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ABSOLUTENESS OF INTUITIONISTIC LOGIC

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Preface

The present treatise is a corrected version of the author's Ph.D. dissertation, written at the University of Amsterdam in 1974/75 under the direction of Professor A.S. Troelstra. My research, as well as the production of the dissertation, were generously supported by the Mathematical Center, and as is customary for such dissertations, it now appears as a Mathematical Centre Tract.

The text that follows is divided into two parts. Part A deals with theories whose arithmetical fragment is part of $\text{IA}^* := \text{Heyting's arithmetic}$ $\text{IA}$ extended with transfinite induction over all recursive well-orderings. Such theories (as well as some others closely related to them) are named "regular". It is shown that fairly strong intuitionistic theories, and - in particular - the intuitionistic impredicative theory of types, are regular.

In part B we treat maximality (or "absoluteness") properties of intuitionistic (Heyting's) propositional and predicate logic $\text{IL}_0$ and $\text{IL}_1$ for regular theories. Here, $L$ is said to be maximal (or "absolute") for $T$ if

$$\forall L, P_1, \ldots, P_k \forall T, A_1, \ldots, A_k$$

for some arithmetical relations $A_i$ of the same arity as $P_i$ ($i=1, \ldots, k$). The maximality is uniform if the $A_i$'s are independent of $F$. We are also interested in having the substituted relations $A_i$ as low as possible in the arithmetical hierarchy.

Refined versions of the results of part A are incorporated in LEIVANT [A], while the theorems of part B together with several other maximality results are proven in LEIVANT [B]. Nevertheless, the present exposition might still be useful to the interested reader. In contrast to the aforementioned papers, we use here natural deduction systems, and the proofs, especially in part B, illustrate the convenience of using natural deduction to straightforwardly formalize one's intuitive ideas. The sections in part B motivating the proofs are particularly relevant here. The sequential calculi used in our [A] and [B] allow more succinct presentations, but at the cost of concealing to some extent the motivating ideas. Also, our exposition here is more leisurely, so that, in conjunction with the use made of natural deduction, the effect is to reduce the effort required from the reader.
To do justice to the reader, we should place the maximality results proven here amongst other similar results.

Since we prove "absoluteness" of IL, the interest in treating propositional logic $\Pi_0$ lies only in reducing the complexity of the substituted sentences (0-ary relations). D.H.J. de Jongh and C. Smoryński [73] have proved that there exist uniform arithmetical substitutions, and also - locally - $\Sigma_0^0$ substitutions for $\Pi_0$ and $T = \Pi$. Theorem I of part B below improves this by making the $\Sigma_1^0$ substitutions depend only on the number of propositional letters in the schema $\Phi$. However, by a uniformization lemma proved in Leivant [B] §1.2, already the local $\Sigma_1^0$-absoluteness implies completely uniform absoluteness with $\Sigma_1^0$ substitutions. From this, using metamathematical properties of $\Pi_0$, one easily derives uniform absoluteness also with (binary) disjunctions of $\Pi_1^0$ sentences as substitutions (idem, §1.6). Similar statements are also true when $\Pi$ is replaced by any regular $T$, and also for $T$ extended with either Church's Thesis $\Delta_0$ or the Independence-of-Premiss Principle $\Pi_0^0$ (cf. Troelstra [73] for their statement).

When Markov's Principle $\Pi$ is added, $\Sigma_1^0$ absoluteness fails (since $\vdash_{\Pi + \Pi} \exists A \exists A \forall A$ for any $\Sigma_1^0$ sentence $A$), but $\Pi_0$ is uniformly $\Sigma_2^0$ absolute for $T + \Pi_0^0 + \Pi$ (cf. idem).

Turning to Intuitionistic Predicate Logic $\Pi_1$, we should start by mentioning a proof of De Jongh [73] of a relativized version of absoluteness. Theorem II of part B below states the uniform $\Pi_0^0$ absoluteness of $\Pi_1$ for any regular theory $T$. $\Pi_1$ is also uniformly $\Sigma_2^0$ absolute for such $T$ (Leivant [B] §2.4) but not even locally $\Sigma_1^0$ absolute for $\Pi$ (Leivant [76]; this was also proved in §B.6 of the original version of the dissertation). Nevertheless, $\Sigma_1^0$-absoluteness does hold for certain fragments of $\Pi_1$ (Leivant [B] thm.2.VII).

All theories mentioned above, for which $\Pi_1$ is proved absolute, are r.e., and the "regular" ones are in $\Pi_{1*}$. Allowing for more complex substitutions, one obtains in one stroke maximality of $\Pi_1$, for all regular theories, with the substitutions independent also of the theory; namely - one proves the uniform maximality of $\Pi_1$ for $\Pi_{1*}$ (Leivant [B] thm.2.VIII). The complexity of the substitutions may be somewhat reduced for $\Pi_0$.

For logic with equality, we have the $\Pi_0^0$ (and $\Pi_2^0$) uniform absoluteness, for regular theories, of $\Pi_1$ extended with the following axioms:

\[
\forall x \forall y (x = y \lor \neg x = y)
\]

\[
\exists x_1 \exists x_2 \ldots \exists x_n [ \forall 0 < i < j < n \forall x_i = x_j ] \quad n = 2, 3, \ldots
\]

(idem §2.6).
As noted in section 2 of the introduction below, there is no straightforward connection between classical and intuitionistic absoluteness. For Classical Predicate Logic $\text{CL}_1$, local $\Delta^0_2$ absoluteness (for sound arithmetic theories) is an immediate consequence of the refinement obtained by HILBERT and BERNAYS [39] to Gödel's Completeness Theorem. The uniformization technique of LEIVANT [B] mentioned above may be applied here, to yield uniform $\Delta^0_2$ (and $\Pi^0_2$) absoluteness of $\text{CL}_1$ (idem, thm.2II).
INTRODUCTION

(For unexplained terminology see the preliminary section P below.)

Int. 1. The concept of absoluteness and the main results

Our central aim in this treatise is to prove that the formal systems of intuitionistic propositional and predicate logics \( L_0 \) and \( L_1 \) resp.) are "schematically complete" for intuitionistic (Heyting's) arithmetic \( A \), as well as for certain extensions of \( A \). Let us first describe these results as cases of a general type of problems.

Let \( L \) be a system of logic, and let \( M \) be a system of mathematics based on the language of \( L \); i.e., the language of \( M \) contains all the logical constants of the language of \( L \) as well as constants or definable objects for each type of parameter of that language. (Examples: (1) \( L \) is first order predicate logic and \( M \) is ZF set theory; (2) \( L \) is second order logic and \( M \) is second order arithmetic; (3) \( L \) is first order logic without first-order parameters (but with first-order "bound" variables, of course), and \( M \) is second order logic.)

Let \( C \) be a class of defined constants in the language of \( M \). A schema \( F \) in the language of \( L \) is \( C \)-absolute for \( M \) if for each instance \( F^* \) of \( F \) which comes from \( F \) by substituting constants of \( C \) for parameters of the corresponding type,

\[
(1) \quad \vdash_{M} F^*.
\]

\( L \) is said to be \( C \)-absolute for \( M \) if

\[
(2) \quad L = \{ F \mid F \text{ is } C\text{-absolute for } M \},
\]

i.e., if

\[
(3) \quad \vdash_{L} F \iff (\forall \text{C-inst. } F^* \text{ of } F) \vdash_{M} F^*
\]

(Several other alternatives have been proposed to name the above property of \( L \): "\( L \) is maximal (schematically complete, saturated) w.r.t. \( M \)," and "\( M \) is faithful to \( L \).")

When \( M \) is based on \( L \), the implication from left to right in (3) is
trivial, the interesting part being of course the converse direction. This has been occasionally expressed in a contrapositive-like form:

\[(4) \quad \vdash_{\mathcal{L}} F \Rightarrow (\exists C\text{-inst. } F^* \text{ of } F) \quad \vdash_{\mathcal{M}} F^*.\]

In our treatment below we prove cases of (4), which is intuitionistically independent of (3). Assuming however Markov’s principle for prim. rec. predicates,

\[\vdash_{\mathsf{MPR}} \forall \exists x A(x) \Rightarrow \exists x A(x) \quad (A \text{ arithmetical, quantifier free}).\]

(4) clearly implies (3) (by contraposition).

Actually the results proved below give the instance $F^*$ of $F$ for (4) constructively, independently of the premiss, and quite uniformly. Namely, for a large class of "regular" theories $T$, which includes $A$ (cf. A.1):

**THEOREM I.** Given a regular theory $T$, $L_0$ is $\Sigma^0_1$-absolute for $T$. For any given schema $F$ of $L_0$ the substitutions depend only on the number of propositional variables in $F$.

**THEOREM II.** For $T$ as above, $L_1$ is $\Pi^0_2$-absolute for $T$, with substitutions which depend on $T$ only.

These theorems are stated in more detail in 8.0 below.

D.H.J. de Jongh had proved already in 1969 the absoluteness of $L_0$ for intuitionistic arithmetic $A$ (and extensions $A^\ast$ of $A$ with transfinite induction over some prim. rec. well ordering $\langle \rangle$). SMORYNSKI [72] shows that the meta-substitutions may be taken to be $\Sigma^0_1$ (though depending on each schema), and H. FRIEDMAN [73] proves that by allowing the meta-substitutions to be $\Pi^0_2$ one gets uniform absoluteness. This last result is a corollary of our theorem II.

All the results just mentioned were obtained by classical methods. Uniform absoluteness is however formalized as a $\Pi^0_2$ statement, since it has roughly the form

\[(\forall \text{ schema } F) \left[ \forall x \neg \mathsf{Pr}_L(x, r^F_n) \Rightarrow \forall x \neg \mathsf{Pr}_T(x, r^{F^*}_n) \right].\]

where $\mathsf{Pr}_L$ and $\mathsf{Pr}_T$ are prim. rec. proof predicates for $L$ and $T$ resp., and
where $f$ is a fixed prim. rec. function. For $\Pi_2^0$ sentences, however, provability in classical arithmetic implies provability in intuitionistic arithmetic (cf.Troelstra [73]).

So the main novelty of theorem I is the "locally-uniform" $\Sigma_1^0$ substitution. We nevertheless present this result in some detail, for two reasons. Firstly, it may be used as an expositional introduction to the proof of theorem II; secondly, the method employed might turn out to be helpful in solving a number of other problems concerning the relation between $\mathbb{L}_0$ and $A$.

As to predicate logic, de Jongh has proved (unpublished) the (local) absoluteness of $\mathbb{L}_1$ for $A$, but where in each formula all quantifiers are restricted to a fixed unary predicate. This restriction allows a model theoretic treatment using Kripke models with a constant universe, and a special notion of "forced realizability" which utilizes results from the theory of Turing degrees.

Int. 2. Absoluteness in relation with some well-known results

For classical first order logic $\mathbb{L}_1^C$ absoluteness is an immediate corollary of Hilbert-Bernays [39]'s proof of Gödel's completeness theorem, where one has:

(5) \[ \vdash_{\mathbb{L}_1^C} F \implies \text{there is a } \Delta_2^0 \text{ instance } F^* \text{ of } F \text{ s.t. } \neg F^*. \]

Hence, if $\vdash_{\mathbb{L}_1^C} F$ and $\not\vdash_{\mathbb{M}_1} F^*$ then $\mathbb{M}$ is not (classically) sound for $\Delta_2^0$ sentences, and thus $\mathbb{L}_1^C$ is $\Delta_2^0$-absolute for any theory $\mathbb{M}$ (in a language which extends the language of Peano's arithmetic) which is sound for $\Delta_2^0$ sentences.

The same situation occurs not only for $\mathbb{L}_1^C$, but even for classical simple type theory $\mathbb{L}_\omega^C$.

It seems here the right place to note that absoluteness results for classical systems are hardly related to absoluteness of (the corresponding) intuitionistic systems. Given (4) for classical $\mathbb{L}_1^C$, $\mathbb{M}_1^C$, nothing is said even about the propositional rule of excluded third, $p \lor \neg p$ (which is intuitionistically invalid and unprovable). Conversely, if (4) is given for intuitionistic systems $\mathbb{L}_1^I$ and $\mathbb{M}_1^I$, then $\vdash_{\mathbb{M}_1^I} F$ does not necessarily imply $\vdash_{\mathbb{M}_1^I} F$ for the classical completion $\mathbb{M}_1^C$ of $\mathbb{M}_1^I$. Hence the easy proof of absoluteness for $\mathbb{L}_1^C$ is of no help in solving the problem for $\mathbb{L}_1^I$, while the uniform result for $\mathbb{L}_1^C$ does not imply the uniform $\Pi_2^0$ absoluteness of $\mathbb{L}_1^C$. 


Another blind alley is to try to imitate the method of the proof of (5) in the treatment of the intuitionistic case. Of course, there is a completeness result for $L_1$ relative to Kripke's semantics which is analogous to (5), i.e.:

\[(6) \quad \vdash_{L_1} F \Rightarrow \text{there is a } \Delta^0_2 \text{ Kripke model } K \text{ in which } F \text{ is not valid}\]

(cf. e.g. THOMASON [68]). Here, however, $K$ is not necessarily a Kripke model for any numerical instance $F^*$ of $F$ since the models standing at each node are not necessarily models of arithmetic. Therefore every use of Kripke's semantics must refer here directly to Kripke's models for arithmetic, as done in de Jongh's and Smorynski's proofs mentioned above, but this is a totally different method.

Let us finally compare absoluteness with a conservative extension result. Let $\bar{A}$ denote arithmetic $A$ with predicate variables and with an axiom schema of arithmetical comprehension:

\[
\begin{align*}
\text{ACA} & \quad \exists x \forall x \left[ Ax \leftrightarrow Xx \right] \\
\end{align*}
\]

(A in the language of $A$). Then $\bar{A}$ is a conservative extension of $A$ (cf. TROELSTRA [73] 1.9.8), and $L_1$ is trivially contained in $\bar{A}$. $\bar{A}$ is also conservative over $L_1$ (compare MAEHARA [58] thm.1), i.e.,

\[
L_1 \not \vdash A[P_1, \ldots, P_n] \Rightarrow \bar{A} \not \vdash A[P_1, \ldots, P_n]
\]

for any schema $A[P_1, \ldots, P_n]$ of $L_1$. Absoluteness of $L_1$ for $A$ reads on the other hand

\[
L_1 \not \vdash A[P_1, \ldots, P_n] \Rightarrow A \not \vdash A[B_1, \ldots, B_n]
\]

for every $B_1, \ldots, B_n$ in the language of $A$. Notice that the absoluteness of $L_1$ for $A$ implies the absoluteness of $L_1$ for $\bar{A}$; this can be easily derived from the fact that $\bar{A}$ is conservative over $A$.

Int. 3. Infinitary derivations and the subformula property.

The central method of proof used below in establishing absoluteness of $L_0$ and $L_1$ is an analysis of infinitary derivations, i.e. (roughly), of
derivations with the "\(\omega\)-rule".

SCHÜTTE [51] seems to have been the first one to notice the usefulness of systems of infinitary derivations for the metamathematical study of arithmetic. He was chiefly interested in extending to arithmetic one of the main advantages of Gentzen's systems for logic, namely - their subformula property. We say that a proof-figure \(\pi\) satisfies the subformula property if every formula which occurs in \(\pi\) is a subformula of the formula derived by \(\pi\). A calculus \(\mathcal{C}\) of proof-figures satisfies the subformula property if the subsystem \(\mathcal{C}_0\) of \(\mathcal{C}\) containing only those proof-figures of \(\mathcal{C}\) which satisfy the subformula property is complete for \(\mathcal{C}\), i.e., if for every proof-figure \(\pi\) of \(\mathcal{C}\) deriving a formula (or a sequent) \(\sigma\) there is a proof \(\pi_0\) of \(\mathcal{C}\) which derives \(\sigma\) and satisfies the subformula property. Gentzen proved that his sequential systems for first order classical and intuitionistic logic satisfy the subformula property (by "cut elimination"; cf. GENTZEN [35]), and PRAWITZ [65] proved that the same holds for GENTZEN's [35] system of natural deduction (by "normalization").

Although cut elimination and normalization for the corresponding calculi for arithmetic can also be carried out with some important metamathematical consequences (such as consistency and the "existential definability" property), the subformula property for these calculi is not implied.

Call a calculus \(\mathcal{C}\) of proof-figures for arithmetic good, if there is a predicate \(\text{Inf}(x,y)\) such that the following hold, for each sequence of formulas \(F_1,\ldots,F_k,G\) \((k \geq 0)\).

1. If \(\prod\limits_{i=1}^{k} F_i G\) is an instance of an inference rule of \(\mathcal{C}\), then
   \[\mathcal{C}_0 \vdash \text{Inf}(\langle F_1,\ldots,F_k \rangle,G),\]
2. \[\mathcal{C}_0 \vdash \text{Inf}(\langle F_1,\ldots,F_k \rangle,G) \Rightarrow F_1 \land \cdots \land F_k \Rightarrow G.\]

Thus, e.g., if \(\mathcal{C}\) generates HA and has all sentences of HA as axioms, \(\mathcal{C}\) is not good. But all standard calculi for arithmetic are good.

**Theorem.** If an r.e. good calculus \(\mathcal{C}\) of finitary proof figures is complete for Heyting's arithmetic \(\mathcal{A}\) and satisfies the elementary derivability conditions (cf. TN1 below, or e.g. SMORYNSKI [75]) then \(\mathcal{C}\) proves that \(\mathcal{C}\) does not satisfy the subformula property.

**Proof.** Let \(\mathcal{C}\) be a calculus as above; the subformula property of \(\mathcal{C}\) is formally expressible as an arithmetical (actually a \(\mathcal{H}_2^0\)) sentence, \(\text{Sp}_{\mathcal{C}}\) say. Since the proof figures of \(\mathcal{C}\) are finite, we can prove in \(\mathcal{A}\) by induction on the length of proof figures that \(\text{Sp}_{\mathcal{C}}\) implies the local reflection principle for \(\mathcal{C}\); i.e.,
for each specific arithmetical \( F \) (cf. TNI). Taking in (7) in particular \( F := \neg \text{Sp}_\varepsilon \) we get (since \( \varepsilon \) is complete for \( A \))

\[
\text{Sp}_\varepsilon \vdash \exists p \ \text{Prf}_\varepsilon(p, \neg \text{Sp}_\varepsilon) \rightarrow \neg \text{Sp}_\varepsilon
\]

and so by propositional logic

\[
\vdash \exists p \ \text{Prf}_\varepsilon(p, \neg \text{Sp}_\varepsilon) \rightarrow \neg \text{Sp}_\varepsilon.
\]

But this implies by the theorem of L"OB [55]

\[
\vdash \neg \text{Sp}_\varepsilon
\]

since \( \varepsilon \) is assumed to satisfy the elementary derivability conditions. QED.

Sch"utte's idea was to restore the subformula property for the systems of arithmetic by giving up the finiteness condition. The reason that cut-elimination for arithmetic does not imply the subformula property is the presence of the induction rule; hence this rule, which for a natural deduction system may be given by

\[
\begin{array}{c}
\text{[A(a)]} \\
\Delta(a) \\
\hline
\forall x A(x)
\end{array}
\]

(\text{using the notations of GENTZEN [35], PRAWITZ [65]}), is replaced by an instance of an infinitary \( \forall \)-introduction rule (\( \varepsilon \)-rule):

\[
\begin{array}{c}
\Gamma \\
A(\bar{a}) \\
\Delta(\bar{a}) \\
\hline
\forall x A(x)
\end{array}
\]

(compare PRAWITZ [71]). Obviously, this infinitary \( \forall \)-introduction rule may
take over the role of the finitary \( \text{VI} \). Similarly the [3E] inference rule

\[
\begin{array}{c}
[A(a)] \\
\Gamma \\
\exists xA(x) \\
\hline
B
\end{array}
\]

may be replaced by a corresponding infinitary rule

\[
\begin{array}{c}
[A(\bar{\delta})] \\
[A(\bar{\iota})] \\
\Gamma \\
\Delta(\bar{\delta}) \\
\Delta(\bar{\iota}) \\
\hline
\exists xA(x) \\
B
\end{array}
\]

By iterating these translations each finitary derivation \( \Delta \) is mapped into a well-founded infinitary derivation \( \Delta^* \) having the same derived formula and the same open assumptions as \( \Delta \). (For a formal definition of the infinitary derivations see A.1).

The mapping above may be described as one which replaces (hereditarily) each expression (i.e., formula or derivation) \( e(\vec{p}) \) which "depends" on a list \( \vec{p} \) of parameters, and where the parameters range implicitly over the natural numbers, by an explicit enumeration \( (e(\bar{n}))_{n \in \mathbb{N}} \) of the closed expressions which correspond to a substitution \( [\bar{n}/\bar{\vec{p}}] \) of numerals for those parameters.

For the system of infinitary proof figures obtained in this manner, a normalization theorem can be proved (cf. e.g. A.3 below), and in this case the subformula property does follow. The method leads also to a number of interesting applications (cf. e.g., KREISL-LEVY [68], PARIKH [73], PARSONS [60], LEIVANT [A]). The general pattern of these applications consists in embedding the (finitary) formal system to be investigated into a ("semi-formal") system of infinitary proof figures, which then allows a smoother proof theoretic analysis.

It is precisely this method which is used in the proof of absoluteness in part B below. We treat those theories whose arithmetical fragment can be embedded as above in a system of infinitary proofs of arithmetic. These proofs are subsequently transformed ("normalized") to ones which satisfy a number of structural properties, the most important of which is the subformula property described above.
An infinitary derivation of the kind described in Int. 3 may be viewed formally as an assignment of sequents (or their Gödel codes) to certain nodes of the universal spread; the assignment may be made total by attaching 0 to the rest of the nodes (A.1.1 below). But while for a calculus $C$ of finitary proof figures (based on an r.e. set of inference rules) we may formally define a prim. rec. proof predicate $\text{Prf}_C(p, F')$, this obviously cannot be done for the arithmetization of infinitary proofs. If $\text{Prf}^\omega(q, F')$ should be a formal proof predicate for the proofs described in Int. 3 above, then we should have (in elementary analysis $V_0$ plus $\mathcal{A}_{00}$ as defined in section P below)

$$F \rightarrow \exists \phi \text{ Prf}^\omega(q, F')$$

for every prenex arithmetical $F$ (compare A.2.2.1). So the system of infinitary proof figures is classically complete, and $\text{Prf}^\omega$ cannot even be arithmetical.

The completeness of the infinitary systems for classical truth expressed by (8) illustrates the potential generality of the analysis of infinitary proof figures as a technique in meta-arithmetic: whatever classically sound theory $T$ is given, an embedding of its arithmetical fragment into infinitary proofs is guaranteed. On the other hand one may wish to utilize the recursive enumerability of the embedded theory $T$, as we do in part B below, and so one has to restrict the class of infinitary derivations into which $T$ is embedded. There are several simple methods for doing this, all having more or less equal merits. We find it particularly convenient to restrict the image of the embedding by requiring that each infinitary proof figure in it is proved to be a correct proof in a given (r.e.) theory $T_1$ (in a language extending the language of analysis $V_0$). I.e., one considers the derivations which are shown in $T_1$ to be well-founded and to respect the inference rules. An enumeration of these derivations can easily be extracted from an enumeration of $T_1$, and so the class of infinitary derivations considered is r.e. in $T_1$.

To sum up, we wish to exploit two properties of a given theory $T$: firstly, that the arithmetic fragment $\mathcal{A}[T]$ of $T$, i.e., the arithmetic sentences provable in $T$, is r.e.; and secondly, that $\mathcal{A}[T]$ can be investigated through an analysis of the structure of infinitary derivations. Both conditions are indeed satisfied if
for some r.e. \( T_1 \), where \( \hat{A}^\omega[T_1] \) is (roughly, cf. A.1.2) the system of infinitary derivations proved in \( T_1 \) to be correct and normal, and where the inclusion refers to the derived sentences. When this is the case, we say that \( T \) is regular (A.1.2). For the proof of theorem II in B.4 below, the infinitary derivations examined have to be recursive, so for that proof (9) is strengthened to

(10) \[ A[T] \subseteq A^{\omega}_{\text{rec}}[T_1] \]

where \( A^{\omega}_{\text{rec}}[T_1] \) are (roughly) the recursive derivations of \( \hat{A}^\omega[T_1] \). When \( T \) satisfies (10) (and another minor condition, cf. A.1.2) we say that \( T \) is strongly regular.

Our feeling now is that regularity (as well as strong regularity) are conditions which are quite general and natural. There are a number of arguments supporting this feeling.

[a] By (8) above we have for any theory \( T \supseteq V_0 + \text{AC}_0 \)

(11) \[ A_p[T] \subseteq A^\omega[T] \]

where \( A^\omega_p[T] \) is the fragment of prenex formulae of \( A[T] \). As \( A^\omega_p[T] \) is classically complete for \( A[T] \), (11) implies that any classical r.e. theory \( T \) satisfies

(12) \[ A[T] \subseteq A_p[T + V_0 + \text{AC}_0] \subseteq A^\omega[T + V_0 + \text{AC}_0], \]

(where the first inclusion is trivial) and so any such \( T \) which is consistent with \( V_0 + \text{AC}_0 \) is regular.

[b] Obviously, regularity and strong regularity are preserved under restriction. It is therefore quite satisfactory to know that some strong theory, in which a large part of current intuitionistic mathematics can be formalized, is (strongly) regular. We indeed show in A.4 below that intuitionistic type theory \( I_{\omega_0} \) is strongly regular.

[c] If \( T \) is (strongly) regular and sound, then so is \( T \) extended with any schema of transfinite induction over some prim.rec. well-ordering (cf. A.2.4 for a precise statement, a proof and a discussion of its significance).
[d] The class of regular theories is closed under the operation of adding self-consistency (A.2.2.4), and so this class is closed under transfinite progressions along $\Sigma^0_1$ paths in Kleene's $\mathcal{O}$.

Int. 5. The method of proof

The proofs of theorems I and II are both composed of two parts.

(i) A reduction of the problem, using proof-theoretic methods. For theorem I we show, roughly, that given a regular theory $T$ and a schema $F$ of $L_0$, if $F^*$ is a $\Sigma^0_1$ meta-substitution of $F$ then

$$\forall F \quad \text{and} \quad \not\exists^* \quad \Rightarrow \quad \not\exists^*$$

where $U$ is a specific schema and where $U^*$ comes from $U$ by the same meta-substitution (B.2.0). The proof theoretic reduction of theorem II is similar, with $L_1$ in place of $L_0$, with $\not\exists^*$ strongly regular and with $\Pi^0_2$ meta-substitutions. In the proof of theorem I the schema $U$ is fixed for all schemata with a certain bound on the number of propositional letters used, while in the proof of theorem II $U$ is fixed altogether.

(ii) A solution of the reduced problem. We find instances $U^*$ of the corresponding $U$ and of the kind required, for which $\not\exists U^*$ is impossible.

Step (ii) uses the recursive enumerability of $T$, and is (in both proofs) a generalization of Gödel's first incompleteness theorem (B.1,b.5). On the other hand the proof of step (i) in each case utilizes the embedding of $A[T]$ in the set of normal infinitary derivations. (Here we define "normality" in a somewhat broader sense which renders the arguments a bit simpler).

The idea of the proof-theoretic reduction is the following. Let $T = \lambda^a[T_1]$. Then (13) is a consequence of the provability in $T_1$ of

$$\forall F \quad \text{and} \quad \not\exists^* \quad \Rightarrow \quad \not\exists^*$$

Assuming the premiss of (14), one analyses the structure of $\phi$ and, using $\not\exists U^*$, shows how to "extract" a derivation $\psi$ (for $U^*$) from $\phi$.

The precise nature of this "extraction" will be clear from the heuristic discussions (B.2.1, B.4.2) and from the technical details of the proofs (B.2.2-6, B.4.3-11). There is however a difference between the proofs of the two theorems which should be noted outright. In contrast to $L_0$, $L_1$ is not
decidable, and as a consequence one has to weaken the proof-theoretic reduction of theorem II to

\[(\exists \Gamma \in \mathcal{F}) \quad \text{Prf}_{\text{rec}}^{\omega} (d, \Gamma^{\bullet}, \gamma) \rightarrow \gamma \subseteq \text{Prf}_{\text{rec}}^{\omega} (\Theta, \zeta^{\bullet}) \]

where \(\text{Prf}_{\text{rec}}^{\omega}\) is the proof predicate for recursive infinitary derivations. Compared to (14), the premiss here is strengthened and the conclusion is weakened. Furthermore, (15) is not proved in the r.e. theory \(T\), but in a certain \(\mathcal{F}_{\omega}\)-enumerated extension of it (cf. B.3).

Int. 6. Normalization of infinitary derivations; regularity of the theory of types

In Int. 3 above we have quoted Schütte's result stating that every infinitary derivation (of arithmetic) can be brought into a "normal" ("cut free") form which satisfies the subformula property; this in turn is used in our proof of absoluteness as indicated in Int. 5 (the additional structural requirements we are using are inessential to the proof of normalization). The traditional proofs of normalization of infinitary derivations (SCHÜTTE [51][60], FEferMAN [68], TAIt [68], MARTIN-LOF [68]) all use ordinal assignments, following GENTZEN's [36][38] consistency proofs. This evolution is quite evident: ordinals can be assigned to well-founded infinitary trees in a natural way, so extending Gentzen's idea was the first thing which came to mind while passing from finite to infinitary proof figures.

In part A below we present however a new proof of normalization which does not use the technique of ordinal assignments. We do so simply to permit a certain generalization which will be explained below, and for which the technique of ordinal assignment is not so adequate.

Cut elimination for (a sequential calculus for) the classical theory of types \(L_{\omega}^{\omega}\) is known since TAKAHASHI [67] (for the theory of species \(L_{\omega}^{\omega}\) proofs were discovered independently also by PRawitz [68] and TAIT [66]). From the work of GIRARD [71][72] (as expounded in detail in MARTIN-LOF [73]) we also know an effective procedure which transforms each proof of \(L_{\omega}^{\omega}\) into a normal one; and like for Gentzen's systems for first order logic \(L_{\omega}^{\omega}\), we get for \(L_{\omega}^{\omega}\) (and ipso facto for \(L_{\omega}^{\omega}\)) normal proofs which do satisfy the subformula property. However, when a normal proof \(\tau\) of \(L_{\omega}^{\omega}\) proves a formula \(F\) in which a second order quantifier occurs, then the subformula property of \(\tau\) is of limited interest: suppose e.g. that \(\exists X G[X]\) is a subformula of \(F\),
then so is $G[H]$ for every formula $H$ including e.g. $F$ itself. This is an evident drawback if one refers to the interpretation of arithmetic in $L_2$, as given by PRAWITZ [65], since under this interpretation first order sentences of arithmetic are always mapped to second order formulae.

Consequently, a system of type theory which does satisfy the subformula property for arithmetical sentences must be built up firstly by extending the language to include the language of arithmetic, and secondly by expanding the first order parametric expressions into explicit infinitary proof figures (as in Int. 3 for first order arithmetic). I.e., a system $L$ is adopted for the union of the languages of $A$ and of $L_1$, whose inference rules are those of the infinitary system for arithmetic, plus the rules of $L_1$ for higher order quantification.

We are now ready to justify our abandoning the technique of ordinal assignment. We wish to prove a normalization theorem for $L_1$, because then we may conclude that $L_1$ is regular: $L_1$ is embedded in $L_1$ in an obvious manner, and every normal derivation in $L_1$ of an arithmetical sentence must actually be a purely arithmetical derivation, because it must satisfy the subformula property. So, if $T$ is a theory in which these facts are provable, then

$$\mathcal{A}[L_1] \subseteq \mathcal{A}^m[T]$$

(cf. A.4.9), and so $L_1$ is regular.

It is easily seen however (cf. TN 3) that if the normalization theorem for $L_1$ was to be proved by assigning an ordinal notation to each proof figure, then notations should be available for all "provable ordinals" of $L_1$. Such notations are unfortunately not known at present.

There remains the possibility of assigning ordinals (in place of ordinal notations) to the proof figures, as done e.g. by SCARPELLINI [71] p.156; the proof of normalization is then carried out in some formal set theory (ZF say). But then it seems unrealistic to expect either an optimal result, or a proof within $L_1$ of a normalization theorem for the systems obtained by restricting $L_1$ to languages with a bound on formula-complexity. The method described in part A below does have, on the other hand, the properties just mentioned, in analogy to the well-known normalization proofs for arithmetic.

In proving the normalization theorem for $L_1$ (A.4) we combine the ideas of PRAWITZ's [71] "validity" argument, the work of GIRARD [71][72] and the "geometrical" treatment of infinitary proof figures of LEIVANT [A]. For another application of the normalization theorem for $L_1$ see TN 4.
TECHNICAL NOTES TO THE INTRODUCTION

TN 1. The elementary derivability conditions for an r.e. system \( T \) and a provability predicate \( \text{Pr}_T \) for it are

\[
\text{D}_1. \quad \left| T \frac{F}{\text{Pr}_T(F^\gamma)} \right|
\]

\[
\text{D}_2. \quad \left| T \frac{\text{Pr}_T(F^\gamma)}{\text{Pr}_T(F^\gamma)} \text{Pr}_T(F^\gamma) \right|
\]

\[
\text{D}_3. \quad \left| T \frac{\text{Pr}_T(G) \& \text{Pr}_T(F^\gamma)}{\text{Pr}_T(G^\gamma)} \right|
\]

The local reflection principle is proved in \( \mathfrak{E} \) by a straightforward induction on the length of derivations as follows. Each inference step of \( \mathfrak{E} \) is of the form

\[
< \sigma_1 > \frac{i<n}{\tau} [p]
\]

where \( \sigma_1 (i<n), \tau \) are formulae or sequents whose validity is equivalent to certain sentences \( \sigma_1 (i<n), \Gamma \) (resp.). Assume now that a proof figure \( \Delta \) of \( \mathfrak{E} \) is given, with

\[
\Delta \equiv < \sigma_1 > \frac{i<n}{\tau}.
\]

\( \Delta \) is finite, and so there is a (restricted) truth definition \( \text{Tr} \) in \( A \) which applies to all formulae occurring in \( \Delta \) (cf. e.g. TROELSTRA [73] 1.5.4). By ind. hyp. we have \( \text{Tr}(F^\gamma) \) \((i<n)\), and since \( \bigwedge \sigma_1 \sigma_1 G \) is simply a rule of \( \mathfrak{E} \), we thus get \( \text{Tr}(G^\gamma) \).

The predicate \( \text{Tr} \) above depends on \( \Delta \), but if \( \Delta \) is known to satisfy the subformula property, then \( \text{Tr} \) depends only on the derived formula of \( \Delta \). So we actually have, in \( A \) (and hence in \( \mathfrak{E} \))

\[
\exists p [ \text{Prf}_A(p,R^F) \& "p satisfies the subformula property" ] \rightarrow \text{Tr}(F^\gamma)
\]

for each sentence \( F \). But

\[
\exists p [ \text{Prf}_A(p,R^F) \& "p satisfies the subformula property" ] ;
\]

\[
\exists p [ \text{Prf}_A(p,R^F) \& "p satisfies the subformula property" ]
\]

\[
\exists p [ \text{Prf}_A(p,R^F) \& "p satisfies the subformula property" ]
\]
so (1) implies

$$\text{Sp}_q \vdash \exists p \Prf(p,^cF) + F$$

for each sentence $F$.

TN 2. KREISEL [65] proves that no r.e. system $C$ of finitary proof figures built up from derived rules of $A$ and which is complete for $A$ can be proved in $A$ to satisfy the subformula property. Our statement is stronger since the subformula property is simply false (not only unprovable) provided $C$ is sound for $\Sigma^0_2$ sentences (i.e., for $\neg \text{Sp}_q$).

The reference to the reflection principle made in Kreisel's proof mentioned above is redundant, since the result quoted is obvious already from Gödel's second incompleteness theorem: one proves trivially in $A$ that no derivation of $\tilde{0} = \tilde{1}$ may satisfy the subformula property, and so if

$$\vdash_{T_E} \text{Sp}_q$$

then $C$ proves its own consistency.

TN 3. Suppose that we can prove in $L_w$ for a certain prim. rec. well-ordering $\prec$

$$T\prec \ := \ \forall x \ [ \forall y \prec x \ P(y) \rightarrow P(x) ] \ \rightarrow \ \forall x \ P(x)$$

where $P$ is a predicate-parameter. We then have (trivially) a proof $(d^c)$ of $L_w, \text{rec}$ for $T\prec$.

Suppose that $(d^c)$ is normalized into $(d^c_N)$; an analysis of $(d^c_N)$ using the subformula property, shows that $(d^c_N)$ must have a specific structure for which the Brouwer-Kleene well-ordering $\ll$, associated with $(d^c_N)$ is equivalent (in $V_0$) to $\ll$ itself (i.e., $T\ll$ and $\ll\ll'$ are equivalent in $V_0$ for each predicate $P$ in the language of $V_0$). Taking various $\ll$ we find that the ordinal of $(d^c_N)$ may be any "provable ordinal" of $L_w$, i.e., any ordinal over which t.i. is provable in $L_w$.

TN 4. A memo of G. Kreisel from 1973 proposes another application of the normalization of $L_w$. Kreisel's aim there is to answer a question of M.J. Beeson about a possible intuitionistic analogue to the KREISEL-SHOENFIED-WANG [60] completeness result (which reads: Peano's arithmetic
extended with transfinite induction over all prim. rec. well-orderings is complete for classically true sentences). As a partial answer to that question, Kreisel's memosketches a possible proof that Heyting's arithmetic $A$ extended with t.i. over all prim. rec. well-orderings is complete at least for $A(L_2)$ (i.e., the arithmetical fragment of the theory of species). A system similar to $L_2^{\text{rec}}$ (i.e., the recursive derivations in $L_2^{\text{rec}}$) is presented, and it is assumed that the normalization of that system can be proved. But if $d$ is a normal proof of $L_2^{\text{rec}}$ which derives an arithmetical sentence $F$, then $d$ is actually a proof of $A_\text{rec}$ by the subformula property. By t.i. over the Brouwer-Kleene well-ordering corresponding to $d$, and using a restricted truth definition for the subformula of $F$ one gets that $F$ is true.

The missing normalization step is proved in A.4 below. The proof remains however incomplete, since one uses not only t.i. over the proof tree $d$, but also the fact that $d$ describes a correct derivation; this last assumption is a $\Pi^0_1$ sentence which is not necessarily provable in $A$. However, this hiatus may be filled up as follows.

Given a quantifier free unary predicate $E$, define

$$
x <_E y \equiv x < y \land \forall z s y E(z)
\lor y < x \land \exists z s y E(z)
$$

$<_E$ is of course prim. rec., and if $\forall x E(x)$ then $<_E$ is simply $<$, and so it is certainly well founded. Let

$$A^E(x) \equiv \exists s \forall z <_E x z s$$

where

$$z s : z is an element of the finite set of natural numbers encoded by s (via the binary encodement say).

It is easily seen that, in $A$,

$$\forall y <_E x A^E(y) \rightarrow A^E(x)$$

and so by t.i. over $<_E$
But in A one proves outright

(2) \( \neg E(x) \rightarrow \neg A^E(x) \)

Since E is decidable, we get from (1) and (2)

\( \forall x E(x) \).

So, if \( \forall x E(x) \) is true, then \( \prec_E \) is well-founded and \( \forall x E(x) \) is provable by t.i. over \( \prec_E \). This completes now Kreisel's sketch: given a derivation \( \tau \) of \( L_\omega \) which proves an arithmetical sentence \( F \), one maps trivially \( \tau \) into an infinitary derivation \( \{d\} \) of \( L_\omega^{\omega, \rec} \) for \( F \). By the normalization theorem of A.4 below, \( d \) is mapped into a normal derivation \( \{e\} \) of \( L_\omega^{\omega, \rec} \) for \( F \) which is, by the subformula property, a purely arithmetical derivation. Now one looks at \( A \) extended with t.i. over \( \prec_E \) and over \( \prec_E \), where \( \prec_E \) is defined as above if \( \forall x E(x) \) expresses the local correctness of the derivation \( \{e\} \), and where \( \prec_E \) is the Brouwer-Kleene well-ordering associated with the proof-tree \( \{e\} \). In the extended theory we can now conclude as above that \( F \) is true.

We thus have:

**Theorem:** \( L_\omega \) is conservative over \( A \) extended with t.i. over prim. rec. well-founded orderings.
PRELIMINARIES

P.1. SYNTACTICAL NOTIONS

P.1.1. The propositional constants we use are $\&$, $\vee$, $\rightarrow$, and $\bot$ (for absurdity); negation is definable in terms of $\rightarrow$ and $\bot$:

$$F \equiv F \rightarrow \bot$$

We find it convenient to distinguish syntactically between "free" and "bound" variables. The label "variable" is reserved to "bound" variables, while the "free" variables we call parameters.

P.1.2. We often have to distinguish between different occurrences of the same syntactic object $\sigma$ (usually a formula, sometimes a term or a parameter). An accurate definition of "an occurrence of $\sigma$ in $\tau$" may be found e.g. in STEEN [72] p.13. We shall write $\underline{\sigma}$ (underlined) when referring to an occurrence of $\sigma$; usually the specific occurrence referred to will be either obvious from the context or irrelevant to it.

In a formula $F \rightarrow G$, $F$ as well as all its sub- (occurrences of) formulae are said to be negatively bound by the shown occurrence of $\rightarrow$. $B$ is said to be a negative subformula of $A$ if the number of implications negatively binding $B$ in $A$ is odd; if this number is even then $B$ is a positive subformula of $A$. (compare PRAWITZ [65] p.43).

P.1.3. We shall usually use natural deduction calculi for generating formal theories. In these calculi there are for each logical constant $\kappa$ an introduction rule $[\kappa I]$ and an elimination rule $[\kappa E]$. The natural deduction calculi were invented by G. GENTZEN [35]; good introductions to them may also be found in PRAWITZ [65], [71]. We shall freely use the terminology of these works for dealing with natural deductions.

We also adopt the following convention: if $\Delta$ is a natural deduction deriving a formula $F$ (or a sequent $s$) we write $\Delta \vdash^F$ (resp., $\Delta \vdash^s$) in place of $\Delta$ when we wish to express this fact explicitly. On the other hand $\Delta \vdash G$ (with a separating horizontal line) stands for the deduction which extends $\Delta \equiv^F \Delta$ by deriving $G$ from $F$. 
P.2. FORMAL SYSTEMS

P.2.1. Intuitionistic propositional and first-order predicate logics
(L₀ and L₁ respectively)

The language of L₀ is built up from the propositional parameters
("letters") p₀, p₁, . . . and from the propositional constants &, v, → and 1.
The language of L₁ is built up as usual from predicate parameters
₂ᵢ^(i, n ≥ 0) (₂ᵢ is n-place), first-order parameters a₀, a₁, . . . and first-order
variables x₀, x₁, . . ., the propositional constants and the first-order quantifiers ∀, ∃.

The theories L₀ and L₁ are generated by the corresponding natural de-
duction calculi (GENTZEN [35], PRAWITZ [65]). A rough picture of these
calculi may be obtained by looking at A.1.1. below.

P.2.2. Second-order Logic L₂ (the theory of species)

The language of L₂ contains, in addition to the second order parameters
₂ᵢ of L₁ also second order variables X_i^(i, n ≥ 0) and corresponding second
order quantifiers ∀(n), ∃(n).

The theory L₂ is now generated by a natural deduction calculus as in
version I of PRAWITZ [65] p.65, i.e., without referring to λ-abstraction
(compare A.4.1 below).

P.2.3. The theory of types Lₐ

The simple types are generated inductively by starting with 0 as a
basic type (the type of "first-order objects"), and passing from a sequence
τ₁, . . ., τₙ of types (n ≥ 0) to a new type (τ₁, . . ., τₙ), the type of properties
of tuples (Υ₁, . . ., Υₙ) of terms of types τ₁, . . ., τₙ respectively. In particu-
lar ( ) is the type of propositions.

The language of Lₐ is built up now similarly to the language of L₂,
but with variables and predicates P_i^{τ}, X_i^{τ} (i ≥ 0) for each type τ and with
corresponding quantifiers ∀^{τ}, ∃^{τ}.

The intuitionistic (simple) type theory Lₐ is generated once again by
a natural deduction system in an obvious manner (for details see
MARTIN-LÖF [73]).

The theories Lₖ (k = 0, 1, 2, 3 . . .) may now be defined to be Lₐ restricted
to the types whose definition is of length ≤ k.
P.2.4. Intuitionistic (Heyting's) arithmetic

Here we have in addition to the first order variables and parameters of $L_1$ also a first-order constant $0$ and function symbols $f^n_i$ ($i \geq 0, n \geq 1$). Each $f^n_i$ denotes a function from $\mathbb{N}^n$ to $\mathbb{N}$, and we may take $f^1_0$ to denote the successor function. The first-order terms are now built up in a standard manner. If a term contains occurrences of ("free") variables we shall say that it is a pseudo-term; otherwise it is a pure term.

The language of $A$ contains only a single second order predicate $=$ which is binary; we write of course $t = s$ for $=(t,s)$.

$A$ is now generated by a natural deduction calculus which includes, in addition to the inference rules of $L_1$:

(i) inference rules expressing Peano's third and fourth axioms;
(ii) an inference rule expressing the principle of induction;
(iii) all defining equations for prim.rec. functions, where each $f^n_i$ is interpreted as the $i$'th $n$-place prim.rec. function.

For details cf. PRAWITZ [71].

P.2.5. $L_1 A$: $L_1$ extended to the language of $A$.

The language of $L_1 A$ is the union of the languages of $L_1$ and of $A$, i.e., we extend the language of $A$ with predicate letters $P^n_i$ ($i, n \geq 0$).

$L_1 A$ is now generated by a calculus of natural deductions based on the rules of $L_1$.

Note that the language of $L_1 A$ is more restricted than the language of HAS$_0$ (Heyting's arithmetic with species variables) of TROELSTRA [73] 1.9.3, where quantification over second order variables is also allowed.

P.2.6. Elementary analysis $V_0$

The language of $V_0$ is the extension of the language of $A$ obtained by allowing function parameters $g^n_i$ ($i, n \geq 0$), function variables $\phi^n_i$ ($i, n \geq 0$) and function quantifiers $\forall^n_i, \exists^n_i$ ($i \geq 0$).

The natural deduction calculus for $V_0$ is obtained by joining to the inference rules of $A$ inference rules for function quantification; e.g., the rule of $\forall$-elimination for function-variables:

$$
\frac{\forall^n_i \forall[\phi]}{A[h^n]}
$$

where $h^n$ is either a function constant $f^n_j$ ($j \geq 0$) or a function parameter $g^n_j$ ($j \geq 0$).
Note that we do not have in $V_0$ any comprehension rule, and consequently the function parameters and variables may be interpreted to range over prim.rec. functions. $V_0$ is therefore a conservative extension of $A$ (cf. HOWARD-KREISEL [66], where $V_0$ is denoted by $H$).

P.2.7. Classical theories

For each intuitionistic theory $T$ one obtains the classical completion $T^C$ of $T$ by joining to $T$ either the axiom schema of double negation,

$$\neg\neg A \rightarrow A,$$

or the axiom schema of excluded third,

$$A \lor \neg A.$$

P.3. ARITHMETIZATION OF METAMATHEMATICAL NOTIONS

P.3.1. Finite sets of numbers $\{n_0, \ldots, n_k\}$ may be encoded by

$$\{n_0, \ldots, n_k\} \mapsto \sum_{i=k}^{0} 2^n_i$$

which is a one-to-one prim.rec. function.

The set-theoretical relations $\in$, $\subset$ etc. are then encoded by prim.rec. relations for which we use the same notation ($\in$, $\subset$ etc.).

P.3.2. Let $\{\}$ stand for the coding of finite sequences given in TROELSTRA [73] 1.3.9. We take

$$\{n_0, \ldots, n_k\} = \{n_0, \ldots, n_k\} + 1.$$

The prim.rec. functions for projection $\langle n_i \rangle$, concatenation $u*v$ and length $\mathbb{L}(n)$ corresponding to the coding $\langle \rangle$ are then defined (as for $\langle \rangle$) in an obvious manner.

We shall use the following properties of $\langle \rangle$:

1. $\langle \rangle$ is onto the positive integers.

So an algorithm may produce the code of a node in the universal spread which satisfies a certain property, and may yield 0 when no such node exists.

2. $n_i < \langle n_0, \ldots, n_k \rangle$ (i=k)

3. $\langle n_0, \ldots, n_k \rangle < \langle n_0, \ldots, n_k, n_{k+1}, \ldots, n_{k+m} \rangle$

4. $u < v \Rightarrow u + \{m\} < v + \{m\}$ and $\langle m \rangle + u < \langle m \rangle + v$. 
We let \( u < v \) stand for the prim. rec. relation "\( u \) is (a code of) a proper initial segment of (the sequence encoded by) \( v \)". and we let \( \text{tail} \) be a prim. rec. function which satisfies
\[
\text{tail}(n_0, \ldots, n_k) = (n_1, \ldots, n_k)
\]

P.3.3. We shall frequently use the notations of KLEENE [52][69] for dealing with general recursive functions: the standard prim. rec. predicates \( T, T' \), the result-extracting function \( U \),
\[
\{n\} \text{ (resp. } \{n\}' \text{) for the partial recursive (resp. recursive in } \phi \text{) function with index } n,
\]
\[
\uparrow(n)(x) \text{ for } \exists y T(n,x,y)
\]
\[
\uparrow\uparrow(n) \text{ for } \forall x \uparrow(n)(x) \text{ (i.e., } \{n\} \text{ is a total function)}
\]
\[
t \approx s \text{ for "} t \text{ and } s \text{ are both well-defined and equal, or they are both undefined".}
\]

P.3.4. We shall implicitly assume throughout this treatise that some standard Gödel coding of syntactical objects is given.

For arithmetization of proofs we shall use:
\[
\text{Der}_T(x) \text{ for "} x \text{ encodes a derivation of (a standard calculus generating) the theory } T \text{";}
\]
\[
\text{Prf}_T(x,y) \text{ for "} x \text{ encodes a proof of } T \text{ for the formula (sentence, sequent) encoded by } y \text{";}
\]
\[
\text{Prf}_T(y) \text{ for } \exists x \text{ Prf}_T(x,y).
\]

P.4. MATHEMATICAL SCHEMATA.

P.4.1. The axiom-schema of choice from numbers to numbers \( AC_{00} \) reads:
\[
AC_{00} : \forall x \exists y A(x,y) \rightarrow \exists \phi \forall x A(x,\phi(x))
\]

P.4.2. The schema of transfinite induction \( T\Gamma^\prec \) over a (fixed) binary relation \( \prec \)
\[
T\Gamma^\prec : \forall x [ \forall y \prec x A(y) \rightarrow A(x) ] \rightarrow \forall x A(x)
\]
P.4.3. The axiom-schema of bar induction, or "induction over well founded trees"

\[ BI : \forall \phi \left[ \text{WF}(\phi) \rightarrow \text{Ind} [A, \phi] \right] \]

where

\[ \text{WF}(\phi) := \forall x \exists y \phi(x, y) = 0 \]

\[ \text{Ind}[A, \phi] := \forall u \left( \forall n \left( J[A, \phi, u(n)] \rightarrow J[A, \phi, u] \right) \rightarrow \forall J[A, \phi, u] \right) \]

\[ J[A, \phi, u] := \forall v < u \phi(v) \neq 0 \rightarrow A(u) \]

(\( < \) is the initial-segment relation between codes of finite sequences).

Here the function \( \phi \) is thought of as representing a tree, namely, the set of nodes \( u \) in the universal spread which satisfy

\[ \forall v < u \phi(v) \neq 0 \]

When \( \phi \) is known to satisfy

\[ \phi(u) = 0 \rightarrow \phi(u^*(n)) = 0 \]

then we may replace the \( J \) above by

\[ J[A, \phi, u] := \phi(u) \neq 0 \rightarrow A(u). \]

BI is easily seen to be derivable in \( V_{\omega}^0 + AC_{00} \). It is also not difficult to verify that our schema BI is a special case of the schema of bar induction for monotonic predicates \( BI_m \) of HOWARD-KREISEL [66] p.326 as well as of the schema of bar induction for decidable predicates \( BI_d \) on p.336 there.
PART A. Regular theories and normalization of infinitary derivations.

A.1. DEFINITION OF REGULAR THEORIES

This chapter is a self-contained introduction to part B. The reader who wishes to do so may skip A.2 - A.4.

1.1. DESCRIPTION OF $A^\omega$

By a sentence we mean a closed formula of $A$. A sequent is a syntactical object of the form $a \Rightarrow F$ where $a$ is a finite set of sentences and $F$ is a sentence. $a$ is the precedent, or antecedent of the sequent, and $F$ is the succedent, or the conclusion.

We use here the absurdity symbol $\bot$ though it is definable as $\bot = \top \lor \bot$ (GENTZEN [33] §6) because one of our aims is to get a formal separation between logic and arithmetic.

Propositional rules of $A^\omega$:

[T] \[ a \Rightarrow F \quad \text{where } F \in a \]

[&I] \[ \frac{a \Rightarrow F_0 \quad a \Rightarrow F_1}{a \Rightarrow F_0 \land F_1} \quad \text{[&E]} \quad \frac{a \Rightarrow F_0 \land F_1}{a \Rightarrow F_1} \quad (i=0,1) \]

[\rightarrow I] \[ \frac{a, F \Rightarrow G}{a \Rightarrow F \rightarrow G} \quad \text{[\rightarrow E]} \quad \frac{a \Rightarrow F + G \quad a \Rightarrow F}{a \Rightarrow G} \]

(where $a, F$ stands for $a \cup \{F\}$)

[vI\textsubscript{-}] \[ \frac{a \Rightarrow F_i}{a \Rightarrow F \lor F_i} \quad (i=0,1) \; \text{[vE]} \quad \frac{a \Rightarrow F \lor F_0 \lor F \quad a, F_0 \Rightarrow G \quad a, F \Rightarrow G}{a \Rightarrow G} \]

[\bot] \[ \frac{a \Rightarrow \bot}{a \Rightarrow F} \]
Quantification and arithmetical rules of $\lambda^m$:

\[
\begin{align*}
[\text{TE}] & \quad a \Rightarrow E \quad \text{where } E \text{ is a true equation when every function symbol } i_j \text{ is interpreted as the } j\text{'th } i\text{-place prim. rec. function.} \\
[\text{FE}] & \quad a \Rightarrow E \quad \text{where } E \text{ is a false equation.} \\
[\forall I] & \quad \frac{a \Rightarrow \forall x F(x)}{a \Rightarrow F(t)} \quad \text{(t a term);} \\
[\forall E] & \quad \frac{a \Rightarrow \exists x F(x)}{a \Rightarrow F(t)} \quad \{ a, F(n) \Rightarrow G \}_{n<\omega} \\
[\exists E] & \quad \frac{a \Rightarrow \exists x F(x)}{a \Rightarrow G} \\
\end{align*}
\]

A function $\phi$ is a derivation of $\lambda^m$ (notation: $\text{Der}^m(\phi)$) if

(1) $\{ n \mid \phi^n \neq 0 \}$ is a tree of (codes of) finite sequents under the obvious partial ordering:

\[
\begin{align*}
\phi u = 0 & \rightarrow \phi(u*(n)) = 0, \\
\phi(u*(n)) = 0 & \rightarrow \phi(u*(n+1)) = 0; \\
(\text{where } * \text{ denotes concatenation of sequent numbers}).
\end{align*}
\]

(2) For every $u$ (= the code of a node in the universal spread) $(\phi u)_0$ is the code of one of the inference rules $\rho$ above (under some fixed encoding), while $(\phi u)_1$ and $(\phi(u*(n)))_1$ $(n<\omega)$ are codes of sequents which relate as the conclusion and the premiss sequents of $\rho$ (and when no $n$'th premiss is required, $(\phi(u*(n)))_1 = 0$).

(3) $\phi$ is well-founded: $\forall x \phi(\bar{x}(x)) = 0$.

**Example 1.** The ("informal") derivation

\[
\begin{align*}
[T] & \quad (A) \Rightarrow A \\
[\text{TE}] & \quad (A) \Rightarrow \emptyset = \emptyset \\
[\forall I] & \quad (A) \Rightarrow A \& \emptyset = \emptyset \\
\end{align*}
\]

is formalized by the function $\phi$ defined by
\( \psi(0) := (\tau \sigma \tau, \tau(A) \Rightarrow A \& \varnothing = \varnothing) \)
\( \psi(0) := (\tau \tau, \tau(A) \Rightarrow A) \)
\( \psi(1) := (\tau \tau, \tau(A) \Rightarrow \varnothing = \varnothing) \)
\( \psi(n) := 0 \) for every \( n \in \{0, 1, 2\} \).

**Example 2.** The derivation

\[
\begin{align*}
&T \vdash f_k(\varnothing) = 0 \\
\forall I \quad \varnothing \Rightarrow \forall x f_k(x) = 0
\end{align*}
\]

is formalized by the \( \psi \) defined by

\[
\begin{align*}
\psi(0) := (\forall I, \varnothing \Rightarrow \forall x f_k(x) = 0) \\
\psi(n) := (\tau \tau, \varnothing \Rightarrow f_k(\varnothing) = 0) \quad \text{for} \ n \in \omega \\
\psi(n) := 0 \quad \text{if} \ \text{th}(n) > 1.
\end{align*}
\]

A number \( d \) is a **recursive derivation** of \( A^\omega \) (notation: \( \text{Der}_{\text{rec}}(d) \)) if \( \{d\} \) is a total function (i.e., \( \forall x \exists y T(d, x, y) \)) and clauses (1)-(3) above hold when \( \psi \) and \( \varepsilon \) are replaced by \( \{d\} \) and \( \omega \) respectively.

\[
\begin{align*}
\text{Prf}_{\omega}(\psi, s) & := \text{Der}_{\omega}(\psi) \& (\psi(1))_1 = s \\
\text{Prf}_{\omega}(\psi, \tau F^\omega) & := \text{Prf}_{\omega}(\psi, \tau F^\omega)
\end{align*}
\]

(The formal ambiguity of \( \text{Prf}_{\omega} \) will never cause any trouble.)

A derivation \( \psi \) is **normal** (notation: \( \text{NDer}_{\omega}(\psi) \)) if:
1. No major (i.e., leftmost) premiss of an elimination rule in \( \psi \) is derived by an instance of an introduction rule;
2. No major premiss of an elimination rule nor a premiss of an instance of \( [\exists I] \) or \( [\forall E] \) is derived by an instance of \( [\forall I] \), \( [\exists E] \) or \( [\Lambda] \).

(The reference to \( [\exists I] \) and \( [\forall E] \) in (2) is made for technical reasons: it simplifies a bit the proofs in part B, since it implies that equations may stand only at top nodes of the normal derivations treated there.)

Predicates like \( \text{NDer}_{\text{rec}}(d) \), \( \text{NPrf}_{\text{rec}}(d, \tau F^\omega) \), \( \text{Prf}_{\text{rec}}(d, s) \) etc. are defined now in an obvious manner.
The central property of normal derivations is the *subformula property*: every formula occurring in a normal derivation is a subformula of the derived sequent. Another property of normal derivations which we use is the disjunction instantiation property: If \( \text{Prf}^m(\phi, \Gamma \phi) \) then \( \text{Prf}^m(\phi(\langle 0 \rangle), \Gamma \phi) \) or \( \text{Prf}^m(\phi(\langle 0 \rangle), \Gamma \phi) \), where \( \phi \) is the "main" subderivation of \( \phi \) (\( \phi(\langle 0 \rangle)(u) := \phi(\langle 0 \rangle)u \)) (see A.3.8/9).

**REMARK.** The use of sequents in the formulation of \( A^m \) above should not mislead the reader to view \( A^m \) as a sequential calculus. Sequents are used here only as a convenience in describing a natural-deduction system. In sequential calculi the precedent and the succedent of a sequent play a symmetric role, and there are two introduction rules for each logical constant, while here (as in all natural deduction systems) there is an introduction and an elimination rule for each constant, both operating on the succedent.

### 1.2. REGULAR AND STRONGLY REGULAR THEORIES

Let \( T \) be a theory in the language of analysis. Write

\[
A^m[T] := \{ F \mid T \vdash \exists \phi \ N\text{Prf}^m(\phi, \Gamma \phi) \}
\]

\[
A^m_{\text{rec}}[T] := \{ F \mid \exists d \{ T \vdash \text{NPrf}^m_{\text{rec}}(d, \Gamma \phi) \} \}
\]

or, otherwise stated,

\[
\text{Pr}_{A^m[T]}(\Gamma \phi) := \text{Pr}_{T} \exists \phi \ N\text{Prf}^m(\phi, \Gamma \phi)
\]

\[
\text{Pr}_{A^m_{\text{rec}}[T]}(\Gamma \phi) := \exists d \text{Pr}_{T} \text{NPrf}^m_{\text{rec}}(d, \Gamma \phi).
\]

In A.3 below we prove (in \( V_0 + L_\omega + BI \) say) that each derivation \( \phi \) of \( A^m \) can be mapped into a derivation \( \phi_N \) which is normal, recursive in \( \phi \) and proves the same sequent as \( \phi \). Hence, if \( T \supseteq V_0 + L_\omega + BI \), we can replace in all the definition above \( N\text{Prf} \) by \( \text{Prf} \).

We could formally strengthen the absoluteness results proved in part B below by modifying the definitions of \( \text{Pr}_{A^m[T]} \), \( \text{Pr}_{A^m_{\text{rec}}[T]} \) above in yet another way, namely - by inserting double negations wherever they make sense. We do not see however any natural applications of these refinements.
An r.e. set $A^*$ of sentences of arithmetic closed under Modus Ponens is a regular number theory when for some consistent r.e. theory $T$ in the language of analysis (cf. P), $A^* \subseteq \mathbb{N}[T]$. A theory $T^*$ in a language extending the language of arithmetic is regular if $A[T^*]$, i.e., the arithmetical fragment of $T^*$, is a regular number theory. When referring to $A^*, T^*$ as above, we shall assume that $T \supseteq \mathcal{V}_0 + \text{BI}$. This assumption does not of course affect the generality of the discussion, since anyhow

$$A[T^*] \subseteq A^m[T] \subseteq A^m[T + \mathcal{V}_0 + \text{BI}]$$

and we may replace a given theory $T$ by $T + \mathcal{V}_0 + \text{BI}$. On the other hand, this convention renders the set of infinitary derivations of $A^m[T]$ closed under operations which are proved in $\mathcal{V}_0 + \text{BI}$ to preserve the correctness of proofs.

A theory $T^*$ is strongly regular if there is a theory $T$ (as above) s.t.

$$A[T^*] \subseteq A^m[T]$$

and where

$$T^* := T + \text{AC}_{00}^- + \Pi_1^0$$

is consistent. Here $\text{AC}_{00}^-$ is a "negative" intuitionistic version of the axiom of choice from numbers to numbers:

$$\forall x \forall y \exists z A(x, z) \rightarrow \forall y \exists x A(x, ax)$$

and $\Pi_1^0$ is the set of all true $\Pi_1^0$ sentences. Formally, a proof predicate $\text{Prf}_T$ for $T^*$ may be defined from a proof predicate $\text{Prf}_T$ for $T$ by the $\Pi_1$ predicate

$$\text{Prf}_T^*(p, \Gamma) := \exists x \in \text{imp}(p, \text{Prf}_T(x, \Gamma))$$

where $\text{imp}$ is a prim.rec. function which satisfies

$$\text{imp}("G", "F") = "G \rightarrow F".$$ 

The motivation for the condition on $T^*$ is of a technical nature, and will be clear from the proof of theorem II in B.
A.2. GENERAL PROPERTIES OF THE CLASS OF REGULAR THEORIES

The aim of this chapter is to show that the class of regular theories is quite large, and that it satisfies some natural closure properties. (compare Int.4).

2.1. ARITHMETIC, THE THEORY OF SPECIES AND TYPE THEORY ARE (STRONGLY) REGULAR

There is an obvious embedding of $A$ in $A^w_{\text{rec}}$ (cf. Int.3), and so

$$A \subseteq A^w_{\text{rec}}[V + \text{BI}].$$

Hence $A$ is regular. In A.4 below we also prove that the theory of species $L_2$, as well as type theory are regular; namely,

$$A[L_2] \subseteq A^w_{\text{rec}}[L_2 + V_0 + \text{BI}]$$

$$A[L_\omega] \subseteq A^w_{\text{rec}}[L_\omega + V_0 + \text{BI}]$$

(actually BI is redundant everywhere, and even after dropping it the inclusions are proper. See LEIVANT [A]). So $L_2$ and $L_\omega$ are also regular. Of course, the last assertion implies the first two ones, since $A \subseteq A[L_2] \subseteq A[L_\omega]$, and when $T_1 \subseteq T_2$, then the (strong) regularity of $T_2$ implies trivially that of $T_1$.

Assuming that $L_\omega + AC_{00}^C + \Pi^D_1$ is consistent (or that $L_\omega^C + AC_{00}^C$ is 1-consistent, cf. KREISEL-LEVY [68] §9) we have
(the operation \((\_\_)^{+}\) is defined in A.1.2) must also be consistent. Hence \(A, L_2, L_\omega\) are all strongly regular.

2.2. ADDING SELF-CONSISTENCY TO A REGULAR THEORY

2.2.1. LEMMA. There exists a prim.rec. function \(p\) s.t.

\[ \forall F \rightarrow \text{Prf}_{\text{rec}}^\omega (p(\overline{F}) , \overline{F}) \]

for any prenex \(\Pi^0\) sentence \(F\).

PROOF. If \(E\) is an equation, and \(\{p(\overline{E})\}\) describes the singleton derivation

\[ [TE] \rightarrow E \]

then, clearly,

\[ E \rightarrow \text{Prf}_{\text{rec}}^\omega (p(\overline{E}) , \overline{E}) \].

Next, if \(F\) is \(\Sigma^0_1\), \(F \equiv \exists x E_x\), then

\[ (1) \quad F \rightarrow \text{Prf}_{\text{rec}}^\omega (p(\overline{F}) , \overline{F}) \]

where \(\{p(\overline{F})\}\) describes

\[ [TE] \rightarrow E(\forall x. Ex) \]

\[ [EI] \rightarrow F \]

Finally, if \(F\) is \(\Pi^0_2\), \(F \equiv \forall x F_0 \cdot x\), then \((1)\) where \(\{p(\overline{F})\}\) is defined to be the description of

\[ \forall n < \omega \]

\[ [\forall I] \rightarrow F \]

when \(\phi_n\) is described by \(p(\overline{F_0(n)})\), i.e. -
p is now a prim. rec. function by the s.m.n. theorem. □

REMARK. (1) above is of course uniform, i.e., for a $\Pi^0_2$ open formula $F(x)$,

$$\forall x \ [ F(x) \rightarrow \text{Prf}_{\text{rec}}(p(F(x))) ]$$

(where $F(x)$ is sub($F(a)^n$, num(x)) := the result of substituting the numeral with value x for the parameter a in F). So, for a $\Sigma^0_3$ sentence $\exists x F(x) \equiv G$

$$G \rightarrow \exists x \text{Prf}_{\text{rec}}(p(F(x)), F(x))$$

$$\rightarrow \exists d \text{Prf}_{\text{rec}}(d, G) \quad \text{(trivially).}$$

But here d cannot depend primitive recursively on $G$. (Already for $\Sigma^0_2$ sentences $G \equiv \exists x F(x)$ one cannot have d depending recursively on G, since one can extract recursively from d a number p s.t. $F(p)$. A partial recursive $\phi$ yielding $d = \phi(G)$ would therefore allow one to decide recursively membership in an arbitrary $\Pi^0_1$ set.)

2.2.2. LEMMA. Let $T$ be (strongly) regular,

$$T \preceq A^1[T_1] \quad (T \succeq A^1_{\text{rec}}[T_1])$$

say. If $F$ is a $\Sigma^0_3$ sentence consistent with $T_1^*$ (with $T_1^*$) then $T + \{F\}$ is (strongly) regular.

PROOF. By the remark at the end of 2.2.1.

$$F \vdash A \exists d \text{Prf}_{\text{rec}}(d, \Gamma^0)$$

and since $T_1 \succeq V_0 \succeq A$ we thus have

$$T_1 + \{F\} \vdash A \exists d \text{Prf}_{\text{rec}}(d, \Gamma^0),$$

i.e.

$$T_1 \use \use \text{Prf}_{\text{rec}}[T_1 + \{F\} \exists d]$$
Together with the consistency conditions assumed for $T_1 + \{\varphi\}$ this concludes the proof. \( \square \)

2.2.3. **Lemma.** If $T$ is regular then it is consistent.

**Proof.** We have trivially (in $V_0$)

$$\forall \psi \neg \text{Prf}(\psi, L)$$

and so, if $T_1$ is consistent then $A^\omega[T]$ is also consistent, and so must be every $T \leq A^\omega[T_1]$. \( \square \)

2.2.4. **Proposition.** Let $T$ be (strongly) regular,

$$T \leq A^\omega[T_1], \quad (T \leq A^\omega[T_1])$$

where $T_1$ is sound for negations of $\Pi^0_1$ sentences. If $\text{Con}_T$ is a canonical consistency sentence for $T$ then $T + \text{Con}_T$ is (strongly) regular.

**Proof.** $T$ is regular and therefore r.e. By 2.2.3 $T$ is also consistent, and so $\text{Con}_T$ is a true $\Pi^0_1$ sentence which must therefore be consistent with $T_1$ (with $T_1$). So by 2.2.2 $T + \text{Con}_T$ is (strongly) regular. \( \square \)

By the same token, if $T_1$ is sound for negations of $\Pi^0_2$ sentences, then $T + \text{(global } \omega\text{-consistency of } T)$ is regular, since the statement added is $\Pi^0_2$. Note that the global consistency of $T$ is equivalent to uniform reflection for $\Pi^0_2$ on $T + \text{'(uniform reflection on } T\text{' (cf. SMORYNSKI \[77\] thm.1.1.)}$.}

2.3. **Theories "Generated by Transfinite Induction" Are Regular**

Let $\xi$ be a binary predicate and write $x\xi y$ for $\langle x, y \rangle$.

\[ \text{Step}_\xi[A(x)] := \forall y (x A(y) \rightarrow A(x)) \]

\[ \text{II}_\xi[A(x)] := \forall x \text{ Step}_\xi[A(x)] \rightarrow \forall x A(x). \]
2.3.1. PROPOSITION. Let \( T \) be (strongly) regular, \( T \subseteq \text{A}[T] \) where \( T \) is sound for \( \Delta^1_1 \) sentences. Let \( \prec \) be a prim.rec. well-ordering, then

\[
T':= T + \{ \text{TI}_x[A(x)] \}_{\text{arithmetical}}
\]

is (strongly) regular.

**PROOF.** Let \( x \prec y \) be expressed by an equation \( f_k(x,y)=0 \) (\( f_k \) prim.rec.), and define

\[
x \prec_z y := x \prec y < z
\]

Given an arithmetical formula \( A(x) \) define the partial recursive function \( \phi \) as a formal description of the following derivation of \( A^n \).

\[
\Sigma \text{End}_{n,z} \rightarrow \text{End}[\text{E}] \rightarrow \text{A}(n)
\]

where \( \Sigma \text{End}_{n,z} \) is

(i) the derivation (represented by)

\[
\begin{array}{l}
[T] \quad \bar{n} \prec \bar{z} \\
\text{(TP)} \quad \text{I} \quad \text{(TP)} \quad \text{I} \\
\bar{n} \prec \bar{z} \vee \bar{n} = \bar{z} \\
\text{(1)} \quad \text{A}(\bar{n}) \\
\text{(1)} \quad \text{A}(\bar{n}) \\
\text{[1]} \quad \bar{n} \prec \bar{z} \rightarrow \text{A}(n) \\
\end{array}
\]

if \( \bar{n} \prec \bar{z} \) is false (and where we have skipped the premisses of all sequents);

(ii) the derivation

\[
\begin{array}{l}
[T] \quad \forall x \text{ Step}_x[A(x)] \rightarrow \forall x \text{ Step}_x[A(x)] \\
[\text{VE}] \quad \ldots \rightarrow \forall y < \bar{n} \text{ A}(y) \rightarrow \text{A}(\bar{n}) \\
\text{[1]} \quad \forall x \text{ Step}_x[A(x)] \rightarrow \forall y < \bar{n} \text{ A}(y) \\
\end{array}
\]

\[
\begin{array}{l}
[\rightarrow \text{E}] \quad \forall x \text{ Step}_x[A(x)] \rightarrow \text{A}(\bar{n}) \\
[\rightarrow \text{I}] \quad \forall x \text{ Step}_x[A(x)] \rightarrow \bar{n} \bar{z} \rightarrow \text{A}(n)
\end{array}
\]
if \( \phi \) is true, and where \( A \) is described by \( \phi \).

\( \phi \) is defined here in terms of \( \phi = a \) (via \( \{a\} \)), and so \( \phi \) is well-defined by Kleene's recursion theorem (cf. e.g. KLEENE [52] p.352, thm. XXVII).

We may pick the index \( a \) to be that one given (primitive recursively) by the proof of the recursion theorem, and define \( d(z) \) by \( d(z) = \lambda u.\{a\}(z,u) \).

\( d \) is a prim. rec. function by the s.m.n. theorem.

Further, let \( \{e\} = \{e^{A}\} \) describe the derivation

\[
\Delta_{z} \quad \frac{\forall x \text{ Step}_{x}^{A}(A(x)) \Rightarrow \forall z A(x)}{z \leq \omega}
\]

\[
\forall x \text{ Step}_{x}^{A}(A(x)) \Rightarrow \forall z A(x)
\]

\[
\Rightarrow \forall x \text{ Step}_{x}^{A}(A(x)) \Rightarrow \forall z A(x)
\]

where \( A \) is described by \( \{d(z)\} \). The derived formula of \( \{e\} \) is clearly a variant of \( T_{\omega}[A(x)] \).

Using a suitable instance of \( T_{\omega}[A(x)] \), namely with

\[
B_{A}(y) \equiv \text{Prf}_{\omega}^{A}(e^{A}_{y,A}, T_{\omega}[A(x)])
\]

we find quite directly that \( \{e^{A}\} \) is total, and describes a derivation in \( A_{\omega}^{A} \) of \( T_{\omega}[A(x)] \). Hence

\[
T^{\omega} \subseteq A_{\omega}^{\omega}[T_{1} + (\text{arithm.}) T_{\omega}[A(x)]]
\]

\[
= A_{\omega}^{\omega}[T_{1} + W^{\omega}].
\]

It is easily verified that \( W^{\omega} \) is a set of \( A_{\omega}^{\omega} \) sentences, and since \( T_{1} \) is assumed to be sound for \( A_{\omega}^{\omega} \) sentences, \( T_{1} + W^{\omega} \) must be consistent. Hence \( T^{\omega} \) is regular. The proof for strong regularity is similar. □

2.3.2. The interest in proposition 2.3.1 springs from the proof theoretic power of the schemata \( T_{\omega}[A(x)] \) (with \( A \) a prim. rec. well-ordering, \( A \) arithmetical) which are complete for classically true arithmetical sentences (cf. KREISEL, SHOENFIELD & WANG [60] §7 thm.6).

It should be noted that the converse of 2.3.1 is false: not every
regular theory is an extension of \( A \) (say) by \( \{\text{TT}(\forall x)(A(x))\}_\text{arith.} \) for some \( \xi \).
The main reason for this is simply that a regular theory is not necessarily sound: let \( F \) be a false but \( A \)-independent \( \Pi^0_2 \) sentence; then \( A + F \) is regular by 2.2.2.

If we restrict attention to classically sound regular theories, and relativize the whole discussion to classical systems, then the converse of 2.3.1 does hold, simply by the Kreisel-Shoenfield-Wang theorem mentioned above. For intuitionistic truth and formal systems we do not however yet have an analogue to that theorem (compare TN4).
A.3. NORMALIZATION IN $A^\infty$

3.1. AN INFORMAL DESCRIPTION OF THE "NORMALIZATION STRATEGY"

We prove here that every derivation $\phi$ of $A^\infty$ for a sequent $A \Rightarrow F$ can be transformed into a derivation $\phi_N$ for $A \Rightarrow F$ which is normal in the sense of 1.1, i.e., no major premiss of an instance of an elimination rule in $\phi_N$ is derived by an introduction rule, and no major premise of an instance of an elimination rule, of $[\exists I]$ or of $[\forall E]$ is derived by an instance of $[\forall I]$, $[\exists E]$ or $[I]$. Further, $\phi_N$ is recursive and continuous in $\phi$, that is, the value of $\phi_N$ at any given node $u$ in the universal spread is computed recursively from the value of $\phi$ at a finite number of nodes.

The transformation of $\phi$ into $\phi_N$ uses, as in the treatment of finite natural deductions, "reduction-steps" which eliminate local violations in $\phi$ of the requirements of normality. For lack of a better name we call these violations cuts, in analogy to the traditional nomenclature for sequential calculi.

One unfortunate situation is that a reduction which eliminates one cut may create new ones; this is familiar from the finitary case. Here, in addition, the number of reduction-steps cannot be finite, and their order is important. What is essential to the success of the procedure we shall describe is that for each specific node $u$ we can compute $\phi_N(u)$ by performing only a finite number of reductions on $\phi$. An insight into this can be obtained by looking at the more general treatment given in LEIVANT [A], where it is also shown that the order of the reductions is relevant only up to obvious requirements.

The properties of reduction sequences proved in LEIVANT [A] are in a way analogous to the strong normalization property of finitary proofs, i.e. every reduction sequence starting with a given finitary natural de-
duction terminates. The proofs use however arguments on the geometry of infinitary derivations which are combinatorially tedious. For our purpose here all this is irrelevant, so we confine ourselves to the more modest task of giving one method of obtaining $\phi_N$ from $\phi$.

The normalization strategy we use can be roughly described as follows. Suppose that we have computed already $\phi_N(v)$ for every $v < u$. Under our conventions on the coding of sequences this means in particular that $\phi_N(v)$ is given for every $v < u$. These values have been computed by constructing a certain reduction sequence

$$(*) \quad \phi \vdash \phi_1 \vdash \ldots \vdash \phi_k$$

(cf. 3.1 below), where for $v < u$ $\phi_N(v) := \phi_k(v)$. To compute $\phi_N(u)$ we show now how to extend $(*)$ by

$$\phi_k \vdash \phi_{k+1} \ldots \vdash \phi_{k+m}$$

so that $\phi_N(v) := \phi_{k+m}(v)$ for $v \leq u$. Examine the inference rules of $\phi_k$ at $u, u*(0), u*(0,0), \ldots$, as long as these are eliminations (or $[SI]$ or $[FE]$); since $\phi_k$ is well-founded, this must come to an end. If no cut occurs immediately above any of the examined nodes, let $m := 0$ (i.e., stop); else let $\phi_{k+1}$ be obtained by a reduction at the uppermost (i.e., maximal) such cut. The process is repeated as long as cuts are found, and our point (to be proved below) is that this may happen only finitely many times.

It should be noted that in $\phi_k$ above more than one cut can occur along $u, u*(0), \ldots$; namely, if we have a chain of instances of $[vE]$ and $[3E]$. If the inference rule in $\phi_k$ at $u$ is not an elimination (or $[3I]$ or $[FE]$) then the sequence $u, u*(0), \ldots$ is empty, and the condition for $m := 0$ is satisfied trivially.

3.2. SOME CONVENTIONS

We slightly modify the formulation of $\mathcal{A}^n$ in 1.1, so as to allow a smoother exposition. Let an indexed formula be a pair $(n,F)$, which we write as $^{n}F$, where $n$ is a natural number and $F$ a formula. A sequent is now a syntactical object of the form $a \Rightarrow G$, where $a$ is a finite set of indexed formulae and $G$ is a formula. The inference rules remain as in 1.1, except
for the "discharging" inferences; i.e., the \([vE]_{k}\) rule for example takes the form

\[
\begin{align*}
& a \Rightarrow A_1 \lor A_2, \quad a, k_A_1 \Rightarrow B, \quad a, k_A_2 \Rightarrow B \\
& \text{[\(vE\)]} \quad a \Rightarrow B
\end{align*}
\]

\([+I_k]\) and \([E_k]\) are defined similarly.

W.l.o.g. we make the convention that two occurrences of the same formula which are "discharged" at distinct nodes of a derivation \(\phi\) are given distinct indices. Note that normalization for the modified \(A^\infty\) implies normalization for the original formulation: indices can be just ignored in \(\phi\) and \(\phi_N\). They are indeed useful only as track-keepers through the normalization proof.

There is one more modification we make in \(A^\infty\), but this one does weaken the results. We add to the rules of \(A^\infty\) the replacement rule

\[
\begin{align*}
& a \Rightarrow F(t) \\
& \text{[R]} \quad a \Rightarrow F(t')
\end{align*}
\]

where \(t' = t\).

The reason we make this modification is of course our concern for the reader's time and patience: its presence allows a simplified formulation of the reduction steps (see 3.3.2 below).

The normalization proof for the original version can be found in LEIVANT [A] (where it is shown that the obvious "term replacing" derivations which take the place of [R] can be inserted into the proof of normalization). In the absoluteness proofs in part B we do use however normalization for the original \(A^\infty\), without [R], because a separation between logic and arithmetic is utilized there, and this separation is destroyed by using [R] in the reductions.

To shorten the discussion of infinitary proof figures we shall use the following notational conventions. Given a derivation \(\phi\) of \(A^\infty\) we shall write \(\rho^{\phi,u}\) and \(s^{\phi,u}\) for the inference rule and the sequent (respectively) standing at the node \(u\) in \(\phi\), i.e.,

\[
\phi(u) = (\rho^{\phi,u}, s^{\phi,u}),
\]

and when \(s^{\phi,u} \equiv b \Rightarrow c\) then \(a^{\phi,u} := b, F^{\phi,u} := c\). Also, we shall freely use geometrical representation of derivations, e.g.,
\[ \phi = \frac{\{ \phi^{(m)} \}_{m \in \omega}}{[\rho \phi, (\cdot)] \phi, (\cdot) \Rightarrow \phi, (\cdot)} \]

(where \( \phi^u := \lambda x.\phi(u^x) \)). \( \phi[a] \) will denote the result of joining the finite set of indexed formulae \( a \) to all the premisses of sequents in \( \phi \), i.e.,

\[ \phi[a](u) := \{ \rho \phi^u, \rho \phi^u, \rho \phi^u, \rho \phi^u \} \]

\( \phi[kA] := \phi[\{kA\}] \)

When \( \phi = \phi[A] \), \( \psi = \psi[A] \), then

\[
\begin{align*}
\psi & \\
\{kA\} & \\
\phi & \\
\psi & \\
\phi & \\
\phi & \\
\end{align*}
\]

is defined as the derivation which comes by replacing (or "grafting on") each top node of \( \phi[b] \) of the form \( \rho \phi, \rho \phi, \rho \phi, \rho \phi \) by \( \psi[c] \), and dropping \( \{kA\} \) from all premisses of the result.

Finally, we shall mark in this chapter by asterisks in the margins those passages which can be omitted when only the negative (i.e., free of \( v \) and of \( \exists \)) fragment of the language of \( A \) is treated: the reader will get a more transparent view of the proof by skipping these sections on a first reading. The beginning of a paragraph to be skipped is marked by //, its end by *//, and isolated phrases by *.

3.3. PROPER REDUCTIONS

3.3.1. The critical inference rules are all the elimination rules

([\&E],[\rightarrow E],[\forall E],[\exists E] and [\exists E]), [\exists I] and [\forall E]. These are the rules which may induce a cut:

\[
\text{Cut}(\phi,u) := \text{"} \rho \phi^u \text{ is an elimination and } \rho \phi^u \text{ is a critical introduction rule, or } \rho \phi^u \text{ is an introduction rule, or } \rho \phi^u \text{ is critical and } \rho \phi^u \text{ is } [\forall E], [\exists E] \text{ or } [\exists I]."
\]
3.3.2. Detour reductions

(i) \(\&_1\)-reduction:

\[
\begin{align*}
\phi^{(0,0)} & & \phi^{(0,1)} \\
\frac{\vdash}{\phi^{(0,i)}}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\frac{A_0 \iff A_1}{\phi^{(i)}} \\
\frac{\vdash}{\phi^{(0,i)}}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\frac{A \iff A_1}{\phi^{(i)}} \\
\frac{\vdash}{\phi^{(0,i)}}
\end{array}
\end{align*}
\]

\(\vdash A \iff A_1 \iff\)

(ii) \(\rightarrow\)-reduction:

\[
\begin{align*}
\phi^{(0,0)} & \\
\frac{A \Rightarrow B}{\phi^{(1)}}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\frac{A \Rightarrow A \iff B}{\phi^{(1)}} \\
\frac{\vdash}{\phi^{(0,0)}}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\frac{A \Rightarrow A \iff B}{\phi^{(1)}} \\
\frac{\vdash}{\phi^{(0,0)}}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\frac{A \Rightarrow A \iff B}{\phi^{(1)}} \\
\frac{\vdash}{\phi^{(0,0)}}
\end{array}
\end{align*}
\]

* (iii) \(\lor_1\)-reduction:

\[
\begin{align*}
\phi^{(0,0)} & \\
\frac{A_1 \iff A_2}{\phi^{(1)}}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\frac{A \iff A_1 \lor A_2}{\phi^{(1)}} \\
\frac{\vdash}{\phi^{(0,0)}}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\frac{A \iff A_1 \lor A_2}{\phi^{(1)}} \\
\frac{\vdash}{\phi^{(0,0)}}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\frac{A \iff A_1 \lor A_2}{\phi^{(1)}} \\
\frac{\vdash}{\phi^{(0,0)}}
\end{array}
\end{align*}
\]
(iv) $\forall$-reduction:

\[
\frac{
\phi^{(0,n)}
}{
\{ \overline{a} \Rightarrow A(\bar{n}) \}_{n<\omega}
}\]

\[\vdash \frac{
\phi^{(0,n_0)}
}{
\overline{a} \Rightarrow A(n_0)
}\]

\[
\text{[VI]} \quad \overline{a} \Rightarrow \forall xA(x)
\]

\[
\text{[VE]} \quad \overline{a} \Rightarrow A(t)
\]

where $n_0$ is the value of the term $t$ (recall that we treat only sentences, and so all terms are closed).

* (v) $\exists$-reduction:

\[
\frac{
\phi^{(0,0)}
}{
\overline{a} \Rightarrow A(t)
}\]

\[
\frac{
\phi^{(n+1)}
}{
\{ \overline{a}, \overline{b} \Rightarrow A(\bar{n}) \}_{n<\omega}
}\]

\[\vdash \frac{
\psi^{(n_0+1)}
}{
\overline{a} \Rightarrow B
}\]

\[
\text{[EI]} \quad \overline{a} \Rightarrow \exists xA(x)
\]

\[
\text{[EE]} \quad \overline{a} \Rightarrow B
\]

where $n_0$ is the value of $t$, and

\[
\psi := \frac{
\overline{a} \Rightarrow A(t)
}{
\text{[R]} \quad \overline{a} \Rightarrow A(n_0)
}\]
3.3.3. Permutative reductions

Let \( \rho \) be a critical inference rule.

\[ \phi(0,0) \]

\[ a \Rightarrow A_1 \lor A_2 \{ a, A_j \Rightarrow B \}_{j=1,2} \]

\[ [vE] \]

\[ a \Rightarrow B \{ \phi(n) \}_{n>0} \]

\[ [\rho] \]

\[ a \Rightarrow C \]

\[ \phi(0,j) \]

\[ a, A_j \Rightarrow B \{ \phi(n) \}_{n>0} \]

\[ [vE] \]

\[ a \Rightarrow C \]

\[ \phi(0,0) \]

\[ \phi(0,j) \]

\[ a, A_j \Rightarrow C \{ \phi(n) \}_{n>0} \]

\[ [\rho] \]

\[ a \Rightarrow C \]

3.3.4. Absurdity reductions

Let \( \rho \) be a critical inference other than [FE].

\[ \phi(0,0) \]

\[ a \Rightarrow 1 \]

\[ [1] \]

\[ a \Rightarrow A \{ \phi(n+1) \}_{n<0} \]

\[ [\rho] \]

\[ a \Rightarrow B \]

\[ [1] \]

\[ a = 1 \]

\[ \phi(0,0) \]

\[ a = 1 \]

\[ [\rho] \]

\[ a = B \]
The absurdity reductions are the converse of the expansion reductions of PRAWITZ [71], 3.3.3. While Prawitz's aim is to show that the intuitionistic absurdity rule can be reduced to a Post rule (more generally, that an intuitionistic first order system is conservative over the system of its Post rules, cf. PRAWITZ [71], 3.5.5), our aim is to get normal derivations where, roughly speaking, breaking into the internal structure of formulae is avoided when possible.

3.3.5. If \( \phi^u \vdash \psi \) and \( \phi_1 \) comes from \( \phi \) by replacing \( \phi^u \) by \( \psi \) (i.e.,

\[
\phi_1 \psi := \begin{cases} 
\psi & \text{if } v \neq u \\
\psi_{uv} & \text{if } v = u = w
\end{cases}
\]

then \( \phi \vdash \phi_1 \), and we also write more specifically \( \phi \vdash_{\psi} \phi_1 \).

\[
\phi \vdash_{\psi} \phi_1 := \exists \theta [ \theta = \psi^{(0)} \& \phi_1 = \psi^{(n)} \& \forall i < n \psi^{(i)} \vdash \psi^{(i+1)} ]
\]

3.4. DEFINITION OF THE NORMALIZATION STRATEGY; THE NORMALIZATION PREDICATE

3.4.1. Write \( (n)^i \) for \( (n, \ldots, n) \),

\[
\text{Infl}(\phi, u, v) := \exists i u [ u = v^{* (0)} \& \forall w \in \phi \& \text{"} \phi, \psi \text{ is critical"} ]
\]

\[
\equiv "u \text{ influences } v \text{ in } \phi."
\]
3.4.2. Define

\begin{align*}
(1) \quad & \text{Clear}(\phi, u) := \forall w \left[ \text{Infl}(\phi, w, u) \rightarrow \neg \text{Cut}(\phi, w) \right] \\
(2) \quad & \{j\}(\phi, v) := \mu w u v. \exists w [ \text{Infl}(\phi, w, u) \& \text{Cut}(\phi, w)].
\end{align*}

I.e., \( u \) is a "clear node" in the derivation \( \phi \) if no cut occurs at a node which "influences" \( u \) in \( \phi \). The function \( \{j\} \) picks out the first node up to the argument \( v \) which is not clear in \( \phi \). Here the natural ordering \( \leq \) is taken in the definition in order to simplify the definition of the normalization procedure defined below. Under our conventions \( u \leq v \) implies \( u \leq v \), but replacing \( \leq \) in the definition of \( \{j\} \) by \( \preceq \) would destroy the linear order of the reduction steps we have in mind, and implied by 3.4.4 below.

We further define

\begin{align*}
(3) \quad & \{k\}(\phi, v) := 0 \quad \text{if Clear}(\phi, v), \\
& := \max \left\{ u \left| \text{Infl}(\phi, u, \{j\}(\phi, v)) \& \text{Cut}(\phi, u) \right. \right\} \quad \text{otherwise}.
\end{align*}

I.e., for the node \( \{j\}(\phi, v) =: w \) defined above \( \{k\} \) picks the maximal node \( y \) which influences \( w \) and where a cut occurs.

3.4.3. We define the derivation \( (r)(\phi, u) \equiv \lambda v. (r_0)(\phi, u, v) \) by

\begin{align*}
\phi \vdash^1_{u} (r)(\phi, u) & \quad \text{if Cut}(\phi, u) \\
(4) \quad & \quad \{r\}(\phi, u) := \phi \quad \text{otherwise}.
\end{align*}
Using this notation we now define the normalization functional

\[ \phi \mapsto \lambda v. \psi(\phi, v) \]

by defining \( \psi \equiv \{n\} \) through

\[ \psi(\phi, v) \equiv \begin{cases} \{n\}(\{r\}(\phi, (k)(\phi, v)), v) & \text{if } (k)(\phi, v) \neq 0, \\ \phi v & \text{otherwise}. \end{cases} \tag{5} \]

It is seen outright that (5) is a correct Herbrand-Gödel definition of \( \psi \) as a partial recursive functional (compare e.g. PETER[67] p.195 or KLEENE[52] sec.54). The reader less familiar with the Herbrand-Gödel definition may prefer to note that \( \psi \) is well-defined by Kleene's recursion theorem (KLEENE[52] p.352, thm. XXVII) and that the index \( n \) is given primitive recursively (from the definition (5)) by the proof of that theorem. The intuitive meaning of \( \psi \equiv \{n\} \) is this: if

\[ \exists w v \ \neg \text{Clear}(\phi, w) \]

then

\[ w \equiv \{j\}(\phi, v) \equiv \mu w v. \neg \text{Clear}(\phi, w) \]

\[ \neq 0 \]

and so

\[ (k)(\phi, v) \equiv u \neq 0 \]

as in the illustration above. Letting

\[ \phi \upharpoonright_u \phi \]

we take

\[ \psi(\phi, v) :\equiv \psi(\phi \downarrow, v). \]
If
\[ \neg \exists w \forall v. \neg \text{Clear}(\phi, w) \]
then \( \{j\}(\phi, v) = 0 \) and so \( \{k\}(\phi, v) = 0 \) and
\[ \forall (\phi, v) \quad :\! : \! : \neg \phi. \]

In other words, \( \forall (\phi, v) \) is obtained by a series of reductions
\[ \phi = \phi_0 \upharpoonright u_0 \phi_1 \upharpoonright u_1 \ldots \upharpoonright u_t \phi_{t+1} \ldots \]
where
\[ u_t := \{k\}(\phi_t, v) \neq 0. \]

If and when for some \( t \) we get
\[ \forall w \forall v. \text{Clear}(\phi_t, v) \]
we stop and set
\[ \forall (\phi, v) \quad :\! : \! : \neg \phi_{t+1} (v). \]

Note that \( \forall (\phi, v) \equiv \{n\}(\phi, v) \) may be defined by different reduction sequences for various values of \( v \), and so it is not evident prima facie that \( \{n\}(\phi, v) \) is at all a derivation.

\[ \text{Norm}(\phi) := \forall \{n\}(\phi, v) \quad \text{WF} \{\{n\}(\phi, v) \}
\]
\[ \equiv \forall x \exists y \{n\}(\phi, x) \quad \forall x \exists y \{n\}(\phi, x(y)) \equiv 0 \]
\[ \equiv "\phi \text{ is normalizable}". \]

We shall also refer below to the function
\[
(\ell)(\phi, v) := \begin{cases} 
(\ell)((r)(\phi,(k)(\phi,v)),v) + 1 & \text{if } \exists w \neg \text{Clear}(\phi,w) \\
0 & \text{otherwise.}
\end{cases}
\]

which, like \(\eta\) above, may be formally defined by a use of the recursion theorem. Intuitively \(\ell\) measures the length of computation of \(\eta\).

3.4.4. When for some \(v\) \(\{k\}(\phi,v) \equiv u \neq 0\) and \(\phi \upharpoonright u \circ \phi_1\), we write \(\phi \upharpoonright u \circ \phi_1\).

If \(v_1 \leq v_2\) and

\[
w := \{j\}(\phi,v_1) := \{u \leq v_1, \neg \text{Clear}(\phi,z)\}
\]

\(\neq 0\) then

\[
\{j\}(\phi,v_2) \equiv \{j\}(\phi,v_1)
\]

trivially, and so

\[
\{k\}(\phi,v_2) \equiv \{k\}(\phi,v_1).
\]

Hence \(\phi \upharpoonright u \circ \phi_1\) for at most one \(\phi_1\).

We further define \(\phi \upharpoonright u \circ \phi_1\) to be the \(t\)-time iteration of \(\phi \upharpoonright u \circ \phi_1\). So \(\{n\}(\phi,v)\) is (when converging) \(\phi_1(v)\), where \(\phi_1\) is uniquely determined by

\[
\phi \upharpoonright \{\ell\}(\phi,v) \phi_1
\]

3.4.5. **Lemma.** Let \(\text{Der}^w(\phi)\).

(a) \(\text{Nable}(\phi) \leftrightarrow \forall \psi [ \phi \upharpoonright u \psi \Rightarrow \text{Nable}(\psi) ]\)

(b) \(\phi \upharpoonright u \psi \Rightarrow s \phi \psi = s \phi_1 \psi_1\)

(c) \(\phi \upharpoonright u \psi \Rightarrow \{n\}^g = \{n\}^{g_1}\)

**Proof.** Obvious from the definitions. \(\Box\)

3.4.6. **Lemma.** Let \(\text{Der}^w(\phi)\). If \(\phi \upharpoonright u \psi\) and \(w \nmid u\) then \(\psi w = \psi u\) and \(\forall u \psi, u^\ast w \equiv \psi, u^\ast w\).

**Proof.** Immediate. \(\Box\)
3.4.7. LEMMA. Let Der⁺(ϕ). If

1. \text{Clear}(ϕ, v),
2. \phi \vdash^1 \psi \text{ and } \\
3. w \not\preceq v

then \text{Clear}(ϕ, v).

PROOF. Assume (1) - (3), and towards proving \text{Clear}(ϕ, v) assume

4. \text{Inf1}(ψ, u, v), \quad u = v*<0>n.

If \ w \not\preceq u*<0> \text{ then we must have by (3) and (4)}

5. \ w = v*<0>k \quad \text{for some } k, 0 \leq k \leq n+1.

We know from 3.4.6 that (2) implies

6. \forall y \forall w \ \psi(y) = \hat{ϕ}(y)

and by (4)

7. \psi \in \text{"}_0^1 \psi, v*<0>1 \text{ is critical".}

From (5), (6) and (7)

8. \psi \in \text{"}_0^1 \psi, v*<0>1 \text{ is critical"}

while (2) implies that \psi, w must also be critical, from which by (8) and (5)

9. \text{Inf1}(ϕ, w, v)

and so by (1) -\text{Cut}(ϕ, w) contradicting (2). Hence

10. \ w \not\preceq u*<0>.

But now we get from (6)
\[
\begin{align*}
\rho^\phi_v u &= \rho^\phi_v u \\
\rho^\psi_v u^{<0>} &= \rho^\phi_v u^{<0>}
\end{align*}
\]

and from (4), (10) and (6)

\[
(12) \quad \text{Infl}(\phi, u, v).
\]

(1) and (12) imply \( \neg \text{Cut}(\phi, u) \), from which by (11) \( \neg \text{Cut}(\psi, u) \), as required.

3.4.8. PROPOSITION. If \( \text{Prt}^\phi(\phi, s) \) and \( \text{Nمب}^\phi(\phi) \) then \( \text{NPr}^\phi(\{n\}^\phi, s) \).

PROOF. Assume the premise; then by the definition of the predicate \( \text{Nمب} \) we have

\[
(1) \quad \forall (\{n\}^\phi & \implies \text{WF}(\{n\}^\phi)).
\]

It remains to prove that \( \{n\}^\phi \) is locally correct and cut free.

Fix a node \( v \). By the argument of 3.4.4 \( \forall (\{n\}^\phi) \) implies that

\[
(2) \quad \phi_{[\{s\}^\phi(\phi, v)]} v
\]

for a certain derivation \( \psi_v \), for which

\[
(3) \quad \forall w \subseteq v \text{ Clear}(\psi_v, w)
\]

\[
(4) \quad (\{n\}^\phi(v) :< \psi_v(v).
\]

Likewise we have for each \( m \) a derivation \( \psi_{v^*<m>} \), s.t.

\[
(5) \quad \phi_{[\{s\}^\phi(\phi, v^*<m>)]} v^*<m>
\]

\[
(6) \quad \forall w \subseteq v^*<m> \text{ Clear}(\psi_{v^*<m>}, w)
\]

\[
(7) \quad (\{n\}^\phi(v^*<m>) :< \psi_{v^*<m>}(v^*<m>).
\]

By 3.4.4 reduction sequence (2) is necessarily a subsequence of (5) (for each \( m \)). Fixing \( m \), we thus have for certain \( x_0, \ldots, x_t, w_1, \ldots, w_t, t \geq 0, \)
We prove by induction on \( t \) that

\[(9) \quad \forall w \leq v \quad \text{Clear}(x_t, w)\]

\[(10) \quad x_t(v) = \psi_v(v)\]

\[(11) \quad s^{x_t(v)} = s^{\psi_v(v)} \quad \text{for the given } m.\]

For \( t = 0 \), \( (9) - (11) \) are trivial (cf. (3)). Assuming \( (9) - (11) \) for \( t \), we have by (9)

\[(12) \quad w_{t+1} \not\equiv v, \quad w_{t+1} \not\equiv v\]

and so by 3.4.7

\[(13) \quad \forall w \leq v \quad \text{Clear}(x_{t+1}, w)\]

while by 3.4.6 \( (12) \) implies

\[x_{t+1}(v) = x_t(v) = \psi_v(v)\]

and

\[s^{x_{t+1}(v)} = s^{\psi_v(v)} = s^{\psi_v(v)} \quad \text{for the given } m.\]

This completes the induction. We thus have from (8)

\[(14) \quad \psi_{\psi_v(v)}(v) = \psi_v(v) \quad \text{(by (10))}\]

\[= \{n\}^\phi(v) \quad \text{(by (4))}\]

\[(15) \quad s^{\psi_v(v)} = s^{\psi_v(v)} = s^{\psi_v(v)} \quad \text{(by (11))}\]

\[= s^{\{n\}^\phi(v)} \quad \text{(by (7))}\]

Hence \( s^{\{n\}^\phi(v)} \) and \( s^{\{n\}^\phi(v)} \) relate according to the inference rule \( \rho^\phi_{\psi_v(v)} = \rho^\phi_{\{n\}^\phi} \) and thus \( \{n\}^\phi \) is a locally correct derivation. Further,
we have

\[(16) \quad \rho(n)^\phi \wedge \vee = \rho \phi \vee \vee \quad \text{(by (4))}\]

\[= \rho \phi \vee \langle 0 \rangle \vee \quad \text{(by (14))}\]

while

\[(17) \quad \rho(n)^\phi \vee \langle 0 \rangle \vee = \rho \phi \vee \langle 0 \rangle \vee \langle 0 \rangle \vee \quad \text{(by (7))}\]

But (6) implies that \(\neg \text{Cut}(\phi \vee \langle 0 \rangle \vee)\) and so (16) and (17) imply \(\neg \text{Cut}(\phi \langle n \rangle \vee)\), since \(\text{Cut}(\chi, \mu)\) depends on \(\rho \chi \mu\) and \(\rho \chi \mu \langle 0 \rangle \vee\) only. Hence \(\{n\}^\phi\) is cut-free. □

3.5. GENERALIZED REDUCTIONS; STABILITY

The concepts defined in this section are analogues of the ones defined for the finitary natural deduction system for \(A\) in LEIVANT [74], i.e., they are based on the ideas of PRWITZ [71]'s "validity argument".

3.5.1. The measure of complexity \(\mu\) on the sentences of the language of \(A\) is defined by recursion on their length as follows.

\[\mu(E) := 0 \quad \text{if} \ E \ \text{is an equation}\]

\[\mu(A \land B) := \mu(A \lor B) := \max[\mu(A), \mu(B)]\]

\[\mu(\forall x A(x)) := \mu(\exists x A(x)) := \mu(A(\overline{0}))\]

\[\mu(A \rightarrow B) := \max[\mu(A) + 1, \mu(B)].\]

We also write, for a derivation \(\phi\) of \(A^\mu\),

\[|\phi| := p^\phi,()\]

i.e. - \(|\phi|\) is the derived sentence of \(\phi\).

3.5.2. We define now simultaneously by (metamathematical) recursion (on \(n\)) two predicates:
\[ \text{St}_n(\phi) \quad \text{(for "}\phi\text{ is stable and }\mu(|\phi|) \leq n\text{")} \]

and
\[ \phi \models_n \psi \quad \text{(for }\phi\text{ s.t. }\mu(|\phi|) \leq n). \]

The metamathematical recursion yields an explicit definition of \( \text{St}_n \) and \( \models_n \) in terms of \( \text{St}_m \) and \( \models_m \) with \( m \preceq n \). When \( \phi \) is recursive, \( \phi x \vdash (d)x \) say, then these predicates are arithmetical (compare LEIVANT [74] §7), and given a (hyperarithmetical) truth definition for the full language of \( A \), one can define (arithmetically in this truth definition) predicates \( \text{St} \) and \( \models \) s.t.

\[
\frac{\text{St}((d))}{\text{St}((d))} \quad \frac{\text{St}((d))}{(d) \models (e)} \quad \frac{n}{(e) \models (d)} \quad \text{where } n := \mu(|\{d\}|).
\]

But there are no arithmetical predicates \( \text{St} \), \( \models \) satisfying (1). Likewise, a truth definition for the full language of \( V_0 \) provides uniform predicates \( \text{St} \) and \( \models \) for arbitrary derivations \( \phi \).

3.5.3. Assume now \( \text{St}_m \), \( \models m \) to be defined for every \( m \preceq n \), and let \( \mu(|\phi|) \leq n \)

(i) If \( \phi \models_{u} \psi \) where \( u := \mu(\{k\}(\phi,<>)) \neq 0 \) then \( \phi \models_n \psi \). \( \langle k \rangle \) is defined in 3.4.3. Note that we may have \( \chi_1 \models_1 \chi_2 \) while \( \{k\}(\chi_1,<>) = 0 \).

(ii) \[ \phi \models (0) \quad \frac{a \Rightarrow A_0 \quad a \Rightarrow A_1}{[\&I]} \quad \frac{a \Rightarrow A_0 \& A_1}{\text{St}_{n}(\phi)} \quad \frac{n}{\text{St}_{n}^{(i)}(\phi)} \quad (i=0,1) \]

(iii) \[ \psi \models (0) \quad \frac{a \Rightarrow A \Rightarrow B}{[\rightarrow I]} \quad \frac{a \Rightarrow A}{\text{St}_{n}^{(0)}(\psi)} \quad \frac{n}{\text{St}_{n}^{(0)}(\psi)} \quad \frac{\mu}{\phi(0)} \quad \frac{\mu}{\phi(0)} \quad \frac{\mu}{\phi(0)} \quad \frac{\mu}{\phi(0)} \quad \frac{\mu}{\phi(0)} \quad \frac{\mu}{\phi(0)} \quad \frac{\mu}{\phi(0)} \quad \frac{\mu}{\phi(0)} \quad (i=0,1) \]

whenever \( \psi = a \Rightarrow A \) and \( \text{St}_{n}(\phi) \).

(iv) \[ \phi \models (0) \quad \frac{a \Rightarrow A_1}{[\vee I]} \quad \frac{a \Rightarrow A_0 \lor A_1}{\text{St}_{n}(\phi)} \quad \frac{n}{\text{St}_{n}^{(0)}(\phi)} \quad (i=0,1) \]
Notice that the reduction is always at () in clauses (ii)-(ix).

We refer to reduction steps (ii)-(ix) as improper reductions. Reductions (vii) and (viii) we label more specifically as simplifications (because of their similarity to Prawitz [71]'s "immediate simplification" reductions).

In the discussion below we write $\overline{\text{St}}_n(\phi)$ for $\overline{\text{St}}_n(\phi)$ and $\phi \models \psi$ for $\phi \models \psi$ where $n := \nu(|\phi|)$. 

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In the discussion below we write $\overline{\text{St}}_n(\phi)$ for $\overline{\text{St}}_n(\phi)$ and $\phi \models \psi$ for $\phi \models \psi$ where $n := \nu(|\phi|)$.
3.5.4. LEMMA.
(a) $\text{St}(\phi) \to \text{Nmble}(\phi)$
(b) $\not\models \phi$ and $\text{St}(\phi) \to \text{St}(\phi)$
(c) $\text{St}(\phi)$ iff $[\text{Nmble}(\phi)$ and $[\phi \models \psi$ by improper reductions $\to \text{Nmble}(\psi)]$.

PROOF. (a) Immediate, since $\phi \models \phi$. (b) and (c) are immediate from the definitions.

3.5.5. $\phi = \frac{k}{\psi}$ is stable at $kA$ if whenever
\[ \begin{array}{c}
\frac{\alpha}{A} \Rightarrow B \\
\frac{\beta}{\gamma} \Rightarrow \delta
\end{array} \]
then $[\psi]_k$ is stable.

$\phi = \frac{\alpha}{A} \Rightarrow B$ is strongly stable (notation: $\text{SSSt}(\phi)$) if when $\alpha = \{ \frac{n_i}{A_i} \}_i$, and $\psi_i, n_i$ are stable derivations with $[\psi_i, n_i] = A_i$, then
\[ \{ \left[ \frac{n_i}{A_i} \right] \} \cap A_i = 0 \]
is stable, where $0$ is defined analogously to the definition at the end of 3.2 for the substitution of a single derivation $\phi$. We write here $\phi \models 0$.

3.5.6. LEMMA. $\text{SSSt}(\phi) \to \text{St}(\phi)$.

PROOF. Immediate, since the singleton derivations
\[ \psi_i, n_i := [ \left[ \frac{n}{A} \right] \Rightarrow A_i ] \]
are trivially stable. □
3.6. THE STABILITY THEOREM; TREATMENT OF THE NON-CRITICAL INFECTION RULES

3.6.1. Our aim is to prove the following

**THEOREM (in $V_0 + BI$)**

(i) $\text{Der}^\omega(\phi) \rightarrow \text{SSt}(\phi)$

(ii) $\text{Der}^\omega(\phi) \rightarrow \text{St}(\phi)$

(iii) $\text{Der}^\omega(\phi) \rightarrow \text{Nmble}(\phi)$.

Here (i) implies (ii) by 3.5.6, and (ii) implies (iii) by 3.5.4. The proof of (i) proceeds by BI on the proof-tree $\phi$. I.e., one proves

$$\forall m \text{ SSt}(\phi^u(m) \rightarrow \text{SSt}(\phi^u))$$

and so by BI $\text{SSt}(\phi^u)$ and $\phi^u = \phi$.

For the top nodes of $\phi$, i.e. where $\rho^\phi,u$ is [T] or [TE], the premise of (i) is satisfied trivially, and the conclusion is immediate from the definitions. So our main concern is to prove (i) for the other cases for $\rho^\phi,u$.

The cases of non-critical inference rules are treated in 3.6.3 below, the proof for the critical rules being postponed to 3.7.

3.6.2. **LEMMA.** Assume $\text{Der}^\omega(\phi)$ and $\text{C1ear}(\phi,\{\})$. Then

(i) $\text{Nmble}(\phi) \rightarrow \forall m \text{ Nmble}(\phi^u(m))$

(ii) If $\text{Nmble}(\phi)$, then

$$\forall n \text{ Nmble}(\phi^u(m))$$

PROOF. [a] Assume $\text{Nmble}(\phi)$. For each $m$ and $v$ we prove

$$\forall n \text{ Nmble}(\phi^u(m), v) = \text{Nmble}(\phi^u(m))$$

i.e., (ii). But then $\forall(n)^{\phi}$ implies that $\forall(n)^{\phi^u}$ and $\forall(n)^{\phi}$ implies $\forall(n)^{\phi^u}$, so $\text{Nmble}(\phi^u(m))$ for every $m$. 

The proof proceeds by induction on \( \{\ell\}(\\dot{\phi}, (m)^*v) \), i.e., we prove by induction \( \forall \phi \forall i \; P(\phi, i) \), where

\[
P(\phi, i) \equiv \{\ell\}(\\dot{\phi}, (m)^*v) \equiv i \rightarrow (n)(\phi^{(m)}(v) \equiv (n)(\phi, (m)^*v)).
\]

Since, by \( \text{Nable}(\phi) \), \( !\{\ell\}(\\dot{\phi}, (m)^*v) \), we get (1).

**Basis.** If \( \{\ell\}(\\dot{\phi}, (m)^*v) \equiv 0 \) then

(2) \; \text{Clear}(\phi, (m)^*v),

and so trivially

(3) \; \text{Clear}(\phi^{(m)}(v)).

Hence

\[
(n)(\phi^{(m)}(v) \equiv \phi^{(m)}(v) \quad \text{by (3)}
\]

\[
= \phi((m)^*v)
\]

\[
= (n)(\phi, (m)^*v) \quad \text{by (2)}.
\]

**Ind. Step.** Let \( \{\ell\}(\phi, (m)^*v) > 0 \), \( \phi \models^1 u \phi \), for some \( u \), where \( \text{Inf}(\phi, u, w) \) for some \( v \leq (m)^*v \).

\( \text{Clear}(\phi, (\dot{\cdot})) \) (which we are assuming from the start) implies \( \text{Clear}(\phi, (\dot{\cdot})) \) by 3.4.7, and also \( u \not\in (\dot{\cdot}) \), which in turn implies that

\[
\begin{cases}
\phi(q) = \phi(q) & \text{for } q \neq (u)_0 \\
\phi((u)_0) = \phi((u)_0) & \text{for } u \not\in (\dot{\cdot})
\end{cases}
\]
So \( \phi_1 \) satisfies the lemma's conditions, while 
\[ \{ \ell \}(\phi, (m) \ast v) = \{ \ell \}(\phi, (m) \ast v) \uparrow 1. \] Hence

\[ \{ n \}(\phi^m, v) = \{ n \}(\phi^m, v), \quad \text{by } (4), \quad \text{in any case } (\text{cf. } 3.4.4) \]

\[ = \{ n \}(\phi, (m) \ast v) \quad \text{by ind.hyp.} \]

\[ = \{ n \}(\phi, (m) \ast v) \quad \text{since by } 3.4.4 \{ n \} = \{ n \} \]

[b] Assume \( \forall m \text{ Noble}(\phi^m) \). First, \( \text{Clear}(\phi, (m)) \) implies \( \{ n \}(\phi, (m)) = \phi \); so to prove \( \text{Noble}(\phi^m) \) it suffices to prove

\[ \{ n \}(\phi, (m) \ast v) = \{ n \}(\phi^m, v). \]

We cannot here use induction on \( \{ \ell \}(\phi, (m) \ast v) \), because \( \{ \ell \}(\phi, (m) \ast v) \) is precisely what we have to prove. So fixing \( m \) and \( v \), we proceed by induction on

\[ \{ p \}(\phi, m, v) := \sum_{w \leq (m) \ast v} \{ \ell \}(\phi^{(w)} \ast \text{tail}(w)) \]

where \( \text{tail}(w) := ((w)_1, \ldots, (w)_{\text{1th}(w)-1}) \).

**Basis.** \( \{ p \}(\phi, m, v) = 0. \) Then for every \( w \leq (m) \ast v \) \( \{ \ell \}(\phi^{(w)} \ast \text{tail}(w)) = 0 \), hence \( \text{Clear}(\phi, (m) \ast v) \) and so \( \text{Clear}(\phi^m, v) \). So we have

\[ \{ n \}(\phi, (m) \ast v) :\Rightarrow \phi((m) \ast v) \]

\[ = \phi^m(v) \]

\[ = \{ n \}(\phi, (m) \ast v). \]

**Ind. Step.** Let \( \{ p \}(\phi, m, v) > 0. \) Then there is at least one \( w \leq (m) \ast v \) s.t.

\( \neg \text{Clear}(\phi^{(w)} \ast \text{tail}(w)) \) and so \( \neg \text{Clear}^+(\phi^m, v) \). Hence, \( \phi \models^+ \phi_1 \); where the reduction occurs at some node \( u \) which influences \( w \), and

\[
\begin{align*}
\phi_1(q) = \phi(q) & \quad \text{for } q \neq (u)_0 = (w)_0 \\
\phi((w)_0)^{\downarrow} & \models^+ \phi_1 \quad \text{at } \text{tail}(u)
\end{align*}
\]
Since clearly \((\{k\}(\phi_{\lambda}),\text{tail}(w)) \cong \text{tail}(u)\) we have 
\[\{\ell\}(\phi_{1,\lambda}),\text{tail}(w)) \cong \ell(\phi_{\lambda}),\text{tail}(u)\] 
\(\ell(\phi_{\lambda}),\text{tail}(w)) = 1\) and so

\[\{p\}(\phi_{1,\lambda},m,v) < \{p\}(\phi_{\lambda},m,v)\]

while (as in [a]) we conclude by 3.4.7 from the assumed \text{Clear}(\phi_{\lambda},\ell)\) that \text{Clear}(\phi_{1,\lambda}).

Hence

\[\{n\}(\phi_{1,m},v) = \{n\}(\phi_{\lambda},m),v)\] 
by ind.hyp.

\[\{n\}(\phi_{1,m},v)\] 
by (5) and 3.4.5. \(\square\)

3.6.3 LEMMA (in \(V_0\)). Let \(\text{Der}^{\omega}(\phi)\).

(i) If \(\rho_{\phi},\ell\) is \([-1]\]

\[\phi^{(0)}\]

\[\phi = \frac{\text{St}(\rho_{\phi}),\text{St}(\ell)}{\text{St}(\rho_{\phi}),\text{St}(\ell)}\]

say, and \(\phi^{(0)}\) is stable at \(k\) then \(\text{St}(\phi)\).
(ii) If \( \rho^{\phi}(\psi) \) is a non-critical inference other than \([\rightarrow I]\), and \( \forall m \text{ St}(\phi^{<m}) \) then \( \text{St}(\psi) \).

**PROOF.** Let \( \phi \models^n \psi \); we have to show \( \text{Nable}(\psi) \).

**Case [a].** \( n = 0, \psi = \phi \). In any case we assume here \( \forall m \text{ St}(\phi^{<m}) \) and so by 3.5.4 \( \forall m \text{ Nable}(\phi^{<m}) \). By 3.6.2 this implies \( \text{Nable}(\phi) \).

**Case [b].** \( n = m+1, \rho^{\phi} = \{1\} \). Here \( \rho^{\phi} \) is not critical, so \( \text{Clear}(\phi,\psi) \) and hence \( \{k\}(\psi,\psi) = 0 \); thus necessarily \( \psi \models^1 \psi \) by an improper reduction. When \( \rho^{\phi} \) is \([R]\) such a reduction is impossible. We are then left with the following cases.

- If \( \rho^{\phi} \) is \([\rightarrow I]\) as in (i) above, then

\[
\psi \models^k \phi \models^1 \text{ for some stable } \psi , \text{ and thus necessarily } \psi \models \psi \text{ by an improper reduction. When } \rho^{\phi} \text{ is } [R] \text{ such a reduction is impossible. We are then left with the following cases.}
\]

1. If \( \rho^{\phi} \) is \([\rightarrow I]\) as in (i) above, then

\[
\psi \models^k \phi \models^1 \quad \text{for some stable } \psi , \text{ and thus necessarily } \psi \models \psi \text{ by an improper reduction. When } \rho^{\phi} \text{ is } [R] \text{ such a reduction is impossible.}
\]

/* 3.7.1. \( \psi \) is stable at \( kA \) under \( \text{A } \rightarrow A \lor B \) if whenever

\[
\psi \models^b \text{ [VI] } b \rightarrow A \lor B
\]

then \( [kA] \) is stable. (A symmetric definition for \( \psi = \text{B } \lor A \).

\( \phi \) is stable at \( kA(t) \) under \( \text{A } \rightarrow \exists x A(x) \) if whenever

\[
\psi \models^b \text{ [EI] } b \rightarrow \exists x A(x) , \quad t = \bar{n}
\]

and
\[
\begin{align*}
\xi \\
\theta &:= \frac{b \Rightarrow A(t)}{[R] \ b \Rightarrow A(n)}
\end{align*}
\]

* then \(\langle k A(n) \rangle\) is stable.

3.7.2. MAIN LEMMA. Let \(\psi\) be a derivation of \(A^\omega\) which satisfies one of (i)-(iii) below; then \(St(\psi)\).

(i) \(\rho \psi, \langle i \rangle\) is a critical inference rule other than \([vE]\) and \([\exists E]\) (i.e. \([-A], [\&E], [\&E], [\exists E]\) or \([IE]\) and \(\bigwedge_{i=0,1} St(\psi, \langle i \rangle)\).

(ii) \(\rho \psi, \langle i \rangle\) is \([vE]\), \(\psi, \langle 0 \rangle\) \(\equiv: A_1 \lor A_2\), \(\bigwedge_{i=0,1} \text{Nmb}(\psi, \langle i \rangle)\), and \(\psi, \langle j \rangle\) is stable at \(k A_1\) under \(\langle 0 \rangle\) (\(j=1,2\)).

(iii) \(\rho \psi, \langle i \rangle\) is \([\exists E]\), \(\psi, \langle 0 \rangle\) \(\equiv: \exists n A(n)\), and for each \(m < \omega\) \(\text{Nmb}(\psi, \langle m \rangle)\).

* and \(\psi^\langle m+1 \rangle\) is stable at \(k A(m)\) under \(\langle 0 \rangle\).

PROOF. (I) Method: the proof proceeds by a primary BI on the tree \(\{n\}^\langle 0 \rangle\). I.e., formally speaking we prove

(1) \(\forall m \ Q(\psi, m) \rightarrow Q(\psi, 0)\)

where

\[Q(\psi, u) := \forall v \left[ \text{Der}\langle v \rangle \& \text{Nmb}(\psi, \langle 0 \rangle) \& \right.\]

\[\left. \text{"}[n]^\langle 0 \rangle\text{ is a subtree of }\{n\}^\langle 0 \rangle\text{"} \right.\]

\[\rightarrow \text{"the lemma holds for }v\text{"} \].\]

Assuming \(WF([n]^\langle 0 \rangle)\), as we do by the lemma's assumptions, we get from (1) by BI \(Q(\psi, \langle 0 \rangle)\), which trivially implies that the lemma holds for \(\psi\).

Further, we shall use a secondary (ordinary) induction on \(\langle \xi \rangle(\psi, \langle 0 \rangle, \langle 0 \rangle)\), i.e., we prove \(\forall m R(\psi, m)\) where

\[R(\psi, m) := \forall v \left[ \text{Der}\langle v \rangle \& \text{"}[n]^\langle 0 \rangle\text{ is a subtree of }\{n\}^\langle 0 \rangle\text{"} \& \right.\]

\[\left. \{\xi\}(\psi, \langle 0 \rangle, \langle 0 \rangle) \otimes m \rightarrow \text{"the lemma holds for }v\text{"} \right.\].\]

Now assuming \(\langle \xi \rangle(\psi, \langle 0 \rangle, \langle 0 \rangle)\) as we do by the lemma's assumptions, we get trivially from \(R(\psi, \xi)(\psi, \langle 0 \rangle, \langle 0 \rangle)\) that the lemma holds for \(\psi\).
(II) A preliminary observation: If \( \phi \) satisfies one of (i)-(iii), and \( \phi \upharpoonright u \vdash \phi_1 \) by a reduction at some \( (0) \rightarrow u \leftrightarrow \{k\}(\psi,()) \), then \( \phi_1 \) satisfies the same condition as \( \phi \) does. The proof is immediate from the definitions.

(III) Let \( \phi \) satisfy one of (i)-(iii), \( \phi \upharpoonright u_1 \upharpoonright u_2 \upharpoonright \cdots \upharpoonright u_n \). We have to prove \( \text{Noble}(\phi_1) \).

If \( n = 0 \) and \( \text{Clear}(\phi,()) \), then \( \text{Noble}(\phi) \) by 3.6.2(i). If \( \neg\text{Clear}(\phi,()) \), \( \{k\}(\psi,()) \approx () \), then \( \phi \upharpoonright \psi \) for some \( \psi \), and we shall see in (IV) below that then \( \text{St}(\psi) \); hence by 3.4.4/5 \( \text{Noble}(\phi) \). Finally, if \( \{k\}(\psi,()) \approx (0) + u \) then \( \phi \upharpoonright \psi \) with \( \psi \) satisfying the lemma's conditions by (II) above, and \( \phi(0) \text{ } \upharpoonright \psi(0) \) where \( u \approx \{k\}(\psi(0),()) \). So \( \{n\}(\psi(0)) = \{n\}(\phi(0)) \) while \( \{\ell\}(\phi(0),()) \approx \ell(\psi(0),()) + 1 \). Hence by the secondary ind.hyp. applied to \( \psi \), \( \text{St}(\phi) \) and so (3.4.4/5) \( \text{Noble}(\phi) \).

Next, if \( n > 0 \), it obviously suffices to prove \( \text{St}(\phi_1) \). If \( \phi \upharpoonright \psi \) we have \( \text{St}(\phi_1) \) as above.

If \( \phi \upharpoonright \psi \) by an improper reduction then \( \rho_{\phi,()} \) must be one of \([31], [vE], [3E]\). For the first one, \( \phi_1 = \phi(0) \) which is assumed stable (case (i) of the lemma); for the last two, \( \phi_1 = \phi(m+1) \) for some \( m < \omega \), which is assumed stable (cases (ii), (iii)). So \( \text{St}(\phi_1) \) in any case.

(IV) It remains to prove that whenever \( \phi \upharpoonright \psi \) and \( () \approx \{k\}(\psi,()) \) then \( \text{St}(\phi) \). We inspect cases for the type of reduction, i.e. - for \( \rho_{\phi,()} \) and \( \rho_{\phi,()} \).

\text{case (a).} \( \rho_{\phi,()} \) is \( \rightarrow \),

\text{subcase (aa).} \( \rho_{\phi,()} \) is \( \rightarrow \).

\[ \begin{align*}
\phi &= \frac{\theta_1}{\varphi(0,0)} \\
\phi(1) &= \frac{\varphi(0,0)}{a \Rightarrow A \Rightarrow B} \\
\phi(1) &= \frac{\varphi(0,0)}{a \Rightarrow A} \\
\phi(1) &= \frac{\varphi(0,0)}{a \Rightarrow B}
\end{align*} \]
Then, since $St(\phi^{(1)})$,

$$\phi^{(0)} \models_I \psi$$

by an improper reduction; and as $St(\phi^{(0)})$ by assumption, $St(\psi)$.

/* subcase (a6). $\phi^{(0)}$ is [vE];

$\phi^{(0,0)}$

\[
\begin{array}{c}
\phi^{(0,j)} \\
\phi^{(0,0)} \\
\text{[vE]} \\
\phi^{(0,j)} \\
\phi^{(0,0)}
\end{array}
\]

\[
\begin{array}{c}
\Rightarrow A_1 \lor A_2 \\
\Rightarrow B \Rightarrow C \\
\Rightarrow A_j \Rightarrow B \\
\Rightarrow A_j \Rightarrow C \\
\Rightarrow A_j \Rightarrow B \\
\Rightarrow A_j \Rightarrow C \\
\end{array}
\]

\[
\begin{array}{c}
\Rightarrow a \Rightarrow B \Rightarrow C \\
\Rightarrow a \Rightarrow B \\
\Rightarrow a \Rightarrow C \\
\Rightarrow a \Rightarrow C \\
\Rightarrow a \Rightarrow C \\
\Rightarrow a \Rightarrow C \\
\end{array}
\]

$\Rightarrow a \Rightarrow B$ (0,0) $\Rightarrow a \Rightarrow B$

\[
\begin{array}{c}
\Rightarrow a \Rightarrow B \Rightarrow C \\
\Rightarrow a \Rightarrow B \\
\Rightarrow a \Rightarrow C \\
\Rightarrow a \Rightarrow C \\
\Rightarrow a \Rightarrow C \\
\Rightarrow a \Rightarrow C \\
\end{array}
\]

$\Rightarrow a \Rightarrow B$

$\Rightarrow a \Rightarrow B$

$\Rightarrow a \Rightarrow C$

$\Rightarrow a \Rightarrow C$

$\Rightarrow a \Rightarrow C$

We assume $(\phi^{(1)}, (1)) = (0)$, and so $\text{Clear}(\phi^{(0)}, (1))$. Hence, by 3.6.2, $St(\phi^{(0)})$ implies $\text{Nmble}(\phi^{(0,0)})$, and $(n)^\phi (0,0) = (n)^\phi (0,0)$ is a proper subtree of $(n)^\phi$. So $\psi$ has a "lower" BI measure than $\phi$, and to conclude that $St(\phi)$ it remains to check that $\phi$ satisfies case (ii) of the lemma.

We have found that $\text{Nmble}(\psi^{(0)})$. We check next that $\text{Nmble}(\psi^{(j)}) (j=1,2)$, also by BI. $\psi^{(j,1)} := \phi^{(1)}, A_j$ is stable by assumption, since $\phi^{(1)}, A_j$ and $\phi^{(1)}$ behave in the same way for all properties concerning reductions. Finally,

$$\phi^{(0)} \models_I \phi^{(0,j)} = \psi^{(j,0)}$$

by a simplification reduction; so $St(\phi^{(0)})$ implies $St(\psi^{(j,0)})$. Hence $\psi^{(j)}$ satisfies case (i) of the lemma. Since $(n)^\phi (j,0) = (n)^\phi (0,0)$ is a proper
subtree of \((n)_0^{(0)}\), we get by BI hyp. \(\text{St}(\psi^{(j)})\) and so \(\text{Nmb}(\psi^{(j)})\) \((j=1,2)\).

It remains to verify that \(\psi^{(j)}\) is stable at \(k\) under \(\psi^{(0)}\). So assume

\[
(1) \quad \psi^{(0)} = \phi^{(0,0)} \parallel \xi \quad a \rightarrow A_j
\]

\[\Rightarrow \quad a \rightarrow A_1 \lor A_2\]

If \(n = 0\), then \(\text{Cut}(\psi^{(0)},())\), so \(\{k\}(\phi,()) \neq ()\), contradicting the assumption of (IV). Else, and

\[
\phi^{(0,0)} \equiv \theta, \quad u = \{k\}(\phi^{(0,0)},())
\]

then

\[
\{k\}(\phi,()) \neq (0,0) \equiv u \neq ()
\]

again a contradiction.

Finally, if \(\phi^{(0,0)} \parallel \emptyset\) by an improper reduction, then this must be a simplification, because reduction (I) preserves the derived formula.

So \(\rho^\phi, (0,0)\) is \([\text{vE}]\) or \([\text{AE}]\), and as \(\rho^\phi, (0)\) is \([\text{AE}]\), we thus have \(\text{Cut}(\phi, (0,0))\), contradiction \(\{k\}(\phi, ()) \neq ()\) once again. Hence (I) is simply impossible under the inspected conditions.

*/ \text{subcase (aV)} \rho^\phi, (0) \text{ is } [\text{AE}]. \text{ Similar to (aS).}

\text{subcase (aS)} \rho^\phi, (0) \text{ is } [\text{L}].

\[
\phi^{(0,0)} \equiv \phi^{(1)} \quad a \rightarrow 1 \\
\phi^\text{[L]} \quad a \rightarrow A \lor B \\
\phi^\text{[AE]} \quad a \rightarrow A \\
\phi^\text{[L]} \quad a \rightarrow B
\]

\[
\Rightarrow \quad \psi
\]
Here $\text{St}(\phi(0))$ is assumed, while $\phi(0) \models \phi(0,0)$ by an improper reduction, so $\text{St}(\phi(0,0))$ outright from the definition of stability.

**Case (b).** $\rho_{\phi(\cdot)}$ is $[\&E]$ — similar to case (a).

**Case (c).** $\rho_{\phi(\cdot)}$ is $[\vee E]$.

**Subcase (ca).** $\rho_{\phi(\cdot)}$ is $[\vee I]$.

\[
\phi = \frac{\phi(0,m)}{[\vee I] \quad a \Rightarrow \forall(x) \quad \phi(0,m)}
\]

\[
\phi(0,m) = \phi(a \Rightarrow A(m)) \quad m < 0
\]

where $m_0$ is the value of $m$. Here $\text{St}(\phi(0))$ by assumption, and $\phi(0) \models \phi(0,m_0) = \phi(0)$. So $\text{St}(\phi(0))$ and by 3.6.3(ii) $\text{St}(\phi)$.

Other subcases of (c) are treated like the analogous subcases of (a).

**Case (d).** $\rho_{\phi(\cdot)}$ is $[\&E]$ or $[\exists I]$; the proof is as for (a8) — (a6).

/* **Case (e).** $\rho_{\phi(\cdot)}$ is $[\vee E]$.

**Subcase (ea).** $\rho_{\phi(\cdot)}$ is $[\vee I]$.

\[
\phi(0,0)
\]

\[
\phi = \frac{\phi(j)}{[\vee E_k] \quad a \Rightarrow B}
\]

\[
\phi(0,0)
\]

Then $\text{St}(\psi)$ outright from the statement of condition (ii).
subcase (a8). $\phi^{(0,0)}$ is $[vE_k]$. 

$$
\frac{\phi^{(0,0)}}{[vE_k]} \frac{\phi^{(0,i)}}{[vE_k]} \frac{\phi^{(0,i)}}{[vE_k]} \frac{\phi^{(j)}}{[vE_k]} \frac{\phi^{(0,i)}}{[vE_k]} \frac{\phi^{(j)}}{[vE_k]} =: \psi
$$

and recall that we assume $k(\phi, \psi) = \phi$. We find that the BI measure of $\psi$ is lower than that of $\phi$ as in (a8), and that $\text{Nuble}(\psi^{(i,j)})$ ($i=1,2; j=0,1,2$) and that the induction measure of $\psi^{(i)}$ is lower than that of $\phi$ ($i=1,2$). To conclude that $\psi^{(i,j)}$ is stable at $[kA_j]$ under $\psi^{(i,j)}$ ($i=1,2; j=1,2$). I.e., we have to verify that when

$$
\psi^{(i,0)} = \phi^{(0,i)} \models \frac{a \land b_1 \Rightarrow A_j}{[vE_k]}
$$

then

$$
\frac{\psi^{(i,0)}}{[kA_j]}
$$

is stable ($i,j=1,2$). But $\phi^{(0)} \models \phi^{(0,1)}$ by a simplification, so with (1) we have $\phi^{(0)} \models \psi^{(i,0)} = \psi^{(0,1)}$, and so, since $\psi$ satisfies case (ii) of the lemma, $\psi^{(i)}$ is stable as required. Now we can apply BI hyp. to $\psi^{(i)}$, and

$$
\psi^{(i,j)}[\ell B_i]
$$
find that $\text{St}(\psi^{(i)})$ $(i=1,2)$.

We conclude from this $\text{St}(\psi)$ as in (a8) (i.e., as in (a8) the stability of $\psi^{(i)}$ at $k_B$ under $\psi^{(0)}$ is satisfied vacuously because $(i)$ $(\psi^{(i)}) = (\psi^{(0)})$.

subcases (e5), (e6). $\varphi^{(0)}$ is [HE] or [L]. Analogous to (e6) (compare (a6)).

Case (f). $\varphi^{(0)}$ is [HE] - like case (e), mutatis mutandis (compare also * case (c) for the use of [R]). □

3.7.3. Lemma (in $V_0 + B_1$). Let $\text{Der}(\psi)$; if $\forall m \text{ SST}(\psi^{(m)})$ then $\text{SST}(\psi)$.

Proof. [a] If $\psi$ is a singleton derivation (i.e., $\varphi^{(0)}$ is [T] or [TE]) then $\text{SST}(\psi)$ outright from the definition.

[b] If $\varphi^{(0)}$ is $[\neg I]$,

$$
\psi^{(0)} = \frac{\phi}{\varphi} \quad \iff \quad \phi_1 = \begin{cases} 
\psi_{i,n_1} \\
\frac{\varphi}{\psi} 
\end{cases}
$$

then by our convention on indexing $k_B \notin \varphi$, so $\psi_1^{(0)}$ is stable at $k_B$, and by 3.6.3(i) $\psi_1$ is stable.

[c] If $\varphi^{(0)}$ is a non-critical inference other than $[\neg I]$, then $\psi \iff \psi_1$ implies outright $\phi^{(m)} \iff \phi_1^{(m)}$, so $\forall m \text{ SST}(\psi^{(m)})$ implies $\forall m \text{ SST}(\psi_1^{(m)})$ and by 3.6.3(ii) $\text{SST}(\psi_1)$.

[d] If $\varphi^{(0)}$ is a critical inference other than [KE], [HE] we obtain that $\text{SST}(\psi)$ as in [c], using 3.7.2(i) in place of 3.6.3(ii).

[e] $\varphi^{(0)}$ is [KE] or [HE]; $\varphi \iff \psi_1$ implies $\phi^{(0)} \iff \psi^{(0)}_1$, so if $\psi^{(0)} \vdash \varphi$ then we get from $\text{SST}(\psi^{(0)})$ that $\text{SST}(\psi)$. Consequently, we find as in [b] above that $\psi_1$ satisfies the conditions of 3.7.2(ii) (respectively, 3.7.2(iii)), and so $\text{SST}(\psi_1)$. □

This concludes the proof of theorem 3.6.1.

3.8. The Subformula Property

For simplicity, we refer to $A^n$ as given in 1.1, i.e., without indexing and without the replacement rule [R].
Ad hoc definitions:

- $F$ sub of $G :\iff F$ is a subformula of $G$ or $F \equiv \top$.
- $F$ sub of $a :\iff F$ sub of some $G \in a$.
- $a$ sub of $b :\iff \text{every } F \in a \text{ is sub of } b$.
- $s_1 = \{F\} \text{ sub of } s_2 = \{G\} :\iff a \cup \{F\} \text{ sub of } b \cup \{G\}$.

**THEOREM** (subformula property). Let $\text{NDer}^\infty(\phi)$, then for every $u$ $\phi, u$ sub of $\phi, u, (0)$.

(Note that the theorem refers also to equations!)

**PROOF** (in $V_0$; BI is not used). The theorem is an immediate consequence (by ordinary induction on the codes of nodes) of

(i) if $u = v \land_1 m$ is a major premiss of an elimination rule, then $\phi, u$ sub $\phi, u, (0)$.

(ii) Otherwise, then $\phi, u$ sub $\phi, v$.

(ii) is clear by inspection of cases. If $\phi, u, (0)$,

\[
\begin{align*}
\text{if } & u \rightarrow a \rightarrow E \\
\text{or } & [\text{FE}] \quad a \rightarrow \top
\end{align*}
\]

say, then $\phi, u, (0)$ cannot be an introduction, since $E$ is an equation, and cannot be $[\top]$ or an elimination - by our definition of normality. So necessarily $\phi, u, (0)$ is $[\top]$, and so $E$ sub $\phi, u$ and (ii) is satisfied.

(i) is proved by induction on the length of the branch $u, u, (0), u, (0), 0, \ldots$ in $\phi$, which by $\text{WF}(\phi)$ must be finite. Since $\phi$ is normal, if $\phi, u, (0)$ is an elimination, then $\phi, u, (0)$ is either an elimination or $[\top]$, so by ind.hyp. (i) holds for $u, (0)$, i.e.,

\[(1) \quad \text{if } \phi, u, (0) \text{ sub of } \phi, u, (0) = \phi, u, (0) \quad \text{and } \quad \phi, u, (0) \text{ sub of } \phi, u, (0), \text{ so by (1) we have (i) for } u.
\]

(a) If $\phi, u$ is $[\land E]$ or $[\lor E]$ then $\phi, u = \phi, u, (0)$ and $\phi, u$ sub of $\phi, u, (0)$, so by (1) we have (i) for $u$.

(b) If $\phi, u$ is $[\land E]$, then $\phi, u = \phi, u, (0) = \phi, u, (0)$ while $\phi, u$ and $\phi, u, (1)$ sub of $\phi, u, (0)$, so by (1) we are done.
(c) If $\phi^u$ is $[\forall \mathbf{E}]$ or $[\exists \mathbf{E}]$, then $s^\phi^u(\mathbb{E}+1)$ sub of $s^\phi^u(0)$, and $s^\phi^u$ sub of $s^\phi^u(1)$; so by (1) $u$ satisfies (i). \[\square\]

3.9. THE DISJUNCTION INSTANTIATION PROPERTY

**Proposition.** If $\vdash A^* A_1 \lor A_2$ (where $A^*$ is $A^w$, $A^w(T)$ or $A^w_{rec}[T]$ for any $T \geq A$) then either $\vdash A^* A_1$ or $\vdash A^* A_2$.

**Proof.** By the normalization theorem (3.4.8, 3.6.1) if $\Pr^w\phi,A_1 \lor A_2$ then for some $\phi$ $\Pr^w\phi,A_1 \lor A_2$. Inspection on the cases for $\phi^u$ shows that the only possible case for this inference is $[\forall \mathbf{E}]$. So $\Pr^w\phi, A_{1}^{i}$ for $i=1$ or 2. \[\square\]
A.4. NORMALIZATION IN \( L_\omega^\infty \); \( L_\omega \) IS REGULAR

4.0. As explained in Int. 6, the chief raison d'être of the departure of the normalization proof in A.3 from the traditional method of ordinal assignments is our wish to smoothly extend the proof to higher order systems: the "abstract" argument used in the proof may be adapted to the infinitary system \( L_\omega^\infty \) which combines \( L_\omega^\infty \) with the language and the inference rules of simple type theory \( L_\omega \). For the reader familiar with GIRARD [71],[72] it is probably clear by now how to combine Girard's proof of normalization of \( L_\omega^\infty \) with the argument of A.3 so as to get a normalization proof for \( L_\omega^\infty \). The more sceptic reader might like some details, and as a compromise we give below a somewhat detailed indication of the proof for the system \( L_\omega^\infty \), which combines \( L_\omega^\infty \) with the language and rules of the theory of species \( L_\omega^\infty \). This should make it clear that the notions of A.3, though applying to infinitary proof figures, may be combined without further ado with Girard's proof for the corresponding finitary systems. A detailed normalization proof for \( L_\omega^\infty \) may then be easily supplied by the patient reader.

The main consequence of the proof is that type theory \( L_\omega \) (and ipso facto also the theory of species \( L_\omega^\infty \)) are regular theories, namely

\[
A[L_\omega] \leq A[\omega \text{ rec }[L_\omega + BI] 
\]

(for refinements cf. LEIVANT [A]). Assuming that \( L_\omega + BI + AC_0^\omega \) is consistent (cf. A.1) we have that \( L_\omega \) is also strongly regular. This last assumption is an immediate consequence of the consistency of \( L_\omega^C + AC_0^\omega \).

As indicated in TN4, another corollary of the normalization of \( L_\omega^\infty \) is that \( L_\omega \) is conservative over \( A \) extended with the schema of transfinite induction over each well-founded p.r. ordering.
4.1. DESCRIPTION OF $L^m_2$

The language of $L^m_2$ is the language of $A$ extended with variables and parameters for species of $n$-tuples, \( \{X^n_i\}_{i<n}, \{P^1_i\}_{i<n}, n<\omega \), and corresponding universal quantification $\forall X^n_1$.

When $V^n$ is a variable or a parameter, $F$ is a formula, $G$ is a pseudo-formula (i.e. some first order variables possibly occur unbounded in $G$) and $x$ is an $n$-tuple of first-order variables, we write $F[G/x]/V$ for the (pseudo-) formula which comes from $F$ by substituting (simultaneously) $G[1/x]$ for every occurrence $P(t)$ in $F$. Usually no confusion occurs if we skip $x$, and so we shall do below.

The inference rules of $L_2$ are those of $A^\omega$, with the addition of the second order quantification rules:

\[
[V^2_1] \quad \begin{array}{c}
\frac{a \Rightarrow A}{a \Rightarrow \forall X A[X/P]} \\
\end{array}
\]

where $P$ does not occur in $a$ (and $X, P$ are of the same type).

\[
[V^2_2] \quad \begin{array}{c}
\frac{a \Rightarrow \forall X A}{a \Rightarrow A[G/X]} \\
\end{array}
\]

Derivations and recursive derivations are defined now as for $A^\omega$. We shall use the notations $\text{Der}^\omega$, $\text{Der}^\omega_{\text{rec}}$, etc. in this chapter for derivations of $L^m_2$.

Without loss of generality we assume that each derivation satisfies the convention on parameters of PRAWITZ [71] 1.2.4, i.e. that no parameter which occurs in the derived sequent of a derivation $\phi$ is the proper parameter of any $[V^2_1]$-inference, and that the same parameter is not the proper parameter of two distinct $[V^2_1]$-inferences. Note that any given derivation $\phi$ may be made to conform to this convention by replacing any occurrence of a parameter $p$ by $P^j_{p^u}$, where $u$ is the code of the node for which that occurrence acts as the proper parameter when there is such a node (else $u := 0$), and where $j$ is the largest index of the parameters which occur in the derived sequent of $\phi$. 
4.2. PROPER REDUCTIONS; NORMALIZATION

The critical inference rules of $L^m_2$ are those of $A^m$ plus $[$V$^2$E$]$. Cuts are defined accordingly as in 3.3.1.

Reductions $\vdash$, $\models$, $\models$ etc. on derivations are defined as in 3.3, with the addition of permutative- and absurdity-reductions with $[$V$^2$E$]$ as the lower (critical) inference rule, and of

\[ \phi (0,0) \]

$V^2$-reductions:

\[ \frac{a \Rightarrow F[P/X]}{[V^2 1]} a \Rightarrow \forall X F \]

\[ \frac{a \Rightarrow F[G/X]}{[V^2 2]} a \Rightarrow F[G/X] \]

Other functions and predicates defined in 3.3 are now adapted to $L^m_2$ by taking into account the above modifications.

4.3. BASES, BASING FUNCTIONS

A basis is a set $B$ of derivations (of $L^m_2$) which satisfies:

$\phi \in B$ and $\phi \mid_1 \phi_1 \vdash \phi \models B$.

A basing function is a finite function $\beta$ from the second order parameters of the language to bases. We write $\beta = \{(P_i, B_i)\}_{i=1,...,k}$ also as

\[ \begin{bmatrix} B_1 \\ P_1 \end{bmatrix} \ldots \begin{bmatrix} B_k \\ P_k \end{bmatrix} \).

This notation makes it easy to denote an extension of a given basing function. (A basing function does not range over occurrences of parameters: our convention on parameters makes this unnecessary.)

Given a formula $F$ and a basing function $\beta$, the basing function $(\beta F)$ is defined to be the restriction of $\beta$ to parameters occurring in $F$. 
4.4. GENERALIZED REDUCTIONS; STABILITY

The measure \( \mu \) on formulae of \( L^n_2 \) is defined as in 3.5.1, with the additional clause:

\[
\mu(\forall x A) := \mu(A).
\]

We refer to triplets \((\phi, \beta, F)\), where \( \phi \) is a derivation, \( \beta \) a basing function and \( F \) a formula, s.t. \( |\phi| := F^\phi,() \) is a substitution instance \( F^\beta \) of \( F \), and where \( \beta = (\beta \cdot F) \) (\( F \) may be thought of as the "skeleton" of \( \phi \)).

We define by metamathematical recursion the predicates \( \models \) and \( \text{St}_n \) over the triplets satisfying \( \mu(F) \leq n \) (\( n \geq 0 \)). The nature of this recursion is the same as in 3.5.2, and we shall omit the index \( n \)

(i) If \( \phi \upharpoonright_1 \bar{\phi}_1 \) where \( u := \{k(\phi,()) \neq 0 \) then

\[
(\phi, \beta, F) \models (\phi_1, \beta, F).
\]

(ii) If \( \text{St}(\phi, \beta, F) \), \( |\psi| = F^* \) then

\[
\frac{\phi(0)}{\text{[\(\forall I\)]}} a \Rightarrow F^* \rightarrow G^*, \beta, F \rightarrow G \]

\[
\frac{\psi}{\text{[\(\forall I\)]}} a \Rightarrow \forall x F^*, \beta, \forall x F
\]

\[
\frac{\phi(0)[P]}{\text{[\(\forall I\)]}} a \Rightarrow \forall x F^*, \beta, \forall x F
\]

\[
\frac{\psi}{\text{[\(\forall I\)]}} \phi(0)[G/P], \beta, \forall x F
\]

where \( G \) is any formula, \( P \) is the proper parameter of the main inference of \( \phi \) and \( \beta \) is any basis.
Other improper reductions are adapted from the corresponding reductions in 3.5.3 in the same manner.

$\models$ is the transitive closure of $\models^1$.

$$\text{St}(\phi, \beta, F) := \forall \phi_1, \beta_1, F_1 \{ \langle \phi, \beta, F \rangle \models \langle \phi_1, \beta_1, F_1 \rangle \rightarrow M(\phi_1, \beta_1, F_1) \}$$

where $M(\phi_1, \beta_1, F_1)$ is the formal predicate expressing

$$\text{Nmble}(\phi_1) \land [ F_1 \equiv P(\cdot) \rightarrow \phi_1 \in \beta_1(P) ] .$$

Here, if $P \not\in \text{Domain}(\beta)$ we let $\phi_1 \in \beta_1(P)$ be definitionally true.

\(\phi\) is strongly stable (notation: $\text{SSt}(\phi)$) if for every derivation $\phi^*$ which comes from $\phi$ by substituting (pseudo-)formulae for parameters, and for every basis function $\beta$, if

$$\psi_i^* := \{ [ n_i^1 A_i ] \}_{i \in I}$$ (like in 3.5.5)

where $\text{St}(\psi_i^*, (\beta^i A_i), A_i)$, $n_i^1 A_i \in a^\phi, i \{ i \in I \}$ then $\text{St}(\psi^* i, (\beta^i | \phi |), | \phi |)$.

4.5. LEMMA. Let $\text{Der}^\text{m}(\phi)$.

(i) If $\text{SSt}(\phi)$ then $\text{St}(\phi, (\beta^i | \phi |), | \phi |)$ for any basis function $\beta$.

(ii) If $\text{St}(\phi, \beta, F)$ for some $\beta, F$, then $\text{Nmble}(\phi)$.

PROOF. (i) Let, in the definition of $\text{SSt}(\phi)$ above, $\phi^* := \phi$ and $\phi^{*'} := \phi^*$ by taking

$$\psi_i := \{ [ T ] a^{\phi^*}, i \rightarrow A_i \}$$

(singleton derivations).

(ii) Take $\models$ of length 0 in the definition of $\text{St}(\phi, \beta, F)$. □
4.6.1. **Lemma.** Given a basing function \( \beta \) and a formula \( F \), let

\[
[\beta,F] := \{ \phi \mid "\phi" \text{ is a substitution instance of } F \}
\]

& \( \text{St}(\phi,(\beta)F),F ) \}.

Then \([\beta,F] \) is a basis.

**Proof.** Immediate from the definitions. \( \square \)

4.6.2. **Substitution Lemma.** Let \( P \) not occur in \( G \).

\[
\text{St}(\phi,\beta,F[G/P]) \leftrightarrow \text{St}(\phi,\beta[\beta[G/P]],F)
\]

(or, put differently,

\[
[\beta,F[G/P]] = [\beta[\beta[G/P]],F] \).

**Proof.** By induction on \( \mu(F) \).

I. Assume

(1) \( \text{St}(\phi,\beta^*,F) \) where \( \beta^* := \beta[\beta[G/P]] \).

If \( F \equiv P \), \( F[G/P] \equiv G \), then (1) implies \( \phi \in \beta^*(P) = [\beta,G] \), and so \( \text{St}(\phi,\beta,F[G/P]) \) outright.

If \( F \not\equiv P \), let

\[
\langle \phi,\beta,F[G/P] \rangle \upharpoonright^k \langle \psi,\gamma,H \rangle.
\]

We prove by (a second) induction on \( k \) that

\[
M(\psi,\gamma,H) := \text{Nbble}(\psi) \& [ H \equiv Q(c) \rightarrow \psi \in \gamma(Q) ].
\]

**Base.** \( k = 0 \), \( \psi = \phi \), and so \( \text{Nbble}(\psi) \) by (1) and 4.5. Further, if \( H \equiv F[G/P] \equiv Q(c) \) then \( F \not\equiv P \) implies \( F \equiv Q(c) \), and so

\[
\psi = \phi \in \beta^*(Q) \quad \text{by (1)}
\]

\[
=: \beta(Q) = \gamma(Q) \quad \text{since } Q \not\equiv P.
\]
Ind. step. Let

$$\langle \phi, \beta, F[G/P] \rangle \models^k \langle \phi_1, \beta_1, F_1[G/P] \rangle$$

and inspect cases for the first reduction. The only non-trivial cases are

the following

[a] \([+\lambda]\)-reduction; \( F = A \rightarrow B, \)

$$\forall x \phi_1 = [^kA[G/P]], \quad \beta_1 = \beta, \quad F_1 = B \quad \phi(\lambda)$$

where \( \text{St}(\chi, \beta, A[G/P]). \) But \( \mu(A) < \mu(F), \) and so by the first ind.hyp.

\( \text{St}(\chi, \beta^*, A). \) Hence

$$\langle \phi, \beta^*, F \rangle \models^1 \langle \phi_1, \beta^*, F_1 \rangle,$$

and so by (1) \( \text{St}(\phi_1, \beta^*, F_1). \) We may therefore apply the second ind.hyp.

to \( \langle \phi_1, \beta_1, F_1[G/P] \rangle \) and conclude \( M(\psi, \gamma, H). \)

[b] \([\forall \lambda]\)-reduction; \( F = \forall \lambda A \)

$$\phi_1 = \phi(\lambda)[D/Q], \quad \beta_1 = \beta[Q], \quad F_1 = A[Q/X].$$

So

$$\langle \phi, \beta^*, F \rangle \models^1 \langle \phi_1, \beta^*, F_1 \rangle,$$

and so by (1)

$$\text{St}(\phi_1, \beta[D/Q][Q^2, F_1]).$$

In order to apply ind.hyp. to \( \langle \phi_1, \beta_1, F_1[G/P] \rangle \) we have to know however that

(1) applies, i.e., that

$$\text{St}(\phi_1, \beta_1^*, F)$$

where \( \beta_1^* = \beta[Q^2, F_1]. \)
But \((\beta_1 G) = (\beta G)\) because \(P\) does not occur in \(G\), and so

\([\beta_1 G] = [\beta G]\);

hence (2) implies (3), as required.

II. Assume

\[(4) \quad \text{St}(\phi, \beta, F[G/P])\]

and let

\[\langle \phi, \beta^*, F \rangle \mid \tau^k \langle \psi, \gamma, H \rangle.\]

As in I above we prove that \(M(\psi, \gamma, H)\) by induction on \(k\). The induction step is symmetric to that in I, while for the induction basis we note that when \(F \in P\), \(F[G/P] \in G\), then

\[\phi \in \beta^*(P) := [\beta, G]\]

by (4). \(\square\)

The reader might note, in connection to the proof above, that GIRARD's [72] proof of the substitution lemma is not quite accurate: the application of the ind.hyp. given at bottom p.II.1.6 should yield \([\alpha, \gamma/A, G]\) within the l.h.s., in place of \([\alpha, A] G\).

4.7. LEMMA. Let \(\langle \phi, \beta, F \rangle\) be a triplet as above, and \(\phi^{(1)}\) be a non-critical inference. If \(\forall m \in \text{Stt}(\phi^{(1)})\) then \(\text{Stt}(\phi)\).

PROOF. The proof is totally analogous to that of 3.6.3. To take as an example the only essentially new case, let \(\phi^{(1)}\) be \([\forall P]_1\),

\[\phi^{(1)}[P]
\]

\[\phi = \frac{a \to F[P/X]}{a \to \forall X F}, \quad a = \{A_1\}^{1}_{1}\]
and let $\phi^{**}$ come from $\phi$ as in 4.4, i.e

$$\phi^{**} = \mathcal{M} \gamma_{\lambda} \sum_{i=1}^{n} [\lambda_{i} A]_{i} \in \phi^*.$$  

In analogy to the proof of 3.6.3, we have to show that $\text{St}(\xi, \beta, F[P/X])$ for any basing function $S$, and where

$$\xi := (\phi^{**})_0 [G]$$

for some formula $G$. $P$ cannot occur in $a$ and therefore, it does not occur in any $\psi_i$ (by the convention on parameters). Hence $\xi$ may be obtained from $\phi^{(0)}_0$ by first substituting $G$ for $P$ and then substituting the derivations $\psi_i$. Since $\text{SSt}(\phi^{(0)}_0)$ is assumed we thus get $\text{St}(\xi, \beta, F[P/X])$ as required. \(\square\)

4.8. LEMMA. Let $(\phi, \beta, F)$ be a triplet as above, $\rho \phi, (\cdot)$ be a critical node. If $\forall m \text{SSt}(\phi^{(m)})$ then $\text{SSt}(\phi)$.

PROOF. Here again the proof is analogous to the proof in 3.7.2-3 for the first order case. Since $\rho \phi, (\cdot)$ is not $[\forall^2 I]$, 

$$(\phi, \beta, F) \models^1 (\phi_1, \beta_1, F_1)$$

must imply that $\beta_1 = \beta$, $F_1 \equiv F$; hence the proof in 3.7.2 for derivations is trivially adapted to triplets. The only exception is the case $\rho \phi, (\cdot) = [\forall^2 E]$, where we have to show (in step (IV)) that if

$$(\phi, \beta, F[G/X]) \models^1 (\phi_1, \beta_1, F[G/X])$$

where

$$\phi = \left\{ \begin{array}{ll}
\xi[P] \\
\frac{a \Rightarrow F^*[P/X]}{a \Rightarrow F^*[G^*/X]} \\
\frac{\forall x F^*}{a \Rightarrow F^*[G^*/X]} \\
\frac{\forall x F}{a \Rightarrow F^*[G^*/X]}
\end{array} \right\}^n \Rightarrow \frac{\xi}{a \Rightarrow F^*[G^*/X]}$$
then $\text{St}(\phi, \beta, F[G/X])$. We have, by an improper reduction,

$$\langle \eta, \beta, \forall X F \rangle \models \langle \eta, \beta, F[G/X] \rangle$$

for any basis $\beta$. Since $\text{St}(\eta, \beta, \forall X F)$ is assumed here, we thus get

(1) $\text{St}(\phi, \beta, F[G/X])$.

But picking up in particular $\beta := [\beta, G]$ we get from (1) by the substitution lemma

$$\text{St}(\phi, \beta, F[G/X])$$

as required. ☐

4.9. THEOREM. Every derivation of $L'$ is normalizable.

PROOF. Assume $\text{Der}(\phi)$. As in 3.6.1, we get from 4.7, 4.8 by BI $\text{St}(\phi)$ and so by 4.5 $\text{Nmble}(\phi)$. ☐.

4.10. THE REGULARITY OF $L'_2$, $L'_w$

Since there is a truth definition for $L'_2$ in $L'_3$ (by Tarski's method, compare e.g. TARKISI [36]), the proof of normalization of $L'_2$ above is easily seen to be formalizable in $L'_3 + BI$, and so


where $L'_2, \text{rec}$ is the system of recursive derivations of $L'_2$. The first inclusion is an immediate corollary of the obvious embedding of $L'_2$ in $L'_2, \text{rec}$. When a derivation $\{d\}$ of $L'_2, \text{rec}$ proves an arithmetical sentence, then the normal form $[n]\{d\}$ of $\{d\}$ is a derivation in $L'_2, \text{rec}$ which satisfies the subformula property, and therefore must actually be a derivation of $A'_\text{rec}$. The local correctness and wellfoundedness of $[n]\{d\}$ are proved in the theory in which the normalization of $\{d\}$ is proved, hence the second inclusion in (1).

Actually the embedding of $L'_2$ in $L'_2, \text{rec}$ mentioned above assigns to each particular (finitary) proof $\pi$ of $L'_2$ a proof $\delta$ of $L'_2$ where there is a bound $n$ on complexity of formulae. Thus for $\delta$ the normalization proof uses the
predicates $|^{m}_n$, $S^m_m$ only for $m \leq n$, and using a truth definition in $\Pi^1_k$-analysis for a suitably large $k = k(n)$ this proof is formalizable in $L_2^\omega + BI$. Hence (1) is refined to

$$A[L_2^\omega] \subseteq A_{rec}^\omega [L_2 + BI].$$

Analogously we have

$$A[L^\omega] \subseteq A_{rec}^\omega [L + BI].$$
B.0. STATEMENT OF THE RESULTS

When $F[p_1,\ldots,p_k]$ is a scheme of $l_0$ with (at most) the $k$ propositional letters shown, and when $A_1,\ldots,A_k$ are arithmetical sentences, write $F[A_1,\ldots,A_k]$ for the sentence which comes from $F[p_1,\ldots,p_k]$ by substituting $A_i$ for every occurrence of $p_i$ ($i=1,\ldots,k$). When $F[p_1,\ldots,p_k]$ is a scheme of $l_1$ with (at most) the $k$ predicate letters shown, where $p_i$ is $n_i$-place, and $A_i^{n_i}$ is an arithmetical formula with $n_i$ free variables ($i=1,\ldots,k$), write $F[A_1,\ldots,A_k]$ for the formula which comes from $F[p_1,\ldots,p_k]$ by replacing every atomic subformula $P_i(x_1,\ldots,x_{n_i})$ by $A_i(x_1,\ldots,x_{n_i})$.

Regular and strongly regular number theories are defined in A.1 above.

THEOREM I (Locally uniform $\Sigma^0_1$ absoluteness of $L_0$).

Let $A^*$ be a regular number theory. For every $k < \omega$ there are $\Sigma^0_1$ sentences $A_1,\ldots,A_k$ s.t.

$$L_0 \not\models F[p_1,\ldots,p_k] \Rightarrow A^* \not\models F[A_1,\ldots,A_k].$$

Or more precisely: there is a quantifier-free (q.f.) formula $E_0(x)$ s.t.

$$\forall x \forall y_L \overline{\text{Fm}}(x) \left[\neg \text{Pr}_{L_0}(x) \land \forall(x) \leq k \Rightarrow \neg \text{Pr}_{A^*}(\text{sub}_k \Sigma^0_1(x, E_0))\right]$$

is provable in $A + "A^* is regular"$, where

$$\overline{\text{Fm}}(x) \equiv "x is the g.n. of a schema in the language of } L_0";$$

$$\forall(F^p) \equiv "the number of propositional letters occurring in } F",$$
and \( \text{sub}^{k}_{0} \) is a prim. rec. function which satisfies
\[
\text{sub}^{k}_{0} (\langle F[p_{1}, \ldots , p_{k}], \Gamma \rangle) = \Gamma F[\exists x E_{0}(k, x), \ldots , \exists x E_{0}(k, x)]^{\top}.
\]

**THEOREM II** (Globally uniform \( \Pi_{2}^{0} \) absoluteness for \( L_{1} \)).

Let \( \Lambda^{*} \) be a strongly regular number theory. There are \( \Pi_{2}^{0} \) predicates \( \{ A_{i}^{j}\}_{i, j < \omega} \) s.t.
\[
L_{1} \not\models F[p_{1}^{i_{1}}, \ldots , p_{k}^{i_{k}}] \Rightarrow \Lambda^{*} \not\models F[\Lambda_{1}^{i_{1}}, \ldots , \Lambda_{k}^{i_{k}}].
\]

Or more precisely: there is a q.f. formula \( E_{1}(x) \) s.t.
\[
\forall x \exists ! \text{ Pel}(x) [\neg \text{Pr}_{L_{1}}(x) \rightarrow \neg \text{Pr}_{\Lambda^{*}} (\text{sub}^{k}_{0}(x, \Gamma))]\]

where \( \text{sub}^{k}_{0} \) is a prim. rec. function which satisfies
\[
\text{sub}^{k}_{0} (\langle F[p_{1}^{i_{1}}, \ldots , p_{k}^{i_{k}}], \Gamma \rangle) = \Gamma F[Q_{1}^{n_{1}}, \ldots , Q_{k}^{n_{k}}]^{\top}.
\]

where
\[
Q_{i}^{n_{i}}(\vec{z}) := \forall x \exists y E_{1}(x, y, i, n_{i}, \vec{z}).
\]
B.1. RECURSION-THEORETIC SOLUTION OF A REDUCED FORM OF THEOREM I

1.0. We wish to find \( \Sigma_1^0 \) sentences \( A_1, \ldots, A_k \) s.t.

\[
(*) \quad \mathcal{A}_0 \mathcal{F}[p_1, \ldots, p_k] \rightarrow \mathcal{A}_* \mathcal{F}[A_1, \ldots, A_k].
\]

If the theories \( \mathcal{L}_0 \) and \( \mathcal{A}_* \) are replaced by their classical completions then \( A_1, \ldots, A_k \) may be defined by truth-table arguments using recursion-theoretic methods only, as in Kripke [63] and in Myhill [72]. The complications for the intuitionistic case are the result of the presence of implications in the schema \( \mathcal{F} \), or more precisely - of negative nestings of implications. It is in such cases that the intuitionistic interpretation of the logical constants is expressed in an impredicative manner ("for every construction... there exists a construction...").

As in A.3., let us count the negative nestings of implications by a measure \( \mu \), i.e.,

\[
\begin{align*}
\mu^F & := 0 \text{ for atomic } F, \\
\mu^F \land G & := \max(\mu^F, \mu^G), \\
\mu^F \lor G & := \max(\mu^F + 1, \mu^G), \text{ and for the full language of } L_1, \\
\mu^\exists xF & := \mu^\exists \exists x F := \mu^F.
\end{align*}
\]

We shall see that for schemata \( \mathcal{F} \) s.t. \( \mu^F \leq 1 \) the classical recursion-theoretic methods work. The complexity involved in the growth of the \( \mu \)-measure is further illustrated by the fact (cf. Leivant [74]) that the consistency of \( A_k \) is provable in \( A_{k+1} \) for every \( k \), where

\[
A_k := A \text{ restricted to formulae } \mathcal{F} \text{ s.t. } \mu^F \leq k.
\]
1.1. STATEMENT OF THE REDUCED SOLUTION

We define a sequence $U_k$ of propositional schemata, where
$U_k \equiv U_k[p_1, \ldots, p_k]$ and $\mu^r U_k \leq 1$ as follows.

\[
U_0 \equiv 1 \\
U_1[p] \equiv pv \rightarrow p.
\]

Assuming $U_k$ to be defined, let

\[
U_k^i[p_1, \ldots, p_{i+1}] \equiv U_k[p_1, \ldots, p_i, p_{i+1}, \ldots, p_{k+1}]
\]

\[
U_{k+1}[p_1, \ldots, p_{k+1}] \equiv \bigvee_{i=1}^{k+1} [p_i \rightarrow U_k^i].
\]

We shall solve in this section (*) for the schemata $U_k$, i.e.,

**PROPOSITION.** We can uniformly construct $\Sigma_0^0$ sentences $A_k^1, \ldots, A_k^k$ s.t.

\[
\vdash_{A^*} U_k[A_1^k, \ldots, A_k^k] \quad (k<\omega).
\]

Here $A^*$ may be taken to be any consistent r.e. extension of $A$ which satisfies disjunction instantiation (the so-called "disjunction property"), i.e.,

\[
\vdash_{A^*} A \rightarrow B \Rightarrow \left[ \vdash_{A^*} A \text{ or } \vdash_{A^*} B \right].
\]

1.2. Actually proposition 1.1 gives a solution of (*) for all schemata $F$ s.t. $\mu^r F \leq 1$, on account of the following

**PROPOSITION.** For any schema $F$ of $L_0$ s.t. $\mu^r F \leq 1$,

\[
\vdash_{L_0} F[p_1, \ldots, p_k] \Rightarrow \vdash_{L_0} F \rightarrow U_k.
\]

**SKETCH OF PROOF.** Use a primary induction on $k$ (= the number of propositional letters occurring in $F$), secondary induction on the length of $F$, and ternary induction on the length of the left main subformula of $F$. □

1.3. LEMMA (propositional logic. Compare Kleene [52] §3).

[a1] If $G$ is a positive occurrence of a subformula of $F$, then
\[ E \vdash L_0 G \rightarrow H \iff E \vdash L_0 F \rightarrow F[H/G] \]

(where \( F[H/G] \) comes from \( F \) by replacing the occurrence \( G \) by \( H \)).

[a2] If \( G \) is a negative occurrence in \( F \), then
\[ E \vdash L_0 G \rightarrow H \iff E \vdash L_0 F[H/G] \rightarrow F. \]

[b] Let \( F^q \) be the propositional schema which comes from \( F \) by replacing (simultaneously) every occurrence of some (fixed) propositional letter \( p \) in \( F \) by \( pvq \), where \( q \) is a fixed propositional letter. Then
\[ \neg q \vdash L_0 F^q \rightarrow F. \]

PROOF. [a]: Straightforward by induction on the length of \( F \) (simultaneously for [a1] and [a2]).

[b]: Since \( q \vdash L_0 pvq \), we get by repeated application of [a2]
\[ (*) \vdash L_0 F^{q^*} \rightarrow F, \]
where \( F^{q^*} \) comes from \( F \) by replacing only negative occurrences \( p \) in \( F \) by \( pvq \). But \( \neg q \vdash L_0 pvq + p \), so we get by iterated application of [a1]: \((**) \vdash L_0 F^q + p^{q^*} \). \((*) \) and \((**) \) yield [b]. □

1.4. SIMPLIFIED DEFINITION OF EFFECTIVELY INSEPARABLE R.E. SETS

It is just to smoothen the exposition that we use the following

**Lemma.** Two disjoint r.e. sets \( A, B \) are effectively inseparable iff there is a (total) recursive function \( f \) s.t.
\[ \begin{align*}
W_i \cap A & = \emptyset \\
W_j \cap B & = \emptyset
\end{align*} \implies f(i,j) \notin W_i \cup W_j. \]

**Proof.**

I. The "if" direction is trivial, since the function \( f \) satisfies more than what is required from a function of effective inseparability (cf. e.g. Rogers [67] p.94).

II. Let, on the other hand \( f_1 \) be a (partial) recursive function for the effective inseparability of \( A \) and \( B \), and let \( i, j \) satisfy
By the reduction principle (cf. ROGERS [67], p.72) there are functions \( g, h \) s.t.

\[
\begin{align*}
(2) & \quad g(i) \subseteq W_i \land h(j) \subseteq W_j, \\
(3) & \quad g(i) \cup h(j) = W_i \cup W_j \quad \text{and} \\
(4) & \quad g(i) \cap h(j) = \emptyset.
\end{align*}
\]

Take now

\[
\begin{align*}
5. & \quad g'(i) = \begin{cases} g(i) & \text{if } i \in A, \\
                    B & \text{otherwise}, \\
\end{cases} \\
    & \quad h'(j) = \begin{cases} h(j) & \text{if } j \in B, \\
                    A & \text{otherwise}.
\end{cases}
\end{align*}
\]

Then

\[
\begin{align*}
6. & \quad g'(i) \subseteq B; \quad h'(j) \subseteq A
\end{align*}
\]

while by (4), (2), (1) and the assumed \( A \cap B = \emptyset \),

\[
\begin{align*}
7. & \quad g'(i) \cap h'(j) = \left[ [g(i) \cap h(j)] \cup [g(i) \cap A] \cup [h(j) \cap B] \cup [A \cap B] \right] = \emptyset.
\end{align*}
\]

For the \( f \) defined by

\[
\begin{align*}
f(i,j) := f_1(g'(i), h'(j))
\end{align*}
\]

we have now, by (6) (7) and the choice of \( f_1 \) that \( f(i,j) \notin W_i \cup W_j \) as required. It is easily seen, in addition, that \( f \) may be extended to a total function. \( \square \)

1.5. DEFINITION OF THE DESIRED \( \Sigma^0_1 \) SENTENCES

The following construction generalizes the method of MYHILL [72]. Let \( A, B \) be r.e. sets, effectively inseparable (in the sense of 1.4) through the function \( f \), and let \( A^* \) be any consistent r.e. extension of \( A \) which satisfies
the disjunction instantiation property. Following SHEPHERDSON [60] we may define (explicitly) a $\Sigma^0_1$ formula $F(a) \equiv \exists x F_0(x, a)$ s.t.

(1)

\[ A = \{ m \mid A^* \vdash F(m) \}; \quad B = \{ m \mid A^* \vdash \neg F(m) \} \]

(To see that this holds also intuitionistically, either inspect Shepherdson's proof, or observe that the equations above are formalizable as $\Sigma^0_2$ statements and recall that for such sentences derivability in classical arithmetic implies derivability in intuitionistic arithmetic.)

We construct now, by recursion on $k$, an infinite sequence $(A^k_i)_{i < \omega}$ s.t.

(2)

\[ \not\vdash_{A^*} \bigcup_{i < \omega} [A^k_i, \ldots, A^k_k] \quad \text{for every distinct } i_1, \ldots, i_k. \]

Basis: By the assumed properties of $A^*$ there is a $\Sigma^0_1$ Rosser sentence $R$ for $A^*$; set $A^k_i := R$ for every $i$.

Recursion step: Assume $A^k_i$, $i < \omega$ to be defined and to satisfy (2). We define a sequence of $\Sigma^0_1$ sentences $(G^k_j)_{j < \omega}$ s.t. no finite boolean combination of the $G^k_j$'s implies in $A^*$ $\bigcup_{i < \omega} [A^k_i, \ldots, A^k_k]$ for some distinct $i_1, \ldots, i_k$. (By a boolean combination we mean here a set $\{ H_j \}$ where $H_j$ is either $G^k_j$ or $\neg G^k_j$.)

Sub-basis: Let

(3) $W_g(1,k) := \{ m \mid \exists \text{ distinct } i_1, \ldots, i_k \text{ for which } F(m) \not\vdash_{A^*} \bigcup_{i < \omega} [A^k_i, \ldots, A^k_k] \} \]

(4) $W_h(1,k) := \{ m \mid \exists \text{ distinct } i_1, \ldots, i_k \text{ for which } \neg F(m) \not\vdash_{A^*} \bigcup_{i < \omega} [A^k_i, \ldots, A^k_k] \} \]

Now $W_g(1,k) \cap A = \emptyset$ and $W_h(1,k) \cap B = \emptyset$ by (1) and (2). Hence

\[ f(g(1,k), h(1,k)) \not\in W_g(1,k) \cup W_h(1,k). \]

Define

$G^k_1 := F(f(g(1,k), h(1,k)));$

then

(5) $\not\vdash_{A^*} \bigcup_{i < \omega} [A^k_i, \ldots, A^k_k] \quad \text{as required.}$
Sub-recursion step: Assume that $G^k_1, \ldots, G^k_k$ are defined, and satisfy

\[(6) \quad G^* \not\models_{A^*} U_k[A^k_{i_1}, \ldots, A^k_{i_k}] \text{ for every boolean combination } G^* \text{ of } G^k_1, \ldots, G^k_k \text{ and every distinct } i_1, \ldots, i_k. \]

Define

\[
W_{g(1+1,k)} := \{ m \mid \exists \text{ distinct } i_1, \ldots, i_k \text{ s.t. } F(m), \quad G^* \not\models_{A^*} U_k[A^k_{i_1}, \ldots, A^k_{i_k}] \\
\text{for some boolean combination } G^* \text{ of } G^k_1, \ldots, G^k_k \}
\]

\[
W_{h(1+1,k)} := \{ m \mid \exists \text{ distinct } i_1, \ldots, i_k \text{ s.t. } F(m), \quad G^* \not\models_{A^*} U_k[A^k_{i_1}, \ldots, A^k_{i_k}] \}. \]

As in the treatment of the sub-basis we have here

\[W_{g(1+1,k)} \cap A = \emptyset; \quad W_{h(1+1,k)} \cap B = \emptyset.\]

So, defining

\[G^k_{1+1} := F(f(g(1+1,k), h(1+1,k))), \]

we have

\[G^* \not\models_{A^*} U_k[A^k_{i_1}, \ldots, A^k_{i_k}] \text{ for every boolean combination } G^* \text{ of } G^k_1, \ldots, G^k_k. \]

Main recursion step continued: Define now $A^k_{i+1}$ to be (the purely $^0_{1+1}$ equivalent of) $A^k_{i+1} \lor G^k_{1+1}$. To conclude the proof, assume

\[\models_{A^*} U_k[A^k_{i_1}, \ldots, A^k_{i_{k+1}}] \text{ for some distinct } i_1, \ldots, i_{k+1}. \]

By the disjunction instantiation property of $A^*$ we get, w.l.o.g.,

\[\models_{A^*} A^k_{i_1} \lor U_k[A^k_{i_2}, \ldots, A^k_{i_{k+1}}]. \]

But recalling the definition of $A^k_{i+1}$, this implies
\[
G^k_{i_1} \vdash \forall A_k \left[ A^k_{i_2}, \ldots, A^k_{i_{k+1}} \right]
\]

which by 1.3 [b] implies

\[
G^k_{i_1}, \exists \bar{G}^k_{i_2}, \ldots, \exists \bar{G}^k_{i_{k+1}} \vdash \forall A_k \left[ A^k_{i_2}, \ldots, A^k_{i_{k+1}} \right],
\]

contradicting the construction of the sequence \( G^k_{ij} \). Hence

\[
| \forall A_k \left[ A^k_{i_1}, \ldots, A^k_{i_{k+1}} \right]
\]

as required. \( \square \)

Note, finally, that the above construction can be rendered totally uniform. That is, every \( A^k_{i_j} \) can be presented as \( \exists \bar{S}(f'(i,k),x) \) for a suitable total recursive function \( f' \). This formula does not belong, strictly speaking, to the formalism of \( A \). But it is equivalent to the following formula of prim. rec. arithmetic:

\[
\exists z \left( T(e, \langle i, k \rangle, (z)_0) \land F_0(U((z)_0), (z)_1) \right),
\]

where \( e \) is the g.n. of the function \( f' \), \( T \) and \( U \) are Kleene's computation-predicate and result-extracting function respectively. We have thus proved theorem I for schemata \( F \) s.t. \( u'F \leq 1. \square \)
B.2. PROOF-THEORETIC REDUCTION OF THEOREM I

2.0. Here we prove, for a regular number theory $A^* \subseteq A^*\{T\}$,

**PROPOSITION.** If $\vdash_{\Pi_0} F[p_1, \ldots, p_k]$ and $\vdash_{A^*} F[A_1, \ldots, A_k]$, then

$$\vdash_{A^*\{T\}} U_k[A_1, \ldots, A_k]$$

for any $\Sigma_1^0$ sentences $A_1, \ldots, A_k$.

To simplify notations we shall actually prove the proposition for $A^* \equiv A_{\text{rec}}^*\{T\}$ (in place of $A^*\{T\}$). A proof of the general version stated above is obtained trivially mutatis mutandis. Combined with the solution given in section 2 for the schemata $U_k$, this implies theorem I, since $A_{\text{rec}}^*\{T\}$ as well as $A^*\{T\}$ are r.e. and satisfy the disjunction instantiation property.

The proposition is proved as follows. In 2.1 - 2.7 below we prove (for some prim.rec. $f$)

1. $\forall_{\Pi_0} \forall_{\Pi_0} \exists_{\Pi_0} (\exists_{\Pi_0} (d, \exists_{\Pi_0} (A_1, \ldots, A_k)) \rightarrow \exists_{\Pi_0} (d, \exists_{\Pi_0} (U_k[A_1, \ldots, A_k]))$.

So, for a theory $T \equiv \forall_{\Pi_0} \forall_{\Pi_0}$ and a proof-predicate $\Pr_T$ for it which is proved in $A$ to be closed under Modus Ponens,

2. $\forall_{\Pi_0} \forall_{\Pi_0} \exists_{\Pi_0} (\exists_{\Pi_0} (d, \exists_{\Pi_0} (A_1, \ldots, A_k)) \rightarrow \exists_{\Pi_0} (d, \exists_{\Pi_0} (U_k[A_1, \ldots, A_k]))$.
But $Pr_{L_0}$ is a prim. rec. predicate, so (2) implies

$Pr_{L_0}(\neg F) \& Pr_A(\neg F[A_1, \ldots, A_k]) \rightarrow Pr_{A^*_{rec}[T]}(\neg F[A_1, \ldots, A_k])$

for any $A^* \subseteq A^*_{rec}[T]$.

(3) is proved in any extension of $A$ where $A^* \subseteq A^*_{rec}[T]$ is proved.

2.1. HEURISTICAL CONSIDERATION LEADING TO THE REDUCTION

2.1.1. Assume the premiss of 2.0(1). It means that a normal derivation $d$ of $F$ in $A_{rec}$ is given where some quantification or arithmetical rule must occur, because $Pr_{L_0}(\neg F)$. We "climb up" in the proof-tree $d$ in search for such an occurrence, starting at the root $()$.

To allow a smoother semi-formal exposition, let us write $-\rho d,u$ for the inference rule encoded by $((d)u)_0$, and

$d,u \models (d)u$

for the sequent coded by $((d)u)_1$.

At every stage of our search in $d$ we arrive at some node $u$ where the sentence $\rho d,u$ is a $\Sigma_0$ substitution of a schema of $L_0$, and where

$Pr_{L_0}(\neg F d,u)$, i.e. $\models d,u \models \neg F d,u$ cannot be proven using the rules of $L_0$ only.

Suppose now that a node $u$ is "selected" at a given stage of the search. If $\rho d,u$ is a propositional rule, then at least one of the premisses $u(n)$, $n < 2$ of $u$ in $d$ must satisfy $Pr_{L_0}(\neg F d,u(n))$, because $Pr_{L_0}(\neg F d,u)$ since $u$ is "selected". We "climb up" to the leftmost of these premisses.

$\neg F d,u$ cannot be $[\forall I]$ or $[\forall E]$, by the subformula property of $d$, because $\forall$ does not occur in $F[A_1, \ldots, A_k]$.

If $\rho d,u$ is $[\exists E]$, and $Pr_{L_0}(\neg F d,u(0))$ (i.e., the major premiss is not provable using propositional rules only), then we climb up to $u(0)$. Else, we proceed simultaneously to all minor premisses $u(n+1)$, $n < \omega$. The major premiss $F d,u(0) \vdash \exists xCz$ must be a $\forall^0_1$ sentence, by the subformula property. So for every $n$
\[ s^{d,u_*(n)} \equiv A_{r_{d,u}} C_n \Rightarrow F^{d,u} \]

where \( C_n \) is an equation, and \( F^{d,u} \) is a \( \xi_1 \) substitution of a propositional schema. It is easy to see (2.3 below) that if \( \text{Pr}_{L_0}(r_{d,u}, C_n \Rightarrow F^{d,u}_n) \) for some \( n \), then \( \text{Pr}_{L_0}(r_{d,u} \Rightarrow F^{d,u}_n) \), which contradicts our assumption that the node \( u \) is selected. It follows that all nodes \( u^{(n+1)} \) corresponding to the minor premises of \( r_{d,u} \) satisfy our conditions on "selected" nodes.

Now since \( d \) is a well founded tree, any successive selection of nodes as above must terminate. Such a "search" cannot terminate at a top-node of the derivation \( d \), because

(i) if \( r_{d,u} = \text{[TE]} \) then \( r_{d,u} \) is an equation, and so \( u \) is not selected;
(ii) if \( r_{d,u} = \text{[T]} \) then \( \text{Pr}_{L_0}(r_{d,u} \Rightarrow F^{d,u}_n) \).

Hence the search determined by any successive choice of minor (or major) premises of instances of \([\exists E]\) must stop at some node \( u \) s.t. \( r_{d,u} \) is either \([\exists I]\) or \([\forall E]\).

2.1.2. Let us now consider how this information on the "search" described above may be used to construct a proof in \( A^{\omega}_{\text{rec}} \) for \( U_k[A_1, \ldots, A_k] \). To start with, take the simplest case, where \( k = 1 \), \( F = \text{F}^{[\exists \forall \exists]} \), and let \( u \) be some terminating node of the search.

**Case 1.** \( r_{d,u} = \text{[EI]} \)

\[
\begin{align*}
\rho_{d,u}^{(0)} & \Rightarrow a \Rightarrow \text{Et} \\
\text{the node } & \uarrow u \Rightarrow \text{[EI]} & a \Rightarrow \exists \forall \exists
\end{align*}
\]

Obviously, the inference rule \( \rho_{d,u}^{(0)} \) cannot be an introduction rule. If \( \rho_{d,u}^{(0)} \) is \([\forall E]\), then we have the configuration

\[
\begin{align*}
& \Rightarrow a \Rightarrow \text{Et} & a \Rightarrow G \\
& \uarrow u \Rightarrow \text{[EI]} & \text{[EI]}
\end{align*}
\]

But no subformula of \( F[A_1, \ldots, A_k] \) has the form \( G + \text{Et} \) where \( \text{Et} \) is an equation. So \( \rho_{d,u}^{(0)} \) cannot be \([\forall E]\), and the cases \([\& E]\) and \([\forall E]\) are ruled out likewise. \( \rho_{d,u}^{(0)} \) cannot be one of \([L]\), \([\forall E]\), \([\exists E]\), by our definition of normality. We are thus left with the case that \( u^{(0)} \) is a top node of \( d \), and \( \rho_{d,u}^{(0)} \) is \([\text{TE}] \) or \([T]\). In the first case we may construct
So we have obtained a derivation for \( U_1[3xEx] \).

On the other hand, the case \( d, u^*(0) = [T] \) is ruled out as follows. Assume that \( d, u^*(0) = [T] \). Then \( E_t \in \bar{a} \), and since \( d \) derives a sequent \( \rightarrow \) with an empty precedent, \( E_t \) must be "discharged" in \( d \) somewhere below the node \( u \). Again by the subformula property of \( d \), this discharge cannot be at an instance of \([+1]\) or of \([v\forall] \), and so it must be at an instance of \([3\forall] \), and we should have the following configuration (where \( t = \bar{n} \)).

\[
\begin{align*}
\text{a} &\Rightarrow \bar{E}_n \\
(\text{u}) &\Rightarrow \text{a} \Rightarrow 3xEx \\
\ldots &\ldots \\
\text{b} &\Rightarrow 3xEx \\
(\text{v}) &\Rightarrow [3\forall] \quad \text{b} \Rightarrow B
\end{align*}
\]

Here the two indicated occurrences of \( \Sigma_i^0 \) formulae must be identical for the case considered. Since the node \( u \) is selected, so must be \( v \), but not \( v^*(0) \). This means that \( \neg Pr_{L_0}("3xEx") \), but \( Pr_{L_0}("b\Rightarrow 3xEx") \). From the configuration just shown we must have, however, \( b \leq \bar{a} \), and this is a contradiction.

Case 2. \( d, u = [F\exists] \), \( a \Rightarrow E \) say.

\[
\begin{align*}
[F\exists] &\quad \text{a} \Rightarrow 1
\end{align*}
\]

As in case 1, we find that \( u^*(0) \) must be a top node of \( d \), and since \( E \) here is a false equation, we are left with the case that \( d, u^*(0) \) is \([T]\); so we must find in \( d \) the following configuration:

\[
\begin{align*}
[T] &\quad \text{a} \Rightarrow E \\
(\text{u}) &\Rightarrow \text{a} \Rightarrow 1 \\
\ldots &\ldots \\
\Sigma_n &\quad \Sigma_{n+1} \\
\text{b} &\Rightarrow 3xEx \\
(\text{v}) &\Rightarrow [3\forall] \quad \text{b}, \bar{E}_n \Rightarrow B
\end{align*}
\]

\[
\begin{align*}
[3\forall] &\quad \text{b} \Rightarrow B
\end{align*}
\]
and we may assume w.l.o.g. (by the well-foundedness of \( d \)) that the configuration of the type shown does not repeat itself within any of the subderivations \( \Sigma_m \). Since \( u \) is selected, so must be \( v \), and hence \( v = \langle m + 1 \rangle \) for every \( m < \omega \). Each search in a subderivation \( \Sigma_m \) must come to an end at some node \( u_m \), and the argument of case I (about ruling out \( d, u = \langle 0 \rangle \) = [T]) shows that since \( v = \langle 0 \rangle \) is not selected, \( \rho^{d, u} = \langle m \rangle \) is not [BI], and must therefore be [FE]. Hence we can extract from the configuration above the derivation:

\[
\begin{align*}
[T] & \quad \exists x E x \Rightarrow \exists x E x \\
[FE] & \quad \exists x E x, E n \Rightarrow 1 \\
\end{align*}
\]

\( \exists x E x \Rightarrow 1 \)

\( \Rightarrow \neg \exists x E x \)

\( \Rightarrow \exists x E x \lor \neg \exists x E x \)

and again we find a derivation in \( A^w_{\text{rec}} \) for \( \mathcal{U}_1[\exists x E x] \). This concludes our observation on the case that \( k = 1 \), \( F = F[\exists x E x] \).

2.1.3. Consider now the case \( k = 2 \), i.e., \( F = F[\exists x E_0 x, \exists x E_1 x] \). Here the following configuration may occur

\[
\begin{array}{c}
\text{a} \Rightarrow \exists x E_0 x \\
\ h_{\Sigma_n} \ \ n < \omega \\
\ h_{\Sigma_n} \ \ n < \omega \\
\ u \Rightarrow [\exists E] \ a \Rightarrow b
\end{array}
\]

where the node \( u \) is selected, and the search continues to the minor subderivations \( \Sigma_n \) (i.e., \( \frac{\text{Fr}_{\Sigma_0}}{\Sigma_0} \) \( \frac{\text{Fr}_{\Sigma_0}}{\Sigma_0} \) \( \frac{\text{Fr}_{\Sigma_0}}{\Sigma_0} \)). But now, from our argument for the case \( k = 1 \) it is clear that, for the node \( u_m \) at which the search in the minor subderivation \( \Sigma_m \) terminates \( P^{u_m} \not \models \exists x E_0 x \) (\( m = \omega \)). So we may apply the argument for the case \( k = 1 \) to each of the minor subderivations separately, and extract from each of these a derivation \( \Sigma_m^* \) for \( \exists x E x \lor \neg \exists x E x \). Since the method of doing this is uniform, we can actually collect the derivations \( \Sigma_m^* \) to yield the following derivation of \( A^w_{\text{rec}} \).
Subordinated \((d,u,v)\) := \(\exists w,n<\varnothing: (w(n+1)<u \& (d,w) = [\exists Ex])\]

"v is a major premiss node of an instance of \([\exists E]\) in d, and u is a node in one of the minor sub-derivations of this instance".

Here \(<\) stands for the initial-segment relation (between sequent-numbers).

Selected \((d,u)\) := "\(F^{d,u}\) is not an equation" \& \(\neg \Pr_{[0]}(r^{d,u}\varnothing)\) \& \(\forall \omega<u: \text{Subordinated}(d,u,\omega(0)) \rightarrow \Pr_{[0]}(r^{d,u,\omega(0)}\varnothing)\)\].

When \(\text{NPr}_{\text{rec}}(d,r^{F[A_1,...,A_k]})\) \((A_1,...,A_k)\) sentences) write

\(\ll^{d,u}\) := \(\{F^{d,v} \mid \text{Subordinated}(d,u,v)\}\)

\(a^{d,u}_0\) := \(\{E \in \ll^{d,u} \mid E \text{ an equation}\}\)

\(v^{d,u}\) := \(U_m[A_{i_1},...,A_{i_m}]\) where \(\{A_{i_1},...,A_{i_m}\} := \{A_1,...,A_k\} \setminus \ll^{d,u}\) (set-theoretic difference)
2.3. LEMMA. Let \( A_1, \ldots, A_k \) be \( \Sigma_1^0 \) sentences, let \( a \Rightarrow G \) be formed of subformulas of \( F[A_1, \ldots, A_k] \) only, where \( G \) is not an equation, and let \( E \) be an equation. Then

\[
\Pr_{l_0} \left( \Gamma, E \Rightarrow G \right) \Rightarrow \Pr_{l_0} \left( \Gamma, a \Rightarrow G' \right).
\]

PROOF. Let \( \Pi \) be a normal proof for \( a, E \Rightarrow G \) which uses propositional inference-rules only, and let \( \Pi^* \) come from \( \Pi \) be eliminating \( E \) from all precedents of sequents in \( \Pi \). Check by inspection on cases for inference rules that \( \Pi^* \) is a correct derivation. (Note that by normality no formula of the form \( E \Rightarrow \Pi \) may occur in \( \Pi \)). □

2.4. LEMMA. (in A) Assume \( \mathsf{NPr}^a_{rec}(d, F[A_1, \ldots, A_k]) \);

(a) \( \text{Selected}(d, u) \rightarrow \left[ \rho^d, u = [E] \lor \rho^d, u = [FE] \lor \exists n \geq 2 \text{ Selected}(d, u*(n)) \right] \)

(b) \( \text{Selected}(d, u) \land \rho^d, u = [E] \land \Pr_{l_0}(\Gamma, s^d, u*(0)^n) \rightarrow \forall n > 0 \text{ Selected}(d, u*(n)) \).

PROOF. Assume \( \rho^d, u \neq [E], [FE] \) and the premiss of (a), and consider cases for \( \rho^d, u \). \( \rho^d, u \) cannot be \([T]\) or \([TE]\), because \( \text{Selected}(d, u) \). \( \rho^d, u \) is not \([VI]\) or \([VE]\) by the subformula property of \( d \). If \( \rho^d, u \) is a propositional inference-rule, the proof is immediate. We are left with the case that \( \rho^d, u \) is \([E]\). If \( \Pr_{l_0}(\Gamma, s^d, u*(0)^n) \) then we are done (for part (a)). Else, then \( \text{Selected}(d, u*(n)) \) for every \( n > 0 \) by 2.3. □

2.5. ASSIGNMENT OF DERIVATIONS TO THE SELECTED NODES.

Assume \( \mathsf{NPr}^a_{rec}(d, F[A_1, \ldots, A_k]) \) as above. We define a function \( \{a(d, u)\} \) recursive in \( d \) and \( u \) by the conditions given below (compare the definition of \( \{n\} \) in A.3.4.3). By the s.m.n.-theorem \( a(d, u) \) is then a prim.rec. function. \( \{a(d, u)\} \) is intended to be the formal description of a derivation of \( A^m \) for \( d^0, u \Rightarrow b^d, u \Rightarrow d^1, u \).

(i) If \( \text{Selected}(d, u) \), then \( \{a(d, u)\} \equiv 0 \).

(ii) Else, and \( \rho^d, u = [E] \), then \( \{a(d, u)\} \) describes the finite derivation
\[ [\rho^d,u(0)] \quad a^d,u \cup b^d,u \Rightarrow f^d,u(0) \]

\[ [3E] \quad a^d,u \cup b^d,u \Rightarrow f^d,u \]

instances of \([\ast I]\) .

and of \([vI]\) .

\[ a^d,u \cup b^d,u \Rightarrow u^d,u \]

Note that, by the argument of 2.1.2, \( f^d,u \not\vdash b^d,u \) and that \( \rho^d,u(0) \) is either \([T]\) or \([TE]\). So the figure above is indeed a derivation.

(iii) Else, and \( \rho^d,u = [FE] \). Let \( \{a(d,u)\} \) describe formally

\[ [T] \quad a^d,u \cup b^d,u \Rightarrow f^d,u(0) \]

\[ [FE] \quad a^d,u \cup b^d,u \Rightarrow \perp \]

\[ [L] \quad a^d,u \cup b^d,u \Rightarrow u^d,u \]

(iv) Else, \( \rho^d,u \) is a propositional inference rule. Let \( u(n) \) be the leftmost premiss of \( u \) in \( d \) s.t. \( \text{Selected}(d,u(n)) \) (cf. 2.4.(a)), and let \( \{a(d,u)\} := \{a(d,u(n))\} \).

(v) Else, and \( \rho^d,u = [3E] \);

\frac{\text{Subcase A: If } \neg \Pr_{L_0}(\tau^d,u(0)) \text{, let } a(d,u) := a(d,u(0)).}{\text{Subcase B: Else, and } \exists x \text{Ex } := f^d,u(0) \not\vdash b^d,u \text{, then let } \{a(d,u)\} \text{ describe the figure}}

\[ \frac{[T] \quad a_0^d,u \cup b^d,u, \exists x \text{Ex } \Rightarrow \exists x \text{Ex} \quad \{a_0^d,u(n) \cup b^d,u, \exists x \text{Ex } \Rightarrow u^d,u(n)\} \quad 0 < n < w}{[3E] \quad a_0^d,u \cup b^d,u, \exists x \text{Ex } \Rightarrow u^d,u(1)} \]

\[ [\ast T] \quad a_0^d,u \cup b^d,u \Rightarrow \exists x \text{Ex } \Rightarrow u^d,u(1) \]

instances of \([vI]\) .

\[ a_0^d,u \cup b^d,u \Rightarrow u^d,u \]
Here, if $\Sigma_n$ is described by \{a(d,u(\alpha(n)))\}, then $\Sigma_n'$ comes from $\Sigma_n$ by joining the formula $\exists x \exists y$ to all precedents. Note that by the case's conditions

\[
\begin{align*}
\frac{\exists x \exists y}{\exists y} & \quad \frac{\exists x \exists y}{\exists x \exists y} \\
& = \frac{a(d,u(\alpha(n)))}{a(d,u(\alpha(n)))} & \text{for } n > 0.
\end{align*}
\]

Subcase C. As subcase B, but $\exists x \exists y \in \Sigma_n$. Then let \{a(d,u)\} describe

\[
\Sigma_n
\]

\[
\frac{\exists x \exists y}{\exists x \exists y} \quad \frac{\exists x \exists y}{\exists x \exists y}
\]

Note that here $u^{d,u}(n) = u^{d,u}$ for every $n$.

2.6.1. LEMMA.

\[
\vdash_{Y_0^{BI}} \text{NP}_{\text{rec}}^\omega (d, F[A_1, \ldots, A_k]) \quad \text{Selected}(d, u) \rightarrow \text{NP}_{\text{rec}}^\omega (a(d, u), u^{b, b, u, d, u}).
\]

PROOF. Straightforward from the definition of $a(d, u)$ above.

2.6.2. LEMMA. For $F, A_1, \ldots, A_k$ as above

\[
\text{Pr}_{L_0} (F[A_1, \ldots, A_k]) = \text{Pr}_{L_0} (F[P_1, \ldots, P_k])
\]

PROOF. Let $\Delta$ be a normal proof of $F[A_1, \ldots, A_k]$ which uses propositional inference-rules only. All formulae occurring in $\Delta$ are subformulae of $F[A_1, \ldots, A_k]$, and a trivial inspection shows that by replacing $A_1, \ldots, A_k$ throughout the proof by $P_1, \ldots, P_k$ respectively we get a correct derivation of $L_0$ for $F[P_1, \ldots, P_k]$. \qed
2.7. PROPOSITION.

\[
\text{nPrf}_{\text{rec}}(d, \overline{F[A_1, \ldots, A_k]^n}) \land \neg \text{PT}_{L_0}(\overline{F}) \rightarrow \text{nPrf}_{\text{rec}}(a(d, ()), \overline{\neg [A_1, \ldots, A_k]^n}).
\]

PROOF. Use 2.6.1 for \(u = ()\), which by the premiss and 2.6.2 must be a selected node. □
B.3. STRUCTURE OF THE PROOF OF THEOREM II

3.1. PRELIMINARIES

3.1.1. Fix a q.f. formula \( E(x) : f(x) = 0 \) (where \( f \) is a fixed prim.rec. function). We shall use the following notations.

\[
\begin{align*}
E^E_1(z_1, \ldots, z_n) &:= \forall x \exists y E(x, y, i, n, (z_1, \ldots, z_n)) \\
E^E &:= \forall i, u, z \forall x \exists y E(x, y, i, u, z) \\
B^E[w] &:= \forall (i, n, z) [\text{Ineq}(i, n, z, w) \rightarrow \forall x \exists y E(x, y, i, n, z)]
\end{align*}
\]

where \( \text{Ineq}(a, b) \) is an equation which expresses the inequality \( a \neq b \). More intuitively,

\[
B^E[(j, m, \bar{a})] := \forall (i, n, \bar{\xi}) (i, n, \bar{\xi}) \neq (j, m, \bar{a}) \quad E^E_{\bar{\xi}}
\]

We further define the sequent

\[
\begin{align*}
s^E[w] := E^E[w] &\Rightarrow \forall x \exists y E(x, y, (w)_0, (w)_1, (w)_2) \\
&\equiv E^E[w] \Rightarrow E(w)_0 ((w)_2, (w)_2) \Rightarrow (w)_1
\end{align*}
\]

i.e.

\[
\begin{align*}
s^E[(i, n, \bar{z})] &:= \forall j, m, \bar{w} \quad (j, m, \bar{w}) \neq (i, n, \bar{z}) \quad E^E_{\bar{w}} \Rightarrow E^E_{\bar{z}}.
\end{align*}
\]

The sequents \( s^E[w] \) play here the same role as the schemata \( U_k \) in the treatment of \( L_0 \) above.
3.1.2. Let \( E \) be an equation as above. An \( E \)-sentence is a sentence built up using the formation rules of \( L_1 \) only, with \( E^n_{i_1} \) taken in place of the predicate letters \( P^n_{i_1} \). An \( E \)-atom is an \( E \)-sentence of the form \( E^n_{i_1}(t_1, \ldots, t_n) \). We call the indicated occurrences of \( t_i \) in the \( E \)-atom above \( (i=1, \ldots, n) \) the formal occurrences in \( E^n_{i_1}(t) \). Since the order of formally-occurring terms in each \( E \)-atom is fixed by the very definition of \( E^n_{i_1} \), it is uniformly decidable whether two \( E \)-atoms are instances of the same \( E^n_{i_1} \).

Let \( d \) be a normal derivation in \( \text{A} \) of an \( E \)-sentence. By the subformula property of \( d \) every formula occurring in \( d \) is either an \( E \)-sentence, an equation \( E(t) = 0 \) or a \( \Sigma^0_1 \) sentence \( \exists y E((p, y, i, n, z)) \). It is easily seen that if we replace every formal occurrence of each term \( t \) (in some formula in \( d \)) by the numeral \( \tilde{n} \) s.t. \( \tilde{n} = t \), we get a correct and normal derivation of the same \( E \)-sentence. We call such a normal derivation an \( E \)-derivation.

Notation: \( E\text{-Der}(d) \); \( E\text{-Prf}(d, \Gamma) \). Since we deal with \( E \)-derivations only, we consider only \( E \)-atoms of the form \( E^n_{i_1}(m_1, \ldots, m_n) \). If \( F[P^n_{i_1}, \ldots, P^n_{i_q}] \) is a schema of \( L_1 \) whose predicate-letters are among those shown, we write \( F^E \) for \( F[E^n_{i_1}, \ldots, E^n_{i_q}] \). So \( \Gamma \subseteq \text{sub}_{\Sigma^0_1}(\forall x, E) \).

3.1.3. We write \( \Theta [\Sigma^n_{i_1}] \) for an instance of \( \Theta \) whose major premiss (i.e. the consequent of the leftmost premiss-sequent) has a q.f. matrix. For an instance of \( \Theta \) which does not satisfy this we write \( \Theta^* \).

3.2. DERIVATION OF \( E \)-SENTENCES IN \( L_1 \)

\( L_1 \) is \( L_1 \) extended to the language of \( \text{A} \) (cf. P.2.5).

3.2.1. LEMMA. Let every formula in \( a, \Phi \) be either an \( E \)-sentence, an open \( \Sigma^0_1 \) formula or an open equation. Let \( \delta \) be a set of closed equations. Then

\[
a \cup \delta \models_{L_1} \Phi \Rightarrow a \models_{L_1} \Phi.
\]

PROOF. Assume \( a \cup \delta \models_{L_1} \Phi \), and let \( \Delta \) be a normal derivation of \( L_1 \Phi \) for \( a \cup \delta \models_{L_1} \Phi \) (cf. PRAVITZ [65]). By induction on the length of \( \Delta \), using the subformula property and the definition of \( E \)-atoms, one proves easily that every formula occurring in \( \Delta \) is either an \( E \)-sentence or an open \( \Sigma^0_1 \) or q.f. formula. Hence formulae in \( \delta \) are actually not used in \( \Delta \), and so \( a \models_{L_1} \Phi \). \( \square \)
3.2.2. **Lemma.** Let $a, F$ be closed formulae of $L_1$. Then

$$a \vdash_{L_1} A \vdash_{L_1} F$$

where

$$a = \{ G^E \mid G \in a \}.$$

**Proof.** The proof of Prawitz [65],[71] for the normalization of $L_I$ applies trivially to $L_1$ and is easily seen to hold also for our definition of normality (for $L_1$ only the trivial $\alpha_1$-reductions have to be considered in addition). So let $\Delta$ be a normal derivation (in the sense of A.1.1) of $L_1$ for $a \vdash_{L_1} F$ and let $G$ be an occurrence of a formula in $\Delta$, $G$ not an $E$-sentence.

By the subformula property of $\Delta$, $G$ must then have one of the forms

(a) $E(u,v,i,n,z)$ or (b) $\exists y E(u,y,i,n,z)$.

By the normality of $\Delta$, $G$ must either

(i) be an equation (case (a)) occurring at a top-node of $\Delta$ (by 2.1.2)

(ii) occur immediately below a top formula, or

(iii) occur as a premiss of $E$ derived by $\forall E$ (in case (b)).

Note now that $E^n_i$ is defined so that the order of variables in each $E$-atom is fixed, so that the two first variables of the matrix are bounded by the $\forall E$ quantifiers preceding it. Furthermore, two $E$-atoms formed from distinct $E^n_i$ are syntactically distinct, and the rule $[FE]$ is not used in $L_1$. Hence every occurrence $G$ as above must occur in a subderivation of $\Delta$ of the form

$$E(u,v,i,n,z) \quad \exists y E(u,y,i,n,z)$$

$$\forall \exists y E(x,y,i,n,z)$$

$$\forall \exists y E(x,y,i,n,z)$$

$$\exists y E(u,y,i,n,z)$$

$$\exists y E(u,y,i,n,z)$$

$$\exists y E(u,y,i,n,z).$$
Replace the subderivation \(\Pi\) of \(\Delta\) by

\[
\Pi^* \equiv \left[ \forall x \exists y E(x, y, i, n, z) \right]_j \Gamma \ H
\]

Note that \(\Pi^*\) is normal. Repeating this operation we get by induction on the number of occurrences of \(\Sigma^0_1\) formulae in \(\Delta\) a derivation \(\Delta^*\) where all occurrences are of \(E\)-formulae. Replace in \(\Delta^*\) every occurrence \(E_i(v)\) of an \(E\)-atom (including occurrences as a subformula) by \(f_i^{E_1}(v)\), and the result is a correct derivation of \(L_1\) for \(a \vdash F\).  

3.3. We wish to prove theorem II, which is trivially implied by the following more formal version.

**THEOREM II (restated).** For any \(T \supseteq \mathcal{V}_0 + BI\) there is a q.f. \(E(x)\) s.t.

\[
\text{AT} \vdash \neg \text{Pr}_{L_1}("F") \rightarrow \neg \text{Pr}_{\text{rec}[T]} \left( \sub_{E_0}("F", "E") \right)
\]

where

\[
\text{AT} := A + \text{CMP}(T) + \text{Rfn}_0(T) + \text{Con}(T^+) \]

\[
\text{CMP}(T) := \forall x, y \left[ \text{Pr}_T(\text{imp}(x, y)) \rightarrow (\text{Pr}_T(x) \rightarrow \text{Pr}_T(y)) \right],
\]

\[
\text{Rfn}_0(T) := \forall x \left[ \text{Pr}_T(x) \& "x \text{ encodes a formula in } C_0" \rightarrow \text{Tr}_{C_0}(x) \right],
\]

and where \(C_0\) is the class of formulae of the form \(\Pi_2^0 \land \Pi_2^0\), and \(\text{Tr}_{C_0}\) is a truth definition for \(C_0\).

\[
\text{Con}(T^+) := \forall x \left[ "x \text{ encodes a conjunction of instances of } AC_{\Pi_0^0}, \text{ of } BI, \text{ and of true } \Pi_1^0 \text{ sentences} \rightarrow \neg \text{Pr}_{T^+}(\neg \text{neg}(x)) \right].
\]
3.4. THE PROOF-THEORETIC REDUCTION

The proof of theorem II proceeds now as follows. Fix an equation \( E \) and a schema \( F \) of \( L_1 \) as above. In sec. 4.3 below we define a (classically) \( \Pi^0_1 \) predicate \( \text{Crit}(d,u) \) for which we prove

\[
\vdash_{\forall_0+\text{BI}} E\text{-Prf}(d,^\gamma F^\gamma) \rightarrow \left[ E^* \land \neg \text{Pr}_{L_1}^\gamma \left( ^\gamma F^\gamma \right) \rightarrow \neg \exists u \text{ Crit}(d,u) \right].
\]

Since \( T \supseteq \forall_0+\text{BI} \) we get from (1)

\[
\vdash_{\Lambda+\text{CMP}(T)} \text{Pr}_{\gamma}^{-}\text{E-Prf}(d,^\gamma F^\gamma) \rightarrow \text{Pr}_{\gamma}^{-}\left[ E^* \land \neg \text{Pr}_{L_1}^\gamma \left( ^\gamma F^\gamma \right) \rightarrow \neg \exists u \text{ Crit}(d,u) \right]
\]

and so

\[
\vdash_{\Lambda+\text{CMP}(T)+\text{Res}_0} \text{Pr}_{\gamma}^{-}\text{E-Prf}(d,^\gamma F^\gamma) \land E^* \land \neg \text{Pr}_{L_1}^\gamma \left( ^\gamma F^\gamma \right) \rightarrow \neg \exists u \text{ Crit}(d,u).
\]

On the other hand we prove in 4.7-4.11 below

\[
\vdash_{\forall_0+\text{BI}+\text{AC}_0} \text{E-Der}(d) \land \text{Crit}(d,u) \land \text{Res}(d,u,x) \rightarrow \neg \exists \psi \text{ NPr}^\gamma (\psi,^\gamma F^\gamma[x])
\]

where

\[
\text{Res}(d,u,x) := \forall y \left[ T(d,u,y) \rightarrow \text{if succedent}(\langle y \rangle) \text{ encodes } E^B_1(\gamma) \text{ then } x=\langle i,n,\langle \gamma \rangle \rangle \right].
\]

Since \( T \supseteq \forall_0+\text{BI}+\text{AC}_0 \) (cf. A.1.2) and \( \text{CMP}(T) \rightarrow \text{CMP}(T^+) \) trivially, we get from (4)

\[
\vdash_{\Lambda+\text{CMP}(T)} \text{Pr}_{\gamma}^{-}\text{E-Der}(d) \rightarrow \text{Pr}_{\gamma^+}\left( \text{Crit}(d,u) \land \text{Res}(d,u,x) \rightarrow \neg \text{NPr}^\gamma (\psi,^\gamma F^\gamma[x]) \right).
\]
But \( \text{Crit}(d,u) \) and \( \text{Res}(d,u,x) \) are \( \Pi^0_1 \) and \( T^+ \) is complete for true \( \Pi^0_1 \) sentences, so

\[
(6) \quad \vdash_{A+\text{CMP}(T)} \text{Pr} \rightarrow \text{E-Der}(d)^\gamma \land \text{Crit}(d,u) \land \text{Res}(d,u,x) \rightarrow \\
\text{Pr} \rightarrow \lnot \text{NP} \rightarrow s_{[x]}^E.
\]

We have however trivially

\[
\vdash_A "(d) \text{ is total}" \rightarrow \exists x \text{ Res}(d,u,x)
\]

and so

\[
\vdash_{A+\text{CMP}(T)+\text{Ref}_{C_0}} \text{Pr} \rightarrow \text{E-Der}(d)^\gamma \rightarrow \exists x \text{ Res}(d,u,x).
\]

Hence we get from (6)

\[
(7) \quad \vdash_{A+\text{CMP}(T)+\text{Ref}_{C_0}} \text{Pr} \rightarrow \text{E-Der}(d)^\gamma \land \exists u \text{ Crit}(d,u) \rightarrow \\
\exists x \text{ Pr} \rightarrow \lnot \text{NP} \rightarrow s_{[x]}^E.
\]

Combining (3) and (7) yields

\[
(8) \quad \vdash_{A+\text{CMP}(T)+\text{Ref}_{C_0}} \text{Pr} \rightarrow \text{Prf}(d,\gamma F)^\gamma \land E^* \land \lnot \text{Pr} \rightarrow \gamma F^\gamma \rightarrow \\
\lnot \exists x \text{ Pr} \rightarrow \lnot \text{NP} \rightarrow s_{[x]}^E.
\]

But from 3.2.2 we have

\[
\vdash_A \lnot \text{Pr} \rightarrow \gamma F^\gamma \rightarrow \lnot \text{Pr} \rightarrow \gamma F^E
\]

\((F \text{ a schema of } L_1)\)

so

\[
(9) \quad \vdash_{A+\text{CMP}(T)+\text{Ref}_{C_0}} \text{Pr} \rightarrow \gamma F^\gamma \land \text{Pr} \rightarrow \gamma F^E \land E^* \rightarrow \\
\lnot \exists x \text{ Pr} \rightarrow \gamma s_{[x]}^E.
\]
This completes the proof theoretic reduction. Note that for any predicate Crit (not necessarily $\Pi^0_1$) for which (1) and (4) hold, we could prove a statement ($7^*$) similar to (7), but with \( \text{Pr}_{T^+} \exists x \neg \text{NP}_{\omega^\omega} (x)^{\sim} \) as the succedent; there is however no way to pull the existential quantifier out of the provability predicate here.

3.5. SOLUTION OF THE REDUCED PROBLEM.

In this part of the proof of theorem II, given in B.5 below, we prove for every $\Sigma^0_2$ theory $S$ the existence of a q.f. $E(x)$ s.t.

\[
\vdash_{\Lambda + \text{Con}(S) + \Sigma^0_2(\lambda_s)} \forall x \neg \text{Pr}_S (x) \land \neg \exists^* E
\]

where $\text{Pr}_S$ is a fixed $\Sigma^0_2$ provability predicate for $S$, and where

\[
\text{Comp}^0_{\Sigma^0_2}(S) := \forall x [ \text{Tr}^0_{\Sigma^0_2}(x) \rightarrow \text{Pr}_S(x) ].
\]

i.e., $S$ is complete for $\Sigma^0_2$ sentences. (Here $\text{Tr}^0_{\Sigma^0_2}(x)$ is a (canonical) truth definition for $\Sigma^0_2$ sentences).

We wish to apply (10) to $S \equiv A^\omega[T^+]$, where $T$ and $T^+$ are as in A.1.2.

First, note

\[
\vdash_{\gamma_0} \neg \text{NP}_{\omega^\omega} ("1")
\]

so

\[
\vdash_{\Lambda} \text{Con}(T^+) \rightarrow \text{Con}(A^\omega[T^+]).
\]

Also, for $\Sigma^0_2$ sentences $F$ we have directly (compare A.2.2.1)

\[
\vdash_{\gamma_0} F \rightarrow \text{NP}_{\omega^\omega} p^\gamma
\]

and since $T^+ \supseteq \gamma_0$, and quite trivially $\text{CMP}(T) \rightarrow \text{CMP}(T^+)$, this implies

\[
\vdash_{\Lambda + \text{CMP}(T)} \text{Pr}_{T^+} p^\gamma \rightarrow \text{Pr}_{A^\omega[T^+]} p^\gamma.
\]
By the definition of $\Pr_{T^+}$ however

$$\vdash_A F \rightarrow \Pr_{T^+} \gamma F$$

for every $\Pi_1^0$ $F$, and so

$$\vdash_A F \rightarrow \Pr_{T^+} \gamma F$$

for every $\Sigma_2^0$ $F$.

Hence we get from (15) and (16)

$$\vdash_{A_T} \text{CMP}(T) \rightarrow \Pr_{A_{T^+}} \gamma F$$

for every $\Sigma_2^0$ formula $F$.

Now observe that steps (15)-(17) can be uniformly formalized (within $A$), i.e., (11) holds for $S \in A_{\{T^+\}}$, as wanted.

From (10) for $S \in A_{\{T^+\}}$, (9) and (13) we now have by predicate logic

$$\vdash_{A_T} \neg\Pr \rightarrow \Pr_{A_{\text{rec} \{T\}}} \gamma E(x)$$

for some fixed quantifier-free $E(x)$.

We proceed now to prove (1) and (4) (the proof-theoretic reduction) and (1) (the recursion-theoretic solution) which together imply as we have just seen theorem II.
B.4. THE PROOF-THEORETIC REDUCTION FOR THEOREM II

4.1. LEMMA. Let the numeral \( \bar{n} \) not occur in \( a, F, \exists x Gx \).

(i) If \( a, G \downarrow \vdash_A F \) then

\[(1) \quad a, Gv \vdash_A F \]

where \( v \) is a parameter which does not occur in \( a, G, F \).

(ii) If \( a \vdash_A G \) then \( a \vdash_A Gv \) (for \( v \) as above).

PROOF. Given a normal derivation of \( L_A \) for (i) replace every occurrence of \( \bar{n} \) by \( v \) and observe, by inspection on cases for the inference rules, that the result is a correct derivation. The proof of (ii) is similar. □

4.2. SEMI FORMAL HEURISTIC OUTLINE OF THE REDUCTION

4.2.1. Preliminary notations.

\[ R_1(d,u) : \text{all equations in } a^d_u \text{ are true}. \]
\[ R_2(d,u) : \text{"all equations in } a^d_u \text{ are true"}. \]
\[ R_3(d,u) : \text{"} \vdash a^d_u \text{ is an } E \text{-sentence"}. \]
\[ R_4(d,u) : \text{"} \vdash a^d_u \text{ is an } E \text{-atom, and } a^d_u \text{ is } [VI]". \]
\[ R_5(d,u) : \text{"} \vdash a^d_u \text{ is a } E^0 \text{-sentence"}. \]

Note that each \( R_j(d,u) \) may be formally defined as a \( \Pi^0_1 \) predicate. Example:

\[ R_3(d,u) : \forall y \left[ T(d,u,y) \rightarrow \text{"succedent}((Uy)_1) \text{ is the g.n. of an } E \text{-sentence"} \right]. \]

\[ \text{Start}(d,u) : \bigwedge_{i=1,2,3} R_i(d,u). \]
\[ \text{Crit}(d,u) : \bigwedge_{i=1,2,4} R_i(d,u). \]
4.2.2. Locating an arithmetical inference in E-derivations (the predicate Crit).

We want to define a predicate Crit and to prove for it 3.4(1),(4). The idea is that when E-Der(d) and Crit(d,u) ("u is a critical node in the proof-tree described by d") then the subderivation d^u of d (where d^u := λx.(d)(u*x)) has sufficiently nice properties so as to enable the extraction from it of a derivation for s^E[w] for some w.

As a first attempt to define such a predicate we try as in the proof of theorem I to look, when E-Prf(d,F^n) and ¬Prf_1(F^n), for a "genuine" use of an arithmetical inference in d. A starting node for such a search upwards may be any node v of d s.t. Start(d,v). When Start(d,v) we can weakly find (i.e., ¬∃) a node v*(n) s.t. Start(d,v*(n)), using lemma 4.1 when ρ^d,v is [VI] or [∃1*], and the truth of E^m and 3.2.1 when ρ^d,v is [∃E^1] (lemma 4.4 below). Thus the search up in d may continue. The only cases where this process stops are when R4(d,v) or when ρ^d,v is [FE]. In the last case, the definition of normality of A.1.1 implies (as in 2.1.2) that a false equation occurs in a^d,v, contradicting R4(d,v). Thus, by the well-foundedness of the proof-tree d, we find a node u > v s.t. Crit(d,u).

When Crit(d,u), we can actually find in each subderivation d^u(m) an inference of the form

\[
\frac{G}{d^u(w)} \quad \{∃1\}
\]

(G is a true equation and P^d,u*w ≤ P^d,u*(m)). So these can be collected to yield a derivation of the form:

\[
\frac{\{∃1\}}{d^u(w)} \quad \{VI\}
\]

\[
B(i,n,\{τ\}) \Rightarrow P^E_{i}(τ)
\]

where P^d,u := P^E_{i}(τ), and each P^E_{i} is (schematically) of the form (\#).

Unfortunately, the crude statement that the situation above occurs is not \(\Pi_1^0\) essentially because there is no bound on the length of the w corresponding to each m<\omega. A certain refinement of the argument is therefore necessary.
4.2.3. Heuristic for the disjunction-free fragment

Assume, again, $E$-$Der(d)$ and $Crit_1(d,u)$. The subderivation $d^u$ of $d$ then takes the form

$$
\begin{array}{l}
\{ \\
\Sigma_m \\
\{ a \Rightarrow \exists y E(m,y,i,n,\langle \tilde{t} \rangle) \} \\
\} m<\omega \\
[VI] a \Rightarrow \forall x \exists y E(x,y,i,n,\langle \tilde{t} \rangle)
\end{array}
$$

where each $\Sigma_m$ is formally described by $d^{u*}(m)$.

From each $\Sigma_m$ we wish to extract a derivation in $A^m$ of

$$
B[(i,n,\langle \tilde{t} \rangle)] \Rightarrow \exists y E(m,y,i,n,\langle \tilde{t} \rangle).
$$

Fix some $m$, and let us analyse the structure of $\Sigma_m$.

We assume first that $d$ is a derivation for a disjunction-free $E$-sentence; this implies by the subformula property that disjunction does not occur in the derivation $d$, and in particular, in the subderivation $\Sigma_m$ we are looking at.

In addition we may assume

$$
\forall w \vdash u^{*}(m) \rightarrow \text{Start}(d,w).
$$

Because if $\text{Start}(d,w)$, $w \vdash u$ then we could start our initial search afresh; this could not be iterated indefinitely, because $d$ is well-founded.

Consider now the main inference rule of $\Sigma_m$, $d^{d,u^*(m)}$. By the subformula property of $d$ we have to consider the following cases only.

Cases (i)-(iii*): contradiction to (3).

(i) $\rho^{d,u^*(m)} = [\perp]$; then $s^{d,u^*(m,0)} = a \Rightarrow \perp$ and so $\text{Start}(d,u^*(m,0))$ contradicting (3).

(ii) \[\forall E\];

$$
a \Rightarrow \rho^k(j(s))
$$

(iii) \[\forall E\] $a \Rightarrow \exists y E(m,y,i,n,\langle \tilde{t} \rangle)$ say.
Recall that $E_j^k(s) \equiv \forall x \exists y E(x, y, j, k, (s))$, and so necessarily $\langle i, n, (\hat{\tau}) \rangle \equiv \langle j, k, (\hat{\sigma}) \rangle$ (syntactical identity). Therefore $\sigma^d, \upsilon^*(m, 0) \equiv s^d, \upsilon$ and so $\text{Start}(d, \upsilon^*(m, 0))$, contradicting (3) once again.

(iiia) \[[\exists E]^1\]; since $d$ is normal, $\exists E_m$ must then have the following form (compare the first part of 4.5.4 below):

\[
\frac{\sigma \Rightarrow \upsilon_j^k(\hat{s})}{[\exists E] \quad \exists \exists Cz} \quad \text{p < \omega}
\]

First, if $\langle j, k, (\hat{\sigma}) \rangle \equiv \langle i, n, (\hat{\tau}) \rangle$ then $\text{Start}(d, \upsilon^*(m, 0, 0))$ as in (ii), contradicting (3).

Cases (iiib), (iv): the search continues.

(iiib) If, in (iiia), $\exists \exists Cz$ is true, let $p := uz.Cz$, and consider — in place of $\exists E_m$ — its subderivation $\Gamma_p$ (formally described by $d^\upsilon(m, p\#1)$).

Before concluding the case $\rho^d, \upsilon^*(m) = [\exists E]^1$ let us turn first to case (iv) If $\rho^d, \upsilon^*(m)$ is \[[\exists E]E^\$\], then guided by lemma 4.1 we pick the first numeral $p$ which does not occur in the sequents $s^d, \upsilon^*(m), d^\upsilon^*(m, 0)$, and we consider (as in case (iiib)) the subderivation $d^\upsilon(m, p\#1)$.

Cases (iiic), (v): happy ending.

(iiic) If $\rho^d, \upsilon^*(m)$ is \[[\exists E]^1\], and (iiia) and (iiib) do not apply, then in (5) $\langle j, k, (\hat{\sigma}) \rangle \not\equiv \langle i, n, (\hat{\tau}) \rangle$ and $\exists \exists Cz$ is false; so we can extract from (5) the following derivation in $A^\omega$ of (2):

\[
\frac{\text{B}(\langle i, n, (\hat{\tau}) \rangle)}{[\exists E] \quad \exists \exists Cz} \quad \quad \text{p < \omega}
\]

(here we dropped the precedents of sequents).

(v) \[[\exists E]1\]; by 2.1.2 $\upsilon^*(m, 0)$ is then a top-node in $d$, and so $\rho^d, \upsilon^*(m, 0)$ is either $[T]$ or $[TE]$. In the first case $\rho^d, \upsilon^*(m, 0) \in \lambda$; but all equations in $\lambda$ are true, so $\rho^d, \upsilon^*(m, 0)$ is in any case a true equation.
These are all the cases in the absence of disjunction. Cases (i),(ii), (iiiia) rule out possible failures of the construction; cases (iiib),(iv) allow the search to continue, while cases (iiic) and (v) yield the required derivation for (2).

Note that if \( E^* \) is true, then \( \exists z Cz \) in (6) is also true, and so case (iiic) is excluded. Our argument here must however be independent of \( E^* \) (cf. 3.4(4)-(6)), and so this case has to be considered throughout.

In order to clarify a bit the form of a search which proceeds through (iiib),(iv), let us consider for example the outcome of case (iv), and suppose that now case (ii) applies to \( \Gamma_p \) (: the derivation formally described by \( d_{\alpha}(m,p+1) \)). I.e., the following configuration occurs:

\[
\begin{array}{c}
\Gamma_p \\
\vdash \exists z Cz \\
\vdash \exists y E(m,y,i,n,(\xi)) \\
\text{the node } (u*(m)) \rightarrow (\exists z E^*) \quad \vdash \exists y E(m,y,i,n,(\xi)) \\
\end{array}
\]

Here (3) implies, as in (i)-(iii),

\[
\begin{align*}
\neg \text{Pr}_A(\exists z Cz^*) & \quad \text{and} \quad \neg \text{Pr}_A(\exists z Cz^*) \\
\end{align*}
\]

which by 4.1(i) and the choice of \( \overline{p} \) give

\[
\neg \text{Pr}_A(\exists z Cz^*)
\]

contradicting \( \text{Crit}_1(d,u) \). So we have adapted the argument of (ii) to the case that a search for a proof of (2) proceeds via case (iv). Other arguments are adapted in about the same way, and this allows the iteration of the search through (iiib),(iv) above.

By the well-foundedness of \( d \) the process must terminate, that is, one of cases (iiic),(v) ultimately appears, and we obtain a proof for (2), as desired.

4.2.4. Disjunction reconsidered

When disjunction does occur in the derivation \( d \) above, we must add to (i)-(v) above another case:

(iii) \( \rho_{d,u*}(m) \) is [vE]. We then consider simultaneously both minor premisses of \( \rho_{d,u*}(m) \), i.e., the nodes \( u*(m,1) \) and \( u*(m,2) \).
As in the last paragraph of 4.2.3, let us see what happens if case (ii) applies now to both \( u^*(m,1) \) and \( u^*(m,2) \). We have then the following configuration:

\[
\begin{align*}
\Delta & \\
\Gamma_1 & = \exists \, g_1 \rightarrow g_1^n(t) \\
\Gamma_2 & = \exists \, g_2 \rightarrow e_1^n(t) \\
\end{align*}
\]

As in the last paragraph of 4.2.3,

\[
\forall \exists \, g_1 \rightarrow [\forall \exists \] \quad \exists \, g_1 \rightarrow \exists y E(...)
\]

As the last paragraph of 4.2.3,

\[
\forall \exists \, g_1 \rightarrow [\forall \exists \] \quad \exists \, g_2 \rightarrow \exists y E(...)
\]

This argument may be generalized to conclude that, at least for one successive choice of minors of \([vE]\) in the search described by (iiib),(iv), (vi) the construction leads to a node falling under one of the cases (iiic), (v), thus allowing a construction of a proof of \( A^w \) (incidentally of \( A^w \)) for (2).

The assertion that this is the case is now seen quite easily to be formalizable as a \( \Pi_1 \) predicate (over \( d,u \)).

4.3. FORMALIZATION OF THE PREDICATE Crit

Step\( (d,w,p) \) :

\[
\begin{align*}
\text{Step}_1(d,w,p) & : = \text{Step}_1(d,w,p) \\
\text{Step}_2(d,w,p) & : = \text{Step}_2(d,w,p) \\
\text{Step}_3(d,w,p) & : = \text{Step}_3(d,w,p)
\end{align*}
\]

where

\[
\begin{align*}
\text{Step}_1(d,w,p) & : = \text{Step}_1(d,w,p) \\
\text{Step}_2(d,w,p) & : = \text{Step}_2(d,w,p) \\
\text{Step}_3(d,w,p) & : = \text{Step}_3(d,w,p)
\end{align*}
\]

1. \( [vE] \) and \( 1 \leq p \leq 2 \).
These three predicates correspond to cases (iiib), (iv) and (vi) in 4.2.3/4, where the search described there proceeds to the p'th premise of the node w. It should be noted that \text{Step} is a $\Delta_i^0$ predicate. For example

\[
\text{Step}_1(d,w,p) \equiv \forall x,y \left[ T(d,w,x) \land T(d,w*(0),y) \rightarrow A(x,y,p) \right]
\]

\[
\equiv \exists x,y \left[ T(d,w,x) \land T(d,w*(0),y) \land A(x,y,p) \right]
\]

where

\[
A(x,y,p) \equiv (\forall x) = \exists y \land \text{Tr}_{QF}(\text{inst}(\text{antecedent}((Uy)_1),p+1))
\]

where

\[
\text{Tr}_{QF} \text{ is a } (\Delta_i^0) \text{ truth predicate for equations, and inst is a prim. rec. function which satisfies inst} (\exists x \forall y \forall z) = \exists x \forall y \forall z.
\]

\[
\text{Selected}(d,v) \equiv \forall i \leq \text{ith}(v) \text{ Step}(d,(v|i),(v)_{1})
\]

where

\[
(v|i) := (v)_0, \ldots, (v)_{i-1} \quad \text{ (for } i \leq \text{ith}(v))
\]

\[
\text{Final}(d,v) \equiv \bigwedge_{i=1,2,3} \text{Final}_i(d,v)
\]

where

\[
\text{Final}_1(d,v) \equiv \text{Selected}(d,v) \land \rho_{d,v} = [L] \text{ or } [\forall E]
\]

\[
\text{Final}_2(d,v) \equiv \text{Selected}(d,v) \land \rho_{d,v} = [\exists L]
\]

\[
\text{Final}_3(d,v) \equiv \text{Selected}(d,v) \land \rho_{d,v} = [\exists I].
\]

These predicates correspond to the cases in 4.2.3 where the construction may stop, whether successfully or not.

\[
\text{Final}^+(d,v,A) \equiv \text{Final}_2(d,v,A) \lor \text{Final}_3(d,v)
\]

where

\[
\text{Final}_2(d,v,A) \equiv \text{Final}_2(d,v) \land F_{d,v}(0,0) \notin A.
\]
When for 4.2.3 \( A \equiv E^{1}_1(t) \equiv F^d,u \) then \( \text{Final}^+(d,v,\bar{A}^-) \) expresses the conclusion of the construction by one of (iiic), (v), or possibly its continuation through (iiib). In any case, a "failure" through one of (i)-(iiia) is excluded. It is important to note that \( \text{Final} \) and \( \text{Final}^+ \) are both \( A_1 \) predicates.

Let us use the binary encodement of finite sets of numbers. The predicates \( n \in x \), \( x = \emptyset \) etc. are then just prim.rec. arithmetical expressions.

\[
\text{Bar}(d,x) := x \neq \emptyset \land \\
\forall x \in \text{Final}(d,w) \land \forall v,y < x \exists \rho_{d,u} = [vE] \land \\
w = u*\frac{1}{2}y \rightarrow \exists w' \exists z < x \wedge w' = u*\frac{2}{3}z .
\]

I.e., a "bar" for \( d \) is a finite non-empty set of "final" nodes, which intersects both minor subderivations of each instance of \( vE \) if it intersects one of them.

\[
\text{Crit}_2(d,u) := \forall m,x [ \text{Bar}(d^u(m),x) \rightarrow \exists x \text{Final}^+(d^u(m),w,F^d,u) ]
\]

\[
\text{Crit}(d,u) := \text{Crit}_1(d,u) \land \text{Crit}_2(d,u).
\]

Note that \( \text{Crit} \) is intuitionistically equivalent to a \( \Pi^0_1 \) predicate.

\[
\text{Final}^{++}(d,v,\bar{A}^-) := \text{Final}^{++}_2(d,v,\bar{A}^-) \lor \text{Final}^{++}_3(d,v)
\]

where

\[
\text{Final}^{++}_2(d,v,\bar{A}^-) := \text{Final}^+_2(d,v,\bar{A}^-) \land \neg \text{Tr}_{x_1}^0(F^d,v^*(0)_v)
\]

\[
\text{Final}^{++}_3(d,v) := \text{Final}^+_3(d,v) \land \text{Tr}_{q_p}^0(F^d,v^*(0)_v).
\]

\( \text{Final}^{++} \) corresponds to a real termination of the search described in 4.2.3. In contrast to \( \text{Final}^+ \) however, \( \text{Final}^{++} \) is a \( \Pi^0_1 \) predicate, and not a \( A_1 \) one.
4.4 - 4.6. PROOF OF 3.4(1): THE EXISTENCE OF A CRITICAL NODE
(first part of the proof theoretic reductions)

4.4. LEMMA.

\[ \forall \nu \exists d [ E^* \& E-Der(d) \& Start(d,u) ] \rightarrow \neg \exists \nu \forall u \text{Crit}_1(d,u). \]

PROOF. Denote the formula to be proven by \( R(u) \). First, we prove below by BI using the well-foundedness of the proof-tree \( d \) the (open) formula

\[ S(u) := [ E^* \& E-Der(d) \& Start(d,u) \& \neg R_0(d,u) ] \rightarrow \neg \exists \nu \forall u \text{Start}(d,w). \]

Assuming \( \forall u S(u) \), \( R(u) \) follows easily by a second use of BI, where \( S(u) \) is to be used for the induction step.

Towards proving \( S(u) \) by BI assume the premiss of \( S(u) \), assume \( \forall u S(u+\langle n \rangle) \), and consider cases for \( \rho^d,u \) which by the normality of \( d \) can only be one of the following:

(i) \( \rho^d,u \) is \([T]\). This contradicts \( R_1(d,u) \). \( \rho^d,u \) is also not \([TE]\) by \( R_3(d,u) \).

(ii) \( \rho^d,u \) is \([FE]\). As in 212 the normality of \( d \) implies then that \( \rho^d,u \langle 0 \rangle \) is \([T]\), and so \( \rho^d,u \in A^d,u \), contradicting \( R_2(d,u) \).

(iii) \( \rho^d,u \) is a propositional rule, \([BI]\) or \([VE]\). If \( \neg \exists n < 3 \neg R_{j+1}(s^d,u \langle n \rangle) \) for all \( n \geq 3 \), then of course \( \neg \exists n < 3 \neg R_{j+1}(s^d,u \langle n \rangle) \), since all the rules considered in this case are rules of \( L_1 \). This contradicts \( R_1(d,u) \). So \( \exists n < 3 \neg R_{j+1}(s^d,u \langle n \rangle) \). For the cases considered the subformula property of \( d \) implies trivially \( R_j(d,u) \rightarrow R_j(d,u \langle n \rangle) \) for \( j = 2,3 \), and so we conclude that \( \exists n < 3 \text{Start}(d,u \langle n \rangle) \).

(iv) \( \rho^d,u \) is \([BE^*]\). Let \( \pi \) be the first numeral which does not occur in \( s^d,u, s^d,u \langle 0 \rangle \), and prove

\[ (\ast) \neg R[ \text{Start}(d,u \langle 0 \rangle) \lor \text{Start}(d,u \langle \pi+1 \rangle) ] \]

like in (iii), using 4.1(i). That is, for the \( u \) considered
\[ \neg \text{Start}(d,u^*(j)) \rightarrow \neg R_1(d,u^*(j)) \]

while by the choice of \( p \) and 4.1(i)

\[ \Pr_{L_A} s, d, u^*(p) \rightarrow \Pr_{L_A} s, d, u^*(p+1) \rightarrow \Pr_{L_A} s, d, u \rightarrow \neg \text{Start}(d,u). \]

Since this contradicts the assumed premise of \( S(u) \), one gets (*) by intuitionistic prop. logic (cf. KLEENE [52], p.119,60i,g).

(v) \( p^*d^*u \) is \([VI]\). Let \( p \) be the first numeral which does not occur in \( s^*d^*u \), and proceed to prove \( \neg \text{Start}(d,u^*(p+1)) \) like in (iii), using 4.1(ii).

(vi) \( p^*d^*u \) is \([\exists E']\), \( F^*d,u^*(0) \in \exists Cz \), where \( Cz \) is q.f.. Since \( R_1(d,u) \), i.e., \( \neg \Pr_{L_A} s, d, u \), we get from 3.2.1 \( \forall m_R_1(d,u^*(m+1)) \). \( R_2(d,u) \) implies \( \forall m_R_2(d,u^*(m+1)) \) trivially. Finally, for each \( m \) \( R_2(d,u) \) and \( Cm \) imply outright \( R_2(d,u^*(m+1)) \). Summing up we hence get

\[ (*) \text{Start}(d,u) \& \exists Cz \rightarrow \exists z \text{Start}(d,u^*(z)). \]

But by the subformula property of \( d \) \( \exists Cz \) is a subformula of the \( \Pi_2^0 \) sentence \( E^* \), and so \( E^* \rightarrow \exists Cz \), while by the assumed \( \forall n_S(u^*(n)) \),

\[ \text{Start}(d,u^*(z)) \rightarrow \neg \exists w \rightarrow u^*(z) \text{Start}(d,w). \]

So we get from (*)

\[ \text{Start}(d,u) \& E^* \rightarrow \neg \exists w \rightarrow u \text{Start}(d,w) \]

as wished. \( \square \)

4.5.1. LEMMA.

\[ \neg \text{Start}(d,u^*(j)) \rightarrow \neg R_1(d,u^*(j)) \]

\[ \rightarrow \neg \Pr_{L_A} s, d, u^*(j) \]

\[ \Pr_{L_A} s, d, u^*(0) \& \Pr_{L_A} s, d, u^*(p+1) \rightarrow \Pr_{L_A} s, d, u \rightarrow \neg \text{Start}(d,u). \]

We prove this lemma as a corollary of
4.5.2. **Lemma.** Let A be an E-sentence. Then

\[ \forall_0 \text{BI } P_{-\text{Der}}(d) \land \text{Crit}_1(d,u) \land w = u^*(m) \ast z \land \text{Selected}(d_{u^*(m)}, z) \]

\[ \land \forall u \lnot \text{Start}(d,v) \land \text{Bar}(d^w, x) \]

\[ \land \forall y \exists x \lnot \text{Final}_+^+(d^w, y, \tau_d^w u_\gamma) \]

\[ \rightarrow \lnot \text{Pr}_1 A(r_d, w_{\text{op}}^d, u_\gamma). \]

4.5.3. **Proof that 4.5.2 implies 4.5.1**

Assume the premiss of 4.5.1. For each \( m < w \) this implies the first five conjuncts of 4.5.2 for \( w = u^*(m), z = \langle \rangle \), and also

\[ \lnot \text{Pr}_1 A(r_d, u^*(m)_{\text{op}}^d, u_\gamma) \]

since \( a_d, u^*(m) = a_d, u \) here. So, by the contrapositive form of 4.5.2, and quantifying over \( m \),

\[ \forall m, x \left[ \text{Bar}(d_{u^*(m)}, x) \rightarrow \forall y \exists x \text{Final}^+(d_{u^*(m)}, y, \tau_d^w u_\gamma) \right] \]

(note that \( \text{Final}_+ \) is decidable); i.e., \( \text{Crit}_2(d,u) \) as required. \( \square \)

4.5.4. **Proof of 4.5.2**

Write \( S(w) \) for the closure over \( x,z \) of the formula to be proved. By \( \text{BI} \) the problem reduces to showing
∀n S(w*(n)) → S(w).

So assume

(1) ∀n S(w*(n)) and
(2) the premise S(w) of S(w).

Note first that the definition of Selected implies, by a trivial induction on \text{lh}(w)

(3) p^d,w \equiv p^d,u^*(m) \equiv \exists y E(m,y,n,(\overline{z}))
(4) R^d_2(d,w) \equiv "all equations in d,w are true".

Consider now cases for d,w.

(i) [\exists]. Then p^d,w \in d,w. But by the subformula property of d no \Sigma^0_1 sentence may be discharged in d, because an E-sentence has no subformula of the form G\vee H, G\rightarrow H or \exists z G where G \in \Sigma^0_1. So this case is ruled out.

A similar argument excludes the cases [\exists\exists] and [\exists\exists\exists].

(ii) [\forall]. Then d,w*(0) = d,w \rightarrow \bot, while \neg\text{Start}(d,w*(0)) implies (by (4))

\neg\text{Pr}_L^d A(\overline{\alpha}^d,w^*(0)\cdot\land),

so

\neg\text{Pr}_L^d A(\overline{\alpha}^d,w^d,u^-).

(iii) [\forall\exists]. Then (3) implies

(5) p^d,w^*(0) \equiv p^d,u.

On the other hand \neg\text{Start}(d,w*(0)) implies

(6) \neg\text{Pr}_L^d A(\overline{\alpha}^d,w^*(0)\cdot\land).

Here \overline{\alpha}^d,w^*(0) = \overline{\alpha}^d,w so (5) and (6) yield \neg\text{Pr}_L^d A(\overline{\alpha}^d,w^d,u^-).

(iv) [\exists\exists^1], p^d,w^*(0) \equiv \exists z Cz. Let \text{Bar}(d^w,x).

Subcase [a]. \langle \rangle \in x. Then \text{Final}^+(d^w,\langle \rangle, p^d,u^-) by S(w), and so by the definition of Final^+ for this case p^d,w^*(0) \equiv p^d,u, and we get as in (iii)

\neg\text{Pr}_L^d A(\overline{\alpha}^d,w^d,u^-).
Subcase [b]. \(\phi \notin x\). Then, since \(x \neq \emptyset\) by the definition of \(\text{Bar}\), we must have for some \(p\) \(\text{Step}(d^w,()_p, p)\). This means that the premiss of \(S(w^*(p))\) is satisfied, and hence by the BI hyp. (1) applied to \(w^*(p)\)

\[
\neg \forall x \forall y \forall z \forall w \forall \bar{d} \forall \bar{u} \forall \bar{v}
\]

But \(C(p)\) is here a true equation, so by 3.2.1

\[
\neg \forall x \forall y \forall z \forall w \forall \bar{d} \forall \bar{u} \forall \bar{v}
\]

(v) \(\exists \bar{d}, \exists \bar{x}, \exists \bar{w} \equiv \exists x \in \mathbb{Z}c\). Let \(\bar{p}\) be the first numeral which does not occur in \(\bar{d}, \bar{x}, \bar{w}\). We have then as in (iv)[b]

\[
\neg \forall x \forall y \forall z \forall w \forall \bar{d} \forall \bar{u} \forall \bar{v}
\]

(7) \(\neg \forall x \forall y \forall z \forall w \forall \bar{d} \forall \bar{u} \forall \bar{v}
\]

and as in (iii) we get

\[
\neg \forall x \forall y \forall z \forall w \forall \bar{d} \forall \bar{u} \forall \bar{v}
\]

which together with (7) and lemma 4.1 yields

\[
\neg \forall x \forall y \forall z \forall w \forall \bar{d} \forall \bar{u} \forall \bar{v}
\]

(vi) \(\forall \bar{z}, \exists \bar{d}, \exists \bar{x}, \exists \bar{w} \equiv G_1 \lor G_2\). Let

\[
\begin{align*}
\{ y \mid (j)*y & \in x \} \\
\end{align*}
\]

Then, by the definition of \(\text{Bar}\), \(S^-\bar{w}\) implies

\[
\text{Bar}(d^w(j), x(j)) \land \forall y \in x(j) \neg \text{Final}^+(d^w(j), y, r^d, u^\gamma)
\]

while trivially

\[
\text{Selected}(d^u, z^*(j)) \\
\]

Apply now, as in (iv) and (v), the BI hyp. (1) to \(w^*(j)\) \((j=1,2)\), to yield

\[
\neg \forall x \forall y \forall z \forall w \forall \bar{d} \forall \bar{u} \forall \bar{v}
\]

On the other hand we get as in (iii)

\[
\neg \forall x \forall y \forall z \forall w \forall \bar{d} \forall \bar{u} \forall \bar{v}
\]
which together with (8) yields

\[ \neg \exists x \in A \left( \frac{d, w \vdash d, u \vdash}{\vdash} \right). \]

(vii) [31]. Then the definition of \text{Bar} implies

\[ \text{Bar}(d, x) \rightarrow x = \emptyset. \]

For this case, Final\(d, (\emptyset, r_{d, u})\) automatically, while by \(S_{w}(\omega)\) (9) implies \(\neg\text{Final}(d, (\emptyset, r_{d, u}))\), so this case is ruled out. \(\square\)

4.6. PROPOSITION.

\[ \vdash \forall x. \text{BI} \quad \text{E-Der}(d) \quad \text{Start}(d, u) \rightarrow \forall \exists \forall v \exists \text{Crit}(d, v). \]

PROOF. Straightforward from 4.4 and 4.5.1 using BI and the well-foundedness of the proof-tree \(d\). \(\square\)

Applying proposition 4.6 to \(u = \emptyset\) we get assertion 3.4.1.

4.7 - 4.11. PROOF OF 3.4(4). (Second part of the proof-theoretic reduction)

4.7. LEMMA.

\[ \vdash \forall x. \text{BI} \quad \text{E-Der}(d) \quad \text{E}_{w}(d, w) \rightarrow \exists x \text{Bar}(d, x). \]

PROOF. Straightforward by BI and the well-foundedness of \(d\). \(\square\)

4.8.1. Let us use the following ad hoc terminology.

"x is a bar" := \(\forall u, v \exists x. u \vdash x \vdash v\)

"x is in d" := \(\forall v \exists (d) \vdash 0\)

\(x \vdash y := \text{"x and y are bars" &} \quad \forall v \exists x (v \vdash u) \quad \forall u \exists x (v \vdash u)\)

\(x \vdash y := \text{"x and y are in d" &} \ x \vdash y. \)
LEMMA (in $\forall^*_0+\text{BI}$). If $d$ is well-founded, i.e.,
$$\forall x \exists n \{ d \upharpoonright (n) \approx 0,$$
then $\|d\|$ is well-founded:
$$\forall \psi \exists n \lceil \psi(n) \|d\| \psi(n+1) \rceil.$$

PROOF. Apply $\text{BI}$ to
$$S(u) := \forall \psi \{ \{ (u) \} \|d\| \psi(0) \rightarrow \exists n \lceil \psi(n) \|d\| \psi(n+1) \rceil \}.$$

4.8.2. PROPOSITION.

$\forall^*_0+\text{BI} \vdash E-\text{Der}(d) \land \text{Crit}(d,u) \rightarrow$

$$\forall m \exists \omega \text{ Final}^{++}(d_u^*(m), \omega, r^d, u^{*m}).$$

PROOF. Assume $E-\text{Der}(d)$ and $\text{Crit}(d,u)$, and fix $m$. We shall apply $\text{BI}$ to the
well-ordering $\|d\|$ of 4.8.1. We prove below that

(1) \( \forall y \exists x S(y) \rightarrow S(x) \)

for

$$S(x) := \text{Bar}(d_u^*(m), x) \rightarrow \exists \omega \text{ Final}^{++}(d_u^*(m), \omega, r^d, u^{*m}).$$

Then, by $\text{BI}$ on $\|d\|$ applied to $S$ we get

(2) \( \forall x S(x) \).

But by 4.7 we have

(3) \( \text{Bar}(d_u^*(m), x) \) for some $x$

and so by (2)

$$\exists \omega \text{ Final}^{++}(d_u^*(m), \omega, r^d, u^{*m})$$
as required.

Towards proving (1) assume \( \forall y \, d_y \mid x \) \( S(y) \) and the premiss of \( S(x) \), \( \text{Bar}(d^{u\ast}(m), x) \). By Crit\(_2\)(d,u) then

\[
(4) \quad \exists w \in x \text{ Final}^+(d^{u\ast}(m), w, \text{F}^{d, u\gamma}).
\]

Observe the two possible cases for \( p \, d, u\ast(m)w =: \rho \).

(1) If \( p = [\exists \xi] \), \( F^{d, u\ast(m)w}(0) \equiv: \exists Cz \), assume

\[
(5) \quad \exists Cz \lor \neg \exists Cz
\]

If \( \neg \exists Cz \) then Final\(_2\)++(\( d^{u\ast(m)}, w, F^{d, u} \)) by definition. If \( \exists Cz \) let \( p := u\ast Cz \).

by 4.7., for some \( y \)

\[
\text{Bar}(d^{u\ast(m)}w\ast(p\ast), y).
\]
Let
\[ y' \text{ encode the set } \{ w*(p+l)*v \mid \forall v \} ; \]
then we have quite trivially
\[ \text{Bar}(d^{u*(m)}, x') \]
for \( x' := (x\{w\}) \cup y' \) (set theoretic operations), and also \( x' \subseteq x \).
So by BI hyp.

Here (6) depends on (5), but since (6) is negated (5) is eliminable (cf. KLEENE [52], p.119 *58b-c,*51a).

(ii) If \( \rho \) is [SI] by our definition of normality (cf. A.1.1) \( \rho^{d,u*(m)}*w*(0) \)
cannot be other than [T]. But we have \( R_2(d,u*(m))*w*(0) \) because of
\[ R_2(d,u*(m)) \text{ and } \text{Selected}(d^{u*(m)},w) . \]
Hence \( TR_{QF}(d^{u*(m)}*w*(0)) \) and so \( \text{Final}^{++}(d^{u*(m)}, w, f^{d,u}) \).

4.9. LEMMA. There are prim.rec. functions \( f_j \) (j=2,3) s.t.

\[ \forall \sigma \in E(d) \land \text{Final}^{++}(d,v, E^N_{\downarrow}(\tau)) \land \text{Ineq}(d,v) \Rightarrow E^N_f(\tau). \]

PROOF.

(i) Let \( \{ f_2(d,v,(i,n,(\tau))) \} \) describe the tree
\[ [T] B[.] \Rightarrow B[(i,n,\hat{\tau})] \]

\[ [\text{VE}] B[.] \Rightarrow \text{Ineq}((j,k,\hat{\tau}), (i,n,(\tau))) \Rightarrow f^{d,v}*(0,0) \]

\[ [\text{TE}] B[.] \Rightarrow \text{Ineq}(\hat{\tau}) \]

\[ [-\exists E] B[.] \Rightarrow f^{d,v}*(0,0) \]

\[ [\text{VE}] B[.] \Rightarrow f^{d,v}*(0) \]

\[ \{ f^M \}_{m<\omega} \]

\[ [\exists E] B[(i,n,\hat{\tau})] \Rightarrow f^{d,v} \]
where \( P^{d,v}(0,0) \equiv E^k_j(s) \) and \( P^{d,v}(0) \equiv \exists \exists C \), and where

\[
[T] \quad B[i,n,\langle \vec{c} \rangle], \exists m \Rightarrow \exists m
\]

\[\Gamma \equiv [FE] \quad B[i,n,\langle \vec{c} \rangle], \exists m \Rightarrow 1\]

\[\ll [B[i,n,\langle \vec{c} \rangle], \exists m \Rightarrow P^{d,v}\]

(ii) Let \( \{f_j(d,v,\langle i,n,\langle \vec{c} \rangle \})\} \) describe the tree

\[
[TE] \quad B[i,n,\langle \vec{c} \rangle] \Rightarrow P^{d,v}(0)
\]

\[\ll [B[i,n,\langle \vec{c} \rangle] \Rightarrow P^{d,v}\]

\( f_j(\ldots) \) are indices of functions recursive in \( [d] \), and by the s.m.n.- theorem \( f_j \) are indeed prim.rec. functions. The proof of the lemma for these functions is now straightforward. The only less trivial detail is the correctness of the [TE] inferences in the definition of \( f_2 \). From \( \text{Final}_2^{++}(d,v,\bar{E}^n_i) \) we only know that \( E^k_j(s) \) and \( E^n_i(\bar{c}) \) are not syntactically identical, but this does not exclude, prima facie, that \( s \) and \( \bar{c} \) are numerically equal. Recall, however, that by our definition of \( \text{EDer} \) in 3.1 \( \bar{c} \) and \( s \) are tuples of numerals, and therefore their numerical equality would imply their syntactical identity. \( \square \)

4.10. COROLLARY.

\[ \vdash_0 \text{BI} \quad \text{E-Der}(d) \quad \& \quad \text{Crit}(d,u) \quad \& \quad P^{d,u} \equiv E^u_n(\bar{c}) \]

\[ \rightarrow \forall x \exists m \forall \text{Prf-rec}(x, B[i,n,\langle \vec{c} \rangle]) \forall P^{d,u}(m) \].

PROOF. Immediate from 4.8 and 4.9. \( \square \)

4.11. PROPOSITION (\( = 3.4.(4) \)).

\[ \vdash_0 \text{BI-AC} \quad \text{E-Der}(d) \quad \& \quad \text{Crit}(d,u) \quad \& \quad P^{d,u} \equiv E^u_n(\bar{c}) \]

\[ \rightarrow \forall x \exists \phi \forall \text{Prf}(\phi, B[i,n,\langle \vec{c} \rangle]) \].
PROOF. Assume the premise; then by 4.10

$$\forall m \exists x \text{NPref}^\omega \text{rec} \left( x, \text{B}(i,n,(\vec{c})) \Rightarrow d, u \ast (m) \ast \gamma \right)$$

and so by AC\text{C}^-\text{rec} \nobreak\smallskip

(1) $$\forall \exists \text{NPref}^\omega \text{rec} \left( \psi m, \text{B}(i,n,(\vec{c})) \Rightarrow d, u \ast (m) \ast \gamma \right).$$

Define now $\phi$ recursively in $\phi$:

$$\phi() := \left( \forall \exists \text{S}(i,n,(\vec{c})) \right)^\gamma$$

$$\phi((m) \ast u) := \{ \psi m \}(u).$$

The matrix of the positive form of (1) for $\psi$ obviously implies

$$\text{NPref}^\omega (\psi, \text{S}(i,n,(\vec{c})))^\gamma,$$

and so (1) implies the succedent of the proposition. □
B.5. SOLUTION OF THE REDUCED PROBLEM FOR L₁ (proof of 3.5(10))

5.1. PROPOSITION (≡ 3.5(10)). Let S be a $\Sigma^0_2$ enumerated theory (with provability-predicate $\exists x y \text{Prf}_S(x,y,F)$; say) which is $\Sigma^0_2$ complete. Then there is a q.f. formula $E(x)$ s.t., in the notation of 3.5,

$$\vdash \text{A} \Rightarrow \text{Con}(S) \Rightarrow \text{Comp} \Sigma^0_2(S) \Rightarrow \forall x \neg \text{Prf}_S \Rightarrow E(x) \Rightarrow \neg \Rightarrow .$$

The proof given below is based on Kripke [63].

5.2. LEMMA. For S as above, there exists a $\Sigma^0_2$ predicate $J(x)$ s.t.

(i) $\vdash \forall x,y [ J(x) \land J(y) \rightarrow x = y ]$

(ii) $\vdash \forall x \neg J(\tilde{x})$ for every numeral $\tilde{x}$.

PROOF. Let $\text{neg}$ and $\text{sub}_2$ be prim.rec. functions s.t. for every formula $F$

$$\text{neg}('F') = ' \neg F$$

$$\text{sub}_2('F',x,y) = 'F[x/a][\tilde{y}/b]'$$

where $\tilde{x}$ is the numeral with value $x$, and where $F[t/a]$ is the formula which comes from $F$ by replacing every occurrence of the parameter $a$ by (the closed term) $t$. Define

$$K(x,n,m) :\equiv \forall y \text{Prf}_S \Rightarrow (x,y,\text{neg}(\text{sub}_2(n,n,m)))$$

$$L :\equiv L(a,b) :\equiv \exists x [ K(x,a,b) \land \forall z < x \forall w < z \neg K(z,a,w) ]$$
J(m) := L("L",m)  (here the g.n. "L" is the code of the fixed formula L(a,b), while the defining symbol L is understood as a predicate)

We may assume w.l.g. that the g.n. of a proof is larger than the g.n. of the formula it derives, because \( \text{Prf}_S \) can be replaced by \( \text{Prf}'_S(x,y,z) := \exists x' < x \text{Prf}^S(x',y,z) \land x = 2^{x'} \cdot 3^y \). This change is harmless in all other respects. Hence

(1) \( L(m,n) \leftrightarrow L^*(m,n) \)

is provable in A, where \( L^* \) is defined like \( L \) except that the bounded quantifier \( \forall w < z \) is replaced by an unbounded \( \forall w \); and so

(2) \[ A \forall x,y \{ J(x) \land J(y) \rightarrow x = y \} \].

Next suppose

(3) \[ \neg J(m) \]

for some \( m \), i.e.,

\[ \exists m \exists x \forall y \text{Prf}^S(x,y,\neg L("L",m)) \].

Then

(4) \[ \forall m \exists x [ \forall y \text{Prf}^S(x,y,\neg L("L",m)) \land \forall z < x \forall w \forall y \neg \text{Prf}^S(z,y,\neg L("L",w)) ] \]

which is just \( \forall m \ L("L",m) \) by (1) and the definition of \( L \).

But by \( \text{Comp}_2(S) \)

(5) \[ \forall m [ L("L",m) \rightarrow \exists x \forall y \text{Prf}^S(x,y,\neg L("L",m)) ] \],

while the definition of \( L \) implies

(6) \[ \forall m [ L("L",m) \rightarrow \exists x \forall y \text{Prf}^S(x,y,\neg L("L",m)) ] \],

so (4), (5), (6) together imply \( \forall m \forall y \text{Prf}^S(x,y,\neg L) \), contradicting \( \text{Con}(S) \).
5.3.1. LEMMA. For $S$ as above there is a $\Sigma^0_2$ predicate $M(x)$, s.t. for every q.f. predicate $P(x)$

$$\vdash_S \neg \forall x [ M(x) \iff P(x) ].$$

PROOF. Let $U(n,x)$ be a binary q.f. predicate which enumerates all unary q.f. predicates (by Kleene's enumeration theorem, cf. e.g. KLEENE [52], §58), and let $J$ be as in 5.2. Define

$$M(x) := \exists y [ J(y) \land U(y,x) ].$$

By 5.2(i) then

$$J(m) \vdash_A \forall x [ M(x) \iff U(m,x) ] \quad \text{for every numeral } m.$$

But by 5.2(ii)

$$\vdash_S \neg J(m),$$

so

$$\vdash_S \neg \forall x [ M(x) \iff U(m,x) ] \quad \text{for every } m, \text{ as desired.}$$

5.3.2. LEMMA. Lemma 5.3.1 holds also when $M$ is required to be $\Pi^0_2$.

PROOF. Replace the $M \equiv \exists y \forall z M_0(x,y,z)$ defined above by $\forall y \exists z M_0(x,y,z)$. □

5.4. PROOF OF 5.1 (concluded). Let $M(x)$ be given by 5.3.2, and write $M(z)$ as $\forall x \exists y E(x,y,z)$.

(i) Assume now $\Pr_S s^F(n)$ for some $n$ (i.e., $\exists x \forall y \Pr_S (x,y,s^E(n))$).

By the form of the sequent $s^F(n)$ we have then

$$\vdash_S \forall z \neg \forall u M(z) \rightarrow M(n)$$

and therefore

$$\vdash_S \forall z [ \neg z \neq n \rightarrow M(z) ]$$

contradicting 5.3.2.

(ii) Assume $\neg E^*$, i.e., $\neg \forall z M(z)$. Then, by $\text{Comp} \ thwart(S)$, $\neg \Pr_S (\neg \forall z M(z))$.

But taking $P(z) := z \neq z$ in 5.3.2 we get $\vdash_S \neg \forall z M(z)$, a contradiction. So $\neg E^*$. □
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INDICES

References are to sections. When a number of references is given for the same item the most relevant one is occasionally underlined.

Index of notions

Here are given the ad hoc notions of this dissertation together with some terms of general use.

Absoluteness
  absolute schema
  absolute logic
antecedent
base, basing function
completeness theorem
conclusion (of a sequent)
critical inference rules
cut
cut elimination
derivability conditions
derivation (infinitary)
disjunction instantiation property
effective inseparability
E-sentence, E-atom, E-derivation
formal occurrence
incompleteness theorem
indexed formula
influence
inference rules
  arithmetical
  critical
  propositional
  quantification
  second order
Kreisel-Shoenfield-Wang theorem
Kripke models
Löb theorem
normal derivation
normalization

ordinals, ordinal notations, ordinal assignments

reduction steps

absurdity
detour
generalized
improper
permutative
proper
second order
simplification

reflection principle

regular theory

strongly regular theory

replacement rule

sentence

sequent

stable derivation

strongly stable

stable under...

stable at...under...

subformula (negative, positive)

subformula property

succedent (of a sequent)

transfinite induction

transfinite progression

truth definition

well-foundedness

ω-rule

Index of Formal theories (script majuscules)

A
A^m
A^n
A^ω
A^ω^n
A^1
A^k
A^l

Int.3, Int.6, A.3, A.4

Int.6
A.3.1
A.3.3.4.
A.3.3.2.
A.3.5, A.4.4.
A.3.5.3.
A.3.3.3.
A.3.3.
A.4.2.
A.3.5.3.
Int.3, TN1
Int.4, A.1.2.
Int.4, A.1.2
A.3.2.
A.1.1.
A.1.1, A.3.2.
A.3.5.
A.3.5.4.
A.3.7.1.
A.3.7.1.
P.1
Int.3, Int.6, TN1, TN2,
A.1.1, A.3.8.
A.1.1.
Int.4, TN3, TN4, A.2.3.
Int.4
TN1, A.3.5.2, B.4.3.
A.1.1.
Int.3

P.2.
A.1.1.
Int.1
Int.2
B.1.0.
B.3.3.
\[
\begin{align*}
A[T] & \quad \text{Int.4} \\
A_p[T] & \quad \text{Int.4} \\
A^{\infty}[T] & \quad \text{Int.4, A.1.2.} \\
A_{rec}[T] & \quad \text{Int.4, A.1.2.} \\
L_l & \quad \text{P.2.} \\
L_2 & \quad \text{A.4.0, A.4.1.} \\
L_{2, rec} & \quad \text{A.4.9.} \\
L_\omega & \quad \text{Int.6, A.4.0.} \\
\gamma & \quad \text{P.2.} \\
\end{align*}
\]

**Index of formal schemata** (bold-face majuscules)

- \(AC_{00}\) \quad P.3.
- \(AC_{00}\) \quad A.1.2.
- ACA \quad Int.2.
- BI \quad P.3.
- \(M_{PR}\) \quad Int.1.
- TI \quad P.3, TN3, A.2.3.

**Index of formal sentences, predicates and functions**

(Standard lettertype, underlined when more than one letter is used. Predicates and sentences start with a capital letter, functions do not – with the exception of Kleen's result-extracting function \(U\)).

- \(A_n^S\) \quad B.6.1.
- Abs \quad B.6.3.
- Bar \quad B.4.3.
- \(C_n\) \quad B.6.1.
- Clear \quad A.3.4.3.
- CMP \quad B.3.3.
- Comp \quad B.3.5.
- Con \quad A.2.2, B.3.3.
- Crit \quad B.3.4, B.4.2.2, B.4.3.
- Crit \quad B.4.2.
- Crit \quad B.4.3.
- Cut \quad A.3.3.
- Der \quad P.4.
\[
\begin{align*}
\text{Der}_m^0, \text{Der}_m^\text{rec} & \quad \text{A.1.1}, \text{A.4.1} \\
E_{m,n}, E[F_m,F_n,P,Q] & \quad \text{B.6.1} \\
E-\text{Der}, E-\text{Prf} & \quad \text{B.3.1.2} \\
\text{Eq} & \quad \text{B.6.1} \\
\text{Final}, \text{Final}^*_1, \text{Final}^*_2, \text{Final}^*_3, \text{Final}^{**}_1, \text{Final}^{**}_2 & \quad \text{B.4.3} \\
\end{align*}
\]

\[
\begin{align*}
\text{imp} & \quad \text{A.1.2} \\
\text{Infl} & \quad \text{A.3.4.1} \\
\{j\} & \quad \text{A.3.4.3} \\
\{k\} & \quad \text{A.3.4.3} \\
\{l\} & \quad \text{A.3.4.3} \\
l_0-\text{Fml}, l_1-\text{Fml} & \quad \text{B.0} \\
\text{Ith} & \quad \text{P.4} \\
\text{M} & \quad \text{A.4.4} \\
\{n\}, \{n\} & \quad \text{A.3.4.3} \\
\text{NDer}_m^0, \text{NDer}_m^\text{rec}, \text{NPrf}_m^0, \text{NPrf}_m^\text{rec} & \quad \text{A.1.1} \\
\text{neg} & \quad \text{B.5.2} \\
\text{Nmble} & \quad \text{A.3.4.3} \\
\text{Pr}, \text{Prf} & \quad \text{P.4} \\
\text{Prf}_m^0 & \quad \text{Int.4, A.1.1} \\
\text{Prf}_m^\text{rec} & \quad \text{A.1.1} \\
\{r_0\}, \{r\} & \quad \text{A.3.4.3} \\
\text{R}_1 & \quad \text{B.4.2} \\
\text{Res} & \quad \text{B.3.4} \\
\text{Rfn} & \quad \text{B.3.3} \\
S & \quad \text{B.6.1} \\
\text{Sp}_1 & \quad \text{Int.3, TN1} \\
\text{St} & \quad \text{B.6.1} \\
\text{St}, \text{St}_n & \quad \text{A.3.5}, \text{A.4.4} \\
\text{Start} & \quad \text{B.4.2} \\
\text{Step}, \text{Step}_1 & \quad \text{B.4.3} \\
\text{Step}_2 & \quad \text{A.2.3} \\
\text{Selected} & \quad \text{B.2.2, B.4.3} \\
\end{align*}
\]
Subordinated

\[ T, T^\phi \]
\[ \text{tail} \]
\[ \text{TeQP} \]
\[ U \]
\[ U_k \]
\[ \omega_i \]
\[ \text{WF} \]
\[ Z \]
\[ \mu \]
\[ \nu \] (counting propositional letters)
\[ \nu \] (enumerating an \( w \)-model)

Index of special symbols

\[ [T], [\&I], [\&E_i], [\&I], [\rightarrow], [\rightarrow E_i], [\forall I_i], [\forall E], [i] \] A.1.1.
\[ [\&E], [\&E], [\forall I], [\forall E], [\exists I], [\exists E] \]
\[ [\forall^2 I_i], [\forall^2 E] \] A.4.1.
\[ [\exists^2 E_i], [\exists^2 E_i] \]
\[ [R] \] A.3.2.
\[ \vdash \]
\[ \vdash \]
\[ (,) , *, \neg, (,)^\iota, !, !!, \epsilon, (), ()^\emptyset, \approx \]
\[ (n)^\iota \]
\[ x \{j\} \]
\[ (v_i) \]
\[ (\delta F) \]
\[ [ ] \]
\[ \phi \]
\[ \phi_i, \phi_a \]
\[ \equiv \]
Index of metamathematical operations

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$\rho^t$, $s^t$, $\rho^t_u$, $s^t_u$, $\rho^t_e$, $s^t_e$

$\rho$, $s$, $\rho_u$, $s_u$, $\rho_e$, $s_e$

$\rho^d$, $s^d$, $\rho^d_u$, $s^d_u$, $\rho^d_e$, $s^d_e$

$\delta^d$, $\delta^d_u$, $\delta^d_e$, $\delta^d_{n^d}$

$\epsilon^E$, $\epsilon^F$

$\epsilon_t^n$, $\epsilon_1^*$

$E^t[w]$, $s^t[w]$

$\beta^*$

$\tau^*$

$\gamma^C$

$G^{\omega}$, $G^e$, $N_e$, $G^e_N_e$

B.6.3.