1.6 MODAL DEFINABILITY

This chapter is concerned with the question which is complementary to the one of chapter I.2, viz. which $L_0$-formulas are modally definable?

6.1 Definition

$P_1 = \{ \alpha \mid \alpha \text{ is an } L_0\text{-formula with one free variable such that, for some modal formula } \varphi, E(\varphi, \alpha) \}.$

$\bar{P}_1 = \{ \alpha \mid \alpha \text{ is an } L_0\text{-sentence such that, for some modal formula } \varphi, \bar{E}(\varphi, \alpha) \}.$

The first results of this chapter are about $P_1$, but the main emphasis will be on $\bar{P}_1$, for which an algebraic characterization is "almost" available.

6.2 Lemma

If $\alpha$ and $\beta$ are $L_0$-formulas with one and the same free variable $x$, then

(i) if $\alpha \in P_1$ and $\beta \in P_1$, then $\alpha \wedge \beta \in P_1$

(ii) if $\alpha \in P_1$ and $\beta \in P_1$, then $\alpha \vee \beta \in P_1$

(iii) if $\alpha \in P_1$, then $(\forall y)(Rxy + [y/x] \alpha) \in P_1$, provided that $y$ does not occur in $\alpha$. 
Proof: (i) follows from lemma 4.1, and so does (ii). (If $E(\phi, \alpha)$ and $E(\psi, \beta)$ for modal formulas $\phi$ and $\psi$, then change the proposition letters in $\phi$ and $\psi$ so that none occur in both $\phi$ and $\psi$. This amounts to a change of bound variables in an $L_2$-formula. After such a change lemma 4.1 is directly applicable.) (iii) follows from lemma 4.2(iv). QED.

6.3 Lemma

$P_1$ is not closed under $\neg$.

$P_1$ is not closed under restricted existential quantification.

Proof: $Rxx \in P_1$, because of $E(\Box p \to p, Rxx)$, but $\neg Rxx \not\in P_1$.

For, $\langle \text{IN}, \langle \rangle \rangle \models \neg Rxx \langle 0 \rangle$ and $f$ defined by $f(n) = 0$ for all $n \in \text{IN}$, is a $p$-morphism from $\langle \text{IN}, \langle \rangle \rangle$ onto $I = \langle 0 \rangle, \langle 0, 0 \rangle$, but $\models \neg Rxx \langle 0 \rangle$, and corollary 2.12 can be applied.

An argument similar to that proving $(\forall x)(\exists y)(Rxy \land Ryy)$ to be outside of $\bar{P}1$ (cf. the example after lemma 2.18) shows that $(\exists y)(Rxy \land Ryy) \not\in P_1$, from which the second assertion follows. QED.

An algebraic characterization result for $L_0$-formulas modally definable in the local sense could be extracted from the proof of theorem 6.15, but, since $\bar{P}1$ is our main object of interest in this chapter, this is omitted. Instead, a preservation result is given for the main semantic notions of chapter I.2. (Cf. the Lyndon homomorphism theorem in Chang & Keisler [2], or the main result of Feferman [4].) In the statement and the proof of this as well as later results of this chapter $\bot$ and $T$ will be abbreviations for $(\forall x)\neg(Rxx \to Rxx)$ and $(\forall x)(Rxx \to Rxx)$, respectively. Formal languages $L$ will be used consisting of $L_0$ with added individual constants.
6.4 Definition

If $L$ is a first-order language containing the binary predicate constant $R$, then the restricted positive formulas of $L$ are the $L$-formulas belonging to the smallest class $RF1(L)$ containing $\bot$ and all atomic formulas of the forms $Rt_1t_2$ and $t_1 = t_2$, where $t_1$ and $t_2$ are variables or individual constants, which is closed under $\wedge$, $\lor$, restricted universal quantification of the form $(\forall y)(Rty \rightarrow \phi)$ and restricted existential quantification of the form $(\exists y)(Rty \land \theta)$, where $t$ is a constant or a variable distinct from $y$.

Formulas of $RF1(L_0)$ contain at least one free variable. As soon as individual constants are present this need no longer be the case.

The following definitions and results up to and including theorem 6.7 are stated for $L_0$-formulas with one free variable, but are easily extended to the case of an arbitrary number of free variables.

6.5 Definition

An $L_0$-formula $\phi$ with one free variable is invariant for generated subframes if, for all frames $F_1 (= <W_1, R_1>)$ and $F_2$ such that $F_1 \subseteq F_2$ and all $w \in W_1$, $F_1 \models \phi [w] \Rightarrow F_2 \models \phi [w]$.

6.6 Definition

An $L_0$-formula $\phi$ with one free variable is preserved under p-morphisms if, for all frames $F_1 (= <W_1, R_1>)$ and $F_2$, all p-morphisms $f$ from $F_1$ onto $F_2$ and all $w \in W_1$, $F_1 \models \phi [w] \Rightarrow F_2 \models \phi [f(w)]$. 
6.7 Theorem

An $L_0$-formula with one free variable is invariant for generated subframes and preserved under p-morphisms iff it is equivalent to a restricted positive $L_0$-formula with the same free variable.

Proof: Any restricted positive formula $\phi$ of $L_0$ with the free variables $x_1, \ldots, x_k$ is invariant for generated subframes. Any restricted positive formula $\phi$ of $L_0$ with the free variables $x_1, \ldots, x_k$ is preserved under p-morphisms. Both of these results are proved by a simple induction on the complexity of $\phi$.

Now let the $L_0$-formula $\phi$ with the one free variable $x$ be invariant for generated subframes and preserved under p-morphisms. An argument rather analogous to the one used in the proof of theorem 1.9 shows that $\phi$ is equivalent to a restricted positive formula with the one free variable $x$:

Let $1(\phi) = \{ \psi \mid \psi \in RF1(L_0), \psi$ has the one free variable $x$, and $\phi \models \psi \}$. It will be shown that $1(\phi) \models \phi$, from which the conclusion follows by the compactness theorem. Let $F_1 \models 1(\phi)[w]$. After adding an individual constant $w$ to $L_0$ to obtain $L_1 F_1$ is expanded to an $L_1$-structure $F_1$ by interpreting $w$ as $w$. In the remainder of this chapter "$L_1$" will be used to denote this language or a similar one: the notational convention of chapter 1.2 regarding the use of "$L_1$" is hereby dropped.

Each finite subset of $\{ [w/x] \phi \} \cup \{ \neg \psi \mid \psi$ is a sentence in $RF1(L_1)$ and $F_1 \models \neg \psi \} has a model. Otherwise, $[w/x] \phi \models \neg(\neg \psi_1 \land \ldots \land \neg \psi_m)$ for some $\psi_1, \ldots, \psi_m$ as described, so $[w/x] \phi \models \psi_1 \lor \ldots \lor \psi_m$, contradicting the fact that $RF1(L_1)$ is closed under $\lor$ and $F_1 \models \neg(\psi_1 \lor \ldots \lor \psi_m)$. It follows that the above set has a model, say $C_1$. (From now on the capital
letter $G$, possibly with subscripts and/or superscripts, will also denote frames.) This yields the following situation:

frames: $F_1^1, F_1$

languages: $L_0, L_1$

where $G_1 \models \left[ w/x \right] \phi$

and $G_1 \models 1(L_1) - F_1$, where "$G - 1(L) - F$" abbreviates "for all sentences $\phi$ in $RF1(L)$, if $G \models \phi$, then $F \not\models \phi$".

Elementary chains $F_1, F_2, \ldots$ and $G_1, G_2, \ldots$ will now be constructed using the following general method. Let a language $L_n$ and $L_n$-structures $F_n$ and $G_n$ be given such that $G_n \models 1(L_n) - F_n$. For each $c$ and $w$, where $c$ is an individual constant in $L_n$, $w$ is in the domain of $G_n$ and $G_n \models Rc \left[ w \right]$, add a new constant $\bar{w}$ to $L_n$ to obtain $L_n^1$. Expand $G_n$ to an $L_n^1$-structure $G_n^1$ by interpreting each $\bar{w}$ as $w$.

We claim that each finite subset of $\Delta = \{ \psi \mid \psi \text{ is a sentence in } \}$ $RF1(L_n^1)$ and $G_n^1 \models \psi$ has a model which is an expansion of $F_n$. For, let $\psi_1, \ldots, \psi_k \in \Delta$, containing the constants $\bar{w}_1, \ldots, \bar{w}_l$ from $L_n^1$. There are constants $c_1, \ldots, c_l$ of $L_n$ such that

$G_n^1 \models Rc_1x_1 \land \ldots \land Rc_lx_l \land [x_1/\bar{w}_1, \ldots, x_l/\bar{w}_l](\psi_1 \land \ldots \land \psi_k)[\bar{w}_1, \ldots, \bar{w}_l]$, where $x_1, \ldots, x_l$ are variables not occurring in $(\psi_1 \land \ldots \land \psi_k)$. Therefore,

$G_n \models (\exists x_1)(Rc_1x_1 \land \ldots \land (\exists x_1)(Rc_lx_l \land [x_1/\bar{w}_1, \ldots, x_l/\bar{w}_l](\psi_1 \land \ldots \land \psi_k))\ldots)$ and so this $RF1(L_n)$-sentence (!) holds in $F_n^1$. From this the claim easily follows, and a standard model-theoretic argument will even establish
there is a model $F_n^1$ for $\Delta$ such that

$F_n^1$ is an $L_n^1$-structure

$F_n \prec_{L_n} F_n^1$ (i.e., $F_n$ is an $L_n$-elementary substructure of $F_n^1$)

$G_n^1 = 1(L_n^1) - F_n^1$.

Picture this as:

frames:

```
\[
\begin{array}{c}
F_n \prec_n F_n^1 \\
\downarrow\quad\downarrow
\end{array}
\]
```

languages:

$L_n$, $L_n^1$, $L_n^1$

For each $c$ and $w$, where $c$ is an individual constant in $L_n^1$, $w$ is in the domain of $F_n^1$ and $F_n^1 \models Rcw [w ]$, add a new constant $k_{cw}$ to $L_n^1$ to obtain $L_{n+1}$. Expand $F_n^1$ to an $L_{n+1}$-structure $F_{n+1}$ by interpreting each $k_{cw}$ as $w$.

Each finite subset of $\Gamma = \{ \psi \mid \psi$ is a sentence of $RFL(L_{n+1})$ and $F_{n+1} \models \neg \psi \}$, $\cup \{ Rck_{cw} \mid k_{cw}$ is a constant in $L_{n+1}^1 - L_n^1$ such that $F_{n+1} \models Rck_{cw}$ has a model which is an expansion of $G_n^1$. To see this, let $\neg \psi_1, \ldots, \neg \psi_k \in \Gamma$ and consider $Rc_1 k_{c_1 w_1}, \ldots, Rc_1 k_{c_1 w_1}$.

(If $\neg \psi_1, \ldots, \neg \psi_k$ contain other constants from $L_{n+1}^1 - L_n^1$ besides $k_{c_1 w_1}, \ldots, k_{c_1 w_1}$, then add the relevant $Rck_{cw}$'s. So one may as well suppose that $k_{c_1 w_1}, \ldots, k_{c_1 w_1}$ are all the constants from $L_{n+1}^1 - L_n^1$ occurring in $\neg \psi_1, \ldots, \neg \psi_k$.) If $\{ \neg \psi_1, \ldots, \neg \psi_k, Rc_1 k_{c_1 w_1}, \ldots, Rc_1 k_{c_1 w_1} \}$ is not satisfiable in an expansion of $G_n^1$, then, for any sequence of variables $x_1, \ldots, x_1$ not occurring in $\neg \psi_1, \ldots, \neg \psi_k$,
\[ G_n^1 \models (\forall x)(Rc_1x_1 + \ldots + (\forall x_1)(Rc_1x_1 + [x_1/k_{c_1}w_1, \ldots, x_1/R_{c_1k_{c_1}w_1}]) (\psi_1 \lor \ldots \lor \psi_k)) \ldots. \]

Moreover, since this RF1(\(L_n^1\))-sentence (\(\psi_1\)) holds in \(G_n^1\), it also holds in \(F_n^1\), as \(G_n^1 \models 1(L_n^1) \rightarrow F_n^1\). This contradicts the fact that
\[ F_n^1 \models Rc_1x_1 \land \ldots \land Rc_1x_1 \land [x_1/k_{c_1}w_1, \ldots, x_1/k_{c_1}w_1] (\neg \psi_1 \land \ldots \land \neg \psi_k) [w_1, \ldots, w_1]. \]

Two remarks should be made at this point. As the reader will no doubt have noticed, there was a slight inexactness in the construction of \(F_n^1\). Constants \(w_1, \ldots, w_1\) were considered, occurring in \((\psi_1 \land \ldots \land \psi_k)\), and \(c_1, \ldots, c_1\) such that \(G_n^1 \models (Rc_1x_1 \land \ldots \land Rc_1x_1 \land [x_1/w_1, \ldots, x_1/w_1]) (\psi_1 \land \ldots \land \psi_k) [w_1, \ldots, w_1]\). It was then concluded that \(G_n \models (\exists x_1)(Rc_1x_1 \land \ldots \land (\exists x_1)(Rc_1x_1 \land [x_1/w_1, \ldots, x_1/w_1]) (\psi_1 \land \ldots \land \psi_k))\ldots.\) But suppose that, e.g., \(w_1\) and \(w_2\) are the same element, i.e., \(w_1 = w_2\), but \(c_1\) and \(c_2\) are different. (In other words, \((c_1)G_n^1\) and \((c_2)G_n^1\) have the R-successor \(w_1\) in common.) Then the above sentence should start with \((\exists x_1)(Rc_1x_1 \land Rc_2x_1 \land \ldots \ldots\). Here this inexactness is harmless, since the new sentence is in RF1(\(L_n^1\)) as well. But with \(F_{n+1}\) this would be serious. For \((Rc_1w, Rc_2w, \neg \psi(w))\) the same construction would lead to \((\psi_1)(Rc_1y_1 \rightarrow (Rc_2y_1 \rightarrow (\psi(y_1))))\) which is not in RF1(\(L_n^1\)).

The \(k_{cw}\)-complication serves to avoid this in a similar way as explained after the proof of theorem 1.9.

The second remark concerns \(\bot\). If no \(\neg \psi_1, \ldots, \neg \psi_k\) are present in the previous argument, then \((\forall x_1)(Rc_1x_1 \rightarrow \ldots (\forall x_1)(Rc_1x_1 \rightarrow \bot) \ldots)\) is to be considered. Here is, where we need \(\bot\) essentially. (In fact, what is needed is the existence of at least one sentence \(\psi\) in RF1(\(L_{n+1}\)) such that
\( F_{n+1} \models \forall \psi. \bot \) is such a sentence, and in some cases it may be the only one, e.g., if \( F_{n+1} = \langle 0 \rangle, \langle 0, 0 \rangle \).

Again a standard model-theoretic argument establishes the existence of an \( L_{n+1} \)-structure \( G_{n+1} \) satisfying

\[
G_n \prec_{L_n} G_{n+1} \ precursor \ G_{n+1} - 1(L_{n+1}) - F_{n+1}.
\]

Picture this as:

frames:

\[
\begin{array}{c}
F_n \\
\downarrow \ni n \\
G_n, G_n^1 \prec_{L_n} G_{n+1} \\
\end{array}
\begin{array}{c}
F_n^1 \\
\uparrow \ni n+1 \\
F_{n+1}
\end{array}
\]

languages:

\([ L_n, L_n^1, L_n^1, L_{n+1} ]\)

It will be clear now how the two elementary chains \( F_1, F_2, \ldots \) and \( G_1, G_2, \ldots \) are constructed, together with the languages \( L_1, L_2, \ldots \).

Several applications of the fundamental theorem on elementary chains, in combination with the initial assumptions on \( \phi \), will yield the required conclusion. \([ w/x ] \phi \) holds in the limit \( G \) of the chain \( G_1, G_2, \ldots \). By the invariance of \( \phi \) for generated subframes, \( TC(G, w^G) \models [ w/x ] \phi \). This generated subframe of \( G \) is exactly the substructure of \( G \) with a domain consisting of the \( c^G \)'s, where \( c \) is a constant in \( \bigcup L_n \). For \( w = c^G \) in the domain of \( TC(G, w^G) \) put \( f(w) = c^F \), where \( F \) is the limit of the chain \( F_1, F_2, \ldots \). We claim that \( f \) is a \( p \)-morphism from \( TC(G, w^G) \) onto \( TC(F, w^F) \).
That f is well-defined follows from the fact that if \( c_1^G = c_2^G \), then, for a suitably large \( n, c_1 \in L_n \) and \( c_2 \in L_n \), \( G_n \models c_1 = c_2 \) and so, since \( G_n - 1(L_n) - F_n, F_n \not\models c_1 = c_2 \) and, therefore, \( F \not\models c_1 = c_2 \). That f is onto follows from the observation that \( TC(F, w^F) \) consists exactly of the interpretations of the \( \bigcup_n L_n \)-constants in F. \( R_{c_1}^G c_2^G \) implies \( R_{c_1}^F c_2^F \), by an argument similar to the one showing f to be well-defined. This proves the first condition in the definition of a p-morphism. For the second one, if \( R_{c_1}^F v \) in \( TC(F, w^F) \), then \( v = c_2^F \) for some \( \bigcup_n L_n \)-constant \( c_2 \) (one of the \( k_{cw} \)'s will serve), so \( v = f(c_2^G) \).

\( \phi \) is preserved under p-morphisms and, therefore, \( TC(F, w^F) \models [w/x] \phi \). It follows from this, by the invariance of \( \phi \) for generated subframes, that \( F \models [w/x] \phi \), and so \( F_1 \models [w/x] \phi \), i.e., \( F_1 \models \phi [w] \). QED.

6.8 Corollary

Each formula in P1 is equivalent to a restricted positive formula with the same free variable.

Proof: Each formula in P1 is invariant for generated subframes and preserved under p-morphisms, because its defining modal formula is. (Cf. corollaries 2.6 and 2.12.) QED.

The final result on P1 is a constructive one, showing how modal definitions may be obtained for certain \( L_0 \)-formulas.

6.9 Definition

A \( \bar{V} \)-formula is an \( L_0 \)-formula with one free variable, which is of the form \( U \psi \), where \( U \) is a (possibly empty) sequence of restricted
universal quantifiers and \( \psi \) is an \( L_0 \)-formula in which only atomic formulas, \( \wedge \) and \( \vee \) occur.

Many relational conditions occurring in the literature are of this form, e.g., reflexivity, transitivity and symmetry, but also the oft-mentioned property of having no more than a given number of \( R \)-incomparable \( R \)-successors at any given point.

6.10 Lemma

Each \( \bar{\psi} \)-formula is in \( P1 \), and its modal definition can be obtained constructively from it.

Proof: Let \( \phi \) be a \( \bar{\psi} \)-formula \( U \psi \). Using the propositional distributive laws, write \( \psi \) as a conjunction \( \prod_{i=1}^{n} \psi_i \) of disjunctions \( \psi_i \) of atomic formulas.

Since \( \phi \) is equivalent to \( \prod_{i=1}^{n} U \psi_i \), it suffices to consider the conjuncts \( U \psi_i \), by lemma 6.2. Rewrite \( U \psi_i \) to a formula of the form "\( \neg \)-sequence of restricted existential quantifiers-conjunction of negated atomic formulas". Remove repetitions from this conjunction, and also drop one of each pair \( \neg x = y \), \( \neg y = x \) in it. Take a different bound variable for each quantifier.

A tree \( T_y \) is constructed inductively for each variable \( y \) occurring in \( \psi_i \). If no restricted quantifiers of the form \( (\exists z)(Ryz \wedge \psi) \) occur in \( \psi_i \), then \( T_y \) consists of a single node \( y \). If not, then \( T_y \) is constructed from \( T_{z_1}, \ldots, T_{z_m} \), where \( z_1, \ldots, z_m \) are the variables \( z \) such that \( (\exists z)(Ryz \wedge \psi) \) occurs in \( \psi_i \), by joining their topnodes to a new topnode \( y \).

For each node \( y \) in the tree \( T_x \), where \( x \) is the one free variable of
\(\psi_i\), a modal formula (y) is defined inductively as the conjunction of
\(\Box(z)\), for each immediate descendant z of y,
\(\Box p_{yz}\), for each \(\neg Ryz\) occurring in the propositional matrix of \(\psi_i\),
\(\neg p_{zy}\), for each \(\neg Rzy\) occurring in the propositional matrix of \(\psi_i\),
\(q_{yz}\), for each \(\neg y=z\) occurring in the propositional matrix of \(\psi_i\),
\(\neg q_{zy}\), for each \(\neg z=y\) occurring in the propositional matrix of \(\psi_i\)
(or T, if the conjunction is empty).

\(\neg (x)\) is the modal formula defining \(U\psi_i\). This is easily shown by
noting that, for all frames \(F = (W, R)\) and each \(w \in W\), \(F \models \neg U\psi_i [w]\)
iff, for some valuation \(V\) on \(F\), \(<F, V> \models (x) [w]\).

QED.

\((\psi y)(Rxy \rightarrow (\psi u)(Rxu \rightarrow (\psi v)(Ruv \rightarrow Ryv)))\) will serve as an example.
Rewriting it as \(\neg (\exists y)(Rxy \land (\exists u)(Rxu \land (\exists v)(Ruv \land \neg Ryv)))\) yields the
tree \(T_x:\)

![Diagram](image)

\((y) = \Box p_{yv}\)
\((v) = \neg p_{yv}\)
\((u) = \Box \neg p_{yv}\)
\((x) = \Box \Box p_{yv} \land \Box \Box \neg p_{yv}\).

\(\neg (x)\) is equivalent to \(\Box \Box p_{yv} + \Box \Box p_{yv}\).
The second part of this chapter is devoted to \( \bar{P}1 \) and to \( L_0 \)-sentences in general.

6.11 Lemma

\( \bar{P}1 \) is closed under conjunctions, but not under disjunctions or negations.

(Note that there is no natural formulation for clauses involving restricted quantification in \( \bar{P}1 \). Compare the difficulty in explaining \( F \models \Box \phi \) in terms of \( F \models \phi : F \models \Box \phi \) iff \( (\forall w \in W) F \models \Box \phi [w] \) iff \((\forall w \in W)(\forall v \in W)(R_{vw} \rightarrow F \models \phi [v])\), but this does not help.)

Proof of lemma 6.11: If \( \alpha \) and \( \beta \) are \( L_0 \)-sentences in \( \bar{P}1 \), then, for some modal formulas \( \phi \) and \( \psi \), \( \bar{E}(\phi, \alpha) \) and \( \bar{E}(\psi, \beta) \). Then also \( \bar{E}(\phi \land \psi, \alpha \land \beta) \), for \( F \models \phi \land \psi \) iff \( F \models \phi \) and \( F \models \psi \). The corresponding result for disjunction does not hold, even if \( \phi \) and \( \psi \) have no proposition letters in common. \((\forall x)R_{xx} \in \bar{P}1 \) \((\bar{E}(\Box p \rightarrow p, (\forall x)R_{xx}))\) and \((\forall x)(\forall y)(R_{xy} \rightarrow R_{yx}) \in \bar{P}1 \) \((\bar{E}(\Diamond p \rightarrow p, (\forall x)(\forall y)(R_{xy} \rightarrow R_{yx}))\), but \((\forall x)R_{xx} \lor (\forall x)(\forall y)(R_{xy} \rightarrow R_{yx}) \notin \bar{P}1 \), for this sentence is not preserved under disjoint unions. E.g., it holds in both \(<\{0\}, \emptyset \) and \(<\{0, 1\}, \{<0, 0>, <0, 1>, <1, 1>\}\), but not in their disjoint union. Finally, \( (\forall x)R_{xx} \in \bar{P}1 \), but \( \forall (\forall x)R_{xx} \notin \bar{P}1 \), since it is not preserved under generated subframes. E.g., it holds in \(<\{0, 1\}, \{<0,0>\}\), but not in \(<\{0\}, \{<0, 0>\}\).

QED.

A very general result is found in Goldblatt & Thomason [9], which gives an algebraic characterization of the classes of frames definable by a set of modal formulas (i.e., as \( FR(\Gamma) \) for a set \( \Gamma \) of modal formulas).
If only $\Sigma^0$-elementary classes of frames are considered their result assumes a particularly elegant form as stated below. This last result is proved here in a non-algebraic fashion. This does not yield a complete characterization for $\mathcal{P}I$, though, since the $L_0$-sentences characterized by it are exactly those which are definable by a class of modal formulas, rather than by a single one. Such difficulties were not encountered in chapter I.3, because modal formulas defined by a class of $L_0$-sentences are definable by a single $L_0$-sentence already, as a simple argument showed. The analogous question for the present case is still open. Theorem 20.10 in Goldblatt [8] does characterize $\mathcal{P}I$ algebraically, but the additional notion involved ("completed ultraproduct") is not as (elegant and) natural as the ones occurring in theorem 6.15 below.

6.12 Definition
If $F = \langle W, R \rangle$ is a frame, then the ultrafilter extension $F^*$ of $F$ is the frame $\langle W^*, R^* \rangle$ with the set $W^*$ of all ultrafilters on $W$ as its domain and the relation $R^*U_1U_2$ between ultrafilters $U_1$ and $U_2$ defined by $(\forall X \subseteq W)(X \in U_2 \Rightarrow \{w \in W | (\exists v \in W)(Rwv & v \in X)\} \in U_1)$.

6.13 Definition
If $M = \langle W, R, V \rangle$ is a model and $\phi$ a modal formula, then $V(\phi) = \{w \in W | M \models \phi [w]\}$.

Recall definition 3.9: for a model $M$, $Th_m(M) = \{\phi | \phi$ is a modal formula such that $M \models \phi\}$, and $Th_m(F)$ is defined similarly. The obvious extension to a class $\mathcal{K}$ of frames is:

$Th_m(\mathcal{K}) = \bigcap_{F \in \mathcal{K}} Th_m(F)$. 

If \(F^*\) is the ultrafilter extension of \(F\), then \(\text{Th}_m(F^*) \subseteq \text{Th}_m(F)\).

Proof: This is shown by an argument much like the standard completeness proofs in modal logic. Suppose that, for some valuation \(V\) on \(F\) and some modal formula \(\phi\), \(\langle F, V \rangle \models \neg \phi [w]\), where \(w \in W\). It will be proved that, for some valuation \(V^*\) on \(F^*\) and some ultrafilter \(U\), \(\langle F^*, V^* \rangle \models \neg \phi [U]\).

Define \(V^*\) by \(V^*(p) = \{U \mid U\text{ is an ultrafilter on } W\text{ and } V(p) \in U\}\). It follows that, for all modal formulas \(\phi\), \(V^*(\phi) = \{U \mid V(\phi) \in U\}\). This is proved by induction on the complexity of \(\phi\), where the cases \(\phi\) is a proposition letter, \(\phi = \psi\) and \(\phi = \psi \land \chi\) are trivial. Consider the case \(\phi = \Box \psi\). If \(U \in V^*(\Box \psi)\), then, for some \(U'\) with \(R_{\text{U}'U'U'}\), \(U' \in V^*(\psi)\), so, by the induction hypothesis, \(V(\psi) \in U'\) and, therefore, by the definition of \(R^*\), \(\{w \in W \mid (\exists v \in W)(R_{\text{w}v} \land v \in V(\psi))\} \subseteq U\). \(V(\Box \psi)\) is exactly this set, so it belongs to \(U\). The converse is the only serious step. Let \(V(\Box \psi) \in U\), i.e., \(\{w \in W \mid (\exists v \in W)(R_{\text{w}v} \land v \in V(\psi))\} \subseteq U\). It is to be shown that, for some \(U'\) with \(R_{\text{U}'U'U'}\), \(V(\psi) \in U'\). Such a \(U'\) is found by noting that \(\{X \subseteq W \mid (w \in W \mid (\forall v \in W)(R_{\text{w}v} \Rightarrow v \in X)) \subseteq U\} \cup \{V(\psi)\}\) has the finite intersection property, and then applying the basic theorem on ultrafilters to this set, yielding a \(U'\) with \(V(\psi) \in U'\) and \(R_{\text{U}'U'U'}\). That the finite intersection property holds is shown as follows. Suppose that, for \(X_1, \ldots, X_k\) as described, \(X_1 \cap \ldots \cap X_k \cap V(\psi) = \emptyset\), i.e., \(X_1 \cap \ldots \cap X_k \subseteq W - V(\psi)\). Then \(\bigcap_{i=1}^k \{w \in W \mid (v \in W)(R_{\text{w}v} \Rightarrow v \in X_i)\} = \bigcap_{i=1}^k \{w \in W \mid \forall v \in W)(R_{\text{w}v} \Rightarrow v \in V(\psi))\} \subseteq \{w \in W \mid (\forall v \in W)(R_{\text{w}v} \Rightarrow v \notin V(\psi))\}\). But the first set is in \(U\) and therefore the second would be, contradicting the assumption that \(\{w \in W \mid (\exists v \in W)(R_{\text{w}v} \land v \in V(\psi))\} \subseteq U\).

So, starting with \(\langle F, V \rangle \models \neg \phi [w]\), i.e., with \(w \in V(\neg \psi)\), \(\{V(\neg \psi)\}\)
is extended to an ultrafilter $U$, and then, by the above, $U \in V^*(\neg \phi)$, so $\langle F, V \rangle \models \neg \phi \{U\}$. QED.

6.15 Theorem (R.I. Goldblatt & S.K. Thomason)

A class of frames closed under elementary equivalence is of the form $\text{FR}(\Gamma)$ for a set $\Gamma$ of modal formulas iff it is closed under generated subframes, disjoint unions, p-morphisms and its complement is closed under ultrafilter extensions.

Proof: The original proof used algebraic notions, which made it possible to apply Birkhoff's theorem on equational classes of algebras. Here the argument is purely modal.

A class of frames of the form $\text{FR}(\Gamma)$ for a set $\Gamma$ of modal formulas satisfies the four closure properties mentioned above because of corollaries 2.6, 2.9, 2.12 and lemma 6.14, respectively.

Now let $\mathcal{K}$ be a class of frames closed under elementary equivalence, generated subframes, disjoint unions and p-morphisms, while its complement is closed under ultrafilter extensions. The first three closure properties imply that $\mathcal{K}$ is $\Delta$-elementary, by theorem 3.4. So, for some set $\Sigma$ of $L_0$-sentences, $\mathcal{K} = \text{FR}(\Sigma)$.

For an arbitrary frame $F$ with $F \models \text{Th}_m(\mathcal{K})$ it will be shown that $F \in \mathcal{K}$, and, therefore, since, quite trivially, each $F \in \mathcal{K}$ satisfies $\text{Th}_m(\mathcal{K})$, $\mathcal{K} = \text{FR}(\text{Th}_m(\mathcal{K}))$, which proves the above assertion.

For each $X \subseteq W$ take a proposition letter $p_X$ and set $V(p_X) = X$ to obtain a model $M(F) = \langle F, V \rangle$. For each modal formula $\phi$ such that $\phi \not\in \text{Th}_m(M(F))$, a frame $F_\phi$, $w_\phi \in W_\phi$ and a valuation $V_\phi$ on $F_\phi$ exist satisfying $\langle F_\phi, V_\phi \rangle \models \text{Th}_m(M(F)) \{w_\phi\}$, but $\langle F_\phi, V_\phi \rangle \not\models \phi \{w_\phi\}$. 
This is so, because otherwise, for some $\phi \not\in \text{Th}_m(M(F))$, 
$\Sigma \cup \text{ST}(\text{Th}_m(M(F))) \models \text{ST}(\phi)$, whence, by compactness, $\Sigma \cup \{\text{ST}(\psi)\} \models \text{ST}(\psi)$ 
for some $\psi \in \text{Th}_m(M(F))$. It follows that $\Sigma \models \text{ST}(\psi) \rightarrow \text{ST}(\phi)$, so 
$\psi \rightarrow \phi \in \text{Th}_m(\mathfrak{K})$, $F \models \psi \rightarrow \phi$, $M(F) \models \psi \rightarrow \phi$, and, since $M(F) \models \psi$, 
$M(F) \models \phi$, contradicting the original assumption about $\phi$. By confining 
attention to $\text{TC}(F_\phi, w_\phi)$ (a frame in $\mathfrak{K}$, because $\mathfrak{K}$ is closed under 
generated subframes) and noting that, for all modal formulas $\alpha$ in 
$\text{Th}_m(M(F))$, $\Box \alpha \in \text{Th}_m(M(F))$, it may be supposed without loss of generality 
that $\langle F_\phi, V_\phi \rangle \models \text{Th}_m(M(F))$ and $\neg \langle F_\phi, V_\phi \rangle \models \phi$ (use lemma 2.5). The dis-
joint union of $\{\langle F_\phi, V_\phi \rangle \mid \phi \not\in \text{Th}_m(M(F))\}$ is a model $M_1 = \langle F_1, V_1 \rangle$ 
such that $F_1 \in \mathfrak{K}$ (because $\mathfrak{K}$ is closed under disjoint unions of frames and it 
is obvious how a disjoint union of models is defined in a completely 
analogous fashion) and $\text{Th}_m(M_1) = \text{Th}_m(M(F))$.

Starting from this frame $F_1 \in \mathfrak{K}$ with a valuation $V_1$ such that the 
resulting model has the same modal theory as $M(F)$, a series of further 
models is constructed:

6.16 Definition (Fine [5])

A model $M = \langle W, R, V \rangle$ is 1-saturated if, for all sets $\Gamma$ of modal 
formulas such that for each finite subset $\Gamma_0$ of $\Gamma$ a $w \in W$ exists with 
$M \models \Gamma_0 \{w\}$, there is a $w \in W$ with $M \models \Gamma \{w\}$.

A model $M = \langle W, R, V \rangle$ is 2-saturated if, for all sets $\Gamma$ of modal 
formulas and all $w \in W$ such that for each finite subset $\Gamma_0$ of $\Gamma$ a 
v $\in W$ exists with $Rw$ and $M \models \Gamma_0 \{v\}$, there is a $v \in W$ with $Rw$ and 
$M \models \Gamma \{v\}$.

Familiar model-theoretic arguments will give a 1- and 2-saturated
elementary extension for $M_1$, say $M_2 (= <F_2, V_2>)$. (Note that the continuum hypothesis is not needed in this case, because $M_2$ need not be saturated in the full model-theoretic sense of the term.) $F_2 \in \mathcal{X}$, because $\mathcal{X}$ is closed under elementary equivalence, and $\text{Th}_m(M_2) = \text{Th}_m(M_1)$, since $M_2$ may be taken to be an $L_1$ (in the sense of chapter I.2) -elementary extension of $M_1$.

The following defines a p-morphism $h$ from $F_2$ onto $F^\mathcal{X}$. For $w \in W_2$, let $h(w) = \{V(\phi) \mid \phi$ is a modal formula such that $M_2 \models \phi[w]\}$, where $V$ was the valuation of $M(F)$. It will be shown that

(i) $h(w) \in W^\mathcal{X}$

(ii) $h$ is onto

(iii) $(\forall w \in W_2)(\forall v \in W_2)(R_2wv \Rightarrow R^\mathcal{X}h(w)h(v))$

(iv) $(\forall w \in W_2)(\forall v \in W^\mathcal{X})(R^\mathcal{X}h(w)v \Rightarrow (\exists u \in W_2)(R_2wu \& h(u) = v))$.

(i): Clearly, each $V(\phi)$ is a subset of $W$. $h(w)$ is a filter on $W$, for, if $V(\phi_1)$ and $V(\phi_2) \in h(w)$, then so is $V(\phi_1) \cap V(\phi_2) (= V(\phi_1 \& \phi_2))$, and if $V(\phi) \in h(w)$ and $V(\phi) \subseteq Y$, then $M(F) \models \phi \Rightarrow \phi_Y$, so $M_2 \models \phi \Rightarrow \phi_Y$

($\text{Th}_m(M_2) = \text{Th}_m(M_1) = \text{Th}_m(M(F))$) and $M_2 \models \phi \Rightarrow \phi_Y[w]$. Then $V(\phi_Y) = Y \in h(w)$. $h(w)$ is also an ultrafilter, because for each $Y \subseteq W$, either $M_2 \models \phi_Y[w]$, or $M_2 \models \phi_{\overline{W}-Y}[w]$, since $\neg \phi_Y \Rightarrow \phi_{\overline{W}-Y} \in \text{Th}_m(M(F))$. So, either $V(\phi_Y) = Y \in h(w)$, or $V(\phi_{\overline{W}-Y}) = \overline{W}-Y \in h(w)$.

(ii): Let $U$ be an ultrafilter on $W$ and consider $r = \{p_X \mid X \in U\}$. For each finite subset $r_0$ of $r$, $M_2 \models r_0[w]$ for some $w \in W_2$, because otherwise $M_2 \models \neg \Pi r_0$, so $M(F) \models \neg \Pi r_0$, contradicting the finite intersection property for $U$. By 1-saturatedness a $w \in W_2$ exists such that $M_2 \models r[w]$, and clearly $h(w) = U$.

(iii): If $w$ and $v \in W_2$ with $R_2wv$, and $X \in h(v)$, then $M_2 \models \phi[v]$ for some modal formula $\phi$ such that $V(\phi) = X$. It follows that $M_2 \models \Box \phi[w]$,
so $V(\diamond \phi) = \{ w \in W \mid (\exists v \in W)(Rwv \& v \in V(\phi) (= X)) \} \in h(w)$. By definition 6.12, this shows that $R^* h(w) h(v)$.

(iv): If, for some $w \in W_2$ and $U \in W^*$, $R^* h(w) U$, then consider $\Delta = \{ \phi \mid \phi$ is a modal formula such that $V(\phi) \in U \}$. If $\Delta_0$ is a finite subset of $\Delta$, then $V(\prod \Delta_0) = \bigcap_{\delta \in \Delta_0} V(\delta) \in U$, so $\{ w \in W \mid (\exists v \in W)(Rwv \& v \in V(\prod \Delta_0)) \} \in h(w)$, by the definition of $R^*$. This set is $V(\diamond \prod \Delta_0)$, so $M_2 \models \diamond \prod \Delta_0 [w]$. By 2-saturatedness, a $v \in W_2$ exists such that $R_2 w v$ and $M_2 \models \Gamma [v]$. Clearly, $h(v) = U$.

Since $\mathcal{X}$ is closed under $p$-morphisms, we have proved that $F^* \in \mathcal{X}$, so, since the complement of $\mathcal{X}$ is closed under ultrafilter extensions, $F \in \mathcal{X}$. QED.

As an example consider purely existential $L_0$-sentences. These are preserved under ultrafilter extensions, because it is easy to see that $f$, defined for each $w \in W$ by $f(w) = \{ X \subseteq W \mid w \in X \}$, is an isomorphism between $F$ and a subframe of $F^*$. (e.g., $R^* f(w)f(v)$ iff $(\forall X \subseteq W)(v \in X \Rightarrow w \in \{ u \in W \mid (\exists s \in W)(Rus \& s \in X) \})$ iff $Rwv$.) We shall return to this subject at the end of this chapter. Now let $\phi$ be an $L_0$-sentence of the form $(\forall x)\psi$, where $\psi$ is a $\bar{V}$-formula (as described in definition 6.9). It is easy to see that $FR(\phi)$ satisfies the conditions of theorem 6.15, so $\phi$ is modally definable. This proves a version of lemma 6.10, but for the global correspondence only, and a little less constructive.

We conclude with a series of preservation results for the semantic notions of theorem 6.15.
6.17 Definition

If $L$ is a first-order language obtained from $L_0$ by adding a (possibly empty) set of individual constants, then

$\text{RF2}(L)$ is the class of $L$-formulas constructed using atomic formulas with variables and/or constants, $\bot, \wedge, \vee, \forall, \exists$ and restricted universal quantifiers of the form $(\forall y)(Rty \rightarrow t)$, where $t$ is an individual constant or a variable distinct from $y$.

$\text{RF3}(L)$ is the class of $L$-formulas constructed using atomic formulas with variables and/or constants, negations of such formulas, $\wedge, \vee, \forall$ and restricted existential quantifiers of the form $(\exists y)(Rty \land t)$, where $t$ is an individual constant or a variable distinct from $y$.

$\text{RF4}(L)$ is the class of $L$-formulas constructed using atomic formulas with variables and/or constants, negations of such formulas, $\land, \land, \exists$ and restricted quantifiers of the forms $(\forall y)(Rty \rightarrow t)$ and $(\forall y)(Ryt \rightarrow t)$, where $t$ is an individual constant or a variable distinct from $y$.

The task of finding more appropriate names for these classes is left to the imaginative reader.

For convenience, we restate definitions 6.5 and 6.6 for the case of $L_0$-sentences and add a new notion.

6.18 Definition

An $L_0$-sentence $\phi$ is preserved under p-morphisms if, for all frames $F_1$ and $F_2$, and all p-morphisms $f$ from $F_1$ onto $F_2$, $F_1 \models \phi$ only if $F_2 \models \phi$.

An $L_0$-sentence $\phi$ is preserved under generated subframes if, for all frames $F_1$ and $F_2$ such that $F_2 \subseteq F_1$, $F_1 \models \phi$ only if $F_2 \models \phi$.

An $L_0$-sentence $\phi$ is preserved under disjoint unions if, for all sets of frames $\{F_i \mid i \in I\}$ with $F_i \models \phi$ for all $i \in I$, $\bigoplus \{F_i \mid i \in I\} \models \phi$. 
6.19 Theorem (R.I. Goldblatt)

An \( L_0 \)-sentence is preserved under \( p \)-morphisms iff it is equivalent to a sentence in \( RF2(L_0) \).

Proof: This proof will be similar to that of theorem 6.7, and therefore details will be omitted, wherever possible. The same holds for the remaining proofs in this chapter.

Formulas in \( RF2(L_0) \) are preserved under \( p \)-morphisms, as an easy induction on the complexity of formulas shows. More precisely, if \( \phi \) is a formula in \( RF2(L_0) \) with the free variables \( x_1, \ldots, x_k \), and \( f \) is a \( p \)-morphism from \( F_1 \) onto \( F_2 \), then, for all \( w_1, \ldots, w_k \in W_1 \), \( F_1 \models \phi \{ w_1, \ldots, w_k \} \iff F_2 \models \phi \{ f(w_1), \ldots, f(w_k) \} \).

Let \( \phi \) be preserved under \( p \)-morphisms. Define \( 2(\phi) = \{ \psi \mid \psi \) is a sentence in \( RF2(L_0) \) and \( \phi \models \psi \} \). We will show that \( 2(\phi) \models \phi \), and again the conclusion follows by compactness, for \( RF2(L_0) \) is closed under \( \land \).

Starting with \( F_0 \) such that \( F_0 \models 2(\phi) \), elementary chains \( F_0, F_1, F_2, \ldots \) and \( G_0, G_1, G_2, \ldots \) are constructed. The only salient points are the construction principle and the final reasoning.

First, each finite subset of \( \{ \phi \} \cup \{ \forall \psi \mid \psi \) is a sentence in \( RF2(L_0) \) and \( F_0 \models \neg \psi \} \) has a model, as a by now familiar argument shows. Let \( G_0 \) be a model for this set. The starting point for the construction is

frames:

\[
\begin{array}{c}
F_0 \\
\downarrow
\end{array}
\]

\[
G_0,
\]
where $F_0$ and $G_0$ are $L_0$-structures and

$$G_0 \models 2(L_0) - F_0,$$ i.e., for any sentence $\phi$ in $RF2(L_0)$, if $G_0 \models \phi$, then $F_0 \models \phi$. (The general notion $G - 2(L) - F$ is defined in the same way.)

Now let $F_n$, $G_n$, and $L_n$ be given such that $F_n$ and $G_n$ are $L_n$-structures satisfying $G_n \models 2(L_n) - F_n$. Add new constants $w$, for each $w$ in the domain of $G_n$, to obtain the language $L_n^1$. Expand $G_n$ to an $L_n^1$-structure $G_n^1$ by interpreting each $w$ as $w$. Then each finite subset of $\Gamma = \{ \psi \mid \psi$ is a sentence in $RF2(L_n^1)$ and $G_n^1 \models \psi \}$ has a model which is an expansion of $F_n$. (To see this, let $\psi_1, \ldots, \psi_k \in \Gamma$ contain $w_1, \ldots, w_l$ from $L_n^1 - L_n$. Then, for any $x_1, \ldots, x_l$ not occurring in $(\psi_1 \land \ldots \land \psi_k)$,

$G_n \models (\exists x_1) \ldots (\exists x_l) \{ x_1/w_1, \ldots, x_l/w_l \} (\psi_1 \land \ldots \land \psi_k),$ so this sentence, being in $RF2(L_n^1)$, holds in $F_n$. ) It follows that $\Gamma$ has a model $F_n^1$, which is an $L_n$-elementary extension of $F_n$, with $G_n^1 \models 2(L_n^1) - F_n^1$. The situation is now:

frames: $\begin{array}{c}
F_n \\
\downarrow \\
G_n, G_n^1
\end{array}$

languages: $L_n, L_n^1, L_n^1$

For each $c$ and $w$, where $c$ is a constant in $L_n^1$, $w$ is in the domain of $F_n^1$ and $F_n^1 \models Rc_x[w]$, add a new constant $k_{cw}$ to $L_n^1$. Also add a new constant $w$ for each $w$ in the domain of $F_n^1$. These additions yield a language $L_{n+1}$, and $F_n^1$ is expanded to an $L_{n+1}$-structure $F_{n+1}$ by
interpreting each $k_{cw}$ as $w$ and each $w$ as $w$. Each finite subset of $\Delta = \{ \neg \psi \mid \psi \text{ is a sentence in } RF2(L_{n+1}) \text{ and } F_{n+1} \models \neg \psi \} \cup \{ Rck_{cw} \mid k_{cw} \in L_{n+1} \cdot L_n^1 \text{ and } F_{n+1} \models Rck_{cw} \}$ has a model which is an expansion of $G^1_n$.

For, consider $\neg \psi_1, \ldots, \neg \psi_k \in \Delta$, as well as $RC_1^{1}k_{c_1w_1}, \ldots, RC_1^{1}k_{c_1w_1}$.

By adding $Rck_{cw}$'s it may be supposed that all constants of the form $k_{cw}$ belonging to $L_{n+1} \cdot L_n^1$ which occur in $\neg \psi_1, \ldots, \neg \psi_k$ are among $k_{c_1w_1}, \ldots, k_{c_1w_1}$. Let $\neg \psi_1, \ldots, \neg \psi_k$ also contain $w_1, \ldots, w_m$ from $L_{n+1} \cdot L_n^1$.

If $\neg \psi_1 \land \ldots \land \neg \psi_k \land RC_1^{1}k_{c_1w_1} \land \ldots \land RC_1^{1}k_{c_1w_1}$ were not satisfiable in an expansion of $G^1_n$, then, for some variables $x_1, \ldots, x_m, y_1, \ldots, y_1$ not occurring in this formula,

$G^1_n \models (\forall x_1) \ldots (\forall x_m)(\forall y_1)(RC_1^{1}y_1 \Rightarrow \ldots (\forall y_1)(RC_1^{1}y_1 \Rightarrow

[ x_1/w_1, \ldots, x_m/w_m, y_1/k_{c_1w_1}, \ldots, y_1/k_{c_1w_1} ] (\psi_1 \land \ldots \land \psi_k) \ldots).

But then this $RF2(L^1_n)$-sentence ($) would be true in $F^1_n$, which contradicts the definition of $\Delta$. It follows that $\Delta$ has a model $G_{n+1}$ which is an $L_n^1$-elementary extension of $G^1_n$ satisfying $G_{n+1} - 2(L_{n+1}) - F_{n+1}$.

The situation is now:

frames:

$$
\begin{array}{c}
F_n & F^1_n & F_{n+1} \\
\downarrow & \downarrow & \downarrow \\
\bullet & \bullet & \bullet \\
G_n & G^1_n & G_{n+1}
\end{array}
$$

languages: $L_n, L^1_n, L^1_n, L_{n+1}$

Once the elementary chains are constructed, the limits $F$ and $G$ are taken. Since $C_0 \models \phi$, $G \models \phi$. $f$ defined as before is a $p$-morphism from $G$ onto $F$ this time. (Each element of $F$ and $G$ is the interpretation of
some constant of $\bigcup_n L_n$. Note how the second clause in the definition of a p-morphism holds because of the $k_{\text{cw}}$'s used in the construction.) By the assumption on $\phi$, $F \models \phi$ and, therefore, $F_0 \models \phi$. QED.

6.20 Theorem (R.ı. Goldblatt)

An $L_0$-sentence is preserved under generated subframes iff it is equivalent to a sentence in $RF3(L_0)$.

Proof: The argument is similar to the preceding one. The three main differences are:

In the construction of $F_n^1$ constants $w$ are added for each $w$ in the domain of $G_n$ such that $G_n \models Rcx[w]$ for some $L_n$-constant $c$. This is because only restricted existential quantifiers are available now, so it is impossible to take a $w$ for each $w$ in the domain of $G_n$.

In the construction of $G_{n+1}$ it suffices to take constants $w$ for each $w$ in the domain of $F_n^1$. One then considers the set $\Delta = \{ \neg \psi \mid \psi$ is a sentence in $RF3(L_{n+1})$ and $F_{n+1}^1 \models \neg \psi \} \cup \{ Rc_w \mid c$ is a constant in $L_n^1$ and $F_n^1 \models Rcx[w]\}$. For $\neg \psi_1, \ldots, \neg \psi_k \in \Delta$ and $Rc_1^{w_1}, \ldots, Rc_i^{w_i} \in \Delta$, where $\neg \psi_1, \ldots, \neg \psi_k$ contain no additional ($L_{n+1}$-$L_n^1$)-constants (this may be ensured by adding $Rc_w$'s) the argument goes as follows. Suppose that $\neg \psi_1 \land \ldots \land \neg \psi_k \land Rc_1^{w_1} \land \ldots \land Rc_1^{w_i}$ is not satisfiable in an expansion of $G_n^1$. Let $x_1, \ldots, x_i$ be variables not occurring in this formula. Then, for $r \leq i$, $G_n^1 \models (\forall x_1) \ldots (\forall x_r) (\neg Rc_1^{x_1} V \ldots V \neg Rc_r^{x_r} V V \alpha V [x_1/w_1, \ldots, x_i/w_i, \ldots, x_r/w_r]) (\psi_1 V \ldots V \psi_m)$, where in this $RF3(L_n^1)$-sentence $r$ may be smaller than $l$, because of the following. In case $w_1 = w_2$ and $Rc_1^{w_1}$ and $Rc_2^{w_2}$ occur, just take one $x_1$, write $\neg Rc_1^{x_1}$ in front and put $\neg Rc_2^{x_1}$ in the disjunction $\alpha$. Etc.
The final reasoning concerns the limits $G$ and $F$ of the chains obtained in this fashion. $G_0 \models \phi$ and, therefore, $G \models \phi$. The substructure $G^1$ of $G$ with the interpretations of the $\bigcup_n L_n$-constants in $G$ as its domain is a generated subframe of $G$, by the manner of choosing constants in the construction of $F^1_n$. So $G^1 \models \phi$. The function $f$ defined as before is an isomorphism now from $G^1$ onto $F$. (RF3($L_n$) contains negations of atomic formulas as well, so $f$ becomes a 1-1 p-morphism, i.e. an isomorphism.)

So $F \models \phi$ and $F_0 \models \phi$. QED.

In order to deal with disjoint unions it becomes necessary to complicate these proofs by constructing systems of elementary chains simultaneously. This method proves theorems 6.23 and 6.28, but it has not led to a proof of the expected result:

An $L_0$-sentence is preserved under disjoint unions iff it is equivalent to a sentence of the form $(\forall x)\phi$, where $\phi$ is in RF4($L_0$).

(Note that a double universal quantifier cannot be allowed. E.g., $(\forall x)(\exists y)Rxy$ is preserved under disjoint unions, as is $(\forall x)(\exists y)Ryx$, but $(\forall x)(\forall y)Rxy$ is not.)

The next theorem is the main result about $\bar{P}1$, comparable to theorem 6.7 for $P1$.

6.21 Theorem

An $L_0$-sentence is preserved under $p$-morphisms, generated subframes and disjoint unions iff it is equivalent to a sentence of the form $(\forall x)\phi$, where $\phi$ is in RF1($L_0$).
Proof: One direction follows directly from previous observations. For the other, suppose that the $L_0$-sentence $\phi$ is preserved under $p$-morphisms, generated subframes and disjoint unions. It will be shown that $\mathfrak{I}(\phi) = \{ (\forall x)\psi \mid \psi \text{ is a formula in } RFL(L_0) \text{ with the one free variable } x \}$ and $\phi \models (\forall x)\psi \models \phi$. Then, by the compactness theorem and the law $(\forall x)(\alpha \land \beta) \models (\forall x)\alpha \land (\forall x)\beta$, the conclusion follows.

Let $F_0^1 \models I(\phi)$. Take constants $w$ for each $w$ in the domain of $F_0^1$. For each $w$, $L_w = \text{def } L_0 \cup \{ w \}$. Expand $F_0^1$ to $F_0$ by interpreting each $w$ as $w$. For each $w$, $F_0$ is an $L_w$-structure. Each finite subset of $\Gamma_w = \{ \phi \} \cup \{ \neg \psi \mid \psi \text{ is a sentence in } RFL(L_w) \text{ and } F_0 \models \neg \psi \} \text{ has a model. If not, then, for } \neg \psi_1, \ldots, \neg \psi_k \in \Gamma_w, \{ \phi, \neg \psi_1, \ldots, \neg \psi_k \} \text{ has no model, i.e. } \phi \models \psi_1 \lor \ldots \lor \psi_k \text{ and so } \phi \models (\forall x)[x/w](\psi_1 \lor \ldots \lor \psi_k)$. (Minor troubles with bound variables may always be avoided by taking suitable alphabetic variants, so they will not be mentioned.) But then

$F_0^1 \models (\forall x)[x/w](\psi_1 \lor \ldots \lor \psi_k)$, contradicting

$F_0^1 \models [x/w](\neg \psi_1 \land \ldots \land \neg \psi_k)[w]$. So $\Gamma_w$ has a model $G_w$. Defining $L_0(G_w)$ as $L_w$ and $G_0$ as $\{ G_w \mid w \text{ and } G_w \text{ as described above} \}$ the following situation is reached:

For each $G \in G_0$, $G-1(L_0(G))-F_0$, where $G-1(L)-F$ was defined in the proof of theorem 6.7

For each $G \in G_0$, $F_0$ is an $L_0(G)$-structure

For different $G's \in G_0$, the languages $L_0(G)$ have disjoint sets of individual constants.

Again elementary chains will be constructed, according to the following principle.

Let $G_n, F_n$ and, for each $G \in G_n$, $L_n(G)$ be given such that, for each $G \in G_n$, $F_n$ is an $L_n(G)$-structure and $G-1(L_n(G))-F_n$, while different
languages $L^1_n(G)$ have disjoint sets of individual constants. Consider any $G \in _= G_n$ and add, for each $w$ in the domain of $G$ such that, for some constant $c$ in $L^1_n(G)$, $G \models Rcx [w]$, a new constant $w$ to obtain $L^1_{n_1}(G)$. $G$ is then expanded to an $L^1_n(G)$-structure $G^1$ by interpreting each $w$ as $w$. Each finite subset of $\Delta_n(G) = \{ \psi \mid \psi$ is a sentence of $RF_1(L^1_n(G))$ such that $G^1 \models \psi \}$ has a model which is an expansion of $F_n$. To prove this, let $\psi_1, \ldots, \psi_k \in \Delta_n(G)$ contain the $(L^1_n(G)-L_n(G))$-constants $w_1, \ldots, w_i$ such that $G \models R_{c_i}x_i[w_i]$ for each $i (1 \leq i \leq l)$, where $c_1, \ldots, c_1$ are $L_n(G)$-constants. For variables $x_1, \ldots, x_l$ not occurring in $\psi_1, \ldots, \psi_k$,

$G \models (\exists x_1)(R_{c_1}x_1 \wedge \ldots \wedge (\exists x_1)(R_{c_1}x_1 \wedge \ldots \wedge \psi_k))$. This is a sentence in $RF_1(L_n(G))$, so it holds in $F_n$, since $G\models L_n(G)$.

A similar argument establishes that each finite subset of $\bigcup_{G \in _= G_n} \Delta_n(G)$ has a model which is an expansion of $F_n$.

(The above argument can be given for finitely many $G$'s at the same time, because the languages $L_n(G)$ involved have disjoint sets of individual constants.) So $\bigcup_{G \in _= G_n} \Delta_n(G)$ has a model $F^1_n$ satisfying for each $G \in _= G_n$,

$F^1_n$ is an $L^1_n(G)$-structure $F_n \prec L_n(G)$ $F^1_n$ $G^1\models L_n(G)$-$F^1_n$.

Now for the other direction:

Consider any $L^1_n(G)$. Add, for each $c$ and $w$ such that $c$ is a constant in $L^1_n(G)$, $w$ is in the domain of $F^1_n$ and $F^1_n \models Rcx [w]$, a new constant $k_{cw}$ to obtain $L^2_n(G)$. Note that different $L^2_n(G)$'s get disjoint sets of individual constants. $F^1_n$ is expanded to $F^2_n$ by interpreting each $k_{cw}$ from each $L^2_n(G)$ as $w$. In this way $F^2_n$ becomes an $L^2_n(G)$-structure for each $G \in _= G_n$. 
Each finite subset of $\Sigma_n(G) = \{ \neg \psi \mid \psi \text{ is a sentence in } RFL(L^2_n(G)) \}$ such that $F^2_n \models \neg \psi \cup \{ R_{k_{cw}} \mid k_{cw} \in L^2_n(G) - L^1_n(G) \}$ has a model which is an expansion of $G^1$. The argument showing this is the same as in previous proofs. So $\Sigma_n(G)$ has a model $G^2$ satisfying

$$G^1 \prec L^1_n(G) \preceq G^2$$

and

$$G^2 - 1(L^2_n(G)) - F^2_n.$$

Next, take new constants $w$ for elements $w$ in the domain of $F^2_n$ not named by any constant in any $L^2_n(G)$. Expand $F^2_n$ to $F_{n+1}$ by interpreting each $w$ as $w$. Since, by our construction, $F_{n+1} \models \bar{I}(\phi)$, the procedure followed in the construction of $G_0$ may be repeated with respect to these constants to obtain models $G_w$ with corresponding languages $L_{n+1}(G_w) = L_0 \cup \{ w \}$.

Defining $G_{n+1}$ as \{ $G^2 \mid G \in G_n$ $\} \cup \{ G_w \mid G_w$ constructed in the preceding paragraph $\}$ and $L_{n+1}(G^2)$ as $L^2_n(G)$, the original situation applies again. For each $G \in G_{n+1}$, $F_{n+1}$ is an $L_{n+1}(G)$-structure and $G - 1(L_{n+1}(G)) - F_{n+1}$, while different $L_{n+1}(G)$'s have disjoint sets of individual constants.

This construction yields a set of elementary chains, each beginning with a member $G$ of some $G_n$, as well as the chain $F_0 \prec F_1 \prec F_2 \ldots$. Call the limit of the last chain $F$ and that of a chain starting with $G$, $C(G)$. Then $C(G) \models \phi$, since $G \models \phi$. As in previous proofs, define a $p$-morphism $f_G$ from the generated subframe of $C(G)$ consisting of the interpretations of the constants in the language of $C(G)$, onto a generated subframe of $F$. (It should be clear from the construction that this is possible.) If $C$ is the disjoint union of the $C(G)$'s, then $C \models \phi$, for $\phi$ is preserved under disjoint unions. The union of the $p$-morphisms $f_G$ is a $p$-morphism from a generated subframe $C'$ of $C$ onto $F$. $\phi$ is preserved under generated subframes, so
C' ⊨ φ, and φ is preserved under p-morphisms, so F ⊨ φ. It follows that 
F_0 ⊨ φ and F_0 ⊨ φ. QED.

In the preservation result involving disjoint unions restricted 
quantifiers of the form (∀y)(Ryt → ) were mentioned. This motivates the 
formulation of a number of similar results for tense-logical formulas, 
i.e., modal formulas with restricted quantifiers of this kind as well.

6.22 Definition
If F = <W, R> is a frame and w ∈ W, then TC(F, w) is the smallest 
subframe <W_1, R_1> of F with a domain satisfying w ∈ W_1 and  
(∀w ∈ W_1)(∀v ∈ W)((Rvw V Rvw) → v ∈ W_1).

It is easy to see that any frame F is (isomorphic to) a disjoint 
union of subframes of the form TC(F, w), called the components of F.

6.23 Definition
An L_0-sentence φ is invariant for disjoint unions if, for all sets  
{F_i | i ∈ I} of frames, ⊕{F_i | i ∈ I} ⊨ φ iff (∀i ∈ I)F_i ⊨ φ.

6.24 Definition
A p-morphism from a frame F_1 onto a frame F_2 is a p-morphism from F_1 
onto F_2 satisfying the additional property  
(∀w ∈ W_1)(∀v ∈ W_2)(R_2vf(w) → (∃u ∈ W_1)(Ruw & f(u) = v)).
6.25 Definition

An $L_0$-sentence $\phi$ is preserved under $\tilde{p}$-morphisms if, for all
$\tilde{p}$-morphisms $f$ from a frame $F_1$ onto a frame $F_2$, $F_1 \models \phi$ only if $F_2 \models \phi$.

6.26 Definition

$\forall$ stands for a restricted universal quantifier of the form $(\forall y)(Rty \rightarrow$.
$\exists$ stands for a restricted existential quantifier of the form $(\exists y)(Rty \land$.

Reviewing our previous results concerning $L_0$-sentences now using this
notation yields:

(preserved under) (syntactic form)

$\tilde{p}$-morphisms atomic formulas, $\bot, \land, \lor, \forall, \exists, \forall$

generated subframes (negations of) atomic formulas,
$\land, \lor, \forall, \exists$

? $\forall x$: (negations of) atomic formulas,
$\land, \lor, \exists, \forall, \forall$

$\tilde{p}$-morphisms and generated subframes
and disjoint unions $\forall x$: atomic formulas, $\bot, \land, \lor, \exists, \forall$.

We will now add to these:
p-morphisms

atomic formulas, \( \bot, \land, \lor, \forall, \exists, \forall, \exists \).

(invariant for) disjoint
unions

\( \forall x: \) atomic formulas, \( \neg, \land, \lor, \exists \).

\( \tilde{p} \)-morphisms and (invariant
for) disjoint unions

\( \forall x: \) atomic formulas, \( \bot, \land, \lor, \exists, \exists \).

More precisely,

6.27 Definition

If \( L \) is a first-order language obtained from \( L_0 \) by adding a (possibly empty) set of individual constants, then

\( RF5(L) \) is the set of formulas constructed from atomic formulas using \( \neg, \land, \lor, \forall, \exists \) and \( \exists \).

\( RF6(L) \) is the set of formulas constructed from atomic formulas and \( \bot \), using \( \land, \lor, \forall, \exists, \forall \) and \( \forall \).

\( RF7(L) \) is the set of formulas constructed from atomic formulas and \( \bot \), using \( \land, \lor, \forall, \exists, \forall \) and \( \forall \).

6.28 Theorem

An \( L_0 \)-sentence is invariant for disjoint unions iff it is equivalent to a sentence of the form \( (\forall x)\phi \), where \( \phi \) is in \( RF5(L_0) \).

An \( L_0 \)-sentence is preserved under \( \tilde{p} \)-morphisms iff it is equivalent to a sentence in \( RF6(L_0) \).

An \( L_0 \)-sentence is invariant for disjoint unions and preserved under \( \tilde{p} \)-morphisms iff it is equivalent to a sentence of the form \( (\forall x)\phi \), where \( \phi \) is in \( RF7(L_0) \).
Proof: Only a sketch of a proof will be given, and that for the first assertion only. The second one is proved almost like theorem 6.19 and the third follows by a combination of the arguments for the first two.

One direction is easy, so consider the other one, and let \( \phi \) be an \( L_0 \)-sentence invariant for disjoint unions. Let \( \bar{5}(\phi) = \{(\forall x)\psi \mid \psi \in RF5(L_0) \} \) with the one free variable \( x \) and \( \phi \models (\forall x)\psi \). We shall show that \( \bar{5}(\phi) \models \phi \), which yields the required conclusion.

Let \( F_0^1 \models \bar{5}(\phi) \). Write \( F_0^1 \) as a disjoint union of its components in some way. E.g., \( F_0^1 = \bigoplus \{ F_{0w}^1 \mid w \in I \} \), where each \( F_{0w}^1 \) is of the form \( TC(F_0^1, w) \) for a \( w \) in the domain of \( F_0^1 \). Consider any \( F_{0w}^1 \). Add a constant \( w \) to \( L_0 \) to obtain \( L_w \) and expand \( F_{0w}^1 \) to \( F_{ow}^1 \) by interpreting \( w \) as \( w \). Doing this for all \( w \in I \) yields \( F_0 = \bigoplus \{ F_{ow}^1 \mid w \in I \} \). Each finite subset of \( \{ \psi \mid \psi \) is a sentence in \( RF5(L_w) \) such that \( F_0 \models \psi \} \cup \{ \phi \) has a model, and so the whole set has a model \( G_w \). Defining \( G_0 \) as the set of all \( G_w \)'s obtained in this way and \( L_0(G_w) \) as \( L_w \), the following situation arises:

For each \( G \in G_0 \), \( G-5(L_0(G))-F_0 \) (where \( G-5(L)-F \) has the by now familiar meaning),

For different \( G \)'s in \( G_0 \), the languages \( L_0(G) \) have disjoint sets of individual constants,

All constants from \( L_0(G) \) are interpreted in one component of \( F_0 \), in which no interpretations of constants from different languages \( L_0(G') \) occur.

The general construction starts from this situation, but now with subscripts \( n \) instead of \( 0 \).

For each \( G \in G_n \), add constants \( w \) to \( L_n(G) \) for those \( w \)'s in the domain of \( G \) which satisfy \( G \models Rc x V Rc x [w] \) for some \( L_n(G) \)-constant \( c \). This yields \( L_n^1(G) \) and \( G \) is expanded to an \( L_n^1(G) \)-structure \( G^1 \) by interpreting
each \( w \) as \( w \). (Take different \( w \)'s for elements from different \( G \)'s, so as to keep the languages disjoint.) Each finite subset of \( \bigcup_{G \in G_n} \{ \psi \mid \psi \text{ is a sentence in } RF5(L_n^1(G)) \text{ such that } G^1 \models \psi \} \) has a model which is an expansion of \( F_n \), so the whole set has a model \( F_n^1 \) satisfying, for each \( G \in G_n \),

\[
F_n < L_n^1(G) \quad F_n^1 \\
G_n^1 - 5(L_n^1(G)) - F_n^1
\]

all constants of \( L_n^1(G) \) are interpreted in one component of \( F_n^1 \), viz. that where those of \( L_n(G) \) were interpreted.

For the other direction, take constants \( w \) for those elements \( w \) in the domain of \( F_n^1 \) which satisfy \( F_n^1 \models \text{Rcx } V \text{ Rxc } [w] \) for some \( L_n^1(G) \)-constant \( c \), to obtain \( L_n^2(G) \). Also take, for each component of \( F_n^1 \) in which no interpretation of any constant occurs as yet, an element \( w \) in that component and a corresponding constant \( w \) to obtain new languages \( L_w \) = \( L_0 \cup \{ w \} \). Expand \( F_n^1 \) to \( F_{n+1} \) by interpreting each \( w \) as \( w \). Repeat the procedure of the beginning of this proof with respect to the last-mentioned \( w \)'s. For the first-mentioned, consider \( \{ \psi \mid \psi \text{ is a sentence in } RF5(L_n^2(G)) \text{ such that } F_{n+1} \models \psi \} \). Each finite subset of this set has a model which is an expansion of \( G^1 \), and therefore the whole set has a model \( G^2 \) satisfying

\[
G^1 < L_n^1(G) \quad G^2 \\
G^2 - 5(L_{n+1}(G^2)) - F_{n+1}, \text{ where } L_{n+1}(G^2) = \text{def } L_n^2(G).
\]

Finally, define \( G_{n+1} \) as the union of \( \{ G^2 \mid G \in G_n \} \) and the set of \( G_w \)'s obtained for the new languages \( L_w \).

This procedure yields elementary chains starting from \( G \)'s in some \( G_n \), with chain limits \( C(G) \). The constants interpreted in \( C(G) \) form a
component \( C'(G) \) of it. From the construction it can be seen that the disjoint union \( G^\ast \) of these components is isomorphic to the limit \( F \) of the chain \( F_0, F_1, F_2, \ldots \). Now \( G \models \phi \), for each \( G \) in each \( G_n \), and so \( C(G) \models \phi \). \( C'(G) \models \phi \), by the invariance of \( \phi \) for disjoint unions, and, for the same reason, \( G^\ast \models \phi \). It follows that \( F \models \phi \) and \( F^1_0 \models \phi \). QED.

Tense-logical formulas are invariant for disjoint unions and preserved under \( \tilde{\mathcal{p}} \)-morphisms, so theorem 6.28 is applicable to \( L_0 \)-sentences defined by tense-logical formulas.

No preservation result has been given for disjoint unions, so an obvious open question remains. The same question is open for ultrafilter extensions. This chapter ends with the few results we have on this subject.

First recall that a frame \( F \) is isomorphic to a subframe of its ultrafilter extension \( F^\ast \). The reason is that for \( w^\ast = \{ X \subseteq W \mid w \in X \} \) and \( v^\ast = \{ X \subseteq W \mid v \in X \} \), where \( w \) and \( v \in W \), \( R^w w^\ast v^\ast \) iff \( Rwv \) and \( w^\ast = v^\ast \) iff \( w = v \). (The second assertion is trivial and the first follows easily using the definition of \( R^\ast \).) So, for all practical purposes, we may consider \( F \) as a subframe of \( F^\ast \). This implies that existential \( L_0 \)-sentences are preserved under ultrafilter extensions, but such a result is hardly exciting. A little more information is provided by lemma 6.30 below.

6.29 Definition

For a fixed variable \( u \), the \( r(u) \)-formulas are the \( L_0 \)-formulas obtained by starting with atomic formulas of the forms \( Rux, Rxu, u = x \) and \( x = u \), where \( x \) is a variable different from \( u \), and applying \( \neg, \land \) and
two kinds of restricted existential quantification, forming \((\exists y)(Ruy \land \lbrack y/u \rbrack \phi)\) or \((\exists y)(Ryu \land \lbrack y/u \rbrack \phi)\) from \(\phi\), if \(y\) does not occur in \(\phi\).

E.g., for a variable \(x\) different from \(u\) and \(i \in \mathbb{IN}\), the formula \(R^i u x\) is an \(r(u)\)-formula (cf. definition 4.10).

6.30 Lemma

If \(\phi = \phi(u, x_1, \ldots, x_k)\) is an \(r(u)\)-formula, then, for any frame \(F = <W, R, \rangle\), any \(w_1, \ldots, w_k \in W\) and \(U \in W^\ast\),

\[
F^\ast \models \phi \lbrack U, w_1^\ast, \ldots, w_k^\ast \rbrack \iff \{v \in W \mid F \models \phi \lbrack v, w_1, \ldots, w_k \rbrack \} \subseteq U.
\]

Proof: We argue by induction on the complexity of \(\phi\).

\(\phi\) is \(Rux\): \(F^\ast \models Rux \lbrack U, w_1^\ast \rbrack\) iff \(R^\ast w_1^\ast \iff \{v \in W \mid Rw_1 \} \subseteq U\) (by an easy deduction) iff \(\{v \in W \mid F \models Rux \lbrack v, w_1 \rbrack \} \subseteq U\).

\(\phi\) is \(Rxu\): this is proved analogously, using the fact that \(R^\ast w_1^\ast U \iff \{v \in W \mid Rw_1 v \} \subseteq U\).

\(\phi\) is \(u = x\): \(F^\ast \models u = x \lbrack U, w_1^\ast \rbrack\) iff \(U = w_1^\ast \iff \{w \in W \mid F \models u = x \lbrack v, w_1 \rbrack \} \subseteq U\).

\(\phi\) is \(x = u\): this is proved analogously.

\(\phi\) is \(\neg \psi\) or \(\phi_1 \land \phi_2\): these cases follow by standard arguments, using the characteristic properties of ultrafilters.

\(\phi\) is \((\exists y)(Ruy \land \lbrack y/u \rbrack \phi)\): \(F^\ast \models (\exists y)(Ruy \land \lbrack y/u \rbrack \phi) \lbrack U, w_1^\ast, \ldots, w_k^\ast \rbrack\) iff, for some \(V \in W^\ast\), \(R^\ast UV\) and \(F^\ast \models \phi \lbrack V, w_1^\ast, \ldots, w_k^\ast \rbrack\) iff (by the induction hypothesis), for some \(V \in W^\ast\), \(R^\ast UV\) and \(\{v \in W \mid F \models \phi \lbrack v, w_1, \ldots, w_k \rbrack \} \subseteq V\). Now apply the following general principle:

If \(\phi = \phi(y, y_1, \ldots, y_k)\), then, for any \(w_1, \ldots, w_k \in W\) and any \(U \in W^\ast\),
\{v \in W \mid (\exists z \in W)(Rvz \land F \models \phi [z, w_1, \ldots, w_k])\} \subseteq U \iff \text{for some } V \in W^*,
R^*UV \text{ and } \{v \in W \mid F \models \phi [v, w_1, \ldots, w_k]\} \subseteq V.

(The standard deduction leading to this principle is omitted. Use the fundamental theorem on ultrafilters.)

The list of equivalences continues with
\{v \in W \mid (\exists z \in W)(Rvz \land F \models \phi [z, w_1, \ldots, w_k])\} \subseteq U, \text{ i.e.,}
\{v \in W \mid F \models (\exists y)(Ruy \land [y/u] \phi) [v, w_1, \ldots, w_k]\} \subseteq U.

\phi \text{ is } (\exists y)(Ruy \land [y/u] \phi): \text{ this is proved analogously, but now using the principle:}

If \phi = \phi(y, y_1, \ldots, y_k), \text{ then, for any } w_1, \ldots, w_k \in W \text{ and any } U \in W^*,
\{v \in W \mid (\exists z \in W)(Rvz \land F \models \phi [z, w_1, \ldots, w_k])\} \subseteq U \iff \text{for some } V \in W^*,
R^*UV \text{ and } \{v \in W \mid F \models \phi [v, w_1, \ldots, w_k]\} \subseteq V.

QED.

6.31 Corollary

If \phi = \phi(u, x_1, \ldots, x_k) \text{ is an } r(u)-\text{formula, then, for any frame } F = \langle W, R \rangle \text{ and any } w, w_1, \ldots, w_k \in W,
F^* \models \phi [w^*, w_1^*, \ldots, w_k^*] \iff F \models \phi [w, w_1, \ldots, w_k].

Proof: By lemma 6.30, \(F^* \models \phi [w^*, w_1^*, \ldots, w_k^*] \iff \{v \in W \mid F \models \phi [v, w_1, \ldots, w_k]\} \in \text{iff}
\{v \in W \mid F \models \phi [v, w_1, \ldots, w_k]\} \in W^* \iff \text{iff}
w \in \{v \in W \mid F \models \phi [v, w_1, \ldots, w_k]\} \text{ iff } F \models \phi [w, w_1, \ldots, w_k]. \text{ QED.}

The corollary implies that any sentence obtained from an } r(u)-\text{formula by existential quantification is preserved under ultrafilter extensions, which extends our result about existential formulas. Yet this result does not exhaust the class of sentences preserved under ultrafilter extensions.
E.g., $(\forall x)Rxx$ and $(\forall x)(\forall y)Rxy$ have this property as well (although $(\forall x)\neg Rxx$ does not). Let us treat the first and the third formula.

If $F \models (\forall x)Rxx$, $U \in W^*$ and $X$ is any set in $U$, then

$\{ w \in W \mid (\exists v \in W)(Rwv \land v \in X) \} \in U$, for it contains $X$; and so $R^*UU$.

But although $<IN,< > \models (\forall x)\neg Rxx$, $\neg <IN,< > \models (\forall x)\neg Rxx$. For any free ultrafilter $U$ on IN and $X \in U$, $\{ w \in IN \mid (\exists v \in IN)(w < v \land v \in X) \} \in IN$, since $X$ is infinite, and, since $IN \in U$, $R^*UU$. 