhtman [120].

(3) There are more connections between relation algebras and modal logics. In Olsowska [92, 93] similarity types have some additional algebraic structure, namely that of a relation algebra. The idea is that if □a and □b are diamonds, then so are □a⁻¹, □⁻a, □a; = b, etc. In the intended frames one should find R□⁻a = R□a, etc. This formalism can be interpreted in "CC§", cf. [131]. Note that, extending this relation algebra with the Kleene star, we would obtain an enriched dynamic logic.

(4) An intriguing question (at least to the author): is AtRRA elementary?

Finally, many of the remarks made in the conclusions of chapter 4 apply to the modal version "CC§" of (representable) relation algebras as well.

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3For a more detailed discussion of the relation algebraic case, the reader is referred to the conclusions of the next chapter, where an analogous result on higher-dimensional cylindric algebras is treated.

4cf. also 5.5.2(ii).