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There exist exactly two Maximal Strictly Relevant Extensions of the Relevant Logic $R^*$

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There exist exactly two maximal strictly relevant extensions of the relevant logic $R^*$

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Abstract
In [60] N. Belnap presented an 8-element matrix for the relevant logic $R$ with the following property: if in a implication $A \rightarrow B$ the formulas $A$ and $B$ do not have a common variable then there exists a valuation $\nu$ such that $\nu(A \rightarrow B)$ does not belong to the set of designated elements of this matrix. Below we present a 6-element matrix with the same properties and prove that the logics generated by these two matrices are all maximal extensions of the relevant logic $R$ which have the relevance property: if $A \rightarrow B$ is provable in such a logic then $A$ and $B$ have a common propositional variable.

1 Preliminaries. $C_R$-matrices.

Let a set of propositional variables $p, q, r, \ldots$ be given and let $F$ be the set of propositional formulae built up from propositional variables by means of the connectives: $\rightarrow$ (implication), $\land$ (conjunction), $\lor$ (disjunction) and $\neg$ (negation). The Anderson and Belnap logic $R$ with relevant implication (cf. A. R. Anderson, N. D. Belnap [75]) is defined as the subset of propositional formulae of $F$ which are provable from the set of axiom schemes indicated below, by application of the rule of Modus Ponens (MP; $A, A \rightarrow B \rightarrow B$) and the Rule of Adjunction ($A, B / A \land B)$:

$A1. \ A \rightarrow A$
$A2. \ (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
$A3. \ A \rightarrow ((A \rightarrow B) \rightarrow B)$
$A4. \ (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
$A5. \ A \land B \rightarrow A$
$A6. \ A \land B \rightarrow B$
$A7. \ (A \rightarrow B) \land (A \rightarrow C) \rightarrow (A \rightarrow B \land C)$
$A8. \ A \rightarrow A \lor B$
$A9. \ B \rightarrow A \lor B$

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A10. \((A \rightarrow B) \land (C \rightarrow B) \rightarrow (A \lor C \rightarrow B)\)
A11. \((A \land (B \lor C)) \rightarrow ((A \land B) \lor C)\)
A12. \((A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)\)
A13. \(\neg \neg A \rightarrow A\).

**Lemma 1** The formulae listed below are theses of \(R\):

- \((p \rightarrow q) \land (r \rightarrow s) \rightarrow (p \land r \rightarrow q \land s)\),
- \((p \rightarrow q) \land (r \rightarrow s) \rightarrow (p \lor r \rightarrow q \lor s)\),
- \((p \lor q \rightarrow r) \rightarrow (p \rightarrow r)\),
- \((p \rightarrow q \land r) \rightarrow (p \rightarrow r)\),
- \((p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))\).

A **matrix** is a pair \(\langle A, \nabla_A \rangle\) where \(A\) is an algebra while \(\nabla_A\) is a subset of the domain of \(A\). To the logic \(R\) and its extensions we can associate a set of so-called \(C_R\)-matrices (cf. W. Dziobiak [83]); their characterization is given by the following

**Theorem 2 (W. Dziobiak (83), L. Maximowa (73))** Let 
\(A = \langle A, \rightarrow, \land, \lor, \neg \rangle\) be an algebra similar to \(F\) and let \(\nabla_A\) be a subset of \(A\). Then the following conditions are equivalent:

(i) \(\langle A, \nabla_A \rangle\) is a \(C_R\)-matrix,
(ii) \(\langle A, \land, \lor \rangle\) is a distributive lattice with \(\land\) and \(\lor\) as its meet and join, respectively, and \(\nabla_A\) is a filter on \(A\) with the property: for all \(a, b \in A, a \land b = a\) iff \(a \rightarrow b \in \nabla_A\); and moreover, the following conditions are satisfied for all \(x, y, z\) of \(A\),

- \((c1) \ (x \rightarrow y) \leq (y \rightarrow z) \rightarrow (x \rightarrow z)\),
- \((c2) \ x \leq (x \rightarrow y) \rightarrow y\),
- \((c3) \ x \rightarrow (x \rightarrow y) \leq x \rightarrow y\),
- \((c4) \ (x \rightarrow y) \land (x \rightarrow z) \leq x \rightarrow (y \land z)\),
- \((c5) \ (x \rightarrow z) \land (y \rightarrow z) \leq (x \lor y) \rightarrow z\),
- \((c6) \ x \rightarrow \neg y \leq y \rightarrow \neg x\),
- \((c7) \ \neg \neg x = x\),

where \(\leq\) is ordering of the lattice \(\langle A, \land, \lor \rangle\).

Let us add some additional properties of \(C_R\)-matrices:

**Lemma 3 (L. Maximowa (73))** Let \(\langle A, \nabla_A \rangle\) be a \(C_R\)-matrix and let the relation \(\leq\) be defined as follows: \(x \leq y \) iff \(x \rightarrow y \in \nabla_A\).

Then the relation \(\leq\) satisfies the following implications and inequalities:

(i) \(\text{if } x \in \nabla_A \text{ then } x \rightarrow y \leq y\)
(ii) \(\text{if } x \leq y \text{ then } y \rightarrow z \leq x \rightarrow z\)
(iii) \(\text{if } x \leq y \text{ then } z \rightarrow x \leq z \rightarrow y\)
(iv) \(x \rightarrow \neg x \leq \neg x\).

Let us moreover quote a lemma and a proposition proved by W. Dziobiak, which are important for our further investigations. Let \(\langle A, \nabla_A \rangle\) be a \(C_R\)-matrix and let \(X \subseteq A\). By \([X]\) we shall denote the least filter on \(A\) containing \(X\). Moreover, a filter \(\nabla\) on \(A\) will be called **normal** iff \(\nabla \subseteq \nabla_A\).

We have first
Lemma 4 (W. Dziobiak (83)) Let $A = (A, \nabla_A)$ be a $C_R$-matrix. Then

(i) $\nabla_A = \{\{a \rightarrow a : a \in A\}\}$,

(ii) If $A$ is generated by elements $a_0, \ldots, a_{n-1}$ then

$\nabla_A = \{\bigwedge_{i < n} (a_i \rightarrow a_i)\}$.

It is known that the set of all $C_R$-matrices forms a variety (cf. W. Dziobiak [83])$^1$; algebras which belong to this variety can be called $R$-algebras. Observe that (cf. Lemma 4) each $R$-algebra determines a $C_R$-matrix. Moreover, the logic $R$ is algebraizable (cf. W. J. Blok and D. Pigozzi [89]), thus in particular the lattice of congruences of each $R$-algebra is isomorphic to the lattice of its normal filters (cf. also W. Dziobiak [83]).

However, the notion of a filter of designated elements plays a fundamental role in this paper and thus we decided to exercise the notion of $C_R$-matrix rather than the notion of $R$-algebra.

Now we have the following

Theorem 5 (W. Dziobiak, unpublished) Each finitely generated $C_R$-matrix has the least and the greatest element which form a $C_R$-matriz isomorphic to the two-element matrix $2$.

Proof: Let us denote by $R$ the variety of $C_R$-matrices and by $2$ the two-element $C_R$-matrix. It is known that $2 \in R$. Let $F_R(n)$ be the $R$-free algebra over the set $n$ of free generators. Of course $2 \in H(F_R(n))$ for each natural $n$. Thus there exists a normal filter $\nabla$ on $F_R(n)$ such that $2 \cong (F_R(n))/\nabla$. Since $2$ is finite, by the Rival-Sands Theorem (cf. I. Rival and B. Sands [78]) the filter $\nabla$ is a principal filter, e.g. $\nabla = \{\{a\}\}$ for some $a$. But $\{\{a\}\}$ is a proper normal filter (because $2$ is not trivial) thus there exists a $b$ such that $b \notin \{\{a\}\}$ and in consequence $b \land a \leq a$, but $b \land a \neq a$. Thus there exist an element below the element $a$. It is easy to observe that there exists exactly one such an element, because $2 \cong (F_R(n))/\nabla$. Let us denote it by $0$. Without any difficulties we can prove that $0$ is the least element in $F_R(n)$; similarly the element $1 = \neg 0$ is the greatest element. Now let $A$ be a finitely generated (e.g. $n$-generated) $C_R$-algebra. Of course, $A \in H(F_R(n))$, i.e $A \cong (F_R(n))/\nabla$, thus $A$ must contain 1 and 0.

2 The Belnap matrix $M_8$.

Denote by $M_8$ the matrix $\langle\{0, a, \neg a, b, \neg b, a \land b, \neg a \lor \neg b, 1\}, \rightarrow, \land, \lor, \neg, \{a, b, a \land b, 1\}\rangle$ whose lattice operations $\land$ and $\lor$ are defined here as it is shown in the following diagram:

$^1$An equational characterization of this variety can be found e.g.in: J. M. Font and G. Rodriguez [90].
and whose operations $\rightarrow$ and $\neg$ are defined by the following tables:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>$a$</th>
<th>$\neg a$</th>
<th>$b$</th>
<th>$\neg b$</th>
<th>$a \land b$</th>
<th>$\neg a \lor \neg b$</th>
<th>1</th>
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</thead>
<tbody>
<tr>
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<td>1</td>
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<td>$a$</td>
<td>0</td>
<td>$a$</td>
<td>$\neg a$</td>
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<td>$\neg a$</td>
<td>1</td>
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<tr>
<td>$\neg a$</td>
<td>0</td>
<td>$a$</td>
<td>$a$</td>
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<td>$a$</td>
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<tr>
<td>$b$</td>
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<td>$b$</td>
<td>0</td>
<td>$\neg b$</td>
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<td>$\neg b$</td>
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<td>$b$</td>
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<td></td>
</tr>
<tr>
<td>$a \land b$</td>
<td>0</td>
<td>$a$</td>
<td>$\neg a$</td>
<td>$b$</td>
<td>$a \land b$</td>
<td>$\neg a \lor \neg b$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\neg a \lor \neg b$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$a \land b$</td>
<td>1</td>
<td></td>
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<td>1</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\neg x$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
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<tr>
<td>$a$</td>
<td>$\neg a$</td>
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<tr>
<td>$b$</td>
<td>$\neg b$</td>
</tr>
<tr>
<td>$a \land b$</td>
<td>$\neg a \lor \neg b$</td>
</tr>
</tbody>
</table>

It is not difficult to prove that $\mathcal{M}_8$ is a $C^R$ matrix; it is just the matrix presented by N. Belnap in [60]. We have changed, however, the symbols used by N. Belnap in [60]; since the algebra of this matrix is 2-generated, however, we have reduced the number of basic symbols to two symbols to make the lattice connections and negation connections in this matrix more suggestive.

For this matrix the following is true:

**Theorem 6 (Belnap (60))** If in $A \rightarrow B$ the sets of variables of $A$ and $B$ are disjoint then there exists a valuation $h^v$ such that $h^v(A \rightarrow B)$ does not belong to the set of designated elements of $\mathcal{M}_8$.

To prove this Theorem it suffices to note that the valuation function $h^v$ can be defined as the homomorphic extension of a function $v$ defined as follows: if $p_i$ occurs in $A$ then we put $v(p_i) = a$ or $v(p_i) = \neg a$, and if $p_i$ occurs in $B$ then we put $v(p_i) = b$ or $v(p_i) = \neg b$. It is easy to check that $h^v(A \rightarrow B)$ cannot belong to the set of designated elements of $\mathcal{M}_8$. 
Let us observe that this proof is based on the fact that \( M_8 \) has two disjoint and \( \leq \)-incomparable submatrices with universes \( \{ a, \neg a \} \) and \( \{ b, \neg b \} \), respectively (they are isomorphic to the matrix 2, of course).

This observation justifies introducing the following notion. Let \( \langle A, \sqcup_A \rangle \) be a \( C_R \)-matrix and let \( B_1, B_2 \) be subalgebras of the algebra \( A \); let \( \leq \) be the partial order which defines the lattice of the algebra \( A \). The subalgebras \( B_1, B_2 \) will be called \( \leq \)-incomparable (i.e. incomparable with respect to the relation \( \leq \)) if for any \( a, b \), if \( a \in B_1, b \in B_2 \), then neither \( a \leq b \) nor \( b \leq a \).

An implication \( A \rightarrow B \) will said to be a relevant implication if the intersection of the set of variables occurring in \( A \) and the set of variables occurring in \( B \) is nonempty; in the opposite case this implication is said to be non-relevant. Moreover, we will say that the relevance principle hold for a logic \( L \) if \( L \) does not contain any non-relevant implication. Thus Belnap’s Theorem just quoted above states that the matrix \( M_8 \) falsifies all non-relevant implications and that if a logic \( L \) is contained in the logic determined by the Belnap’s matrix \( M_8 \) then the relevance principle holds for \( L \).

The definition of the next notion needs some assumptions. Let \( C \)-matrix \( \langle C, \sqcup_C \rangle \) contain two \( \leq \)-incomparable submatrices: \( \langle C_1, \sqcup_{C_1} \rangle \) and \( \langle C_2, \sqcup_{C_2} \rangle \). Moreover, let each non-relevant implication \( A \rightarrow B \) be falsified in \( \langle C, \sqcup_C \rangle \) by any valuation \( h^v \) which is the homomorphic extension of a valuation \( v \) such that \( v(p_i) \in C_1 \) if \( p_i \) is a variable which occurs in the formula \( A \) and \( v(p_j) \in C_2 \) if \( p_j \) occurs in the formula \( B \). Then the submatrices \( \langle C_1, \sqcup_{C_1} \rangle \) and \( \langle C_2, \sqcup_{C_2} \rangle \) will be called falsifying submatrices of the matrix \( C \)-matrix \( \langle C, \sqcup_C \rangle \).

**Lemma 7 (on the matrix \( M_8 \))** Let \( C = \langle C, \sqcup_C \rangle \) be a \( C_R \)-matrix. Let \( \leq \) be the partial ordering relation which defines the lattice of the algebra \( C \) and let the algebra \( C \) has two \( \leq \)-incomparable subalgebras \( A, B \) with units and zero’s; let us denote by \( a, \neg a \) the unit and the zero of \( A \) and by \( b, \neg b \) - the unit and the zero of \( B \) and let \( a \neq \neg a \) and \( b \neq \neg b \). Then if \( C \) satisfies the equalities:

\[
(a \rightarrow b) = (b \rightarrow a) = (b \rightarrow \neg a) = (\neg a \rightarrow b) = \neg a \land \neg b
\]

then the submatrix of the matrix \( C \) generated by the elements \( a, b \) (i.e. the matrix \( \langle [a, b]_C, \sqcup_{[a, b]_C} \rangle \)) is isomorphic to the matrix \( M_8 \).

**Proof:** Let \( [a, b]_C \) be the subalgebra of the algebra \( C \) generated by \( a, b \). By the assumptions it is known that \( \neg a < a, \neg b < b \) and moreover the following equalities hold in \( C \) (thus in \( [a, b]_C \) as well):

\[
\begin{align*}
(a \rightarrow a) & = a, \\
(a \rightarrow \neg a) & = a, \\
(b \rightarrow a) & = \neg a, \\
(b \rightarrow b) & = b, \\
(a \land b) & = a \land b
\end{align*}
\]

By the W. Dziołak’s theorem (cf. Lemma 4, above) the filter \( \sqcup_{[a, b]_C} \) of designated elements of the matrix \( \langle [a, b]_C, \sqcup_{[a, b]_C} \rangle \) is the filter \( \langle (a \rightarrow a) \land (b \rightarrow b) \rangle \).

The algebra \( [a, b]_C \) contains, of course, the elements \( a, \neg a, b, \neg b, a \land \neg b, a \land b, \neg a \land b, a \lor b, a \lor \neg b, \neg a \lor b, \neg a \lor \neg b, (a \land b) \). Since the subalgebras \( [a]_C, [b]_C \) of the algebra \( C \) are two-element \( \leq \)-incomparable algebras, the lattice operations and the operation \( \neg \) reduce the number of elements of the algebra \( [a, b]_C \) to the following eight ones: \( a, \neg a, b, \neg b, a \land

\text{\footnote{It is not excluded that two } \( C_R \)-subalgebras of a } \( C_R \)-\text{algebra are disjoint, but } \leq \text{-comparable; cf. e.g. the subalgebras } \{ 1, 0 \} \text{ and } \{ a, \neg a \} \text{ of the algebra of the matrix } M_8 \)
$\neg b, a \land b, \neg a \lor \neg b, a \lor b$; the remaining ones from the list above are equal to some of those eight, e.g. $a \land \neg b = \neg a \land b = \neg a \land \neg b; \neg a \lor \neg b = \neg (a \land b)$ etc. It can be shown very easily that the lattice (with the "negation" $\neg$) of the algebra $[a, b]_C$ is isomorphic to the lattice of the algebra of the matrix $M_8$.

We will show that if the equalities listed in the Lemma are satisfied by the elements $a, b$ then the set $\{a, \neg a, b, \neg b, a \land \neg b, a \lor b, \neg a \lor \neg b, \neg (a \land b)\}$ is closed under the operation $\to$ and the "table of values" for the operation $\to$ is just the "table of values" of the operation $\rightarrow$ in the matrix $M_8$.

To abbreviate the calculations let us denote by 0 the element $\neg a \land \neg b$ and by 1 the element $a \lor b$; of course, we will sometimes write $a \to b, b \to a$ etc. instead of 0, if necessary; similarly, we will sometimes write $a \lor b, \neg (a \to b)$ etc. instead of 1.

Let us begin with proving the following useful equality:

(*$1 = (1 \to 1) = (0 \to 0$)
i.e. the equality

(*)$a \lor b = a \lor b \rightarrow a \lor b = (a \to b) \rightarrow (a \to b) = \neg (a \land b)$.

Proof of (*): Since $a \rightarrow b \leq a \rightarrow b, a \rightarrow b \leq a \rightarrow (a \rightarrow b)$, (by (15), Lemma 1)

$a \leq (a \rightarrow b) \rightarrow (a \rightarrow b), i.e. a \leq a \lor b \rightarrow a \lor b$. Similarly we get $b \leq (a \rightarrow b) \rightarrow (a \rightarrow b), i.e. b \leq a \lor b \rightarrow a \lor b$. These inequalities entail $a \lor b \leq (a \lor b) \rightarrow (a \lor b), i.e. 1 \leq (1 \to 1)$. For the converse inequality, let us observe that $(a \lor b) \rightarrow (a \lor b) \leq (a \lor b)$ because $a \lor b \in \nabla [a, b]_C$

(cf. Lemma 3), i.e. $(1 \rightarrow 1) \leq 1$. Since the De Morgan laws are valid in $R, 1 \rightarrow 1 = 0 \rightarrow 0$.

Now we will fill the "table of values" for the operation $\rightarrow$.

a) Values for 0 $\rightarrow x$.

1. $0 \rightarrow 0 = 1$.
   Proof: Cf. (*) above.

2. $0 \rightarrow \neg a = 1$.
   Proof: (i) By the Principle of Transposition we have $0 \rightarrow \neg a = a \rightarrow 1$, thus $a \rightarrow 1 \leq 1$ because $a \in \nabla [a, b]_C$.

(ii) By the proof of (*), $a \leq 1 \rightarrow 1$, thus $1 \leq a \rightarrow 1$, and it is the second inequality we need.

3. $0 \rightarrow \neg b = 1$
   Proof: See the case 2.

4. $0 \rightarrow a = 1$.
   Proof: (i) By the Principle of Transposition we have $0 \rightarrow a = \neg a \rightarrow 1$. Since $\neg a \leq a$, by the proof of (*) we have $\neg a \leq 1 \rightarrow 1$, thus $1 \leq \neg a \rightarrow 1$.

(ii) Since in each $C_R$-algebra the inequality $x \land \neg y \leq \neg (x \rightarrow y)$ is satisfied, $0 = 0 \land \neg a \leq \neg (0 \rightarrow a)$, thus $(\neg a \rightarrow 1) \leq 1$.

5. $0 \rightarrow b = 1$.
   Proof: See the case 4.

6. $0 \rightarrow (a \land b) = 1$.
   Proof: (i) Since $a \land b \leq a, 0 \rightarrow a \land b \leq 0 \rightarrow a$ (cf. Lemma 3), i.e. (by 4. above) $0 \rightarrow a \land b \leq 1$.

(ii) By Lemma 1, (f2) we have $(0 \rightarrow a) \land (0 \rightarrow b) \leq (0 \rightarrow a \land b)$, thus by 4. and 5. (cf. above) $1 \leq (0 \rightarrow a \land b)$.

7. $0 \rightarrow (a \land b) = 1$.  

6
Proof: (i) By the Principle of Transposition we have $0 \rightarrow \neg(a \land b) = a \land b \rightarrow 1$. Since $a \land b \in \neg_{[a,b]_{C}}, a \land b \rightarrow 1 \leq 1$.

(ii) The inequalities proved in the proof of (*) imply the inequality $a \land b \leq 1 \rightarrow 1$, thus $1 \leq a \land b \rightarrow 1$.

8. $0 \rightarrow 1 = 1$.

Proof: (i) As in the proof of 4. (cf. above) we have $\neg a \leq 1 \rightarrow 1, \neg b \leq 1 \rightarrow 1$, thus $\neg a \land \neg b \leq 1 \rightarrow 1$, thus $1 \leq 0 \rightarrow 1$.

(ii) We have $0 = (a \rightarrow b) \land (a \rightarrow b) \leq \neg((a \rightarrow b) \rightarrow \neg(a \rightarrow b)) = \neg(0 \rightarrow 1)$, i.e. $0 \leq \neg(0 \rightarrow 1)$, thus $0 \rightarrow 1 \leq 1$, and it finishes the proof of part a).

b) The values for $\neg a \rightarrow x$. The first five cases are obvious.

1. $\neg a \rightarrow 0 = \neg a \rightarrow (b \rightarrow \neg a) = b \rightarrow a = 0$.
2. $\neg a \rightarrow \neg a = a$.
3. $\neg a \rightarrow \neg b = 0$.
4. $\neg a \rightarrow a = a$.
5. $\neg a \rightarrow b = 0$.
6. $\neg a \rightarrow a \land b = 0$.

Proof: (i) By thesis (t1) of Lemma 1, $(\neg a \rightarrow a) \land (\neg a \rightarrow b) \leq (\neg a \rightarrow a \land b)$, thus $0 \leq \neg a \rightarrow a \land b$.

By (t3), $(\neg a \rightarrow a \land b) \leq \neg a \rightarrow b$, i.e. $\neg a \rightarrow a \land b \leq 0$.

7. $\neg a \rightarrow \neg(a \land b) = a \land b \rightarrow a = a$.

Proof: (i) Since $a \land b \in \neg_{[a,b]_{C}}, a \land b \rightarrow a \leq a$.

Since $a \land b \leq a, a \rightarrow a \leq a \land b \rightarrow a$, i.e. $a \leq a \land b \rightarrow a$ (cf. Lemma 3).

8. $\neg a \rightarrow 1 = 0 \rightarrow a = 1$.

Proof: Cf. a) 4.

c) The values for $\neg b \rightarrow x$.

1. $\neg b \rightarrow 0 = \neg b \rightarrow (a \rightarrow \neg b) = a \rightarrow b = 0$.
2. $\neg b \rightarrow \neg a = a \rightarrow b = 0$.
3. $\neg b \rightarrow \neg b = b$.
4. $\neg b \rightarrow a = 0$.
5. $\neg b \rightarrow b = \neg b$.
6. $\neg b \rightarrow a \land b = 0$.

Proof: See the the case b) 6.

7. $\neg b \rightarrow \neg(a \land b) = b$.

Proof: See the case b) 7.

8. $\neg b \rightarrow 1 = 0 \rightarrow b = 1$.

Proof: Cf. a) 5.

d) The values for $a \rightarrow x$.

1. $a \rightarrow 0 = a \rightarrow (b \rightarrow a) = b \rightarrow (a \rightarrow a) = 0$.

2. $a \rightarrow \neg a = \neg a$.
3. $a \rightarrow \neg b = 0$.
4. $a \rightarrow a = a$.
5. $a \rightarrow b = 0$.
6. \(a \rightarrow a \land b = 0\).

Proof: (i) \(a \land b \leq b\) implies \(a \rightarrow a \land b \leq a \rightarrow b\), i.e., \(a \rightarrow a \land b \leq 0\).

(ii) By thesis (i) of Lemma 1 we have \(0 = (a \rightarrow a) \land (a \rightarrow b) \leq (a \rightarrow a \land b)\).

7. \(a \rightarrow \neg(a \land b) = a \land b \rightarrow \neg a = \neg a\).

Proof: (i) Since \(a \land b \in \nabla_{[a,b]}\), \(a \land b \rightarrow \neg a \leq \neg a\).

(ii) Since \(a \land b \leq a\), \(a \rightarrow \neg a \leq a \land b \rightarrow \neg a\) (cf. Lemma 3).

8. \(a \rightarrow 1 = 1\).

Proof: (i) \(a \in \nabla_{[a,b]}\), thus \(a \rightarrow 1 \leq 1\).

(ii) By an inequality used in the proof of (*) \(a \leq 1 \rightarrow 1\), thus \(1 \leq a \rightarrow 1\).

f) The values for \(b \rightarrow x\).

1. \(b \rightarrow 0 = 0\).

2. \(b \rightarrow \neg a = 0\).

3. \(b \rightarrow \neg b = \neg b\).

4. \(b \rightarrow a = 0\).

5. \(b \rightarrow b = b\).

6. \(b \rightarrow a \land b = 0\).

Proof: Similar to d) 6.(cf. above).

7. \(b \rightarrow \neg(a \land b) = \neg b\).

Proof: Similar to d) 7.

8. \(b \rightarrow 1 = 1\).

Proof: As in the case d) 8.

The values for \(a \land b \rightarrow x\).

1. \(a \land b \rightarrow 0 = 0\)

Proof: (ii) Since \(a \land b \in \nabla_{[a,b]}\), \(a \land b \rightarrow 0 \leq 0\).

(ii) By the inequalities used in the proof of (*), \(a \land b \leq 0 \rightarrow 0\), thus \(0 \leq a \land b \rightarrow 0\).

2. \(a \land b \rightarrow \neg a = a \rightarrow \neg(a \land b) = \neg a\).

Proof: By d) 7.

3. \(a \land b \rightarrow \neg b = \neg b\).

Proof: Cf. e) 7.

4. \(a \land b \rightarrow a = \neg a \rightarrow \neg(a \land b) = a\).

Proof: Cf. b) 7.

5. \(a \land b \rightarrow b = b\).

Proof: Cf. c) 7.

6. \(a \land b \rightarrow a \land b = a \land b\).

Proof: (i) \(a \land b \in \nabla_{[a,b]}\), thus \(a \land b \rightarrow a \land b \leq a \land b\).

(ii) Since \(a \land b\) is the generator of the filter of the designated elements of the submatrix we consider, \(a \land b \leq a \land b \rightarrow a \land b\).

7. \(a \land b \rightarrow \neg(a \land b) = \neg(a \land b)\).

Proof: (i) \(a \land b \in \nabla_{[a,b]}\), thus \(a \land b \rightarrow \neg(a \land b) \leq \neg(a \land b)\) (cf. Lemma 3).

(ii) By the inequality \(a \land b \leq a \land b \rightarrow a \land b\) (cf. the proof of the previous equality) we have \(a \land b \leq \neg(a \land b) \rightarrow \neg(a \land b)\), thus \(\neg(a \land b) \leq a \land b \rightarrow \neg(a \land b)\).

8. \(a \land b \rightarrow 1 = 1\).

Proof: (i) \(a \land b \rightarrow 1 \leq 1\), because \(a \land b \in \nabla_{[a,b]}\).
(ii) We have $a \land b \leq 1 \rightarrow 1$ (cf. the proof of (*)), thus $1 \leq a \land b \rightarrow 1$.

g) The values for $\neg(a \land b) \rightarrow x$.
1. $\neg(a \land b) \rightarrow 0 = 1 \rightarrow a \land b = 0$.
Proof: (i) By (t2) (Lemma 1) d) 6. and e) 6. we have $0 = (a \rightarrow a \land b) \land (b \rightarrow a \land b) \leq (a \lor b) \rightarrow (a \land b) = 1 \rightarrow a \land b$.
(ii) By Lemma 1, (t4) and d) 6. we have $1 \rightarrow (a \land b) \leq (a \lor b) \rightarrow (a \land b) \leq a \rightarrow a \land b = 0$.
2. $\neg(a \land b) \rightarrow \neg a = a \rightarrow a \land b = 0$.
Proof: Cf. d) 6.
3. $\neg(a \land b) \rightarrow \neg b = 0$.
Proof: Cf. e) 6.
4. $\neg(a \land b) \rightarrow a = 0$.
Proof: Cf. b) 6.
5. $\neg(a \land b) \rightarrow b = 0$.
Proof: Cf. c) 6.
6. $\neg(a \land b) \rightarrow (a \land b) = (\neg a \lor \neg b) \rightarrow (a \land b) = 0$.
Proof: (i) Using the same inequalities as in g) 1., by c) 6. we have $0 = (\neg a \rightarrow a \land b) \land (\neg b \rightarrow a \land b) \leq (\neg a \lor \neg b) \rightarrow a \land b$.
(ii) $(\neg a \lor \neg b) \rightarrow a \land b \leq \neg a \rightarrow a \land b = 0$.
7. $\neg(a \land b) \rightarrow \neg(a \land b) = a \land b$.
Proof: Cf. f) 6.
8. $\neg(a \land b) \rightarrow 1 = 0 \rightarrow a \land b = 1$.
Proof: Cf. a) 6.

h) The values for $1 \rightarrow x$.
1. $1 \rightarrow 0 = 0$.
Proof: (i) $1 \rightarrow 0 \leq 0$, because $1 \in \nabla_{[a,b]}$.
(ii) Since by (*) $1 \leq 0 \rightarrow 0$, $0 \leq 1 \rightarrow 0$.
2. $1 \rightarrow \neg a = a \rightarrow 0 = 0$.
Proof: Cf. d) 1.
3. $1 \rightarrow \neg b = 0$.
Proof: Cf. e) 1.
4. $1 \rightarrow a = 0$.
Proof: Cf. b) 1.
5. $1 \rightarrow b = 0$.
Proof: Cf. c) 1.
6. $1 \rightarrow a \land b = 0$.
Proof: Cf. g) 1.
7. $1 \rightarrow \neg(a \land b) = 0$.
Proof: Cf. f) 1.
8. $1 \rightarrow 1 = 1$.
Proof: Cf. (*),
and it finishes the proof of the Lemma.
3 The matrix $\mathcal{M}_6$.

Denote now by $\mathcal{M}_6$ the matrix $\langle\{0, a, b, a \wedge b, c \vee b, 1\}, \rightarrow, \wedge, \vee, \neg\rangle, \{a, b, a \wedge b, a \vee b, 1\}$ whose lattice operations $\wedge$ and $\vee$ are defined as it is shown in the following diagram:

\[
\begin{array}{c}
1 \\
a \vee b \\
\diamond \\
\downarrow \\
as \wedge b \\
\downarrow \\
0
\end{array}
\]

and whose operations $\rightarrow$ and $\neg$ are defined by the following tables:

\[
\begin{array}{c|cccccc}
\rightarrow & 0 & a \wedge b & a & b & a \vee b & 1 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
a \wedge b & 0 & a \wedge b & a & 0 & a \vee b & 1 \\
a & 0 & 0 & a & 0 & a & 1 \\
b & 0 & 0 & 0 & b & b & 1 \\
a \vee b & 0 & 0 & 0 & 0 & a \wedge b & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c}
x & \neg x \\
\hline
0 & 1 \\
a & a \\
b & b \\
a \wedge b & a \vee b
\end{array}
\]

We have the following

**Theorem 8** (i) $\mathcal{M}_6$ is a $C_R$-matrix;
(ii) If in $A \rightarrow B$ sets of variables of $A$ and $B$ are disjoint then there exists a valuation $v$ such that $v(A \rightarrow B) = 0$ in $\mathcal{M}_6$.

To prove part (ii) it suffices to note that the valuation function $h^v$ can be defined as the homomorphic extension of a function $v$ defined as follows: if $p_i$ occurs in $A$ then we put $v(p_i) = a$ and if $p_i$ occurs in $B$ then we put $v(p_i) = b$. It is easy to check that $h^v(A \rightarrow B) = 0$.

Similarly as in the case of the matrix $\mathcal{M}_8$ the proof of the property (ii) is based on the fact that $\mathcal{M}_6$ has two one-element (i.e. trivial)\(^3\) $\leq$-incomparable submatrices: $\{a\}$ and $\{b\}$.

\(^3\)A similar 6-element $C_R$-matrix (called "crystal") was used by P. B. Thistlewaite, M. A. Mc Robbie and R. K. Meyer in their book [88]. However, the "negation" operation in the "crystal" is defined in a quite different way: $\neg a = b$ and $\neg a = b$, thus their "crystal" does not contain trivial subalgebras.
Moreover, the matrix $\mathcal{M}_6$ is the least matrix which has two one-element $\leq$-incomparable falsifying submatrices. Namely, we have

**Proposition 9** Let $\mathcal{A} = \langle A, \vee_A \rangle$ be a $C_R$-matrix and let the lattice of the algebra $A$ be defined by the partial ordering relation $\preceq$. Then if $A$ contains two trivial $\leq$-incomparable falsifying submatrices then the matrix $\mathcal{M}_6$ is a submatrix of $\mathcal{A}$.

**Proof:** Let us denote the universes of these trivial submatrices by $\{a\}$, $\{b\}$, respectively. Of course, $a = \neg a, b = \neg b, a \rightarrow a = a, b \rightarrow b = b$. Let us consider the subalgebra of $\mathcal{A}$ generated by the elements $a, b$. Of course, it will contain (besides of $a, b$) at least the following elements: $a \lor b, a \land b, a \rightarrow b, b \land a, \neg(a \rightarrow b), \neg(b \rightarrow a)$. It is clear that (cf. Lemma 4 (W. Dziobiak [83]), see above) the filter of designated elements of this submatrix is generated by $a \land b$. Since (by the Principle of Transposition) $a \rightarrow b = b \rightarrow a$ and $a \rightarrow b \leq a, b \rightarrow a \leq a$ (cf. Lemma 3, see above), $a \rightarrow b < a \land b$, because (by the assumption that the trivial submatrices $\{a\}$ and $\{b\}$ are falsifying submatrices) $a \rightarrow b$ does not belong to the filter of designated elements of the algebra $\mathcal{A}$.

In the following we will show that (i) the submatrix of $\mathcal{A}$ generated by $a, b$ consists of the following elements: $a, b, a \land b, a \lor b, a \rightarrow b, \neg(a \rightarrow b)$, (ii) the element $a \rightarrow b$ is the least and the element $\neg(a \rightarrow b)$ is the greatest element of the algebra of this submatrix and (iii) the submatrix in question is isomorphic to $\mathcal{M}_6$.

Let us observe first that the following inequalities hold for our elements $a, b$ of the matrix $\mathcal{A}$:

1. $a \leq (a \rightarrow b) \rightarrow (a \rightarrow b)$,
2. $b \leq (a \rightarrow b) \rightarrow (a \rightarrow b)$,
3. $a \land b \leq (a \rightarrow b) \rightarrow (a \rightarrow b)$,
4. $a \lor b \leq (a \rightarrow b) \rightarrow (a \rightarrow b)$,
5. $\neg(a \rightarrow b) \leq (a \rightarrow b) \rightarrow (a \rightarrow b)$.

For the proof of (1) and (2) - cf. the proof of the equality (*) in Lemma 7; (3) and (4) follow from (1) and (2). The proof for (5) is the following sequence

$a \leq (a \rightarrow b) \rightarrow (a \rightarrow b)$ if $a \rightarrow b \leq (a \rightarrow b)$

iff $(a \rightarrow b) \leq a \rightarrow (a \rightarrow (b \rightarrow b))$

iff $(a \rightarrow b) \leq a \rightarrow (b \rightarrow (a \rightarrow b))$

iff $(a \rightarrow b) \leq a \rightarrow (b \rightarrow (a \rightarrow b))$

iff $(a \rightarrow b) \leq (a \rightarrow (a \rightarrow b))$

iff $(a \rightarrow b) \leq (a \rightarrow (\neg(a \rightarrow b) \rightarrow b))$

iff $(a \rightarrow b) \leq (a \rightarrow (a \rightarrow b))$

iff $(a \rightarrow b) \leq (a \rightarrow (a \rightarrow b))$

iff $(a \rightarrow b) \leq (a \rightarrow (a \rightarrow b))$.

The inequalities (1) - (5) imply the following

**Claim.** The element $\neg(a \rightarrow b)$ is an upper bound for the elements $a, b, a \lor b, a \land b$, and $a \rightarrow b$.

To prove Claim let us note that by the inequalities (1) - (5) (cf. above) the element $(a \rightarrow b) \rightarrow (a \rightarrow b)$ is an upper bound for the elements $a, b, a \lor b, a \land b, \neg(a \rightarrow b)$. By the assumptions concerning the matrix $\mathcal{A}$ we have $a \rightarrow b < a \land b$, so (by (3)) we have the following inequality: $(a \rightarrow b) \leq (a \rightarrow b) \rightarrow (a \rightarrow b)$, thus $(a \rightarrow b) \rightarrow (a \rightarrow b)$ is an upper bound for the elements we consider. Now it suffices to prove that $\neg(a \rightarrow b) = (a \rightarrow b) \rightarrow (a \rightarrow b)$. So since $a \rightarrow b \leq a \land b, a \lor b \leq \neg(a \rightarrow b)$, thus $\neg(a \rightarrow b) \in \nabla_A$ and in consequence $\neg(a \rightarrow b)$
is a designated element of the submatrix we consider. It is known (cf. Lemma 3, see above) that if \( x \in \nabla \) then \( x \rightarrow y \leq y \) thus \( \neg(a \rightarrow b) \rightarrow \neg(a \rightarrow b) \leq \neg(a \rightarrow b) \) and by the Principle of Transposition we get \((a \rightarrow b) \rightarrow (a \rightarrow b) \leq \neg(a \rightarrow b)\). Now we join the last inequality with the inequality (5) from the previous Lemma and get \( \neg(a \rightarrow b) = (a \rightarrow b) \rightarrow (a \rightarrow b) \). It finishes the proof of our Claim.

Let us return to the proof of the Proposition. Since - as we just proved - \( a \rightarrow b \) is the lower bound and \( \neg(a \rightarrow b) \) the upper bound of this set, we will sometimes denote them by 0 and 1, respectively. It is clear that the set \( \{a, b, a \lor b, a \land b, a \rightarrow b, \neg(a \rightarrow b)\} \) is closed under the lattice operations \( \land \) and \( \lor \) and under the operation \( \neg \). We will prove now that this set is closed under the operation \( \rightarrow \) and that the submatrix generated by \( a, b \) is isomorphic to \( \mathcal{M}_6 \). To show it we will fill row by row the "table of values" for the operation \( \rightarrow \).

\[ \begin{align*}
\text{a) Values for } 0 \rightarrow x. \\
1. & \ 0 \rightarrow 0 = 1. \\
\text{Proof: Cf. above.} \\
2. & \ (c) 0 \rightarrow (a \land b) = 1. \\
\text{Proof: (i) } (a \rightarrow b) \rightarrow (a \land b) \leq a \lor b \rightarrow \neg(a \rightarrow b) \leq \neg(a \rightarrow b) \ (\text{because } a \lor b \in \nabla_A), \ i.e. \\
& (a \rightarrow b) \rightarrow a \land b \leq \neg(a \rightarrow b). \\
(ii) & \ \text{Since } a \lor b \leq \neg(a \rightarrow b), \neg(a \rightarrow b) \rightarrow \neg(a \rightarrow b) \leq a \lor b \rightarrow \neg(a \rightarrow b) \ (\text{cf. Lemma 3}), \ \text{the} \\
& \ \text{Principle of Transposition and the inequality (5) (cf. above)} \neg(a \rightarrow b) \leq (a \rightarrow b) \rightarrow (a \land b). \\
3. & \ a = 1. \\
\text{Proof: (i) } a \rightarrow b \leq a, \ \text{thus } (a \rightarrow b) \rightarrow (a \rightarrow b) \leq (a \rightarrow b) \rightarrow a \ (\text{cf. Lemma 3}) \ i.e. \\
& \neg(a \rightarrow b) \leq (a \rightarrow b) \rightarrow a. \\
(ii) & \ (a \rightarrow b) \rightarrow a \leq a \rightarrow \neg(a \rightarrow b) \leq \neg(a \rightarrow b) \ (\text{because } a \in \nabla_A), \ \text{thus } (a \rightarrow b) \rightarrow a \leq \\
& \neg(a \rightarrow b). \\
4. & \ 0 \rightarrow 1 = 1. \\
\text{Proof: As in the case 3.} \\
5. & \ 0 \rightarrow a \lor b = 1. \\
\text{Proof: (i) Since } a \land b \in \nabla_A, \ a \land b \rightarrow \neg(a \rightarrow b) \leq \neg(a \rightarrow b) \ i.e. \ (a \rightarrow b) \rightarrow a \lor b \leq \neg(a \rightarrow b). \\
(ii) & \ \text{We have } a \land b \leq \neg(a \rightarrow b), \ \text{thus by (5), cf. above, } a \land b \leq \neg(a \rightarrow b) \rightarrow \neg(a \rightarrow b), \ \text{thus} \\
& \neg(a \rightarrow b) \leq (a \rightarrow b) \rightarrow (a \lor b). \\
6. & \ 0 \rightarrow 1 = 1 \\
\text{Proof: (i) Since } x \rightarrow \neg x \leq \neg x \text{ for all } x \text{ of } A, \ (a \rightarrow b) \rightarrow \neg(a \rightarrow b) \leq \neg(a \rightarrow b). \\
(ii) & \ \text{Since } (a \rightarrow b) \leq \neg(a \rightarrow b) \ (\text{cf. (5) and the proof of Claim, cf. above}), \ (a \rightarrow b) \leq \neg(a \rightarrow b) \rightarrow \neg(a \rightarrow b), \ \text{thus } \neg(a \rightarrow b) \leq (a \rightarrow b) \rightarrow \neg(a \rightarrow b). \\
\text{b) The values for } a \land b \rightarrow x. \\
1. & \ a \land b \rightarrow 0 = 0. \\
\text{Proof: (i) } a \land b \rightarrow 0 \leq 0, \ \text{because } a \land b \in \nabla_A. \\
(ii) & \ \text{We have } a \land b \leq 1 \ i.e. \ a \land b \leq (a \rightarrow b) \rightarrow (a \rightarrow b) \ \text{iff } a \rightarrow b \leq a \land b \rightarrow (a \rightarrow b) \ i.e. \\
& 0 \leq a \land b \rightarrow 0. \\
2. & \ a \land b \rightarrow a \land b = a \land b. \\
\text{Proof: (i) Since } a \land b \in \nabla_A, a \land b \rightarrow a \land b \leq a \land b. \\
\end{align*} \]
(ii) Since $a \land b$ is the generator of the filter of designated elements of the submatrix in question, $a \land b \leq a \land b \to a \land b$.
3. $a \land b \to a = a$.

Proof: (i) $a \land b \to a \leq a$ because $a \land b \in \nabla_A$.
(ii) Since $a \land b \leq a$, $a \to a \leq a \land b \to a$ by Lemma 3.
4. $a \land b \to b = b$.

Proof: As for 3., cf. above.
5. $a \land b \to a \lor b = a \lor b$.

Proof: (i) Since $a \land b \in \nabla_A$, $a \land b \to a \lor b \leq a \lor b$.
(ii) Since $a \land b \leq a \land b \to a \land b$ (cf. the part (ii) of the proof of b) 2., cf. above), $a \land b \leq a \lor b \to a \lor b$ (by the Principle of Transposition), thus $a \lor b \leq a \land b \to a \lor b$.
6. $a \land b \to 1 = 1$.

Proof: (i) $a \land b \to 1 \leq 1$, because $a \land b \in \nabla_A$.
(ii) $a \land b \leq (a \to b) \to (a \to b)$ (cf. (5) above), thus $(a \to b) \leq (a \land b) \to (a \to b)$, i.e. $1 \leq a \land b \to 1$.

 c). The values for $a \to x$.
1. $a \to 0 = 0$.

Proof: (i) Since $a \in \nabla_A$, $a \to 0 \leq 0$.
(ii) $a \leq (a \to b) \to (a \to b)$ (cf. (5) above), thus $a \to b \leq a \to (a \to b)$.
2. $a \to a \land b = 0$.

Proof: (i) Since $b \leq (a \lor b), (a \lor b) \to a \leq b \to a$, thus $a \to a \land b \leq a \to b$.
(ii) By $b \to a \leq b \to a$ we get (*) $b \leq (b \to a) \to a$ and by $b \to a \leq a$ (i.e. $b \to a \leq a \to a$) we have (** $a \leq (b \to a) \to a$. By (*) and (** $we get $a \lor b \leq (b \to a) \to a$, i.e. a \lor b \leq a \to (a \to b)$, thus $a \leq a \lor b \leq (a \to b)$ and in consequence $a \leq (a \to b) \to a \land b$ and at last $a \to b \leq a \to a \land b$.
3. $a \to a = a$ - by the assumption.
4. $a \to b = 0$.
5. $a \to a \lor b = a$.

Proof: $a \land b \to a = a$ (cf. b) 2.).
6. $a \to 1 = 1$.

Proof: (i) $a \to 1 \leq 1$, because $a \in \nabla_A$.
(ii) Since $a \leq \neg(a \to b) \to \neg(a \to b)$ (cf. the inequality (1), see above), $\neg(a \to b) \leq a \to \neg(a \to b)$.

d) The values for $b \to x$ - can be determined as in c).

e) The values for $a \lor b \to x$.
1. $a \lor b \to 0 = 0$.

Proof: (i) $a \lor b \to 0 \leq 0$, because $a \lor b \in \nabla_{bFA}$.
(ii) Cf. the inequality (4) above.
2. $a \lor b \to a \land b = 0$.

Proof: (i) Since $(p \lor q \to r) \to (p \to r) \land (q \to r) \in R, a \lor b \to a \land b \leq (a \to a \land b) \land (b \to a \land b)$,
so by c) 2. \(a \lor b \rightarrow a \land b \leq (a \rightarrow b) \land (a \rightarrow b)\), and in consequence \(a \lor b \rightarrow a \land b \leq (a \rightarrow b)\)
i.e. \(a \lor b \rightarrow a \land b \leq 0\).

(ii) Conversely, since \(a \rightarrow b \leq a \rightarrow b\), by c) 2. \(a \rightarrow b \leq a \rightarrow a \land b\), thus \((*)a \leq (a \rightarrow b) \rightarrow a \land b\). Similarly we get \((**)b \leq (a \rightarrow b) \rightarrow a \land b\), so by (*) and (**) we have \(a \lor b \leq (a \rightarrow b) \rightarrow a \land b\), thus \((a \rightarrow b) \leq (a \lor b) \rightarrow a \land b\).

3. \(a \lor b \rightarrow a = 0\).

Proof: \(a \rightarrow a \land b = 0\) (cf. c) 2.); we apply the Transposition Principle.

4. \(a \lor b \rightarrow b = a \rightarrow b\).

Proof: As in the previous case.

5. \(a \lor b \rightarrow a \lor b = a \land b\).

Proof: We apply the Transposition Principle to \(a \land b \rightarrow a \land b = a \land b\) (cf. b) 2.).

6. \(a \lor b \rightarrow 1 = 1\).

Proof: By a) 2.

f) The values for \(1 \rightarrow x\).

1. \(1 \rightarrow 0 = 0\).

Proof: (i) \(1 \in V_A\), thus \(1 \rightarrow 0 \leq 0\).

(ii) Since \(0 \rightarrow 0 = 1\) (cf. a) 1. above), \(1 \leq 0 \rightarrow 0\), thus \(0 \leq 1 \rightarrow 0\).

2. \(1 \rightarrow a \land b = 0\).

Proof: Cf. e) 1.

3. \(1 \rightarrow a = 0\).

Proof: Cf. c) 1.

4. \(1 \rightarrow b = 0\).

Proof: As in case 3.

5. \(1 \rightarrow a \lor b = 0\).

Proof: Cf. b) 1.

6. \(1 \rightarrow 1 = 1\).

Proof: Cf. a) 1.

Thus the set \(\{a, b, a \land b, a \lor b, a \rightarrow b, \neg(a \rightarrow b)\}\) is closed under all basic operations; it suffices now to compare the "table of values" for the operation \(\rightarrow\) we have just filled with the "table of values" for the operation \(\rightarrow\) in the matrix \(M_6\) to state that the matrix we have obtained in the proof of this Proposition is just the matrix \(M_6\). It finishes the proof of this Proposition.

As the last proposition of this section let us note the following

**Proposition 10** The algebras of the matrices \(M_6\) and \(M_8\) are subdirectly irreducible.

**Proof**: Proposition 1.5 of: W. Dziobiak [83].
4 Fundamental result.

Let us begin with the following

**Lemma 11** There does not exist a $C_R$-matrizes $(C, \nabla_C)$ which contains two proper submatrices which satisfy the conditions

(i) the algebras of these submatrices are incomparable with respect to the partial order relation $\leq$ which defines the lattice of the algebra $C$ and

(ii) the first submatriz is the trivial matriz and the second one is the two-element matriz.

**Proof:** Let us assume that such a matrix exists. Let us denote by $a, \neg a$ the elements of the two-element submatriz of the matrix in question (let $a \in \nabla_A, \neg a \notin \nabla_A$) and by $b$ the only element of the trivial submatriz of this matrix ($b \in \nabla_A$, of course). Note now that (since the Principle of Transposition hold for $C_R$-matrices) if $a \lor b = \neg a \lor b$ then $\neg a \land b = a \land b$. However in such a case the set $\{a, \neg a, b, a \lor b, \neg a \land b\}$ forms the well-known lattice $N_5$, thus the lattice of the $C_R$-matriz in question cannot be distributive. From $\neg a < a$ it follows now that $\neg a \land b < a \land b, \neg a \lor b < a \lor b$. Moreover $a \land b < b < \neg a \lor b$. But in this case the elements $\{a, a \land b, b, \neg a \lor b, a \lor b\}$ form the lattice $N_5$ which finishes the proof.

To formulate the next Proposition a new notion is useful.

A variety $V$ of $C_R$-matrices will said to be a variety with the relevance principle if $V$ falsifies all non-relevant implications, i.e. for each non-relevant implication $A \rightarrow B$ there exists a matrix $A$ of $V$ and a valuation $h$ such that $h(A \rightarrow B)$ does not belong to the set of designated elements of $A$.

We have now

**Proposition 12** Let $V$ be a $C_R$-variety with the relevance principle. Then $V$ contains a matrix $C = (C, \nabla_C)$ that falsifies all non-relevant implications and contains two submatrices $A, B$ whose algebras are 1-generated and incomparable with respect to the partial order which defines the lattice of the algebra $C$.

**Proof:** Let $V$ be a variety with the relevance principle. Let us consider the $V$-free algebra $C$ over two generators $a, b$. Of course, the matrix $(C, \nabla_C)$ falsifies all non-relevant implications. Let us denote by $\leq$ the partial order which defines the lattice of the algebra $C$. Let us consider the subalgebras of $C$ generated by $a, b$, respectively; let us denote them by $[a]_C$ and $[b]_C$. By Theorem 5 (cf. W. Dziobiak, unpublished, see above) both of these algebras have units and zero's; let us denote them by $1_a, 0_a, 1_b, 0_b$. Observe now that the algebras $[a]_C, [b]_C$ are $\leq$-incomparable. If not then there exist elements $a_1, b_1$ such that $a_1 \in [a]_C, b_1 \in [b]_C$ and e.g. $a_1 \leq b_1$. But in such a case $0_a \leq 1_b$ and in consequence this free algebra cannot falsify all non-relevant implications. This finishes the proof.

Let us observe here that the submatrices $[a]_C$ and $[b]_C$ described in this Proposition are falsifying submatrices.
Theorem 13 Let $V$ be a $C_R$-variety with the relevance principle. Then $V$ contains either the Belnap matrix $M_8$ or the matrix $M_6$.

Proof: We know from the previous Proposition that there exists in $V$ a matrix $C = (C, \nabla_C)$ such that $C$ contains two submatrices whose algebras are $\leq$-incomparable ($\leq$ denotes here the partial order which defines the lattice of the algebra $C$); let us denote the subalgebras of these submatrices by $A$ and $B$, respectively. We may assume that both of these subalgebras have a greatest and a least element; let us denote these elements as follows: the unit of $A$ by $a$, the zero of $A$ by $\neg a$, and by $b, \neg b$ - the unit and the zero of the algebra $B$, respectively. If $a = \neg a$ and $b = \neg b$ then (cf. Proposition 9) the matrix $M_6$ belongs to $V$; let us assume that $a \neq \neg a$, $b \neq \neg b$.

Let us denote by $[a, b]_C$ the subalgebra of the algebra $C$ generated by the elements $a, b$. The algebra $[a, b]_C$ contains in particular the following eight elements: $a, \neg a, b, \neg b, a \lor b, a \land \neg b, \neg a \land \neg b$; it is easy to observe that the lattice operations $\land, \lor$ on these elements as well as the operation $\neg$ can be described here as in the algebra of the matrix $M_8$; this eight-element set is closed under these lattice operations and under $\neg$. Our further investigations will concern only the operation $\rightarrow$.

Let us write down first the following obvious connections between the elements $a, \neg a$:

\[ a \rightarrow a = a, \neg a \rightarrow a = a, a \rightarrow \neg a = \neg a, \neg a \rightarrow \neg a = a. \]

We have quite similar connections for the elements $b, \neg b$.

By the Lemma 4. (cf. W. Dziobiak (83)) the filter $((a \rightarrow b) \land (b \rightarrow b)) [a, b]_C$ i.e. (by the previous remark) the filter $[a \land b] [a, b]_C$ is the filter of designated elements of the matrix $([a, b]_C, \nabla_{[a, b]_C})$, i.e. $\nabla_{[a, b]_C} = [a \land b]$. By the construction of the algebra $[a, b]_C$ the elements $a \rightarrow b, b \rightarrow a, a \rightarrow \neg b, \neg a \rightarrow b$ cannot belong to the filter $[a \land b]$.

Since $\neg a < a, \neg b < b$, by Lemma 3 we have the following equalities and inequalities:

1. $b \rightarrow \neg a = a \rightarrow \neg b \leq a \rightarrow b \leq \neg a \rightarrow b = \neg b \rightarrow \neg a$, and
2. $b \rightarrow \neg a = a \rightarrow \neg b \leq b \rightarrow a \leq \neg a \rightarrow b = \neg b \rightarrow a$.

The next connections, which are important for our proof we get by the following connection valid for any $C_R$-matrix $(C, \nabla_C)$:

If $z \in \nabla$ then $z \rightarrow x \leq x$ for any $x \in C$ (cf. Lemma 3); thus we have the following useful inequalities:

2. $a \rightarrow b \leq b, b \rightarrow a \leq a, a \rightarrow \neg b \leq \neg b, b \rightarrow \neg a \leq \neg a$,
3. which imply the next important connection

$3. a \rightarrow \neg b = b \rightarrow \neg a \leq \neg a \land \neg b$.

Since the proofs of the two connections which are the basis of our proof are rather long, we present them in the form of two lemmas.

Lemma 14 The following equalities hold in $[a, b]_C$:

4. $a \rightarrow b = b \rightarrow a = \neg a \rightarrow b$. 

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Proof of the Lemma: Since \( \{ (p \rightarrow q) \land p \} \rightarrow q \in R \), we have
\[(\neg a \rightarrow b) \land \neg a \leq b \text{ and } (\neg b \rightarrow a) \land \neg b \leq a,\]
i.e. \( (\neg b \rightarrow a) \land \neg b \lor a = a \) and \( ((\neg a \rightarrow b) \land \neg a) \lor b = b \). Since the equality \( \neg b \lor a = b \lor a \) holds in \([a, b]c\), we have:
\[a = ((\neg b \rightarrow a) \land \neg b) \lor a = [(\neg b \rightarrow a) \lor a] \land (\neg b \lor a) = [(\neg b \rightarrow a) \land b] \lor a,\]
i.e. \( [(\neg b \rightarrow a) \land b] \leq a \). In a quite similar way we get the inequality \( [(\neg a \rightarrow b) \land a] \leq b \).

From these inequalities the following inequalities follow:
\[ [(\neg b \rightarrow a) \land b] \leq [a \land (\neg b \rightarrow a)], \]
\[ [(\neg b \rightarrow a) \land a] \leq [b \land (\neg b \rightarrow a)], \]
thus \( [(\neg a \rightarrow b) \land b] = [(\neg a \rightarrow b) \land a] \), because \( (\neg a \rightarrow b) = (\neg b \rightarrow a) \).

Since in all \( C_R \)-algebras holds the inequality \( x \rightarrow y \leq \neg x \lor y \), we have:\( (\neg a \rightarrow b) \leq (a \lor b) \). Thus we have:\( (\neg a \rightarrow b) = (a \rightarrow b) \land (a \lor b) = [(\neg a \rightarrow b) \land a] \lor [(\neg a \rightarrow b) \land b] = [(\neg a \rightarrow b) \land a] \), i.e. \( (\neg a \rightarrow b) \leq a \). In the same way we get \( (\neg a \rightarrow b) \leq b \).

As we remember, in all \( C_R \)-algebras the following implication holds:
if \( x \leq y \) then \( z \rightarrow x \leq z \rightarrow y \) (cf. Lemma 3, see above).

By this implication and the first inequality from the last two we have: \( b \rightarrow (\neg a \rightarrow b) \leq b \rightarrow a \), i.e. \( \neg a \rightarrow (b \rightarrow b) \leq b \rightarrow a \), i.e. \( \neg a \rightarrow b \leq b \rightarrow a \). The second inequality (i.e. \( (\neg b \rightarrow a) \leq b \)) implies \( (\neg b \rightarrow a) \leq a \rightarrow b \), i.e. \( (\neg a \rightarrow b) \leq b \rightarrow a \). Since the inequalities (1) (cf. above) hold, the Lemma has been proved.

Let us add that the inequalities shown in the proof of this Lemma imply the following useful inequality:
(5) \( \neg a \rightarrow b \leq a \land b \).

The next connection is given by

Lemma 15 Let the algebra \([a, b]_A\) we consider satisfy the equalities
\( (\neg a \land \neg b) = (a \rightarrow b) = (b \rightarrow a) \).

Then this algebra satisfies the equality:
\( (a \rightarrow \neg b) = (\neg a \rightarrow b) \).

Proof of the Lemma: Let us prove the inequality \( b \leq (a \lor b) \rightarrow (a \lor b) \) first; the proof of this inequality is as follows:
\[ \neg a \land \neg b \leq \neg a \land \neg b \quad \text{iff} \]
\[ \neg a \land \neg b \leq a \rightarrow b \quad \text{iff} \]
\[ \neg a \land \neg b \leq \neg b \rightarrow \neg a \quad \text{iff} \]
\[ \neg a \land \neg b \leq \neg b \rightarrow (a \rightarrow \neg a) \quad \text{iff} \]
\[ \neg a \land \neg b \leq a \rightarrow (a \rightarrow b) \quad \text{iff} \]
\[ \neg a \land \neg b \leq a \rightarrow (\neg b \rightarrow b) \quad \text{iff} \]
\[ \neg a \land \neg b \leq a \rightarrow (\neg b \rightarrow (a \rightarrow b)) \quad \text{iff} \]
\[ \neg a \land \neg b \leq a \rightarrow ((\neg a \rightarrow b) \rightarrow b) \quad \text{iff} \]
\[ \neg a \land \neg b \leq a \rightarrow ((\neg a \rightarrow b) \rightarrow b) \quad \text{iff} \]
\[ a \lor b \leq (\neg a \land \neg b) \rightarrow (a \land \neg b) \quad \text{iff} \]
\[ a \lor b \leq a \land b \rightarrow a \lor b. \]
The last inequality also implies \( b \leq a \lor b \rightarrow a \lor b \). It follows from this last inequality that
\( a \lor b \leq b \rightarrow a \lor b \), thus \( a \leq b \rightarrow a \lor b \), i.e. \( a \leq b \rightarrow \neg(a \rightarrow b) \), thus \( a \leq (a \rightarrow b) \rightarrow \neg b \), and in consequence \( (a \rightarrow b) \leq (a \rightarrow \neg b) \). The inverse inequality we get from inequalities (1) (cf. above).

Let us return to the proof of Theorem.

Now arises the question where the "arrow elements" of the algebra \([a, b]_A\) are situated in relation to the remaining elements of this algebra. Since it is known that the following equalities hold in this algebra: \( a \rightarrow b = b \rightarrow c = \neg b \rightarrow a = \neg a \rightarrow b \) (cf. Lemma 14) and \( a \rightarrow \neg b = b \rightarrow \neg a \), we have to consider here only two "arrow elements": \( a \rightarrow b \) and \( a \rightarrow \neg b \).

Let us note now that the inequality (5) entails that \( a \rightarrow b \leq a \land b \), and it is known from the construction of the algebra \([a, b]_A\) that \( (a \rightarrow b) \) does not belong to the filter \([a \land b]\), thus we have the first important inequality

(a) \( a \rightarrow b < a \land b \).

Moreover, we have established before that in the algebra \([a, b]_A\) the two following interesting inequalities hold:

(b) \( a \rightarrow \neg b \leq \neg a \land \neg b < a \land b \)

(c) \( a \rightarrow \neg b \leq a \rightarrow b \).

The remainder of the proof will be devoted to considering the question where the elements \( a \rightarrow b \) and \( a \rightarrow \neg b \) may be situated in the lattice of the algebra \([a, b]_A\). Below we will prove that in each of possible cases (determined by the inequalities (a) - (c), of course) either \( M_8 \in HS([a, b]_A) \) or \( M_6 \in HS([a, b]_A) \) and that will finish the proof of the Theorem.

We have here the following possibilities:

A. \( \neg a \land \neg b < a \rightarrow b < a \land b \)

(it occurs - cf. below - that in this case it is not important (cf. the inequality (c)) whether \( a \rightarrow \neg b = \neg a \land \neg b \) or \( a \rightarrow b < \neg a \land \neg b \) ).

B. \( \neg a \land \neg b = a \rightarrow b \),

(in this case \( a \rightarrow \neg b = a \rightarrow b \), cf. Lemma 15).

C. \( a \rightarrow b < \neg a \land \neg b \),

D. \( a \rightarrow b \) is \( \leq \)-incomparable with \( \neg a \land \neg b \), but, of course \( a \rightarrow \neg b \leq a \rightarrow b < a \land b \).

Thus let us consider now the four cases listed above.

A. Let us assume first that \( \neg a \land \neg b < a \rightarrow b \).

Since \( a \land b \leq (a \rightarrow b) \rightarrow (a \rightarrow b) \leq \neg(a \rightarrow b) \lor (a \rightarrow b) \), by the inequalities we have just assumed, i.e. by \( \neg a \land \neg b < (a \rightarrow b) \leq a \land b \) we have \( \neg a \lor \neg b < \neg(a \rightarrow b) \leq a \lor b \), and the last two inequalities give \( \neg a \lor \neg b \leq (a \rightarrow b) \lor \neg(a \rightarrow b) \leq a \lor b \). The last inequality can be strengthened to the equality \( (a \rightarrow b) \lor \neg(a \rightarrow b) = a \lor b \), because \( a \rightarrow b \leq a \lor b \) implies \( a \rightarrow b \leq a \rightarrow (a \rightarrow b) \) and it implies \( a \leq (a \rightarrow b) \rightarrow (a \rightarrow b) \); similarly we show that \( b \leq (a \rightarrow b) \rightarrow (a \rightarrow b) \), thus \( a \lor b \leq (a \rightarrow b) \rightarrow (a \rightarrow b) \leq (a \rightarrow b) \lor \neg(a \rightarrow b) \).
Let us observe now that if \( \neg a \land b < a \to b < a \land b \) then we have the obvious equalities:
\[
a \land b \land \neg a = \neg a \land \neg b, \quad \neg a \land (a \to b) = \neg a \land \neg a \quad \text{and} \quad \neg a \lor (a \land b) = a.
\]
If moreover \((a \to b) \lor \neg a = a\) then the elements \(\{-a \land \neg a, -a, a, a \land b, a \to b\}\) form the lattice \(N_2\), thus the lattice of the algebra \([a, b]_A\) is not distributive. However, that is not possible, so we must assume that \((a \to b) \lor \neg a < a\) and that \(\neg a < (a \to b) \lor \neg a\). The second inequality is justified as follows. It is obvious that \(\neg a \leq (a \to b) \lor \neg a\). But if \(\neg a = (a \to b) \lor \neg a\) then on one hand \(\neg a \land \neg b = a \land b \land \neg a\), and on the other hand \(\neg a \land \neg b = (a \to b) \lor \neg a = (a \land b \land (a \to b)) \lor (a \land b \land \neg a) = (a \land b \land \neg a) \lor (a \land b \land \neg b) = a \land b \lor \neg a \land \neg b = a\), i.e. \(\neg a \land \neg a = a\). Analogously we prove that between the elements \(b \) and \(\neg b\) there exist elements \(\neg b \lor (a \land b)\) and \(b \land (a \land b)\) which satisfy the inequalities
\[
\neg b < b \land \neg (a \land b) < b, \quad \text{and} \quad \neg b < b \land \neg (a \land b) < b.
\]
Let us denote: \(c := (a \to b) \lor \neg a\); we have, as above, that \(d \lor \neg d = b\) and \(d \land \neg d = \neg b\).

Let us consider now the element \(\neg a \land \neg d\). We have \(\neg c \land \neg d = [a \land (a \to b)] \land [b \land (a \to b)] = a \land b \land (a \to b)\).

On one hand we have \((\neg c \land \neg d) \land (a \to b) = a \land b \land (a \to b) \land (\neg c \land \neg d) = a \land b \land (\neg a \land \neg b) = \neg a \land \neg b\), i.e. \((-c \land \neg d) \land (a \to b) = \neg a \land \neg b\), on the other hand \(- (c \land \neg d) \lor (a \to b) = (a \to b) \lor [a \land b \land (a \land b) \land (\neg a \land \neg b)] = (a \land b) \lor (a \land b) = a \land b\), i.e. \((-c \land \neg d) = a \land b\). In consequence we have the inequalities:
\[
\neg a \land \neg b \leq \neg c \land \neg d \leq a \land b.
\]
and moreover \((a \to b) \neq \neg c \land \neg d\).

Since \(\neg c \land \neg d \leq a \land b\), the filter \(F = [\neg c \land \neg b]\) is a normal filter, thus it determines a congruence relation on the algebra \([a, b]_A\), and moreover does not contain the element \((a \to b)\). Let us investigate the quotient algebra \([a, b]_A / \Theta (F)\). We have in particular
\[
(a) \quad (a \to b) \equiv (\neg a \land \neg b) (\Theta (F)).
\]

To prove it we let us observe that by the assumption we have \(\neg a \land \neg b \leq (a \to b)\), thus \(\neg a \land \neg b \to (a \to b) \in F\). On the other hand we have:
\[
a \to b \leq a \iff (a \lor b) \leq (a \to b) \iff (\neg a \land \neg b) \leq a \lor b \lor \neg (a \lor b) \iff (\neg a \land \neg b) \leq (a \land b) \to (\neg a \land \neg b).
\]
But since \((a \to b) \in F\), \(\neg (a \to b) \land \neg (a \land b) \in F\). Thus \(a \to b \equiv \neg a \land \neg b (\Theta (F))\).

\[
(b) \quad \neg c \land \neg d \equiv a \land b (\Theta (F))
\]

The proof of (b): It it obvious that \(\neg (a \to b) \land a \land b \leq a \land b\), thus \([a \land b \land (a \to b) \lor (a \land b)] \lor (a \land b \land (a \to b)] = a \land b \land (a \to b)\), thus \([a \land b \land (a \to b)] \leq a \land b \lor [a \land b \land (a \to b)] \lor (a \land b \land (a \to b)) \in F\),
because \( F = [a \land b \land \neg(a \rightarrow b)] \).

Moreover, let us note that since \( a \land b \rightarrow (a \rightarrow b) \equiv (a \rightarrow b) \), it is not true that 
\( (a \rightarrow b) \equiv a \land b(\Theta(F)) \), because \( a \land b \rightarrow (a \rightarrow b) \) does not belong to \( F \).

It is easy to prove that \( a \equiv a \land \neg(a \rightarrow b)(\Theta(F)) \), (i.e. that \( a \equiv \neg c(\Theta(F)) \)), \( a \equiv c(\Theta(F)) \), \( b \equiv \neg d(\Theta(F)) \), \( c \equiv d(\Theta(F)) \) and that it is not the case that \( a \equiv \neg a(\Theta(F)) \), \( b \equiv \neg b(\Theta(F)) \).

As concerns the element \( (a \rightarrow \neg b) = (b \rightarrow \neg a) \), we did not consider this element yet; however, since in the quotient algebra \( [a, b]_A/\Theta(F) \) we have \( \neg a \land \neg b \equiv (a \rightarrow b)(\Theta(F)) \), by the connection (*) (cf. the second lemma of this proof) we know that \( (a \rightarrow \neg b) \equiv (\neg a \land \neg b)(\Theta(F)) \).

Let us consider now the subalgebra \( [a/\Theta(F), b/\Theta(F)]_{[a, b]_A/\Theta(F)} \) of the algebra \( [a, b]/\Theta(F) \) (i.e. the subalgebra generated by the elements \( a/\Theta(F), b/\Theta(F) \)). In particular we have in this subalgebra: \( (a \rightarrow b) \equiv (b \rightarrow a) \equiv (a \rightarrow \neg b) \equiv (\neg a \rightarrow b) \equiv \neg a \land \neg b(\Theta(F)) \). By Lemma 7 (on Belnap’s matrix \( M_8 \)) we have that the matrix \( \langle [a/\Theta(F), b/\Theta(F)]_{[a, b]_A/\Theta(F)}, [(a \land b)/\Theta(F)] \rangle \) is isomorphic to Belnap’s matrix \( M_8 \).

This finishes the proof of the fact that if \( \neg a \land \neg b \land (a \rightarrow b) \land (a \rightarrow \neg b) \) then our variety \( V \) contains the Belnap’s matrix \( M_8 \); let us add that our proof of this fact did not depend on the position of the element \( a \rightarrow \neg b \) in the algebra \( [a, b]_A \).

B. Let us assume now that \( \neg a \land \neg b = a \rightarrow b \). Thus, by (*) the second lemma of this proof we have \( a \rightarrow b = a \rightarrow \neg b \), and by the Lemma on Belnap’s matrix \( M_8 \) it follows that \( M_8 \) is isomorphic to the matrix \( \langle [a, b]_A, [a \land b]_{[a, b]_A} \rangle \).

C. Let us assume now that \( (a \rightarrow b) \leq \neg a \land \neg b \). Since \( (a \rightarrow \neg b) \leq (a \rightarrow b) \), \( (a \rightarrow \neg b) \leq \neg a \land \neg b \)\(^4\). Then the filter \( F = \neg a \land \neg b \) is a normal filter in \( [a, b]_A \), i.e. this filter determines a congruence relation on \( [a, b]_A \). Let us consider the quotient algebra \( [a, b]_A/\Theta(F) \). It is obvious that \( b \rightarrow (\neg \neg b) \in F \) and \( (a \rightarrow \neg a) \in F \), thus \( a \equiv \neg a(\Theta(F)), b \equiv \neg b(\Theta(F)) \). However, it is not the case that \( a \equiv b(\Theta(F)) \), because by the assumption \( a \rightarrow b \) does not belong to \( F \). Thus the quotient algebra \( [a, b]_A/\Theta(F) \) contains two one-point falsifying subalgebras, and by Proposition 9 (on the matrix \( M_8 \)) a matrix isomorphic to the matrix \( M_8 \) will be a submatrix of the quotient matrix \( \langle [a, b]_A/\Theta(F), [(a \land b)/\Theta(F)]_{[a, b]_A/\Theta(F)} \rangle \).

D. \( a \rightarrow b \) is \( \leq \)-incomparable with \( \neg a \land \neg b \). Let us note first that \( a \rightarrow \neg b \leq \neg a \land \neg b \) in this case, in general it is known that \( a \rightarrow \neg b \leq a \land \neg b \) (cf. (b) above), but if \( a \rightarrow \neg b = a \land \neg b \), then this case reduces to the case A. As in C. let us consider now the quotient algebra \( [a, b]_A/\Theta(F) \) where \( F = \neg a \land \neg b \). Since neither \( a \rightarrow b \) nor \( a \rightarrow \neg b \) belongs to \( F \), the quotient algebra contains two trivial falsifying subalgebras - and we can argue as in the previous case to show that the matrix \( M_6 \) is a submatrix of the quotient matrix \( \langle [a, b]_A/\Theta(F), [(a \land b)/\Theta(F)]_{[a, b]_A/\Theta(F)} \rangle \).

This finishes the proof of the Theorem.

\(^4\)There exists at least one \( C_F \)-algebra generated by the elements \( a, b \) such that the following equalities and inequalities hold in it: \( a \rightarrow b = a \rightarrow \neg b \leq \neg a \land \neg b \).
From the last Theorem follows the following

**Theorem 16** Let $L$ be an extension of the relevant logic $R$. Then the relevance principle holds for $L$ if and only if $L$ is either a sublogic of the logic determined by $M_6$ or a sublogic of the logic determined by $M_8$.

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**References**


W. Dziobiak, *There are $2^{2^{n^2}}$ Logics with the Relevance Principle between $R$ and $RM$*, Studia Logica XLII,1, 1983, p. 49-60.


