1.2: MODAL REDUCTION PRINCIPLES

1. INTRODUCTION

Modal reduction principles (MRPs) are modal formulas of the following form: $\langle \vec{M}p \rightarrow \vec{N}p \rangle$, where $\vec{M}, \vec{N}$ are (possibly empty) sequences of modal operators (i.e. $\square$ or $\Diamond$). Short-hand notation: $\vec{M}, \vec{N}$. We study a certain semantic correspondence between modal formulas and relational properties and obtain two main results.

(1) On transitive semantic structures every MRP corresponds to a first-order relational property.

(2) For the general case a syntactic criterion exists for distinguishing modal formulas with corresponding first-order properties from the others. The first reference to a problem like this we found in [3], where it is shown that MRPs of the form $\Diamond^i$, $\vec{N}$ or $\vec{M}, \square^i$ have corresponding first-order properties. ($\Diamond^0 = -$: the empty sequence. $\Diamond^1 = \Diamond$. $\Diamond^{i+1} = \Diamond \Diamond^i$. $\square^i$: similarly.) An extension to the case $\Diamond^i \square^j$, $\vec{N}$ was given in [11], as well as a proof that $\square \Diamond$, $\Diamond \square$ has no corresponding first-order property. The methods of the latter paper are used extensively here.
2. PRELIMINARIES

Two formal languages are used: one for modal propositional logic, the other for predicate-logic. Primitive signs for the first: \( \neg \) (not), \( \rightarrow \) (if...then), \( \bot \) (falsum), \( \square \) (necessarily) \(-\) and \( \land \) (and), \( \lor \) (or), \( \Diamond \) (possibly), \( T \) (verum) etc. are defined in the usual manner; for the second: \( \neg \), \( \rightarrow \), \( \bot \), \( \forall \) (for all) \(-\) and \( \exists \) etc. defined in the usual way. Lower case Greek letters are used for formulas. The modal semantic structures are frames: ordered couples \( <W, R> \) with \( R \) a binary relation on \( W \). (These may also be regarded as semantic structures for a predicate-logic with a single binary predicate-letter \( R \).) Notation for frames: \( F = <W, R> \). Ordered couples \( <F, V> \) with \( F \) a frame and \( V \) a valuation, i.e. a function from the set of proposition-letters into the power-set of \( W \), are called models. Notation for models: \( M = <W, R, V> \). The well-known Kripke truth-definition defines the notion \( M \models \alpha [w] \), for a model \( M \), \( w \in W \), \( \alpha \) a modal formula. We define a second notion: \( F \models \alpha [w] \) by means of: for all \( V: <F, V> \models \alpha [w] \). \( F \models \alpha \) is defined by: for all \( w \in W: F \models \alpha [w] \).

The correspondence we consider is the following:

For all \( F, w: F \models \gamma [w] \iff F \models \phi [w] \), where \( \phi \) is a modal formula, \( \gamma \) any formula (but mostly first-order) expressing a relational property of \( R \). (\( \phi \) has exactly one free variable.) Results about this correspondence are given in [2]. We state a few for future reference.

To every \( \phi \) there corresponds a first-order relational property in the following sense. Let \( ST(\phi) \) be the standard first-order translation of \( \phi \):

\[
\begin{align*}
ST(p) &= Px, \ P \text{ a one-place predicate-letter} \\
ST(\bot) &= Px \land \neg Px \\
ST(\neg \alpha) &= \neg ST(\alpha) \\
ST(\alpha \rightarrow \beta) &= ST(\alpha) \rightarrow ST(\beta) \\
ST(\square \alpha) &= (\forall y)(Rxy \rightarrow [y/x] ST(\alpha)), \text{ where } y \text{ does not occur in } ST(\alpha). 
\end{align*}
\]

The only free variable in \( ST(\alpha) \) is \( x \).

Let \( M \) be a model, \( M^R \) the predicate-logical structure corresponding to \( M \) in the obvious way. Then: \( M \models \phi [w] \iff M^R \models ST(\phi_m) [w] \).
This gives us, for a $\phi_m$ with proposition-letters $p_1, \ldots, p_n$,
\[(M = \langle F, V \rangle): F \models \phi_m \models w \iff F \models (\forall p_1) \ldots (\forall p_n) \text{ST}(\phi_m)[w]\.
\]
The $\phi_r$ obtained in this way is second-order. Very often a first-order $\phi_r$ exists, however. The main (and open) problem is to characterize the class $\mathcal{M}$ of modal formulas with corresponding first-order properties. Not much is known about $\mathcal{M}$. [2] contains, amongst others, some closure-conditions ($\mathcal{M}$ is closed under $\Lambda, \Box$, not under $\forall, V, +, \diamondsuit$), but the main result seems to be essentially the following theorem (based directly on a theorem of Sahlqvist's. Cf. [4]).

**Theorem 1**

Every modal formula $\phi_m$ of the form $\alpha \rightarrow \beta$, with:

1. $\alpha$ is constructed from $\bot, T, p$'s and $\neg p$'s using $\Lambda, V, \diamondsuit, \Box$.
2. no unnegated $p$ occurs in $\alpha$ inside the scope of some $\boxdot$ which is itself inside the scope of some $\Box$.
3. no unnegated $p$ occurs in $\alpha$ in a subformula $\gamma V \delta$ which is inside the scope of some $\Box$.
4. $\alpha(\beta)$ is monotone or antitone in its proposition-letters that do not occur in $\beta(\alpha)$.
5. $\beta$ is monotone in all its proposition-letters that occur in $\alpha$ as well.

has a first-order corresponding $\phi_r$, obtainable from it in a constructive manner.

The limits of this theorem are given by the following three formulas:
\[
\Box \diamondsuit p \rightarrow \Box p; (\Box p V p) \rightarrow (\Box (\Box p \land p)); \Box(p V q) \rightarrow (\Box \diamondsuit p \Box \Box q). \text{They do not have corresponding first-order properties.}
\]

For our special formulas, the MRPs, we obtain a full solution of the characterization problem. We need a special case of theorem 1 for that. All MRPs of the form $\Box^i \Box^j, N$ have corresponding first-order properties. This implies the same fact for those of the form $\Box^i \Box^j, N_i, \ldots N_j, \ldots N_m p [w]$ by virtue of the inversion-principle IP: $F \models M_1 \ldots M_k p + N_1 \ldots N_m p [w] \iff F \models N_1 \ldots N_m p \rightarrow M_1 \ldots M_k p [w]$, where $\Box^i = \Box^i$ and $\Box^j = \Box$. 
3. MRPs ON TRANSITIVE FRAMES

We restrict attention to frames with transitive R. On these frames the following formulas hold:

(i) $\Box \Box \Diamond p \leftrightarrow \Box \Diamond p$ (and so, by IP, $\Box \Diamond \Box \Diamond p \leftrightarrow \Box \Diamond p$)
(ii) $\Box \Box \Box \Diamond p \leftrightarrow \Box \Diamond p$ (and so, by IP, $\Box \Diamond \Box \Diamond p \leftrightarrow \Box \Diamond p$).

Proof: E.g. (ii): Let $<F, V> \models \Box \Box \Diamond p [w]$. Consider any $y$ with $Rwy$.
$<F, V> \not\models \Box \Diamond \Box \Diamond p [y]$. So there is a $z$ with $Ryz$ and $<F, V> \not\models \Box \Diamond p [z]$.
R transitive: $Rwz$, so $<F, V> \models \Box \Box \Diamond p [z]$. Therefore there is a $u$ with $Rzu$ and so $<F, V> \models \Box \Diamond p [u]$, and finally a $v$ with $Ruv$ and $<F, V> \models p [v]$. R transitive: $Rvy$, so $<F, V> \not\models p [y]$. So $<F, V> \not\models \Box \Diamond p [w]$.
Conversely, let $<F, V> \models \Box \Diamond \Box \Diamond p [w]$. Consider any $y$ with $Rwy$.
$<F, V> \not\models \Box \Diamond p [y]$ and, by the transitivity of $R$, $<F, V> \not\models \Box \Diamond p [y]$.
So there is a $z$ with $Ryz$ and, again by transitivity, $<F, V> \not\models \Box \Diamond p [z]$.

In other words: $<F, V> \not\models \Box \Box \Box \Diamond p [w]$. QED.

Remark: Although our methods are purely semantic, facts like the above could be proved syntactically as well, using the minimal modal system with $\Box p \rightarrow \Box \Box p$ added.

The above allows us to restrict attention to sequences of modal operators of the following types: $\Diamond^i, \Box^i \Box, \Diamond^i \Box \Diamond, \Box^i, \Box^i \Box, \Box^i \Box \Box$.

The only relevant MRPs then (excluding those that have first-order equivalents by virtue of section 2) are:

1. $\Diamond^k \Box^l \Box$
2. $\Box^k \Box^l \Box$
3. $\Diamond^k \Box^l \Box, \Box^l \Box$
4. $\Box^k \Box, \Box^l \Box$
5. $\Box^k \Box, \Box^l \Box$
   $\Box^k \Box, \Box^l \Box$
   $\Box^k \Box, \Box^l \Box$ ($= \text{type 1, by IP}$)
6. $\Box^k \Box, \Box^l \Box$
   $\Box^k \Box, \Box^l \Box$ ($= \text{type 5, by IP}$)
   $\Box^k \Box, \Box^l \Box$ ($= \text{type 2, by IP}$)
Theorem 2
On transitive frames all MRPs have first-order equivalents.

Proof: We will exhibit first-order equivalents for each of the six types in the above list. Two preliminary results:
(a): If $\phi_m$ has no proposition-letters (so only $\bot$, $T$ and operators may occur) then $F \models \phi_m[w] \iff F \models \text{ST}(\phi_m)[w]$. For these $\phi_m$'s $\text{ST}(\phi_m)$ is a first-order formula in $R$. So giving, for some $\phi_m$, an equivalent formula of this kind is as good as giving a first-order corresponding property.

(b):
(AC) Lemma 1
Let $R$ be a transitive binary relation on a set $X$. If for all $x \in X$ there exists $\eta \in X$ with $\eta \neq x$ and $R_{xy}$ then two disjoint cofinal subsets of $X$ exist. ($\eta$ cofinal in $X$ means: for all $x \in X$ there exists a $y \in Y$ such that $R_{xy}$.)

Proof: Enumerate $X$ as $\{x_0, \ldots, x_x, \ldots\}$ using some initial ordinal number. Consider the set $C$ of pairs $<Y, Z>$ with (i) $Y \cap Z = \emptyset$, (ii) $Y, Z \subseteq X$, (iii) $\forall y \in Y \exists z \in Z : R_{yz}$; $\forall z \in Z \exists y \in Y : R_{zy}$, (iv) $\forall y \in Y \exists y' \in Y : R_{yy'}$; $\forall z \in Z \exists z' \in Z : R_{zz'}$. $C$ is not empty: $<\emptyset, \phi> \in C$. We apply Zorn's lemma to the binary relation $\subseteq$ given by $<Y_1, Z_1> \subseteq <Y_2, Z_2>$ iff $Y_1 \subseteq Y_2$; $Z_1 \subseteq Z_2$. Clearly every chain is bounded. So we are ready if we can show that for a $<\subseteq$-maximal $<Y, Z>$: $Y \cup Z = X$. Suppose $<Y, Z>$ maximal, but $x \in X, x \notin Y, x \notin Z$.

(1) If $R_{xy}$ for some $y \in Y$ we would have: $<Y \cup \{x\}, Z> \in C$ (transitivity of $R$),

(2) $R_{xz}$ for some $z \in Z$: similarly.

In both cases contradiction with the maximality of $<Y, Z>$.

(3) If these cases do not apply we construct $Y_1, Z_1$ such that $<Y \cup Y_1, Z \cup Z_1> \in C$. Put $x$ in $Y_1$. Take the first $y_x$ in $Y$ with $R_{xy}$, $x \neq x$. Put it in $Z_1$. $(x \notin Y \cup Z \forall y_x)$. Repeat this. In the course of this process it may happen that e.g. $u$ is put in $Z_1$, but all $v$ with $R_{uv}$, $v \neq u$ have been put in $Y_1 \cup Z_1$ already at some earlier stage. We may
break off then. (For suppose \( x \) \( \gamma \) is the first in \( X \) with \( R x \gamma, u \neq x \gamma \).
Assume \( x \gamma \in Z_1 \). Since \( x \gamma \neq u \) the process did not stop there and we have
\( R x v, v \in Y_1 \). By transitivity \( R u v \) \( u \in Y_2 \): similarly. If this does
not happen the process may be stopped after \( \omega \) steps. QED.

One more definition: \( R^0 xy: x = y \); \( R^1 xy: Rxy \); \( R^2 xy: (\exists z)(Rxz \land Rzy) \); etc.

We now list the first-order properties \( \phi_r \) corresponding to the MRP\'s in
our previous list. We may assume \( k \geq 1, l \geq 1 \).

\( \phi_m \) of type:

1. \( (\forall y)[(R^k xy \land (\forall z)(Ryz + (\exists u)Rzu)) \rightarrow (\exists v)(R^1 xv \land
(\forall w)(Rvw \rightarrow v = w \land Rwv))] \)

2. \( k \geq 1: T. k < 1: \Box^k \Diamond T \land \Diamond^l \Box \perp \)

3. \( (\forall y)[(R^k xy \land (\forall z)(Ryz + (\exists u)Rzu)) \rightarrow (\forall v)(R^1 xv + 
(\exists w)(Rvw \land (\forall s)(Rws + s = w \land Rwv)))] \)

4. \( \Box^k \Box \perp v (\exists y)(R^1 xy \land (\forall z)(Ryz + z = y)) \)

5. \( \Box^k \Box \perp v \Diamond^l \Box \land T \)

6. \( \Box^k \Box \perp v \Diamond^l \Box \land T \)

We check the cases 6, 4 and 3.

6: Let \( F \models \Box^k \Box p \rightarrow \Diamond^l \Box \Box [w] \). Take \( V(p) = W \). Then either not
\( <F, V> \models \Box^k \Box p [w] \) (and so \( F \models \Diamond^l \Box \perp [w] \)) or \( <F, V> \models \Box^l \Box \Box [w] \)
(so \( F \models \Diamond^l \Box \land T [w] \).)

Conversely, suppose \( F \models \Box^k \Box \perp [w] \). Then trivially \( F \models \Box^k \Box \Box \rightarrow \Diamond^l \Box \Box [w] \). If \( F \models \Diamond^l \Box \land T [w] \) we reason as follows: we have a \( y 
with \( R^1 wy \) such that \( F \models \Box \land T [y] \). Suppose \( <F, V> \models \Box^k \Box \Box [w] \).

We will show that \( <F, V> \models \Box^l \Box \Box [w] \). Consider any \( z \) with \( Ryz \).

\( <F, V> \models \Diamond T [z] \). Because of transitivity: \( <F, V> \models \Box \land T [z] \).

This implies \( <F, V> \models \Diamond^i T [z], \) for every \( i \geq 1 \).

If \( k \geq 1 + l: <F, V> \models \Box^{k+1} \Box [y] \). But also if \( k < 1 + l: 
<F, V> \models \Box \Box p [y] \). (\Box \rightarrow \Box \Box .) So we have in any case \( <F, V> \models 
\Diamond^i \Box \Box [z], \) where \( i \geq 1 \). By the above we get: \( <F, V> \models \Diamond^i \Box \Box [z], \)
which reduces to: \( <F, V> \models \Diamond p [z] \). Using \( <F, V> \models \Box \land T [z] \) once more we get \( <F, V> \models \Diamond p [z] \) and so finally \( <F, V> \models \Diamond p [z] \).
4: It is easy to check that if \( \phi_{r} \) holds so does \( \phi_{m} \). For the converse suppose not \( F \not\models \phi_{r} [w] \). So \( F \not\models \Box^{1} \Diamond T [w] \) and
\[
F \models (\forall y)(R'xy \rightarrow (\exists z)(Ryz \land z \neq y)) [w].
\]
We look for a \( V \) with
\[
<F, V> \models \Box^{1} \Diamond p [w] \quad \text{and} \quad <F, V> \not\models \Box \Diamond \neg p [w].
\]
First we apply the lemma with \( X = \{ z \mid R'xz \} \). Let \( Y \) be one of the cofinal sets obtained. \( V(p) = \text{def} \ Y \cup \{ z \mid R^{k+1}xz \text{ and } z \notin X \} \).

3: Again \( \phi_{r} \) obviously implies \( \phi_{m} \). Now suppose not \( F \not\models \phi_{r} [w] \).
Then there are \( y, v \) with: \( R^{k}wy, F \models \Box \Diamond T [y], R^{1}wv, \) and
\[
F \models (\forall w)(Rzw \rightarrow (\exists s)(Rws \land (s \neq w \land V \neg Ryw))) [v].
\]
A \( V \) is required such that \( <F, V> \not\models \Box^{1} \Box \Diamond p [w] \), \( <F, V> \not\models \Box \Box \Diamond \neg p [w] \).
In fact we want: \( <F, V> \not\models \Box \Diamond p [y] \), \( <F, V> \not\models \Box \Diamond \neg p [v] \).
Apply the lemma with \( X = \{ z \mid Ryz \text{ and } Ryz \} \). Let \( Y \) be one of the cofinal sets obtained. \( V(p) = \text{def} \ Y \cup \{ z \mid R^{2}yz \text{ and } z \notin X \} \).
QED.

Remarks:

(1) Not all modal formulas are equivalent to a first-order property on transitive frames. E.g. \( LF (\Box(\Box p \land p) \rightarrow \Box p) \) expresses well-foundedness of the converse relation of \( R \). More precisely: if \( R \) transitive then
\( F \models LF [w] \) \( \Leftrightarrow \) there is no \( f \in \mathcal{W}^{w} \) such that \( f(0) = w; Rf(i)f(i+1), \) all \( i \in \omega. \) (In fact LF implies transitivity itself.)
(2) Theorem 2 shows that all MRPs have corresponding first-order properties on the basis of most well-known modal logics. For the characteristic axiom of S4: \( \Box p \rightarrow \Box \Box p \) is equivalent to: \( (\forall y)(Rxy \rightarrow (\forall z)(Ryz \rightarrow Rzx)) \).
4. MRPs ON FRAMES WITH SUCCESSORS

We now consider frames with successors, i.e. \( F \models (\forall x)(\exists y)Rxy \).

Definition: a \( \preceq \)-formula is a MRP of the form \( M_1...M_k, N_1...N_k \) with:
\( M_i = \Box \) implies \( N_i = \Box \), for all \( i \). We are going to prove

Theorem 3

On frames with successors the MRPs with corresponding first-order properties are exactly those of the types (1) \( \preceq \)-formulas
(2) \( \Box^i \Box^j, \bar{N} \)
(3) \( \bar{M}, \Box^i \Box^j \)

Proof:

The formulas of the kinds described have corresponding first-order properties. For \( \preceq \)-formulas are universally valid on frames with successors and the others are even in \( M_1 \) (Cf. section 2). So we have to show that MRPs \( \bar{M}, \bar{N} \) that are not \( \preceq \)-formulas and contain an occurrence of \( \Box \Box \) in \( \bar{M} \), and one of \( \Box \Box \) in \( \bar{N} \) have no first-order equivalent on frames with successors. In order to do this we need the following

Lemma 2

MRPs of the following types have no first-order equivalent on frames with successors:
(1) \( \Box \Box \bar{M}, \Box \Box \bar{N} \).
(2) \( \Box \Box \bar{M}, \Box \Box \bar{N}; \Box \Box \) occurs in \( \Box \Box \bar{N}; \bar{M}, \bar{N} \) is not a \( \preceq \)-formula.
(3) \( \Box \Box \bar{M}, \Box \Box \bar{N}; \Box \Box \) occurs in \( \Box \Box \bar{N}; \bar{M}, \bar{N} \) is not a \( \preceq \)-formula.
(4) \( \Box \Box \Box^i \Box^j, \Box \Box \bar{N}; \Box \Box \) occurs in \( \bar{N} \).
(5) \( \Box \bar{M}, \Box \bar{N}; \Box \Box \) occurs in \( \bar{M}, \Box \Box \) occurs in \( \bar{N}; \bar{M}, \bar{N} \) a \( \preceq \)-formula.

Proof:

For the proof of this lemma it is essential to know the method of proof in [1]. We will state the main steps in proving (1), but the remainder of the proof will be as short as possible.

(1): Consider the frame \( F = \langle W, R \rangle \), given by
\( W = \{ q \} \cup \{ r_n, r_{n.1}, r_{n.2} \mid n \in \omega \} \cup \{ p_f \mid f \in \{ 1, 2 \}^\omega \} \).
\( R = \{ <q, r_n>, <q, r_{n.1}>, <q, r_{n.2}>, <r_n, r_{n.1}>, <r_{n.1}, r_{n.1}>, <r_{n.1}, r_{n.2}>, <r_{n.2}, r_{n.2} \mid n \in \omega \} \cup \{ <q, p^*_f \mid f \in \{ 1, 2 \}^\omega \} \cup \bigcup_{f \in \{ 1, 2 \}^\omega} \{ <p^*_f, r_n, f(n) > \mid n \in \omega \} \).
a) $F \models \square \diamond \vec{M}, \diamond \square \vec{N} [q]$.

$W$ is uncountable. Take a countable elementary substructure $F'$ of $F$ with domain containing $q$, $r_n$, $r_{n.1}$, $r_{n.2}$, all $n \in \omega$.

b) $F' \not\models \square \diamond \vec{M}, \diamond \square \vec{N} [q]$.

(Use a $f$ belonging to a $p_f$ that is not in the domain of $F'$ in order to find a suitable counter-example.)

If $\square \diamond \vec{M}, \diamond \square \vec{N}$ were equivalent to a first-order formula it would have to hold in $F'$ at $q$.

From now on we will just give the frames needed for applying this method of proof and some comment.

(2): Let $\square \diamond \vec{M}, \diamond \square \vec{N}$ be $\square M_1 \ldots M_k, \diamond \square N_1 \ldots N_1$. ($k = 0$ means: $\vec{M}$ is empty.)

For some $i$: $N_i = \square (i)$. For some $j$: $M_j = \diamond$, $N_j = \square$ or $k \neq 1$ (ii).

We associate trees $T_n$ with $\vec{M}$ as follows. Start with $r_{n.1}, r_{n.2}$.

Let $R_{r_{n.1}}, R_{r_{n.2}}$. If $M_1 = \square$, add $r_{n.1.1}, r_{n.1.2}$ and let $R_{r_{n.1.1}}, R_{r_{n.1.2}}$.

If $M_1 = \diamond$, add $r_{n.1.1}, r_{n.1.2}, r_{n.2.1}, r_{n.2.2}$ and let $R_{r_{n.1.1}}, R_{r_{n.1.2}}, R_{r_{n.2.1}}, R_{r_{n.2.2}}$. Etc. Let $\{e_1^n, \ldots, e_n^n\}$ be the set of end-points. $N \geq 2$. Now add a point $s_n$ and let $R_{s_n}$ for every end-point $e$. In this way we have obtained a frame $F_n$ where $\diamond \vec{M}_p$ is true at $r_n$, if $p$ is true at some end-point $e$. Also if $p$ is true in only one end-point $e \diamond N_1 \ldots N_1 p$ will not be true in $r_n$, because of (ii).

Also take a copy of this structure, say $F$, with minimal point $r$. Let $i$ be the first-number for which $N_i = \square$. We now describe the frame.

(Identify $F_n$ with the set of its points for convenience.)

$W = \{q, p_1, \ldots, p_i\} \cup \{r_n \mid n \in \omega\} \cup F \cup \bigcup_{n \in \omega} F_n \cup \{p_f, p_{f,i+1}^n, \ldots, p_{f,i+1}^n \mid f \in \{1, \ldots, N\}, n \in \omega\}$.

(If $i = 1$ only $p_f$'s.) Identify $r$ and $p_1$. 

\[ R = \{<q, r_n> \mid n \in \omega\} \cup \text{the structure of the } F_n\text{'s and } F \text{ as explained} \cup \{<q, p_k> \mid \omega \in \{1, \ldots, N, 1\} \cup \{<p_k, p_k> \mid f \in \{1, \ldots, N, 1\} \cup \{<p_f, p_{f.i+1}> \mid f \in \{1, \ldots, N, 1\}, n \in \omega\} \cup \bigcup_{f \in \{1, \ldots, N, 1\}} \{<p_{f.1}, e_{f}(n) > \mid n \in \omega\}. \]

(F is needed for making \(\square \otimes \hat{N}\) true at \(q\) in the countable elementary subframe.)

E.g. \(\square \diamond \square \odot, \diamond \diamond \square \odot\)

(3): Let \(\square \diamond \hat{M}, \square \otimes \hat{N}\) be \(\square \diamond \hat{M}_1 \ldots \hat{M}_k, \square \otimes \hat{N}_1 \ldots \hat{N}_l\). (\(k = 0\) means: \(\hat{M}\) empty.) Let \(i\) be the first number for which \(N_i = \square\).

\[ W = \{q, p_1, \ldots, p_k\} \cup \{r_n, r_{n.1}, r_{n.2} \mid n \in \omega\} \cup \{p_f \mid f \in \{1, 2\} \} \cup F \text{ (as constructed above)} \]

\(W \cup \{p_f \mid f \in \{1, 2\} \} \cup \{<q, p> \mid f \in \{1, 2\} \} \cup \text{the structure of } F \cup \bigcup_{f \in \{1, 2\}} \{<p_{f.1}, r_{n,f}(n) > \mid n \in \omega\}. \]

E.g. \(\square \diamond \diamond \diamond, \square \diamond \diamond \diamond\).
Suppose $\tilde{M}, \tilde{N}$ is a $\leq$-formula, but $\tilde{M} \neq \tilde{N}$. (We need this case in the next section.) Let $\tilde{M}_j$ be the first $\blacksquare$ in $\tilde{M}$ for which $N_j = \Diamond$.

A minor modification in the given frame suffices. Instead of $F$ use $\{u_1, \ldots, u_j\}$ with $\langle p_1, u_1 \rangle, \ldots, \langle u_{j-1}, u_j \rangle \in R$. (This new frame has a point without a successor: $u_j$.)

(4): Let $\tilde{N}$ be $\blacksquare^k \Diamond 1 \Diamond \blacksquare N_1 \ldots N_m$. ($m = 0$ means: $\tilde{N}$ is empty.) So we have: $\blacksquare \Diamond \Diamond \Diamond \blacksquare \blacksquare \blacksquare \blacksquare \Diamond 1 \Diamond \blacksquare N_1 \ldots N_m$.

Case 1: $k < i$.

For every $n \in \omega$ we construct $U_n$ as follows: take $r_{n,1}, r_{n,1}^1, \ldots, r_{n,1}^{1}, r_{n,1}^{2}, \ldots, r_{n,1}^{i}, r_{n,1}^{j}, r_{n,1}^{k}, r_{n,1}^{i+k}$ and let $R$ on $U_n$ be: $\langle r_{n,1}^1, r_{n,1}^{1}, r_{n,1}^{2}, \ldots, r_{n,1}^1, r_{n,1}^{i}, r_{n,1}^{j}, r_{n,1}^{k}, r_{n,1}^{i+k} \rangle$.

$W = \{q, p_1, \ldots, p_{k+1}^2 \} \cup \bigcup_{n \in \omega} U_n \cup \{p_f, p_f^1, \ldots, p_f^m \mid f \in \{1, 2\}^\omega, n \in \omega\}$ (if $m = 0$ only $p_f$'s appear) $\cup \{s\}$.

$R = \{q, r_n \mid n \in \omega\} \cup$ the structure of the $U_n$'s as described $\cup$

$\{q, p_1^\omega, p_1^\omega, \ldots, p_{k+1}^\omega, p_{k+1}^\omega \} \cup \{p_{k+1}^\omega, p_f \mid f \in \{1, 2\}^\omega\}$

$\cup \{p_f, p_f^1, \ldots, p_f^m \mid f \in \{1, 2\}^\omega, n \in \omega\} \cup$

$\bigcup_{f \in \{1, 2\}^\omega} \{p_f^m \mid f \in \{1, 2\}^\omega, n \in \omega\} \cup \{\langle p_1, s \rangle, \langle s, s \rangle, \langle q, s \rangle\}$.

E.g. $\blacksquare \Diamond \Diamond \Diamond \Diamond \Diamond$, $\blacksquare \Diamond \Diamond \Diamond \Diamond \Diamond \Diamond$
Case 2: $k \geq i$.

$U$ is like before, but in the $R$-structure the ordered couples $r^1_{n,1+k}$, $r^2_{n,1+k}$ must be dropped.

$W$ = like before, but with a new element $t$ added.

$R$ = like before, but for the simpler structure of the $U_n$'s and the addition: $\{q, t, \langle r^1_{n,i}, t \rangle, \langle r^2_{n,i+1}, t \rangle \mid n \in \omega\}$.

E.g. $\Diamond \Box \Box \Box$, $\Box \Box \Box \Box \Box$.

\[(5): \text{Let } \Diamond \tilde{M}, \Box \tilde{N} \text{ be } \Diamond M_1, \ldots, M_k, \Box N_1, \ldots, N_k. \text{ Let } \tilde{M}, \tilde{N} \text{ a } \preceq\text{-formula. Let } i \text{ be the first number such that } M_1 = \Box, M_{i+1} = \Diamond. \text{ Let } j \text{ be the first number such that } N_j = \Diamond, N_{j+1} = \Box. \text{ Suppose } i \leq j. (\text{If this is not the case we use the inversion-principle IP and treat } \Diamond \tilde{N}_1, \ldots, \tilde{N}_k, \Box \tilde{M}_1, \ldots, \tilde{M}_k \text{ instead.}) \]

\[W = \{q, r^1, \ldots, r^i, p_1, \ldots, p_j\} \cup \{r_n, r_{n,1}, r_{n,2} \mid n \in \omega\} \cup \{p_f \mid f \in \{1,2\}^\omega\}.\]

\[R = \{\langle q, r^1, \ldots, <i-1, r^i, \langle q, p_1, \ldots, \langle p_{j-1}, p_j \rangle \rangle \mid \langle r^i, r_n, r_{n,1}, \langle r_n, r_{n,2}, <r_{n,1}, r_n, r_{n,2} \rangle \mid n \in \omega\} \cup \{\langle p_j, p_f \mid f \in \{1,2\}^\omega\} \cup \]

\[\bigcup_{f \in \{1,2\}^\omega} \{\langle p_f, r_n.f(n) \rangle \mid n \in \omega\}.\]

(The hard part here is to prove that the MRP considered holds in this frame. Use the fact that $\tilde{M}, \tilde{N}$ is a $\preceq$-formula and note that $N_{i+1} = \Diamond, M_{j+1} = \Box$.)

E.g. $\Diamond \Box \Box \Box$, $\Box \Box \Box$.

QED.
We can finish the proof of theorem 3 now. Start with \( \mathcal{M}, \mathcal{N} \): not a \( \square \) formula, \( \square \) occurs in \( \mathcal{M} \), \( \square \) in \( \mathcal{N} \). Let \( \mathcal{M} = M_1 \ldots M_k, \mathcal{N} = N_1 \ldots N_l \). Let \( i \) be the smallest number for which \( M_i = \square \), \( M_{i+1} = \Diamond \)
or \( N_i = \Diamond \), \( N_{i+1} = \Box \) (\( *) \).

Case 1: \( M_1 \ldots M_k, N_1 \ldots N_l \) is a \( \leq \) formula. Then for some \( n < i \):
\( M_n = \Diamond \), \( N_n = \Box \). Take the largest such \( n \) and consider \( M_n \ldots M_k, N_n \ldots N_k \). (\( 1 = k \))! We use lemma 2(5) now as follows. Take points \( q_1, \ldots, q_{n-1}, q \) with \( R_{q_1} q_2, \ldots, R_{q_{n-1}} q \). Add these to the frame constructed in the proof of lemma 2(5). This gives a frame \( F_0 \) in which:
\[ F_0 \models M, N [q_i] \Rightarrow F_0 \models M_n \ldots M_k, N_n \ldots N_k [q] \Rightarrow F_0 \models M_n \ldots M_k, N_n \ldots N_k [q]. \]
Call this procedure 'fixing the first \( n-1 \) modalities'.

Case 2: \( M_1 \ldots M_k, N_1 \ldots N_l \) is not a \( \leq \) formula.
(a): Suppose \( M_1 \ldots M_k = \square \Diamond M_{i+1} \ldots M_k \). If \( N_1 N_{i+1} = \Diamond \Box \Diamond \Box \Diamond \Box \)
- while \( M_1 \ldots M_k \) is of the form described in lemma 2(4) - we may use the corresponding clauses of the lemma, fixing the first \( i-1 \) modalities. If \( N_1 N_{i+1} = \Box \Box \), but \( M_1 \ldots M_k \) not of the required form, we move on to the right. Let \( i_1 \) be the smallest number > \( i \) for which the situation (\( * \)) occurs and repeat the procedure.
(b): If \( N_1 \ldots N_l = \Diamond \Box N_{i+1} \ldots N_l \) and we are not in case (a), we act just like before, using the dual form of lemma 2 which we did not state. (It amounts to inverting \( M_1 \ldots M_k, N_1 \ldots N_l \) to \( \overline{N}_1 \ldots \overline{N}_l, \overline{M}_1 \ldots \overline{M}_k \)) QED.

Remark:
There is a second notion of correspondence: For all \( F; F \models \phi_m \Rightarrow F \models \phi_r \).
\( (F \models \phi_m \Rightarrow \phi_r \text{ def } \forall w \in W; F \models \phi_m [w].) \) Our results do not imply that formulas without first-order equivalents in our sense of the word have no first-order equivalents in this weaker sense. (Compare the remark at the end of section 5.)

The method of \( [1] \) as used in the preceding proofs allows one to prove the non-existence of first-order equivalents in the weaker sense, provided that \( F \models \phi_m \) in the frame \( F \) given. Although this is true for some of the frames given it does not hold for all of them. Therefore the problem which formulas have no first-order equivalents in the second sense remains open.
5. MRPs ON ARBITRARY FRAMES

In view of the preceding results it now suffices to establish the behaviour of \( \leq \)-formulas on arbitrary frames in order to solve the general problem mentioned in the introduction.

Theorem 4

The \( \leq \)-formulas with first-order corresponding properties are exactly those of the types

1. \( \check{N}, \Diamond^i \Diamond^j \)
2. \( \Diamond^i \Diamond^j, \check{N} \)
3. \( \Box^i \check{N}, \check{N} \check{M}, \) where length \( (\check{N}) \neq i \).
4. \( \check{N} \check{M}, \Diamond^i \check{M}, \) where length \( (\check{N}) = i \).

Proof: Formulas of type (3) are equivalent to \( \Box^i \check{M} T V \check{N} T \). For both \( F \models \Box^i \check{M} T [w] \) and \( F \models \check{N} T [w] \) imply \( F \models \Box^i \check{M}, \check{N} \check{M} [w] \).

Conversely, let \( F \models \Box^i \check{M} T \land \check{N} T [w] \), and \( V(p) = W \). Then \( <F, V> \models \Box^i \check{N} p [w] \) but not \( <F, V> \models \check{N} \check{M} p [w] \).

For the negative part we need:

Lemma 3

\( \leq \)-formulas \( \check{M}, \check{N} \) with an occurrence of \( \Box \Diamond \) in \( \check{M} \) and one of \( \Diamond \Box \) in \( \check{N} \) have no corresponding first-order property if they are of one of the following types:

1. \( \Diamond \Box \check{O}, \Diamond \check{P} \Box \check{R} \), where length \( (\check{O}) = \) length \( (\check{P}) \).
2. \( \Box \check{O} \check{O} \check{O} \Diamond \check{S}, \Diamond \check{P} \Box \check{R} \Diamond \Diamond \check{T} \), where length \( (\check{O}) = \) length \( (\check{P}) \),
   length \( (\check{T}) = \) length \( (\check{R}) \).
3. \( \check{O} \Box \check{O} \check{I} \check{O}, \check{P} \Box \check{I} \check{O} \check{R}, \) where length \( (\check{O}) = \) length \( (\check{P}), i \neq 0 \).

Remark: Using IP it turns out that the same holds for the types:

1. \( \Box \Diamond \check{O}, \check{P} \Box \Diamond \check{R} \), where length \( (\check{O}) = \) length \( (\check{P}) \) and \( \Box \Diamond \) occurs on the right-hand side.
2. \( \check{O} \Diamond \check{O} \check{I} \check{O}, \check{P} \Diamond \check{R} \Diamond \check{T} \), where length \( (\check{O}) = \) length \( (\check{P}) \),
   length \( (\check{T}) = \) length \( (\check{R}) \).
3. \( \check{O} \Diamond \check{I} \check{O}, \check{P} \Diamond \check{I} \check{O} \check{R} \), where length \( (\check{O}) = \) length \( (\check{P}), i \neq 0 \).
Proof: Again we only give the frames, the method being that of [1].
\( \bar{o} = 0_1 \ldots 0_k, \bar{r} = p_1 \ldots p_k, \bar{q} = 0_1 \ldots 0_1, \bar{r} = r_1 \ldots r_1, \bar{s} = s_1 \ldots s_m, \bar{t} = t_1 \ldots t_m. \)

(1) Let \( i \) be the first number such that \( 0_{i+1} = \Box, 0_{i+1} = \Diamond \) (a) or, if there is no such number, the first such that \( Q_i = \Diamond \) (b). Let \( j \) be the first number such that \( P_j = \Box, j = k + 1 \) if no such number exists. We only treat case (a). The solution for case (b) is essentially the same (but easier).

\[ W = \{ q, r^1, \ldots, r^i, p_1, \ldots, p_{j-1} \} \text{ (if } j = 1: \text{ no } p_i\text{'s) } \cup \{ r_n, r_{n.1}, \ldots, r_n.(2+1+k-i), r_n.(2+1+k-i), r_{n.(2+1+k-i)}, r_{n.k-i.1}, r_{n.k-i.1} \} \cup \{ p_f | f \in \{1, 2\}^\omega \}. \]

\[ R = \{ <q, r^1>, \ldots, <r^i, i>, <q, p_1>, \ldots, <p_{j-2}, p_{j-1}> \} \cup \]

\[ \bigcup_{f \in \{1, 2\}^\omega} \{ <p_f, r_f>(n) | n \in \omega \} \cup \{ <p_{j-1}, r_n>, <r_n, r^1>, <r_n, r_{n.1}>, <r_n, r_{n.2}>, \ldots, <r_n, r_{n.(1+k-i)}, r_{n.(1+k-i)}, r_{n.(2+1+k-i)}>, <r_n, r_{n.(1+k-i)}, r_n.(1+k-i)>, <r_n, r_{n.k-i.1}, r_{n.k-i.1}>, <r_n, r_{n.k-i.1}, r_{n.k-i.1}>, <r_n, r_{n.k-i.1}, r_{n.k-i.1}> | n \in \omega \}. \]

E.g. (i) \( \Diamond \Box \Box \Box \Box, \Diamond \Box \Box \Box \)

(ii) \( \Diamond \Box \Box \Box, \Diamond \Box \Box \Box \Box \)

\[ p_f \]

\[ q \]

\[ r_n \]

\[ r_{n.1} \]
(2) We do not prove this case since the idea involved is essentially that of (1). The same trick of adding points without successors, but now at more places, works here as well. (Compare (3).)

E.g. (i) \[ \text{Diagram} \]

(ii) \[ \text{Diagram} \]

(3) Let \( j \) be the first number such that \( P_j = \square \), \( j = k + 1 \), if no such number exists. \( h \) is the number of modalities after the first occurrence of \( \square \) on the right-hand side.

\[
W = \{ q, r_1, \ldots, r_{k+1}, r_{k+1}, \ldots, r_{i+1}, p_1, \ldots, p_{j-1} \} \quad \text{(if } j = 1: \text{no } p_i \text{'s)} \cup \\
\{ p^n_f, p^n_{f,1}, \ldots, p^n_{f,h}, p^n_{h-1-1}, f \in \{1, 2\}^\omega, n \in \omega \} \cup \{ r_n, r_{n.1}, r_{n.2} \mid n \in \omega \}.
\]

\[
R = \{ <q, r_1>, <r_1, r_2>, \ldots, <r^{k+1}, r_{k+1}>, <r^{k+1}, r_1>, <r^{k+1}, r_2>, \ldots, <r^{k+1}, r_{i+1}>, <q, p_1>, <p_1, p_2>, \ldots, <p_{j-2}, p_{j-1}> \} \cup \\
\{ <r^{k+1}, r_n>, <r_n, r_{n.1}>, <r_n, r_{n.2} \mid n \in \omega \} \cup \{ <p_{j-1}, p_f> \mid f \in \{1, 2\}^\omega \}
\]

(if \( j = 1: \) take \( q) \cup \{ <p^n_f, p^n_{f,1}, p^n_{f,2}, \ldots, p^n_{f,h-1}, p^n_{f,h}, <p^n_{f,h}, r_{n.f(n)}> \mid f \in \{1, 2\}^\omega, n \in \omega \} \cup \{ <p^n_{f.h-1}, p^n_{f.h-1.1} \mid n \in \omega, f \in \{1, 2\}^\omega \}.\]
E.g. □ □ ◊ □ ◊ ◊ ◊

Using the lemma in the same way as in the proof of theorem 2 (fixing modalities as needed) we can show that all \( \leq \)-formulas of the form \( \mathcal{M}, \mathcal{N} \) where length \( (\mathcal{M}) = \text{length} (\mathcal{N}) \) and \( \mathcal{T}, \mathcal{U} \) is of one of the types described in the lemma, have no first-order equivalents.

The proof may be completed now by considering what \( \leq \)-formulas \( \mathcal{M}, \mathcal{N} \) with \( □ ◊ \) occurring in \( \mathcal{M} \) and \( ◊ □ \) in \( \mathcal{N} \), are not excluded by the above and the remark in the proof of lemma 2(3) of section 4 (again fixing irrelevant modalities if needed). It turns out that these can only be of the types mentioned in theorem 4.

Since ◊ □ occur in \( \mathcal{N} \) there is an \( i \) with: \( M_i = □, N_i = □ \).

Since □ ◊ occurs in \( \mathcal{M} \) there is an \( j \) with: \( M_j = ◊, N_j = ◊ \).

Take the smallest such \( i, j \) and suppose \( i < j \). (If \( i > j \): use IP.)

We will not describe the elimination-process in full detail, but just give the types that are not excluded.

(a) \( i \neq 0. □^i □ □^j □ □ M, □^i □ N □ ◊ ◊ M, \) with length \( (\mathcal{N}) = j \).

(If we try to evade lemma 2(3) by having ◊ □ on the right-hand side followed by only ◊'s it turns out that \( \mathcal{M} \) should contain only ◊'s in order not to violate lemma 3(3).)

(b) \( i \neq 0. □^i □ □^j □ □ ◊ M, □^i □ N □ ◊ ◊ k M, \) length \( (\mathcal{N}) = j \).

(c) \( i \neq 0. □^i □ □^j □ □ ◊ M, □^i □ N □ ◊ ◊ ◊ ◊ k M, \) length \( (\mathcal{N}) = j \).

(No ◊ may be allowed instead of the last shown □ on the right-hand side because of lemma 3(2).)

(d) □ □^j □ □ M, □ N □ ◊ M, \text{ length } (\mathcal{N}) = j .

(Same comment as under (a).)
(e) $\Box \Box^j \Diamond \Diamond^k, \Box \hat{N} \Diamond \Diamond^k$, length $(\hat{N}) = j.$

(f) $\Box \Box^j \Diamond \Diamond^k \Box \hat{M}, \Box \hat{N} \Diamond \Diamond^k \Box \hat{M}$, length $(\hat{N}) = j.$

(Same comments as under (a).)

(g) $\Box \Box^j \Diamond \Diamond^k \Box \hat{M}, \Box \hat{N} \Diamond \Diamond^k \Diamond \Box \hat{M}$ is excluded, if $\Diamond \Box$ occurs on the right-hand side. (Because of lemma 3(1).)

So the only case that is allowed would be a MRP of the type $\hat{N}, \Box^r \Diamond^s.$

QED.

Corollary:

The MRPs with first-order corresponding properties are exactly those of the types:

(1) $\Diamond \Box^i \Box^j, \hat{N}$

(2) $\hat{M}, \Box^i \Diamond^j$

(3) $\Box^i \hat{M}, \hat{N}, \hat{M}$, where length $(\hat{N}) = i.$

(4) $\hat{N}, \hat{N}, \Diamond^i \hat{M}$, where length $(\hat{N}) = i.$

Remark: (Cf. the remark at the end of section 4.)

Lemma 3(3) provides us with formulas $\phi_m$ for which no first-order $\phi_r$ exists with: $F \models \phi_m [w] \leftrightarrow F \models \phi_r [w]$, all $F, w$, such that do have first-order equivalents in the weaker sense $F \models \phi_m \leftrightarrow F \models \phi_r$, all $F$. E.g. $\Box \Box \Diamond \Box, \Diamond \Box \Diamond$. This formula corresponds to $(\forall x)(\exists y)Rxy$ in the weaker sense. (Of course, if $\phi_m$ is equivalent to $\phi_r$ in the first sense it is equivalent to $(\forall x) \phi_r$ in the second sense.)
6. SOME USES OF MRP

(1) Define the length \( l(M) \) of a modal logic \( M \) as the smallest number \( n \) such that every sequence of modal operators \( \vec{\mathcal{N}} \) is equivalent in \( M \) to such a sequence of length \( \leq n \), if such a number exists; \( l(M) = \omega \), otherwise. We have e.g. \( l(S5) = 1, l(S4.2) = 2, l(S4) = 3 \). As for transitive frames: \( l(\langle \Box, \Box \Box \rangle) = \omega \). For \( \Box^k \) is not reducible to a \( \vec{\mathcal{N}} \) of length \( < k \). (Use a linear order of length \( k \).)

MRPs are especially interesting if they serve to establish the length of a logic. Consider \( S5 \), with characteristic axioms \( \Diamond \Box, \Box \) and \( \Box, \Box \).

A more natural way of obtaining a system with length 1 would be by using: \( \Diamond \Box, \Box ; \Box, \Diamond \Box ; \Box \Box, \Box ; \Box, \Box \Box \). In [2] it is shown, using corresponding first-order properties, that this logic can also be axiomatized as \( \Diamond \Box, \Box ; \Box, \Box \Box \); \( \Diamond T \). This logic is weaker than \( S5 \). Quite generally we have:

For all \( n \in \omega \) there exists a modal logic \( M^n \) such that \( l(M^n) = n \).

Proof: Let \( M^n \) have the characteristic axioms \( \Diamond \vec{\mathcal{N}}, \vec{\mathcal{N}} ; \vec{\mathcal{N}}, \Diamond \vec{\mathcal{N}} ; \Box \vec{\mathcal{N}}, \vec{\mathcal{N}} ; \vec{\mathcal{N}}, \Diamond \vec{\mathcal{N}} \) for all \( \vec{\mathcal{N}} \) of length \( n \). Clearly \( l(M^n) \leq n \). But not \( l(M^n) < n \).

For suppose \( M^n \) implied \( \vec{\mathcal{N}}, \vec{\mathcal{N}} \), where length \( (\vec{\mathcal{N}}) = n \), length \( (\vec{\mathcal{N}}) < n \).

Consider the frame \( F = \langle W, R \rangle \) with \( W = \{1, \ldots, n+1\} \),
\( R = \{i, i+1\} \mid 1 \leq i \leq n \} \cup \{n+1, n+1\} \). \( F \models M^n [1] \), but not \( F \models \vec{\mathcal{N}}, \vec{\mathcal{N}} [1] \) (Let \( V(p) = \{n+1\} \).

\( l(M) = l(N) \) does not imply that \( M \) and \( N \) are deductively equivalent.
E.g. \( l(\langle \Box, \Box \Box ; \Box, \Box \rangle) = 3 = l(S4) \). (Another example was given above.)

(2) Define the degree of a modal formula as follows (degree \( (\alpha) = d(\alpha) \)):
\( d(p) = 0 \); \( d(\neg \alpha) = d(\alpha) \); \( d(\alpha \rightarrow \beta) = \max (d(\alpha), d(\beta)) \); \( d(\Box \alpha) = d(\Diamond \alpha) = d(\alpha) \), if \( \alpha \) is of the form \( \Box \beta \) or \( \Diamond \beta \); \( = d(\alpha) + 1 \), otherwise.

In \( S5 \) there is a theorem about the existence of modal conjunctive normal forms. It states that every formula is reducible to a propositional compound of the types \( \Diamond \alpha, \Box \alpha, \alpha \), where \( \alpha \) is a propositional formula.

In this case two reductions are performed at once: both length and degree are reduced to 1.
We can separate the two notions and concentrate on a reduction of the degree only. If we look at the form an inductive proof for this kind of assertion would have, we find that we need a principle of the form: $$\Box(\Box p \lor \Diamond q \lor r) \implies ?.$$ We study here $$\Box(\Box p \lor \Diamond q \lor r) \implies \Box \Box p \lor \Box \Diamond q \lor \Box r.$$ (Cf. [2], p. 55/6.) One direction of this is trivial so only $$\Box(\Box p \lor \Diamond q \lor r) \implies \Box \Box p \lor \Box \Diamond q \lor \Box r$$ is relevant. By a general form of IP we may just as well consider $$\Diamond \Box p \land \Diamond \Box q \land \Diamond \Box r \implies$$ $$\Diamond (\Box p \land \Box q \land \Box r).$$ This is of the form described in theorem 1. It turns out that its corresponding first-order property is equivalent, after some simplification, to $$(\forall y)(Rx y \rightarrow (\forall z)(Rx z \rightarrow (\forall u)(Rzu \rightarrow Ry u))).$$ By a translation result from certain predicate-logical formulas to corresponding modal ones ([2] also treats the problem, a converse to that of section 2, of determining which relational properties are expressible by means of modal formulas) this is seen to be an equivalent of the MRP $$\Box, \Box. \Box.$$ So in the logic with $$\Box p \rightarrow \Box \Box p$$ as its single characteristic axiom all formulas are reducible to formulas of degree 1. A general result like the one about length would seem to be provable as well.
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