Chapter V.

More Results about Definability

of I-operators. Generalized I-operators.

The first part of this chapter will be devoted to some more results about the definability of I-operators, the most important result being that not all normal I-operators are standard. The last part will be devoted to a generalization of the concept of \( I^n \)-function to \( I^n \)-functions with infinite domains and to the consequent generalization of the concepts of I-operators, characteristic sets, etc. Here we will reach a completeness theorem, but we will have to use classical methods.

The clearest method to prove that not all normal I-operators are standard uses \( I^n \)-functions on trees. As this is also an interesting subject in itself, we will start with an exposition on these \( I^n \)-functions.

**Def.** A tree is a P.O.G.-set such that for all \( t \in T \) the set \( \{ t' \in T : t \leq t' \} \) is finite and linearly ordered.

**Def.** If \( f \in F^n \), then \( f \) is tree-irreducible if \( D_f \) is a tree, and for every \( g \in F^n \), if \( g \) is a reduced form of \( f \), and \( D_g \) is a tree, then \( g \equiv f \).
Lemma 5.1. If $T$ is a tree, then, for all $t \in T$, $T(t)$ is a tree, and $T-T(t)$ is a tree.

Proof. Trivial.

Th.5.1. If $f$ is tree-irreducible, then for all $p \in D_f$, $f_p$ is tree-irreducible.

Proof. By the lemma $D_f(p)$ is again a tree.

If there were a non-isomorphic reduced form $g$ of $f_p$ by $\phi$, then we could construct a non-isomorphic reduced form $h$ (by $\psi$) of $f$ by defining $D_h=(D_f-D_f(p)) \cup D_g$, $q'^{\leq}_h q$ iff $q'^{\leq}_f q$ or $q'^{\leq}_g q$ or $\phi^{-1}(q')^{\leq}_f q$ for all $q', q \in D_h$, $h(q)=g(q)$ for all $q \in D_g$, $h(q)=f(q)$ for all $q \in D_f-D_f(p)$, and $\phi(p')=p'$ for all $p' \in D_f-D_f(p)$, and $\phi(p')=\psi(p')$ for all $p' \in D_f(p)$.

Th.5.2. $f \in \mathbb{F}^n$ is tree-irreducible iff $D_f$ is a tree, and (1) $f$ allows no $\alpha$-reduction, (2) there are no $r, r', t \in D_f$, such that $r \neq r'$, $r'$ is an immediate predecessor of $t$, $r^{\leq}_f t$ and $f_r \neq f_r'$.

Proof. $\Rightarrow$(1) is obvious.

(2) Assume there are $r, r', t \in D_f$ such that $r'$ is an immediate predecessor of $t$ and $r^{\leq}_f t$, and $f_r \neq f_r'$, by $\phi$. Then define $g$ as the restriction of $f$ to $D_f-D_f(r')$. By the lemma $D_g$ is a tree. Now define $\psi$ on $D_f$ as follows: $\psi(p)=p$ iff $p \not\in D_f(r')$, $\psi(p)=\phi(p)$ iff $p \in D_f(r')$. To prove that $\psi$ is strongly isotone we have to prove the properties (i) and (ii) of the definition of strongly isotone. Property (i) is immediately obvious. To prove (ii), assume $\psi(q) \leq \psi(p)$.
Now there are three possibilities:
I. $\psi(q) \notin D_f(r)$, $\psi(p) \notin D_f(r)$. Then $\psi(q) = q$, $\psi(p) = p$, so $q < p$.
II. $\psi(q) \in D_f(r)$, $\psi(p) \notin D_f(r)$. Then $\psi(p) = p$, so $\psi(q) < p$, and $\psi(\psi(q)) = \psi(q)$.
III. $\psi(q) \in D_f(r)$, $\psi(p) \in D_f(r)$. Then there are again two possibilities: (a) $\psi(p) = p$. Then $\psi(q) \leq \psi(p) = p$, and $\psi(\psi(q)) = \psi(q)$.
(b) $\psi(p) \neq p$, so $p = \phi(\psi(p))$ and $\phi(\psi(q)) < p$, and $\psi(\phi(\psi(q))) = \psi(q)$, since on $D_f(r)$ $\phi$ is the inverse of $\psi$.

This we will prove by induction on the depth of $f$. For depth 1 the result is trivial. So assume the depth of $f$ is $m$ and the theorem is valid for depth $< m$, and assume (1) and (2). Obviously (1) and (2) also hold for the sub-$I^n$-functions of $f$. So, if we assume that $p_1, \ldots, p_k$ ($k \geq 1$) are the immediate predecessors of $m_f$ w.r. to $< f$, then the induction hypothesis assures us that $f_{p_1}, \ldots, f_{p_k}$ are tree-irreducible. Now there are two possibilities:

I. $k = 1$. Now, if $g$ is a reduced form of $f$ by $\phi$, then $\phi(D_f(p_1))$ is isomorphic to $D_f(p_1)$, since according to the induction hypothesis $f_{p_1}$ is tree-irreducible. But also $\phi(m_f) \neq \phi(p_1)$, otherwise $\phi$ would be an $\alpha$-reduction-function, contrary to (1). This implies that $f \equiv g$ by $\phi$, and $f$ is tree-irreducible.

II. $k > 1$. Again assume $g$ is a reduced form of $f$ by $\phi$. Assume $r, r' \in D_f$, $r \neq r'$ and $\phi(r) = \phi(r')$. If we assume that $r' = m_f$, then for some $m$ ($1 \leq m < k$) $r \not\equiv_m$. But then
\( \phi(p_m) \leq \phi(m_r) = \phi(r) \), and, since \( \phi \) is strongly isotone, for some \( s \leq r \) \( \phi(s) = \phi(p_m) \). This means that we can assume that \( r \neq m_r \) and \( r' \neq m_r \). In that case for exactly one \( i \) and exactly one \( j \) (\( 1 \leq j, 1 \leq k \)) \( r \leq p_i \) and \( r' \leq p_j \), since \( D_f \) is a tree. Also if \( i \neq j \), otherwise \( g_{\phi(p_1)} \) would be a non-congruent reduced form of \( f \) contrary to the induction hypothesis. But \( \phi(p_i) \leq \phi(p_j) \) or \( \phi(p_j) \leq \phi(p_i) \), otherwise \( D_g \) would not be a tree. Assume \( \phi(p_j) \leq \phi(p_i) \). Then, since \( \phi \) is strongly isotone, \( \phi(p_j) = \phi(q) \) for some \( q \leq p_i \). Since \( f \) is tree-irreducible, \( \phi(D_f(p_j)) \) is isomorphic to \( D_f(p_j) \), and for the same reason, \( \phi(D_f(q)) \) is isomorphic to \( D_f(q) \). Now, if \( s \leq p_j \), then \( \phi(s) \leq \phi(q) \). So for some \( s' \leq q \) \( \phi(s') = \phi(s) \). But the fact that \( \phi(D_f(q)) \) is isomorphic to \( D_f(q) \), then implies that this \( s' \) is unique. The same thing holds inversely, so \( f \) \( \equiv f \), and \( q \leq m_r \), the immediate successor of \( p_j \), contrary to (2). So for all \( r, r' \in D_f \), \( r \neq r' \) implies \( \phi(r) \neq \phi(r') \). The properties of strongly isotone functions then imply that \( \phi \) is an isomorphism. So \( f \) is a tree-irreducible \( I^n \)-function.
An example of an \( I^2 \)-function that is tree-
irreducible, but not irreducible, is:
\[
\begin{array}{c}
(0,0) \\
(0,1) \\
(0,0)
\end{array}
\begin{array}{c}
(0,0) \\
(0,0)
\end{array}
\begin{array}{c}
(0,0) \\
(1,0)
\end{array}.
\]

It allows a \( \beta \)-reduction to the irreducible \( I^2 \)-function:
\[
\begin{array}{c}
(0,0) \\
(0,0) \\
(0,0)
\end{array}
\begin{array}{c}
(0,0) \\
(1,0)
\end{array}.
\]

**Th. 5.3.** If \( f \in F^n \), \( D_f \) is a tree, and \( p_1 \) is the only
immediate predecessor of \( m_f \) w.r. to \( <_f \), then \( f \) is tree-
irreducible iff, \( f_{p_1} \) is tree-irreducible and \( f(m_f) \neq f(p) \).

**Proof.** Immediate from Th. 5.2.

**Th. 5.4.** If \( f \in F^n \), \( D_f \) is a tree, and \( p_1, \ldots, p_k \)
\((k \geq 2)\) are the only immediate predecessors of \( m_f \) w.r. to \( <_f \),
then \( f \) is tree-irreducible iff, for all \( i \) \((1 \leq i \leq k) \), \( f_{p_i} \) is
tree-irreducible and for no \( i, j \) \((1 \leq i, j \leq k, i \neq j) \) \( f_{p_i} < f_{p_j} \).

**Proof.** Immediate from Th. 5.1 and Th. 5.2.

Assume \( f \) not tree-irreducible, and apply
Th. 5.2. As no \( \alpha \)-reduction is possible, there must be \( r, r' \in D_f \)
\((r \neq r')\) such that \( f_r \equiv f_{r'} \) and \( r \leq t \), \( t \) being the direct
successor of \( r' \). Obviously \( r \leq m_f \) and \( r' \leq m_f \), so for some \( i, j \)
\((1 \leq i, j \leq k) \), \( r \leq p_i \) and \( r' \leq p_j \). As \( p_i \) is tree-irreducible \( i \neq j \).
But then \( t = m_f \), \( r' = p_j \) and \( f_{p_j} < f_{p_i} \), contrary to hypthesis.
So \( f \) is tree-irreducible.
We will now prove that in each equivalence-class of $I^n$-functions there is a tree-irreducible $I^n$-function, unique up to congruence. The meaning of this theorem is that in our discussions in the Chapters II and III we could have restricted ourselves to $I^n$-functions on trees instead of P.O.G.-sets. The intuitive interpretation of Chapter I does not give grounds either for or against restricting ourselves to trees. Intuitively that means the choice between excluding or not excluding the possibility that two states incomparable in time both have the same possible future state in common.

**Lemma 5.2.** If $P$ is a finite P.O.G.-set, and for all $p \in P$ $p$ has at most one immediate successor, then $P$ is a tree.

**Proof.** Take any $p \in P$. Then define a sequence $p_0, \ldots, p_m$ for some $m \geq 0$ in the following way: $p = p_0$; for all integers $i$, if $p_{i-1}$ is the maximum element of $P$, then $i-1 = m$. If $p_{i-1}$ is smaller than this maximum element, then $p_i$ is the unique immediate successor of $p_{i-1}$. The sequence thus obtained is the set $\{p' \in P: p \prec p'\}$, and so this set is linearly ordered, and $P$ is a tree.

**Th. 5.5.** For any $h \in F^N$, there is an $f$ such that $h \equiv f$ and $f$ is tree-irreducible. This $f$ is unique up to congruence.
Proof. Assume $g \in F^n$, $g$ irreducible and $g \equiv h$. We will construct a tree-irreducible $f$ such that $f \equiv g$. If $D$ is a tree, our problem is solved. So we assume that $g$ is not a tree. By lemma 5.2 there is then an $r \in D_g$ such that $r$ has more than one immediate successor. Let us assume that $r$ is minimal with respect to this property, and that $s_1, \ldots, s_k$ are the immediate successors of $r$. Then $D(r)$ is a tree, and, for all $r' \in D_g(r)$, if $r' \leq s$, then $s \leq r$ or $r \leq s$. Now we take $k-1$ trees from $A, T_1, \ldots, T_{k-1}$, disjoint from $D_g$ and from each other, such that for all $i (1 \leq i \leq k-1)$, $T_i$ is isomorphic to $D(r)$ by $\phi_i$. Then we define an $\in$-function $g'$ as follows: $D_g' = D_g \cup \bigcup_{i=1}^{k-1} T_i$; for all $p \in D_g g'(p) = g(p)$ and for all $p \in T_i (1 \leq i \leq k-1)$ $g'(p) = g(\phi_i(p))$; and for all $p', p \in D_g$, $p' \leq g', p$ iff, either $p', p \in D_g(r)$ and $p' \leq g, p$ or $p', p \in D_g - D(r)$ and $p' \leq g, p$, or for some $i (1 \leq i \leq k-1)$ $p', p \in T_i$ and $\phi_i(p') \leq g, \phi_i(p)$, or for some $i (1 \leq i \leq k-1)$ $p', p \in T_i$ and $s_i \leq g, p$ or $p' \in D_g(r)$ and $s_i \leq g, p$. Then the function $\phi$ from $D_g'$ onto $D_g$ defined by $\phi(p) = p$ for all $p \in D_g$ and $\phi(p) = \phi_i(p)$ for $p \in T_i$, is strongly isotone. So $g'$ is equivalent to $g$. If $D_g'$ is not a tree, then we repeat the same procedure for $g'$ etc. As the number of elements with more than one immediate successor diminishes each time, the process must end. The end product $f$ has then a tree as domain. We will prove by induction on the depth of $g$ (= the depth of $f$) that $f$ is
tree-irreducible. For depth 1 this is trivial. So we now assume that the depth of \( g \) is \( m \) and that this process applied to any irreducible \( I^n \)-function of depth \( < m \) delivers a tree-irreducible \( I^n \)-function. When we look at the construction of \( f \) above, we see that \( m_f = m_g \), and that the immediate predecessors \( p_1, \ldots, p_k \) of \( m_f \) w.r. to \( <_f \) are also the immediate predecessors of \( m_g \) w.r. to \( <_g \). We also see in that construction that the same process was applied to \( g_{p_i} \) for all \( i \) \((1 \leq i \leq k)\) with the function \( f_{p_i} \) as outcome for all \( i \) \((1 \leq i \leq k)\). The induction hypothesis then states that \( f_{p_i} \) is tree-irreducible for all \( i \) \((1 \leq i \leq k)\). Now we study two cases. (1). \( k=1 \). Then \( f(m_f) = g(m_g) \neq g(p_1) = f(p_1) \), and by Th. 5.3 \( f \) is tree-irreducible. (2). \( k>1 \). Then for no \( i, j \) \((1 \leq i, j \leq k, i \neq j)\) \( f_{p_i} < f_{p_j} \), since that would imply \( g_{p_i} < g_{p_j} \) (see lemma 2.2), which in its turn would imply that \( g \) is not irreducible (by lemma 2.3). Then Th. 5.4 implies that \( f \) is irreducible.

Now we prove that \( f \) is unique, by induction on the depth of \( f \). For depth 1 it is again trivial. If the depth of \( f \) is \( m \), the we assume the theorem for \( I^n \)-functions with depth \( < m \). Assume that \( p_1, \ldots, p_k \) are the immediate predecessors of \( m_f \) w.r. to \( <_f \). Then according to the induction hypothesis \( f_{p_1}, \ldots, f_{p_k} \) are uniquely determined. It is then very easy to see that \( f \) is also uniquely determined.
Lemma 5.3. If \( f, g \) are tree-irreducible, \( f = h_1, g = h_2 \), \( h_1 \) and \( h_2 \) irreducible and \( h_1 \leq h_2 \), then \( f \leq g \).

Proof. Clear from the construction in Th. 5.5.

Th. 5.6. A normal I-operator \( a \) is uniquely characterized set \( C_a^* \) of all tree-irreducible \( f \in \mathbb{F}^n \) in its characteristic set (the tree-characteristic set of \( a \)), and there is a function from \( C_a^* \) onto \( C_a^{**} \) that is an isomorphism w.r. \( \leq \), and if it maps \( f \) onto \( g \) then \( f \leq g \).

Proof. Immediate from Th. 5.5 and Lemma 5.3.

Th. 5.7. For \( n \geq 2 \) not all normal n-ary I-operators are standard.

Proof. We will construct a sequence of \( I^2 \)-functions \( \{u_{ij}\}_{i=1}^{\ldots} \) as follows by induction on \( i \) (it is obvious how to do this in an exact way, but very tiresome, so we will do it with the help of pictures) \( u_{11} = (1,1) \), \( u_{12} = (1,0) \), \( u_{13} = (0,1) \), \( u_{1+1} = (0,0) \), \( u_{1+2} = (0,0) \), \( u_{1+3} = (0,0) \), for all \( i \geq 1 \). Then we can prove by induction on \( i \) (a) for all \( i, j \) \( (i \geq 1, 1 \leq j \leq 3) \) \( u_{ij} \) is tree-irreducible, and (b) for all \( i \leq 1 \), if \( j \neq k \) \( (1 \leq j, k \leq 3) \) \( u_{ij} \neq u_{ik} \). For \( i = 1 \) it is trivial and if (b) is true for \( i = k \), then by Th. 5.4 (a) is true for \( i = k + 1 \), and (b) follows immediately for \( i = k + 1 \).

Now we construct a sequence \( \{v_i\} \) of \( I^2 \)-functions by induction on \( i \) from the sequence \( \{u_{ij}\} \). \( v_1 = (0,0) \), for all \( i \geq 1 \).
Again it is obvious that for all $i \ (1 \leq i < \infty)$ $v_i$ is tree-irreducible, but also for all $i, j \ (i \geq 1, j \geq 1)$ if $i \neq j$ then $v_i \not\leq v_j$. This is obvious if $i > j$, and if $i < j$, then $v_i \not\leq v_j$ would imply $v_i \equiv u_{km}$, for some $k$ and $m \ (1 \leq k < j, 1 \leq m < 3)$, and this is impossible since $D_{u_{km}}$ is a binary tree, and $m_{v_i}$ has three immediate predecessors. Now we can for any set of natural numbers $M \subseteq \mathbb{N}$ ($N$ being the set of all natural numbers) define an operator $a_M$, by its tree-characteristic set, $f \in C^*\!\!\!\!^* a_M$ iff $f \leq v_i$ for some $i \in M$. Now it is obvious from the fact that $i \neq j$ then $v_i \not\leq v_j$ that, if $M \neq L$, then $C^*\!\!\!\!^* = a_M \not\leq a_L$. This implies that there are non-denumerably many normal $I$-operators, and so they cannot all be standard.

Note that not even all primitive recursive normal $I$-operators are standard, since the logic is decidable. This is of course a negative theorem, and we will now prove that no $I$-operator like $a_M$ with $M$ infinite can be a standard $I$-operator.

**Def.** A set $\{f_1, \ldots, f_m\} \subseteq \mathbb{P}^N$ is independent, if for no $i, j \ (1 \leq i, j \leq m, i \neq j)$ $f_i \leq f_j$.

**Def.** A normal $n$-ary $I$-operator $a$ is weakly connected of degree $m$, if for any independent sequence $f_1, \ldots, f_{m+1} \in C^*\!\!\!\!^* a$ there exists a $g \in C^*\!\!\!\!^* a$ of the form $g(m)$ for some $i, j \ (1 \leq i, j \leq m, i \neq j)$. (I.e. if there exists a $g \in C^*\!\!\!\!^* a$, such that $m_g$ has
two direct predecessors \( q_1 \) and \( q_2 \) in \( D_f \), and \( g_{q_1} = f_1 \) and \( g_{q_2} = f_j \).

Note that, if \( a \) is weakly connected of degree 1, then \( a \) is connected. Not even for standard I-operators though are these concepts equivalent; e.g. the unary standard I-operator with tree-characteristic set

\[
\{ 0, 0, 1, 0 \}
\]

is connected like all finite normal I-operators of which the tree-characteristic (or normalized characteristic) set has only one maximal element, but for the I-functions 0 and 1 we cannot find an I-function \( g \) as required by the definition. On the other hand this I-operator is trivially weakly connected of degree 2, since there are no subsets of more than two elements of its tree-characteristic set that are independent.

**Def.** A normal I-operator is *weakly connected*, if it is weakly connected of degree \( m \) for some \( m \). A normal I-operator is *disconnected*, if it is not weakly connected.

From the remarks just made it is easy to conclude that all finite normal I-operators are weakly connected. Examples of disconnected I-operators are the binary I-operators \( a_M \) defined in the proof of Th. 5.7, in the cases that \( M \) is infinite. We will prove that all standard I-operators are weakly connected, but that not all weakly connected normal I-operators are standard. We will prove
the last statement first.

**Th. 5.8.** Not all binary weakly connected normal I-operators are standard.

**Proof:** We consider the I-operators $a_M$ for subsets $M$ of $N$ constructed in the proof of Th. 5.7. We define a binary operation $C$ on the set of all independent couples from $(F^2)^2$ by, $C(f, g) = (0, 0)$. Then we take for all $f \rightarrow g$ $M \triangleleft N$ the closure $S_M$ of $C^{**}$ w.r. to $C$. The set $S_M$ has all the properties required for a tree-characteristic set.

Now define for all $M \triangleleft N$ $b_M$ as the normal I-operator with tree-characteristic set $S_M$. For all $M \triangleleft N$, $b_M$ is weakly connected of degree 1, and for all $M, L \triangleleft N$, if $M \neq L$, then $b_M \neq b_L$. This implies that there are nondenumerably many binary weakly connected normal I-operators, and not all of these can be standard.

The proof of this theorem shows that the sharpest characterization we have as yet been able to give, is by no means sharp enough. It seems that we need a more restrictive concept in the spirit of weakly connected. The set of $n$-ary weakly connected I-operators is not even a closed set, at least for $n \geq 3$. (For binary I-operators we have not been able to prove this, for unary I-operators the whole situation is special, as we will see later).
Th. 5.9. The set of ternary weakly connected normal I-operators is not closed.

Proof. Since \& is a finite normal I-operator (Th. 3.3), \& is weakly connected. In fact \& is weakly connected of degree 1. We will now construct two ternary normal I-operators that are weakly connected of degree 1 of which the conjunction is disconnected. Define $a_N$ as in the proof of Th. 5.7. Then define the ternary I-operator $a'$ by, for all $f \in F^3$, $f \in C'^*$ iff $(f^1, f^2) \in C'^*$ and for all $p \in D_f$, $f^3(p) = 1$. And we define the binary operations $C$ and $C'$ on the set of independent couples from $(H^3)^2$ by

$$C(f, g) = \begin{cases} (0, 0, 0) & \text{if } f \neq a_N \\ (0, 0, 1) & \text{if } f = a_N \end{cases}$$

Then we take the closures $S$ and $S'$ of $C'^*$ w.r. to $C$ and $C'$. $S$ and $S'$ again have all the properties required for tree-characteristic sets and $S \cap S' = C'^*$. This means that for the normal I-operators $b'$ and $c'$ defined by $C_{b'}^* = S$ and $C_{c'}^* = S'$, $b' \land c' = a'$. But is clear that $b'$ and $c'$ are weakly connected, while $a'$ is not weakly connected. This means that the set of ternary weakly connected normal I-operators is not closed.

As \&, $b'$ and $c'$ are connected and $a'$ is not, this proof also shows that the set of all connected normal I-operators is not closed. The proof shows too that we
cannot prove that all standard I-operators are weakly connected by a simple induction over the number of occurrences of the symbols ∧, ∨, ⇒ and ¬ in the definition of the I-operator, since, if a and b are weakly connected, a&b is not necessarily so.

Th. 5. 10. All standard I-operators are weakly connected.

Proof. By induction over the length of the definition of the I-operator a. If a has length 1, then a = u^n_i for some i. u^n_i is weakly connected of degree 1, since, if f_1, f_2 ∈ C^** and f_1, f_2 independent, then the g defined by, g^j(m_i) = 0 for j ≠ i (1 ≤ j ≤ n) and g = g(m_i) has the required properties.

Now we assume the theorem is valid for all standard I-operators with length ≤ k, and we assume a has length k+1. We will treat a as a ⇒ ¬(u^n_1 ⇒ u^n_i). This means that we have to look at the cases a = ¬(u^n_1 ⇒ u^n_i), a = bvc, a = b ⇒ c and a = b & c.

(i). a = ¬(u^n_1 ⇒ u^n_i). In that case C^**_a = ∅, and a is trivially weakly connected of degree 1.

(ii). a = bvc. By the induction hypothesis b and c are weakly connected, assume of degrees k and m. We will prove that then a is weakly connected of degree k+m. Assume f_1, ..., f_{k+m+1} independent and in C^**_a, then, since
C** = C** ∪ C**, either if necessary after renumbering
f_1, ..., f_{k-1} ∈ C**, or f_1, ..., f_{m+1} ∈ C**. In both cases we
find a g as required.

(iii) a = b = c. By the induction hypothesis c is
weakly connected, assume of degree m. We will prove
that a is weakly connected of degree m. Assume f_1, ..., f_{m+1}
independent and in C**_{b= c}, then there are two possibilities:

(1) for all i (1 ≤ i ≤ m + 1) f_i ∈ C**; then the g we find
in C** is also an element of C**_{b= c},

(2) for some i (1 ≤ i ≤ m + 1) f_i ∉ C**; then take j ≠ i
(1 ≤ j ≤ m + 1) and define g = (0, 0, ..., 0). Now f_i ∉ C** implies
f_i ∉ C**, since f_i ∈ C**. But then also g ∉ C**, g ∉ C**. This
implies that for all p ∈ D_g, if g_p ∈ C**, then g_p ∈ C**, so g ∈ C**_{b= c}.

(iv) a = b & c. There are three subcases:

(1) b = u_i^n, c = u_j^n for some i, j ≤ n. Then a is weakly
connected of degree 1. The proof is similar to the one for
case (i).

(2) b = b_1 v b_2, or c = c_1 v c_2. Then since (b_1 v b_2) & c =
(b_1 & c) v (b_2 & c) and b & (c_1 v c_2) = (b & c_1) v (b & c_2) and b_1 & c, b_2 & c,
b & c_1 and b & c_2 are weakly connected by the induction hypothesis,
we can apply (ii) again.

(3) Since we can write (u_i^n > u_i^n) > u_i^n for u_i^n, the only
case left to investigate is a = (a_i > b_i). We will give the
proof for the case that m = 2, it is easily seen that the
proof for the general case is similar. By the induction hypothesis \(b_1, b_2\) and \(b_1 \& b_2\) are weakly connected, assume of respective degrees \(p, q, r\). Assume \(s = \text{Max}(p, q, r)\). Then we will prove that \(a\) is weakly connected of degree \(3s + 1\).

Let us assume \(f_1, \ldots, f_{3s+1}\) is an independent sequence in \(C_a^{**}\). Then (after renumbering if necessary) there are four possible cases.

I. \(f_1, \ldots, f_{s+1} \in C_{b_1}^{**}, f_1, \ldots, f_{s+1} \not\in C_{b_2}^{**}\). Then \(f_1, \ldots, f_{s+1} \in C_{b_1 \& b_2}^{**}\), so we can find a \(g\) with the required properties in \(C_{b_1 \& b_2}^{**}\). Then \(g \in C_{a_1}^{**}\), since \(C_{b_1 \& b_2}^{**} \subset C_{a_1}^{**}\). Now, since \(f_1 \not\in C_{b_2}^{**}, f_1 \not\in C_{a_2}^{**}\), also \(g \not\in C_{b_2}^{**}\) and \(g \not\in C_{a_2}^{**}\).

II. \(f_1, \ldots, f_{s+1} \in C_{b_1}^{*}, f_1, \ldots, f_{s+1} \not\in C_{b_2}^{**}\). Then for some \(i, j\) (\(1 \leq i, j \leq s+1\)) \(g = g(m) \in C_{b_1}^{**}\), so \(g \in C_{a_1}^{**}\). Now, since \(f_1 \not\in C_{b_2}^{**}, f_1 \not\in C_{a_2}^{**}\).

III. \(f_1, \ldots, f_{s+1} \not\in C_{b_1}^{**}, f_1, \ldots, f_{s+1} \in C_{b_2}^{**}\). Proof similar to case II.

IV. \(f_1 \not\in C_{b_1}^{**}, f_1 \not\in C_{b_2}^{**}\). Then take \(g = (0, 0, \ldots, 0)\) and again \(g \in C_{a}^{**}\).

The unary normal I-operators take a very special place in the set of all I-operators. We are able to prove that all unary normal I-operators are standard, since the only infinite normal I-operator is \(u^1 \Rightarrow u^1\). This we will prove now in the following theorem.
Th. 5.11. All unary normal I-operators are standard.

Proof. We define a sequence of the irreducible
I-functions \( w_i (i=1, \ldots, ) \) with the help of pictures in the
following way: \( w_0 = 1 \), \( w_1 = 0 \), \( w_2 = 1 \), for all \( i \geq 3 \)

\[
\begin{array}{c}
\text{w}_1 \\
\text{1-2} \\
\text{w} \\
\text{1-3}
\end{array}
\]

We will prove for all \( i \geq 0 \) that \( w_i \) is tree-irreducible
and that for all \( j \geq 0 \), \( w_j \leq w_i \iff j = i \) or \( j \leq i-2 \), by induction
on \( i \). The statement is clearly true for \( i=0,1,2 \). Let us
assume \( k > 3 \) and the statement is valid for all \( i < k \).

\[
\begin{array}{c}
w_k \\
w_{k-2} \\
w_{k-3}
\end{array}
\]

According to the induction hypothesis
\( w_{k-2} \) and \( w_{k-3} \) are tree-irreducible and \( w_{k-3} \leq w_{k-2} \). If \( j > 1 \),
then clearly \( w_j \leq w_i \), so \( w_{k-2} \leq w_{k-3} \). Then according to Th. 5.2
\( w_k \) is tree-irreducible. Now assume \( j \leq k-2 \), then there are
three possible cases: (1) \( j \leq k-4 \), then \( w_j \leq w_{k-2} \leq w_k \), by the
definition of \( w_k \). (2) \( j = k-3 \), then \( w_j \leq w_k \) by the definition
of \( w_k \). (3) \( j = k-2 \), then \( w_j \leq w_k \) by the definition of \( w_k \).
To conclude, not \( w_{k-1} \leq w_k \), since not \( w_{k-1} \leq w_{k-2} \), \( w_{k-1} \leq w_{k-3} \), and
not \( w_{k-1} = w_k \).

Next we will prove that this sequence is complete
in the sense that all I-functions are equivalent to \( w_i \) for
some natural numbers. According to the Th. 5.5 it is suffi-
cient to prove that all tree-irreducible I-functions are
congruent to \( w_1 \) for some \( i \). We will prove this for tree-irreducible I-functions \( f \) by induction on the depth of \( f \).

If \( f \) has depth 1, then \( f \) is clearly congruent to either \( w_0 \) or \( w_1 \). It is also clear that for all \( i \geq 0 \), \( w_{2i-2} \) and \( w_{2i-1} \) have depth 1. Now assume \( f \) has depth \( n > 1 \), and assume the statement we want to prove is valid for all I-functions with depth \(< n \). Assume further \( m_f \) has immediate predecessors \( p_1, \ldots, p_k \) (\( k \geq 1 \)). Then for all \( i \) (\( 1 \leq i \leq k \)) \( f_{p_i} \) has depth \(< n \), so according to the induction hypothesis \( f_{p_i} \preceq w_j \) for some \( j \) (\( 0 \leq j \leq 2n-1 \)). There are now two possible cases: (1) \( k = 1 \). Then \( f(m_f) = 0 \), \( f(p_1) = 1 \), otherwise there would exist an \( \alpha \)-reduction of \( f \) w.r. to \( p_1 \), \( m_f \). Then \( f_{p_1} \equiv 1 \equiv w_0 \), since, if not \( f_{p_1} \equiv 1 \), then \( f_{p_1} \) is not tree-irreducible and neither is \( f \) (Th. 5.1). So \( f = w_2 \). (2) \( k > 1 \). Since for all \( i, j \) if \( j \leq i - 2 \) then \( w_j \preceq w_4 \), according to Th. 5.4 \( k = 2 \). That the depth of \( f \) is \( n \) implies that \( f \equiv \frac{0}{\frac{w_{2n-4}}{w_{2n-5}}} = w_{2n-2} \) or

\[
\frac{f}{w_{2n-4} \quad w_{2n-5}} = w_{2n-2}
\]

Now we define a sequence \( c_{ij} \) (\( i = 1, \ldots, n \), \( j = 1, 2 \)) of unary normal I-operators by their tree-characteristic sets. \( C^{**} = \emptyset \). \( C^{**} = \{ w_1 \} \), \ldots, \( C^{**} = 0 \). For all \( i \geq 2 \)

\[
C^{**} = \{ f \in F : f \preceq w_1 \text{ or } f \preceq w_1 \} = \{ f \in F : f \preceq w_1 \} = \{ w_0, \ldots, w_{i-1} \} \}
\]

\( C^{**} = \{ f \in F : f \preceq w_1 \} = \{ w_0, \ldots, w_{1-2}, w_1 \} \). It is clear that this sequence contains
all unary normal I-operators. All these I-operators are finite except c_{02}, and c_{02}=u_{1,1}\Rightarrow u_{1,1}, so all unary normal I-operators are standard according to Th. 3.7.

But we will give here a simpler way of defining the c_{ij}. We will prove that c_{01} = \gamma(u_{1,1}\Rightarrow u_{1,1}), c_{02} = u_{1,1}\Rightarrow u_{1,1}, c_{11} = u_{1,1}, c_{i2} = \gamma, c_{21} = u_{1,1}\Rightarrow \gamma(u_{1,1}), c_{22} = \gamma, and for all i \geq 3

c_{i1} = c_{i-1,1,2} \land c_{i-2,2}, c_{i2} = c_{i-1,1,2} \Rightarrow c_{i-2,2}. This is evident for c_{01}, c_{02}, c_{11}, c_{12} and c_{21}. f tree-irreducible and f \in C_{\gamma,\gamma}, iff for no g \preceq f g \in C_{\gamma,\gamma}, i.e. for no g \preceq f, g=0. This is true only for w_0, w_2. And \{w_0, w_2\} = c_{22}^* = c_{22}^{**}, so indeed C_{22} = \gamma. Now we prove the last part by induction on i.

Assume m \geq 3 and assume the definitions are valid for all i < m. Then we have to prove c_{m2} = c_{m-1,1,2} \land c_{m-2,2}, which follows from C_{m2}^{**} = c_{m-1,1,2} \land C_{m-2,2}^{**} = c_{m-1,2}^{**} (reasoning like in lemma 3.1 (c)). And we have to prove c_{m2} = c_{m-1,2} \Rightarrow c_{m-2,1}. To prove this last statement it is sufficient to establish that w_{m} \in C_{m2}^{**} \Rightarrow c_{m-1,1}, w_{m} \in C_{m2}^{**} \Rightarrow c_{m-1,1}, since

\{w_0, \ldots, w_{m-2}, w_{m}\}. According to the induction hypothesis w_{m-1} \in C_{m-1,2}^{**}, w_{m} \in C_{m-2,2}^{**}, w_{m} \in C_{m-2,2}^{**}, w_{m} \in C_{m-2,1}^{*}, w_{m} \in C_{m-1,2}^{**}

In the first place this implies w_{m-1} \in C_{m-1,2}^{**}. Further

w_{m} \in c_{m-2} \Rightarrow c_{m-1,1},

and for all sub-I-functions f of w_{m-2} that
are not congruent to $w_{m-2} \in \mathbb{C}^{**}_{m-1,2}$ and $g \in \mathbb{C}^{**}_{m-2,1}$; and $w_{m-3} \in \mathbb{C}^{**}_{m-1,2}$ and $w_{m-3} \in \mathbb{C}^{**}_{m-2,1}$. All this implies that indeed $w \in \mathbb{C}^{**}_{m-1,2}$, since for all $p \in \mathbb{D}_w$, if $(w_m)_p \in \mathbb{C}^{**}_{m-1,2}$, then $(w_m)_p \in \mathbb{C}^{**}_{m-1,2}$.

Now we are able to give a sequence of formulas that comprises all equivalence-classes of formulas formed from a single atom $A$. The formulas here seem to have the shortest length possible. (See for a very similar result [18]).
In the last part of this chapter we will give a short description of how we can generalize our concepts to \( I^n \)-functions with infinite domains. We will restrict our attention to countable domains. The generalization to higher cardinalities is easy to make.

Let \( \mathcal{B}' \) be the set of all P.O.G.-sets from \( A \).

**Def.** An "\( II \)-function" is a function with domain a P.O.G.-set \( \mathcal{P} \in \mathcal{B}' \) and range the set \( \{0, 1\} \) with the property: for all \( p, p' \in P \), if \( p' \leq p \) and \( f(p) = 1 \), then \( f(p') = 1 \).

We can now define the concepts of \( II^n \)-function, congruence of \( II^n \)-functions, II-operator, ordered II-operator, and characteristic set of an II-operator in exactly the same way as in Chapter II. We write \( D_f \) for the domain of an \( II^n \)-function \( f \), \( I^n \) for the set of all \( II^n \)-functions, \( I \) for \( I^1 \). The generalization of the notions of normal form, equivalence and normal I-operator gives some difficulties. If we define normal form in the same way as for \( I^n \)-functions, by means of reductions, then not all \( II^n \)-functions have a normal form. Th.2.2 is not valid for infinite partially ordered sets. E.g. take \( \langle N, \leq_1 \rangle \) where for all \( m, n \in N \), \( m \leq_1 n \) iff \( n = 0 \), and the set \( \{0, 1\} \) with the normal ordering. Then there is a strongly isotone function from \( N \) onto \( \{0, 1\} \) (for all \( n \phi(n+1) = 0, \phi(0) = 1 \), but \( \{0, 1\} \) cannot be reached from \( \langle N, \leq_1 \rangle \) by reductions. We succeed in
defining a normal form with the help of the strongly isotone functions. We can define the concept of reduced form in the same way as in Chapter II. We then give a definition suggested by Th.2.1. Cor.2.

**Def.** An IIⁿ-function is irreducible, if for any \( \tilde{g} \in \mathcal{I}ⁿ \), if \( g \) is a reduced form of \( f \), then \( g \equiv f \).

**Def.** An IIⁿ-function \( g \) is a normal form of the IIⁿ-function \( f \), if \( g \) is a reduced form of \( f \), and \( g \) is irreducible.

We have not succeeded in giving a direct proof of an equivalent of the uniqueness theorem Th.2.3. But we can give an indirect proof based on Th.4.6 of [9].

**Th.5.12.** (Th.4.6 of [9].) If \( P \) and \( Q \) are partially ordered sets, then \( \overline{Q} \) is a complete subalgebra (i.e. a subalgebra w.r. to \( \cup, n, \Rightarrow, -, \bigcup \) and \( \bigcap \)) of \( \overline{P} \) iff there exists a strongly isotone mapping from \( P \) onto \( Q \). In fact, if \( \phi \) is a strongly isotone function from \( P \) onto \( Q \), then the subalgebra \( A \) of \( \overline{P} \) defined by, \( A = \{ a \in \overline{P} : \forall p, q \in P \text{ if } p \preceq a \text{ and } \phi(p) = \phi(q) \text{ then } q \preceq a \} \), is isomorphic to \( \overline{Q} \) and forms a complete subalgebra of \( \overline{P} \).

**Th.5.13.** If \( f, g \in \mathcal{I}ⁿ \) and \( g \) is a reduced form of \( f \), then \( \overline{Dg} \) is isomorphic to a complete subalgebra of \( \overline{Df} \) that contains for all \( i \) (\( 1 \leq i \leq n \)) the elements \( a_i = \{ p \in D_f : f^i(p) = 1 \} \).

**Proof.** According to Th.5.12, if \( g \) is a reduced form of \( f \) by \( \phi \), then \( \overline{Dg} \) is isomorphic to the complete subalgebra
of $\overline{D_f}$ formed by the set $A$ of elements $\alpha \in \overline{D_f}$ with the property that for all $p \in A$, $q \in D_f$, if $\phi(p) = \phi(q)$, then $q \in A$. Assume for some $i (1 \leq i \leq n)$ $p \in A_i$ and $\phi(p) = \phi(q)$. Then $f(p) = g(\phi(p)) = g(\phi(q)) = f(q)$. So, since $f_i(p) = 1$, also $f_i(q) = 1$, and $q \in A_i$. So we have proved that for all $i (1 \leq i \leq n) A_i \in A$.

The next theorem besides giving us the necessary apparatus to prove that the normal form is unique up to congruence, also gives us some more insight in the results of Chapters II and III.

Th. 5.14. If $f, g \in \mathcal{I}^n$, and $g$ is a normal form of $f$, then $\overline{D_g}$ is isomorphic to the complete subalgebra of $\overline{D_f}$ generated by the elements $A_i = \{ p \in D_f : f_i(p) = 1 \} (1 \leq i \leq n)$ (i.e., the smallest complete subalgebra containing the $A_i$) and if $\psi$ is the isomorphism then for all $r \in D_g$, $g_i^j(r) = 1$ iff $r \in \psi^{-1}(A_i)$.

Proof. $\overline{D_g}$ is isomorphic to a complete subalgebra $A$ of $\overline{D_f}$. If $B$ is the complete subalgebra of $\overline{D_f}$ generated by the $A_i$, then $B \supset A$. Now, according to Th. 5.12 there is a strongly isotone function $\phi$ from $D_g$ onto $B$. There is an $II^n$-function $h$ definable on $B$ by, for all $r \in B$, $h(r) = g(s)$, if $s$ is such that $\phi(s) = r$. This is a proper definition, for assume $s, s' \in D_g, \phi(s) = \phi(s')$, and assume $g_i^j(s) = 1$ for some $i (1 \leq i \leq n)$. Then $s \in A_i$, and by Th. 5.12 applied to $\overline{D_g}$ and $B$, $s' \in A_i$, so $g_i^j(s') = 1$. By the same reasoning, if $g_i^j(s') = 1$ then $g_i^j(s) = 1$, for all $i (1 \leq i \leq n)$. This means that we have proven
g(s) = g(s'). So h is properly defined and h is a reduced form of f. As g was assumed to be irreducible it follows that g h, and \( \overline{D_g} \) is isomorphic to B. The last part of the theorem now follows immediately.

**Th. 5.15.** If \( f, g, h \in I^n \), and g and h are normal forms of f, then \( g \equiv h \).

**Proof.** Immediate from the Th. 5.14, since, in the first place, both \( \overline{D_g} \) and \( \overline{D_h} \) are isomorphic to the same subalgebra B of \( \overline{D_f} \), so \( D_g \) and \( D_h \) are isomorphic (Th. 4.4), and in the second place, both f and h are determined by the partial ordering of the \( \alpha_i \) in B.

Th. 5.15 enables us to define the concepts of equivalence, normal II-operator and normalized characteristic set in the same way as in Chapter II. Also, the concept of standard II-operator can be defined in the same way as in Chapter III.

**Def.** If \( J \) has cardinality \( \kappa \), and \( \{ a_i \}_{i \in J} \) is a set of normal II-operators, then \( \bigcup_{i \in J} (a_i) \) (\( \bigcap_{i \in J} (a_i) \)) is defined as the II-operator with normalized characteristic set \( \bigcup_{i \in J} (C_{a_i}^s) \) (\( \bigcap_{i \in J} (C_{a_i}^s) \)). We call these "generalized" II-operators the \( \kappa \)-disjunction and \( \kappa \)-conjunction.

**Def.** A quasi-standard II-operator is an II-operator the intersection of the sets G of II-operators such that (1) \( G \) contains all standard II-operators, (2) \( J \) has cardinality \( \kappa \leq 2^{|a_i|} \) and for all \( i \in J \), \( a_i \in G \), implies \( \bigcup_{i \in J} (a_i) \in G \).
and \( \bigcap_{i \in J} (a_i) \in G \).

**Def.** For any cardinal \( \kappa \) the set of \( \kappa \)-pseudo-Boolean terms is the intersection of all sets \( T \) such that (1) \( \alpha, \beta, \gamma, \alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots \in T \), (2) if \( U \) and \( V \) are in \( T \), then \( U \cup V \), \( U \cap V \), \( U \rightarrow V \) and \( -U \) are in \( T \), (3) if \( J \) has cardinality \( \leq \kappa \) and for all \( i \in J \), \( U_i \in T \), then \( \bigcup_{i \in J} U_i \in T \) and \( \bigcap_{i \in J} U_i \in T \).

**Lemma 5.4.** If the pseudo-Boolean algebra \( A \) has cardinality \( 2^{\kappa} \) and \( A \) is generated by \( \{\alpha_1, \ldots, \alpha_n\} \), then all elements of \( A \) can be written as \( 2^{\kappa} \)-pseudo-Boolean terms in \( \alpha_1, \ldots, \alpha_n \).

**Proof.** We can define a function from the set of atomic terms onto the set \( \{\alpha_1, \ldots, \alpha_n\} \). Then according to the recursion principle for terms (3.2.1 of [\( \mathcal{A} \)]) there is a homomorphism from the set of \( 2^{\kappa} \)-pseudo-Boolean terms into \( A \) that is an extension of this function. It is clear that the range of this mapping is a complete subalgebra of \( A \) containing \( \alpha_1, \ldots, \alpha_n \). From this the lemma follows immediately.

**Th. 5.16.** All normal II-operators are quasi-standard.

**Proof.** If \( a \) is a normal II-operator, then \( a \) has some normalized characteristic set \( \{g_{i_j} \}_{i \in J} \cdot \{g_i \}_{i \in J} = \bigcup_{i \in J} (f \in \mathbb{N}: f \leq g_i) \), so, if for all \( i \in J \) \( a_i \) are the normal II-operators with \( C_{a_i} = \{f \in \mathbb{N}: f \leq g_i\} \), then \( a = \bigcup_{i \in J} (a_i) \). So we only have to consider the normal II-operators that have a normalized characteristic set with a greatest element. Let us assume then that
\( C_a = \{ f \in \mathbb{R}^n : f \leq g \} \) for some irreducible \( g \). Then \( \overline{D_g} \) is generated by the \( \alpha_i = \{ q \in D_g : g^q = 1 \} \) (1 \( \leq i \leq n \)). This means that 
\( \overline{U_{\alpha_1, \ldots, \alpha_n}} = D_g = U(\alpha_1, \ldots, \alpha_n) \) for some \( \alpha \)-pseudo-Boolean term \( U \) in \( \alpha_1, \ldots, \alpha_n \), according to Lemma 5.4, since \( \overline{D_g} \) cannot have more than \( 2^k \) elements. Assume that \( \{ U_i \}_{i \in J} \) is the set of all pseudo-Boolean terms with this property. \( J \) then has at most cardinality \( 2^{2^k} \). Assume that, for all \( i \in J \), \( a_i \) is the normal II-operator corresponding to \( U_i \). (It is obvious that all quasi-standard II-operators are normal by the same reasoning as in Th.3.6.) Then we will prove that \( a = \bigcap_{i \in J} a_i \).

If we write \( V = U_1 \), then we have to show that \( f \leq g \) iff 
\( V(\beta_1, \ldots, \beta_n) = D_f \) where, for all \( i \), \( \beta_i \) is defined as 
\( \{ p \in D_f : f^p = 1 \} \). First assume \( f \leq g \). Without losing generality we can assume that \( f = g_q \) for some \( q \in D_g \). Then \( D_f = D_g(\beta) \) and \( D_f \) is a relativization of \( D_g \) (see Th.3.5 of \( \lbrack D_g \rbrack \)).

Then there is a complete homomorphism from \( D_g \) onto \( D_f \) ("complete" meaning a homomorphism also w.r. to the infinite operations) defined by, for all \( D_g \), \( \phi(\alpha) = \alpha \cap D_f \).

This implies, for all \( i \) (1 \( \leq i \leq n \)), that \( \phi(\alpha_i) = \beta_i \). And, since \( \phi \) is a complete homomorphism, \( U(\beta_1, \ldots, \beta_n) = D_f \). Now assume \( f \leq g \). Then take an irreducible II-function \( h \) such that \( f \leq h \) and \( g \leq h \). We again assume without losing generality that \( f = h_s \) and \( g = h_t \), for some \( s, t \in D_h \). Let for
all \(1 \leq i \leq n\) \(\gamma_i = \{r \in D_h : h_i^1(r) = 1\}\). Then by lemma 5.4 for some \(2^{\aleph_0}\)-pseudo-Boolean term \(W, W(\gamma_1, \ldots, \gamma_n) \equiv D_g\).

But then in \(\overline{D_g} W(\alpha_1, \ldots, \alpha_n) \equiv D_g\). So for some \(i \in J\), \(W = U_1\).

But in \(\overline{D_f} W(\beta_1, \ldots, \beta_n) \not\equiv D_f\). So also in \(\overline{D_f} V(\beta_1, \ldots, \beta_n) \not\equiv D_f\).

Th.5.17. The unary normal II-operators consist of the standard I-operator, and the II-operator \(\bigcup_{i \in I} c_{i2}\).

Proof. It is easy to check that this system of II-operators is closed under the operations of \&, v, \Rightarrow, \top, \land, \lor.