LIFSHITZ' REALIZABILITY

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ITLI Prepublication Series
for Mathematical Logic and Foundations  ML-88-01
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Received March 1988
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Abstract. In Lifschitz 1979 a variation on Kleene's realizability for HA is presented, with a different clause for the existential quantifier. Lifschitz showed soundness for this realizability, and showed furthermore that $CT_0 \not\models CT_0$. In this paper a formalized version is considered, for which an axiomatization is given. An extension to HAS is given, as well as an analogue for realizability for functions. Finally, the construction of an "effective topos" for this realizability is sketched.

Key words and phrases: realizability, HA, HAS, EL, tripos.

AMS Subject classification: 03F50
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Acknowledgement. This paper grew out of some research I did for my master's thesis, under the guidance of Prof. Troelstra. I am very much indebted to him for supplying many basic ideas, patiently rereading numerous succeeding versions, spotting several grave errors, etc.

Introduction.

In Lifschitz 1979 a realizability interpretation for HA is given which differs from Kleene's realizability only in the clause for the existential quantifier.

A somewhat more complex coding of finite sets of natural numbers by numbers is given: let $V_e$, the finite set coded by $e$, be defined by

$V_e = \{ x \leq j_2 e \mid \{ j_1 e \}(x) \uparrow \}.$

Here $j_1, j_2$ are the first and second projections of the inverse of a bijective primitive recursive pairing-function $j: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, $\uparrow$ denotes partial recursive application.

Lifschitz put $e \in \exists x A x \equiv V_e \neq \emptyset \& \forall g \in V_e (j_2 g \subseteq \{ j_1 g \})$. His aim was to show that the schema $CT_0$ is really stronger than $CT_0!$, where

$\forall x \exists y A x y \rightarrow \exists z \forall x (\{ z \}(x) \downarrow \& A x(z)(x)).$

The basic idea for the proof of this is that there can't be an effective procedure which produces, given that $V_e \neq \emptyset$, an element of $V_e$; on the other hand, there is such a procedure working on all $e$ for which $V_e$ is a singleton.

For, if there were a code $g$ such that $V_e \neq \emptyset \Rightarrow \{ g \}(e) \downarrow \& \{ g \}(e) \in V_e$, and $W_f$ and $W_h$ are two disjoint, recursively inseparable r.e. sets, find a recursive function $F$ such that

$\forall x[(F(x))(0) \equiv f(x) \& (F(x))(1) \equiv h(x)].$

Then always $V_{j(F(x),1)} \neq \emptyset$, so $\{ g \}(j(F(x),1)) \in V_{j(F(x),1)}$ and $g$ serves to construct a recursive separation between $W_f$ and $W_h$.

(If $V_e$ is a singleton then one simply waits until $\{ j_1 e \}(x)$ has been computed for all $x \leq j_2 e$ save one; the remaining one must be the element of $V_e$.)
In this paper we will be concerned with the following questions: Can Lifschitz' realizability be formalized? Can we give an adequate axiomatization? Can we extend it to higher-order systems like HAS? Is there an analogon to Kleene's realizability for functions? Can it be put into the framework of tripos theory?

These questions can be answered affirmatively; however, to formalize the proof of soundness we seem to need to extend these systems somewhat. Lifschitz' proof that HA is sound for his realizability hinges on some lemmas that can't be formalized in HA. For this we seem to need two extra principles. One is Markov's Principle for primitive recursive predicates:

$$\forall n \lnot \exists \alpha \forall n - \exists \alpha \forall n,$$ for A primitive recursive.

The other one is:

$$\psi(e) \quad \forall n - \lnot [\text{th}(n) = j_2 e + 1 \land \forall i \leq j_2 e T(j_1 e, i, (n)_i)] \rightarrow \exists i \leq j_2 e \forall n - T(j_1 e, i, n),$$

which can be read as: if there is no witness for $V_e = \emptyset$, then $V_e$ must contain an element. An equivalent formulation would be:

$$\lnot \exists i \leq y \forall n A(i, z, n) \rightarrow \exists i \leq y \forall n A(i, z, n)$$

for primitive recursive $A$.

(Let us show this. One has to see:

$$\forall n - (\text{th}(n) = j_2 e + 1 \land \forall i \leq j_2 e T(j_1 e, i, (n)_i)) \leftrightarrow \exists i \leq j_2 e \forall n - T(j_1 e, i, n),$$

and use a standard Kleene normal form for $\Pi^0_1$-predicates.

Now $\rightarrow$ is trivial because $(\text{th}(n) = j_2 e + 1 \land \forall i \leq j_2 e T(j_1 e, i, (n)_i))$ of course implies $\exists i \leq j_2 e \forall n - T(j_1 e, i, n)$.

For $\leftarrow$: suppose $\exists i \leq j_2 e \forall n - T(j_1 e, i, n)$, then $\forall i \leq j_2 e - \forall n - T(j_1 e, i, n)$, so

$$\forall i \leq j_2 e - \exists n T(j_1 e, i, n).$$

And this implies $\lnot \exists i \leq j_2 e \exists n T(j_1 e, i, n)$ because of $\lnot \forall i \leq y - \exists n T(z, i, n) \rightarrow \lnot \forall i \leq y \exists n T(z, i, n)$ (induction on $y$).

Now $\forall i \leq j_2 e \exists n T(j_1 e, i, n)$ gives at once $\exists n \forall i \leq j_2 e T(j_1 e, i, (n)_i)$, so

$$\forall n \lnot \exists i \leq j_2 e \exists n T(j_1 e, i, (n)_i),$$

contradiction. Conclusion $\exists i \leq j_2 e \forall n - T(j_1 e, i, n).$)

It is easy to show that, w.r.t. EL, $\forall e \psi(e)$ is equivalent to some form of König's Lemma (see §3).

In the following, $V_e = \emptyset$ will be an abbreviation for $\exists x (x \leq j_2 e \land \forall n - T(j_1 e, x, n)).$

We define formulas $x \in A$, for Lifschitz' $\subset$, in the obvious way.

The formalization of Lifschitz' soundness proof is completely straightforward.

§0. Formalization of Lifschitz' realizability
Lemma 0.1. There is a total recursive function $b$ such that
$\text{HA} \vdash \forall a \forall y (y \in V_{b(a)} \iff y = a)$.

Lemma 0.2. There is a partial recursive function $\phi$ such that
$\text{HA} \vdash \forall c (\exists x \forall y (y \in V_c \iff y = x) \rightarrow \phi(c) \uparrow \& \phi(c) \in V_c)$.

The proofs are easy.

Lemma 0.3. There is a partial recursive function $\Phi$ such that
$\text{HA} + M_{PR} + \forall e \forall \Psi(e) \vdash \forall e, f (\forall x \forall e (f(x) \downarrow \rightarrow \Phi(e, f) \downarrow \& \forall h (h \in V_{\Phi(e, f)} \iff \exists g (g \in V_c (h = f(g)))))$.

Proof. $\exists g \in V_c (h = f(g)) \iff \exists g \leq j_{2e} (\forall n \rightarrow T(j_{1e}, g, n) \& \exists m (T(f, g, m) \& U_m = h))$, which is, given that $\forall g \leq j_{2e} (\forall n \rightarrow T(j_{1e}, g, n) \rightarrow \exists m (T(f, g, m))$, equivalent to $\exists g \leq j_{2e} (\forall n \rightarrow T(j_{1e}, g, n) \& (T(f, g, n) \rightarrow U_n = h))$, or $\exists g \leq j_{2e} (\forall n \rightarrow T(\chi(e, h, f) , g, n)$ for a suitable primitive recursive $\chi$; by $\forall e \forall \Psi(e)$, $\exists g \leq j_{2e} (\forall n \rightarrow T(\chi(e, h, f), g, n)$ is equivalent to $\forall n \rightarrow (1 \leq g = j_{2e} + 1 \& \forall i \leq j_{2e} T(\chi(e, h, f), i, (n, i)))$, or to $\forall n \rightarrow T(\chi(e, h, f), h, n)$ for suitable $\chi(e, f)$; let $\Phi$ be $j(\chi(e, f), \kappa)$ with $\kappa = \max \{Un \mid n = \min z (T(j_{1e}, l, z) \forall T(f, l, z)), l \leq j_{2e}\}$. Note that this is defined, by $M_{PR}$.

Lemma 0.4. There is a total recursive function $\gamma$ such that
$\text{HA} + M_{PR} + \forall e \forall \Psi(e) \vdash \forall e (\forall h (h \in V_{\gamma(e)} \iff \exists g (g \in V_c (h = V_{\gamma(g)})))$.

Proof. $\exists g \in V_c (h \in V_g)$ is $\exists g \leq j_{2e} (\forall n \rightarrow T(j_{1e}, g, n) \& \forall n \rightarrow T(j_{2e}, g, n)$ or $\exists g \leq j_{2e} (\forall n \rightarrow T(\pi(e, h, g), n)$ for suitable $\pi$; which by $\forall e \forall \Psi(e)$ is equivalent to $\forall n \rightarrow (1 \leq g = j_{2e} + 1 \& \forall i \leq j_{2e} T(\pi(e, h, f), i, (n, i)))$ or $\forall n \rightarrow T(\pi(e, h, f), h, n)$ for suitable $\pi$; so if we take $\gamma(e) = j(\pi(e), \max \{j \leq g \leq j_{2e} \})$, then $\gamma$ satisfies the lemma.

Lemma 0.5. For every formula $A$ in the language of $\text{HA}$ there is a $p$-term $\lambda x. \chi_A (x)$ (which may contain variables occurring free in $A$) such that
$\text{HA} + M_{PR} + \forall e \forall \Psi(e) \vdash \forall e (V_e \neq \emptyset \& \forall f \in V_e (f \subseteq A) \rightarrow \chi_A (e) \uparrow \& \chi_A (e) \subseteq A)$.

Lemma 0.6. For every theorem $A$ of $\text{HA}$ there is a number $n$ such that
$\text{HA} + M_{PR} + \forall e \forall \Psi(e) \vdash n \subseteq A$. 
Lemmas 0.5 and 0.6 are immediate formalizations of Lifschitz’ lemmas 5 and 6.

§1. Characterization of Lifschitz’ realizability.

The following lemma gives a more manageable form to Lifschitz’ realizability.

**Lemma 1.1.** Define a realizability \( r' \) by the following clauses:

1) \( x r' t = s \) \( \equiv \forall \chi_x \neq \emptyset \land \forall y \in V_x \quad t = s \quad (y \text{ not in } t = s) \)

2) \( x r' A \& B \) \( \equiv \forall \chi_x \neq \emptyset \land \forall y \in V_x \quad j_{1y} r'^* A \land j_{2y} r'^* B \)

3) \( x r' A \rightarrow B \) \( \equiv \forall \chi_x \neq \emptyset \land \forall y \in V_x \quad \forall w (w r'^* A \rightarrow (y)(w) \downarrow \land (y)(w) r'^* B) \)

4) \( x r' \forall z A z \) \( \equiv \forall \chi_x \neq \emptyset \land \forall y \in V_x \quad \forall n ((y)(n) \downarrow \land (y)(n) r'^* A(n)) \)

5) \( x r' \exists z A z \) \( \equiv \forall \chi_x \neq \emptyset \land \forall y \in V_x \quad j_{2y} r'^* A(j_{1y}) \)

Then for every formula \( A \) in the language of \( HA \) there are recursive functions \( \phi_A \) and \( \psi_A \) (they may contain variables occurring free in \( A \)) such that

\[
HA + M_{PR} + \forall e \psi(e) \vdash \forall e (e \Gamma A \rightarrow \phi_A(e) \downarrow \land \phi_A(e) r'^* A)
\]

\[
HA + M_{PR} + \forall e \psi(e) \vdash \forall e (e r'^* A \rightarrow \psi_A(e) \downarrow \land \psi_A(e) r'^* A),
\]

where \( \Gamma \) denotes Lifschitz’ realizability. (Note the form of the clauses: apart from a prefix \( \chi_x \neq \emptyset \land \forall y \in V_x \), it is just the Kleene clauses.)

**Proof.** Definition of \( \phi_A \) and \( \psi_A \) and proof of the lemma simultaneously by induction on \( A \). The notation is from the lemmas in §0. Following Lifschitz we write \( g^* \) for \( \lambda f. \Phi(f, g) \), where \( \Phi \) is as in lemma 0.3.

1) \( \phi_{\text{true}}(e) \equiv b(e) \)

\( \psi_{\text{true}}(e) \equiv 0. \)

2) \( \phi_{A \& B}(e) \equiv b(j(\phi_A(j_1 e), \phi_B(j_2 e))) \)

\( \psi_{A \& B}(e) \equiv j(\chi_A((\psi_A j_1) (e)), \chi_B((\psi_B j_2) (e))). \)

3) \( \phi_{A \rightarrow B}(e) \equiv b(\lambda h. \phi_B(e(\psi_A(h)))) \)

\( \psi_{A \rightarrow B}(e) \equiv \chi_{A \rightarrow B} (g^*(e)), \) where \( g = \lambda f. \lambda a. \psi_B(f(\phi_A(a))). \)

4) \( \phi_{\forall x A x}(e) \equiv b(\lambda n. \phi_A[n/x](e(n))) \)

\( \psi_{\forall x A x}(e) \equiv \lambda x A x (g^*(e)), \) where \( g = \lambda f. (\lambda n. \psi_A[n/x]((f)(n))). \)

5) \( \phi_{\exists x A x}(e) \equiv g^*(e) \) with \( g = \lambda f. j(j_1 f, \phi_A[j_1 f/x](j_2 f)) \)

\( \psi_{\exists x A x}(e) \equiv g^*(e) \) with \( g = \lambda f. j(j_1 f, \psi_A[j_1 f/x](j_2 f)). \) We trust that the reader
will be able to carry out the proof by himself.

**Definition.** Let $\Gamma$ be the class of formulas inductively generated by the clauses:
1) $\Sigma^0_1$-formulas are in $\Gamma$;
2) Formulas of form $\exists x \leq y A x$, with $A \in \Pi^0_1$, are in $\Gamma$;
3) $\Gamma$ is closed under $\forall$, $\rightarrow$ and $\land$.

As $\Gamma$ will play a role similar to that of the "almost negative" formulas in §3.2 of Troelstra 1973, which could be termed $\Sigma^0_1$-negative, let us call $\Gamma$-formulas "$\Sigma^0_2$-negative".

**Lemma 1.2.** (cf. I.c., 3.2.11) For every $\Sigma^0_2$-negative formula $A(a)$ (with free variables $a$) there is a partial recursive function $\psi_A$ satisfying

i) $HA + M_{PR} + \forall c \psi(c) \vdash \exists u (ur^*A) \rightarrow A$

ii) $HA + M_{PR} + \forall c \psi(c) \vdash A(a) \rightarrow \psi_A(a) \downarrow \land \psi_A(a) r^*A$.

**Proof.** We prove i) and ii) simultaneously by induction on $A$.

1) Suppose $A$ is $\Sigma^0_1$: $A \equiv \exists y B y$, $B$ prime; then $ur^*A$ is $V_u \neq \emptyset \land \forall f \in V_u \forall j \neq \emptyset \land \forall h \in V_{j, h} B(j, f)$ which clearly implies $A$; for ii) take $\psi_A \equiv b(j(x_B, b(0)))$ where $x_B \equiv \mu x B x$. For then, $A$ implies $x_B \downarrow$ and $b(0)r^*B(x_B)$, so $\psi_A \equiv r \exists x B x$.

2) Suppose $A \equiv \exists x \leq y B x$, $x$ not in $t$, $B$ is $\Pi^0_1$; say $B \equiv \forall x C x y$, then $ur^*A$ is equivalent to

$$(\star) V_u \neq \emptyset \land \forall h \in V_u \forall j \neq \emptyset \land \forall k \in V_{j, h} \forall n[(k, n) \downarrow \land (j, h, n) r^* (j, h, n) \land B(j, h, n)] \land C j h n$$

which implies $V_u \neq \emptyset \land \forall h \in V_u \forall j \neq \emptyset \land \forall n C j h n$ which implies $A$. For ii) let $e$ be such that $A = V_e \neq \emptyset$, and let $u$ such that $V_u = (j, h, b(\lambda n.b(0))) \land \forall h \in V_e$; then $V_e \neq \emptyset$ implies $(\star)$ for $u$.

3) We will only do the case $A \equiv \exists B \rightarrow C$; the other cases are left to the reader. $ur^*A$ is $V_u \neq \emptyset \land \forall h \in V_u \forall x (xr^*B \rightarrow ((h)(x) \downarrow \land (h, (x) r^* C))$. Now if $B$ then $\psi_B r^*B$ so $\forall h \in V_u ((h)(\psi_B) \downarrow \land (h)(\psi_B) r^* C)$; so if $C$ is such that $V_C (u) = ((h)(\psi_B) \land \in V_U)$ then $\phi(\chi(u)) r^* C$, so $C$; But if $B \rightarrow C$ then $b(\lambda u. \psi_C) r^* B \rightarrow C$, for suppose $ur^*B$, then $B$, so $C$, so $\psi_C r^* C$.

**Remark.** So the $\Sigma^0_2$-negative formulas are the "self-realizing" formulas for this realizability. As a quick glance reveals that formulas of form $xr^*A$ are $\Sigma^0_2$-negative, this realizability is idempotent.
Furthermore, since $\forall e\Psi(e)$ is also $\Sigma^0_2$-negative, as well as $M_{PR},$ we see that the soundness theorem for HA for this realizability can be extended to $HA+M_{PR}+\forall e\Psi(e)$.

We now introduce a principle analogous to ECT$_0$. Consider

$$\forall x(Ax\rightarrow\exists y Bxy) \rightarrow \exists z \forall x(Ax\rightarrow(z)(x)\downarrow \land V(z)(x)\neq\emptyset \land \forall y e V(z)(x)Bxy),$$

for $A$ $\Sigma^0_2$-negative.

Lemma 1.3. (cf. Troelstra 1973, 3.2.15) ECT$_L$ is $r'$-realizable.

Proof. Suppose $u'r'$ $\forall x(Ax\rightarrow\exists y Bxy).$ This is:

$$V_u\neq\emptyset \land \forall e \in V_u \forall n((f(n)) \downarrow \land V_f(n) \neq \emptyset \land \forall e \in V_f(n) \forall w(wr^*An\rightarrow(h)(w) \downarrow \land V(h)(w) \neq \emptyset \land \forall k \in V_f(n)(j_2kr^*Bn_jk)))$$

Let us simplify a bit. Let $u'$ be such that $\forall n((u')(n) \downarrow \land V_{u'}(n) = (\cup V_{f(n)}(n) \land \forall e \in V_u)$, then $\forall e \in V_{u'}(n) \forall w(wr^*An\rightarrow(h)(w) \downarrow \land V(h)(w) \neq \emptyset \land \forall k \in V_f(n)(j_2kr^*Bn_jk)).$ Put $\beta = (h)(\psi_A(n)), u''$ such that $V_{u''}(n) = \cup (V_{\beta}(n) \land \forall e \in V_{u'}(n))$, then $\forall w(wr^*An\rightarrow(u'')(n) \downarrow \land V_{u''}(n) \neq \emptyset \land \forall k \in V_{u''}(n)(j_2kr^*Bn_jk)).$ It is clear that $u''$ can be obtained recursively in $u$.

Now choose $z$ with $\forall x V_{(z)(x)} = j_1[V_{u''}(x)], \gamma$ such that $V_{\gamma}(m) = (l \land m(k)(m) \in V_{u''}(x)), \gamma''$ such that $V_{\gamma''}(m) = (\lambda y.\phi(\gamma(m))).$ Then we have $V_{\gamma''}(m) \neq \emptyset,$ and if $gr^*meV_{(z)(x)}$ then $m \in V_{(z)(x)}$ (since this is $\Sigma^0_2$-negative), so $V_{\gamma''}(m) \neq \emptyset \land \forall k \in V_{\gamma''}(g)kr^*Bm,$ so $\phi(\gamma(m))r^*Bm.$ Let $\gamma = b(y),$ then

$$V_{\gamma''} \neq \emptyset \land \forall \in V_{\gamma} \forall m((l)(m) \downarrow \land V_{(l)(m)} \neq \emptyset \land \forall p \in V_{(l)(m)} \forall g(gr^*(m \in V_{(z)(x)}) \rightarrow(p)(g) \downarrow \land (p)(g)r^*Bm)),$$

is $\gamma^r \forall h \in V_{(z)(x)}Bhx.$ The rest is easy.

Theorem 1.4. (cf. 1.c. 3.2.18; characterization of $r'$-realizability).

i) $HA+M_{PR}+\forall e\Psi(e)+ECT_L \vdash A \iff \exists x(xr'A);$  

ii) $HA+M_{PR}+\forall e\Psi(e) \vdash \exists x(xr'A) \iff HA+M_{PR}+\forall e\Psi(e)+ECT_L \vdash A.$

Proof. i) is proved by induction on $A.$ As usual, the only non-trivial steps are $A=B\rightarrow C$ and (similar) $A=\forall y By.$

Now $(B\rightarrow C) \iff \forall x(xr'B\rightarrow\exists y yr'C)) \iff \exists z \forall x(xr'B\rightarrow(z)(x)\downarrow \land V_{(z)(x)} \neq \emptyset \land \forall e \in V_{(z)(x)}(yr'C)) \iff \exists x(xr'B\rightarrow C).$ We leave the other case to the reader.
The proof of ii) (using i)) is completely analogous to 3.2.18 of Troelstra 1973.

Remarks on ECTₐ. i) ECTₐ is equivalent to a schema which resembles ECT₀ except for the condition that A can be taken Σ₀₂-negative. We see that this schema is consistent relative to HA, whereas ECT₀ w.r.t. Σ₀₂-negative formulas is not: if Wₑ and Wᵩ are disjoint, recursively inseparable r.e. sets, let F be such that ∀x (F(x))(0) = {e}(x), (F(x))(1) = {i}(x), then Vⱼ(F(x)),₁ ≠ ∅ for all x, so let Aₑ = Vⱼ(F(x)),₁ ≠ ∅ (Σ₀₂-negative), Bₓy = y ∈ Vⱼ(F(x),₁). Any z as in the conclusion of the schema will give a recursive separation between Wₑ and Wᵩ.

ii) The example given in 3.2.20 of Troelstra 1973 (A = ∃yT(xy) v ¬∃yT(xy), B = (x = 0 → ∃yT(xy) & x ≥ 1 → ¬T(xy))) shows that the restriction to Σ₀₂-negative formulas cannot be dropped.

iii) We can define a q'-realizability corresponding to r'-realizability by the clauses:

1) xq' t s = Vₓ ≠ ∅ & ∀y ∈ Vₓ t s
2) xq' A & B = Vₓ ≠ ∅ & ∀y ∈ Vₓ j₁ y q' A & j₂ y q' B
3) xq' A → B = Vₓ ≠ ∅ & ∀y ∈ Vₓ ∀w (w q' A → (y)(w) ↓ & (y)(w) q' B) & A → B
4) xq' s ∀ z A = Vₓ ≠ ∅ & ∀y ∈ Vₓ ∀n ((y)(n) ↓ & (y)(n) q' A(n))
5) xq' s ∃ z A = Vₓ ≠ ∅ & ∀y ∈ Vₓ j₂ y q' A(j₁ y)

Proposition 1.5. HA + Mₚₚ + ∀cψ(c) ⊢ A ⇒ HA + Mₚₚ + ∀cψ(c) ⊢ n q' A for some n; HA + Mₚₚ + ∀cψ(c) ⊢ y q' A → A; if A is Σ₀₂-negative, HA + Mₚₚ + ∀cψ(c) ⊢ A → ψₐ q' A for ψₐ as in lemma 1.2.

Proof. The first statement is proved by a routine induction on lengths of deductions in HA; the reader may wish to consult Theorem 3.2.4 of Troelstra 1973. The other two statements are proved by induction on A.

Corollary 1.6. HA + Mₚₚ + ∀cψ(c) obeys the following rule:

\[ \vdash \forall x (A x \rightarrow \exists y B x y) \Rightarrow \exists z \vdash \forall x (A x \rightarrow (z)(x) \downarrow \& V (z)(x) \neq \emptyset \& \forall h \in V (z)(x) B x h) \], for A Σ₀₂-negative.

§2. Extension of Lifschitz' realizability to HAS.
The extension of Kleene's realizability to HAS, described in Troelstra 1973, is given by the simple clauses:

\[ xr (t_0,\ldots,t_{n-1}) \in X \equiv (t_0,\ldots,t_{n-1}) \in X^* \]
\[ xr \forall X A(X) \equiv \forall X^* xr A(X) \]
\[ xr \exists X A(X) \equiv \exists X^* xr A(X), \]

where \( X \rightarrow X^* \) is an operation that assigns to each \( n \)-ary set variable \( X \) a \( n+1 \)-ary set variable \( X^* \) from a fresh stock of variables.

As a consequence, this extension satisfies the Uniformity Principle:

\[ \forall X \exists n A(X,n) \rightarrow \exists n \forall X A(X,n). \]

Now this cannot work for Lifschitz' realizability, because in that case we would have all realizability clauses equal for both interpretations except for the clause for the numerical existential quantifier; but this quantifier can be eliminated in HAS, because of the equivalence

\[ \exists y A(y) \leftrightarrow \forall X (\forall y (Ay \rightarrow X) \rightarrow X), \]

that holds in systems based on second-order logic with full comprehension. So then these two interpretations would be the same, quod non. However, combined with lemma 1.1, this idea suggests the following extension:

6) \( xr' (t_0,\ldots,t_{n-1}) \in X \equiv V_x \neq \emptyset \land \forall y \in V_x (t_0,\ldots,t_{n-1},y) \in X^* \)
7) \( xr' \forall X A(X) \equiv V_x \neq \emptyset \land \forall y \in V_x \forall X^* yr' A(X) \)
8) \( xr' \exists X A(X) \equiv V_x \neq \emptyset \land \forall y \in V_x \exists X^* yr' A(X). \)

**Theorem 2.1.** \( r' \) is a sound realizability for \( \text{HAS} + \forall e \Psi(e) + M_{PR} \).

**Proof.** The verification of the rules for second-order predicate logic does not pose any problem. For instance, if \( \psi(y) r' A(y) \rightarrow B \), \( y \) not in \( B \), and \( xr' \exists y A(y) \), where \( A \) and \( B \) are arbitrary formulas in the language of HAS, then \( V_x \neq \emptyset \land \forall y \in V_x j_2 yr' A(j_1 y) \), so \( V_x \neq \emptyset \land \forall y \in V_x \forall h \in V_{\psi (j_2 y)} (h)(j_2 y) \downarrow \land (h)(j_2 y) r' B. \) Let \( x \) be such that \( V_x = \{(h)(j_2 y) h \in V_{\psi (j_2 y)}, y \in V_x \} \), then \( \phi (\chi(x)) r' B, \) so \( b(\lambda x. \phi (\chi(x))) r' \exists y A(y) \rightarrow B \), where \( b \) and \( \phi \) are as defined in lemmas 0.1 and 0.4.

For the comprehension schema:

\[ CA \land \exists y (y \in X \leftrightarrow Ay), \]

first note that the following holds:

\[ (*) \ V_k \neq \emptyset \land \forall l \in V_k \exists k'(l \in V_k \land k' r' A) \rightarrow kr' A \quad (\text{Trivial from the definition of } r'-) \]
realizability). Now $x \vdash \exists! y (y \in X \iff Ay)$ means

$$\forall x \neq \emptyset \land \forall f \in \forall x \exists! y (f(y)) \downarrow \land \forall k$$

$$(o) \quad (\forall x \neq \emptyset \land \forall f \in \forall x \exists! y (f(y)) \downarrow \land (j_1((f(y)))(k) \downarrow \land (j_1((f(y)))(k) = k; \text{and if})$$

$$X^* = ((y,l) \exists k (kr^* Ay \land l \in V_k)), \text{then (o) is easily verified for } f, x, \text{and } X^*, \text{using (*)}.$$ The verification of extensionality

$$\text{EXT} \quad Ay \land y = x \rightarrow Ax,$$

is completely trivial, which concludes the proof.

§3. A Lifschitz analogon to realizability for functions.

Description of EL. The language of EL contains, in addition to the
language of HA, variables for functions, an application operator $A_p$, a
recursor $R$ and abstraction operators $\lambda x$. for every number variable $x$, such
that the following hold:

1) function variables are functors (i.e. terms for functions);
2) function constants are functors (for example, the constants for
all primitive recursive functions);
3) If $\phi$ is a functor and $t$ a term then $A_p(\phi, t)$, always written $\phi(t)$, is a
term;
4) $R$ is a functor;
5) If $t, t'$ are terms and $\phi$ is a functor then $R(t, \phi, t')$ is a term;
6) If $t$ is a term and $x$ a number variable then $\lambda x. t$ is a functor.

The non-logical axioms and rules of EL are:

$\lambda$-CON: $(\lambda x. t)(t') = t't/x$, and

$R$-ax: $R(t, \phi, 0) = t$ and $R(t, \phi, S t') = R(t, \phi, t'), t')$. $QF-AC_{00}$:

$\forall x \exists y Axy \rightarrow \exists \alpha \forall x A(x, \alpha x), \text{for A quantifier-free}.$

EL is discussed extensively in Troelstra 1973, as well as Kleene’s
function-realizability for EL, based on partial continuous application. Let
us fix some notation.

$\overline{0} \equiv >; \overline{a}(k + 1) \equiv \overline{a} \overline{k} \overline{a} \overline{(k)} \overline{a}$ where $<>$ denotes the empty sequence, and $\star$
concatenation of finite sequences.
\[<_{n}^{m} = m \land \forall i < m \sigma_i = n.\]
\[\beta(\alpha) \downarrow \text{ means } \exists x (\beta(x) \neq 0), \text{ and } \beta(\alpha) = \beta(\alpha(\beta(x) > 0)) + 1;\]
\[\beta \mid \alpha \downarrow \text{ means } \forall x \beta(<x>_x \alpha) \downarrow, \text{ and } \beta \mid x = \lambda x. \beta(<x>_x \alpha);\]
\[\forall \alpha \text{ will stand for } \lambda x. \alpha;\]
\[\varnothing \rightarrow \tau \text{ means that } \sigma \text{ is an initial segment of } \tau.\]
\[\alpha \in \sigma \text{ says } \forall i < \text{th}(\sigma) (\alpha(i) = \alpha(i));\]
\[\beta \leq \alpha \text{ is } \forall i (\beta(i) \leq \alpha(i));\]
\[j\alpha = \lambda x. j_1(\alpha(x)), \text{ for } i = 1, 2.\]

The obvious analogon in the language of functions of the coding \(V_e\) is to put
\[V_{\alpha} = (\beta \leq \beta_2 \alpha | j_1(\alpha(\beta)) = (\beta \leq \beta_2 \alpha | \forall n \beta(\overline{i_1(\alpha)} = 0).\]
If we read the principle \(\forall e \Psi(e)\) from \(\Sigma^0_1\) as: ((there is no witness \(n\) for
\[V_e = \emptyset \rightarrow V_e = \emptyset\), then the analogous principle in the language of EL is:
\[\forall n \neg \exists \sigma [(\text{th}(\sigma) = n \land \forall i < n \sigma_i \leq \text{th}(\sigma) \rightarrow \exists \sigma \exists \text{th}(\sigma) > 0 \rightarrow \exists \beta \leq \beta_2 \alpha \forall n j_1(\alpha_{\beta(n)} = 0,\]
which amounts to a version of König's Lemma.

In fact, if we put \(P(\sigma) = \forall i < \text{th}(\sigma) \exists m < \text{th}(\sigma) T(j_1e, (\sigma), i, m)\), then \(\Psi(e)\) is in EL
equivalent to \(\forall n \exists \sigma [\text{th}(\sigma) = n \land \forall i < n \sigma_i < \text{th}(\sigma) \land \neg P(\sigma) \rightarrow \exists \beta \leq \beta_2 \alpha \forall n P(\overline{\beta(n)} = 0).\]
(For in EL one has: \(\exists i \leq \beta_2 \forall n \neg T(j_1e, i, n) \iff \exists \beta \leq \beta_2 \forall n \neg P(\beta(n + 1)).\)
And in HA: \(\forall n \neg (\text{th}(n) = j_2 + 1 \land \forall i \leq \beta_2 T(j_1e, i, n)) \iff \exists i \leq \beta_2 \forall n \neg P(\overline{\beta(n)} = 0).\]
From this the equivalence easily follows.)

To prove the appropriate closure properties of the sets \(V_{\alpha}\), we will work in
the theory EL + MP\_QF + KL\_QF, where MP\_QF denotes Markov's Principle w.r.t.
quantifier-free formulas and KL\_QF will be:

\[\forall n \exists \sigma (\text{th}(\sigma) = n \land \forall i < n \sigma_i < \text{th}(\sigma)) \land R(\sigma) \rightarrow \exists \beta \forall n (\beta(n) < \alpha(n) \land R(\overline{\beta(n)}), \text{ for } R \text{ quantifier-free.}\]

We see, using the equivalent formulation of \(\forall e \Psi(e)\) given in \(\Sigma^0_1\) and the
well-known fact that every finitely branching tree can be encoded as a
subtree of e.g. the binary tree, that \(\forall e \Psi(e)\) is actually equivalent (in EL) to
KL\_QF.

Observe that KL\_QF + MP\_QF + FAN\_QF, where FAN\_QF is the schema:

\[\forall \beta \leq \alpha \exists n R(\overline{\beta(n)} \rightarrow \exists \beta \leq \alpha \exists n \leq \alpha R(\overline{\beta(n)}), \text{ R quantifier-free.}\]

Also note that KL\_QF + QF - AC\_00 is sufficient to prove KL for \(\Sigma^0_1\)-formulas R.
(Suppose \(\forall n \exists \sigma[lth(\sigma)=n \& \forall i<n(\sigma) \leq \alpha(i) \& \exists \tau R(\sigma, \tau)]\), so \(\forall n \exists \alpha \exists \sigma[lth(\sigma)=n \& \forall i<n(\sigma) \leq \alpha(i) \& R(\sigma, \tau)]\).)

We will denote \(\text{MP}_Q, \text{KL}_Q\) and \(\text{FAN}_Q\) simply by \(\text{MP}, \text{KL}\) and \(\text{FAN}\), respectively.

We will make use of the expressions "p-term" and "p-functor" as in Kleene 1969.

**Definition.** We define for every formula \(A\) a formula \(\alpha \vdash A\) with \(\alpha \not\in \text{FV}(A)\) and \(\text{FV}(\alpha \vdash A) \subset (\alpha) \cup \text{FV}(A)\) as follows:

1. \(\alpha \vdash A\) \iff \(A\) for \(A\) atomic;
2. \(\alpha \vdash A \& B\) \iff \(j_1 \alpha \vdash A \& j_2 \alpha \vdash B\);
3. \(\alpha \vdash A \rightarrow B\) \iff \(\forall \beta (\beta \vdash A \rightarrow \alpha[\beta] \downarrow \& \alpha[\beta] \vdash B)\);
4. \(\alpha \vdash \forall x A x\) \iff \(\forall n (\alpha[n] \downarrow \& \alpha[n] \vdash A_n)\);
5. \(\alpha \vdash \exists x A x\) \iff \(\forall \alpha \neq \emptyset \& \forall \gamma \in \alpha (j_2 \gamma \vdash A(j_1 \gamma(0))));
6. \(\alpha \vdash \forall \beta A(\beta)\) \iff \(\forall \beta (\alpha[\beta] \downarrow \& \alpha[\beta] \vdash A(\beta))\);
7. \(\alpha \vdash \exists \beta A(\beta)\) \iff \(\forall \alpha \neq \emptyset \& \forall \gamma \in \alpha (j_2 \gamma \vdash A(j_1 \gamma)).\)

The proof that \(\text{EL}\) is sound for this realizability, goes completely parallel to the proof of \(\S 0\).

**Lemma 3.1.** There is a p-functor \(\beta_1\), such that
\[
\text{EL} + \text{KL} + \text{MP} \vdash \forall \alpha (\nu_\alpha = \{\beta \mid \beta \leq j_2 \alpha\})
\]

**Proof.** Write \(B_\alpha = \{\beta \mid \beta \leq j_2 \alpha\}\).

If \(\nu_\alpha = \{\beta\}\) then for every \(n\) and \(m\) such that \(m \leq j_2 \alpha(n)\) and \(m \not\in \beta(n)\), a finite computation suffices to show that \(j_1 \alpha(\gamma) \downarrow\) for every \(\gamma\) such that \(\gamma \in [m] \& \nu_\alpha\).

Now \(\forall \gamma \in B_\alpha (\gamma \in \bar{\nu}_\alpha \rightarrow j_1 \alpha(\gamma) \downarrow)\) holds for every \(m \leq j_2 \alpha(n)\) save one; a finite computation shows this and the remaining \(m \leq j_2 \alpha(n)\) must be equal to \(\beta(n)\).

**Lemma 3.2.** There is a p-functor \(\beta_2\), such that
\[
\text{EL} + \text{KL} + \text{MP} \vdash \forall \alpha (\nu_\alpha = \{\beta_2 \mid \beta_2 \vdash \nu_\alpha = \{\alpha\}\})
\]
Proof. Let $\gamma$ be such that $\forall \alpha (\langle \gamma \alpha \rangle (\sigma) = 0 \leftrightarrow \alpha = \sigma)$; take $\beta_2$ such that $\forall \alpha (\beta_2 \alpha = j(\gamma \alpha, \alpha))$.

The following sublemma, trivial as it may be, greatly simplifies the proofs of the lemmas thereafter, and will be applied frequently.

Sublemma 3.1. Let $A(\beta)$ and $C(\beta, \gamma)$ be formulas such that:
1) there is a $p$-functor $\psi$ such that $A(\beta) \vdash \psi \beta \downarrow \& \forall \gamma (C(\beta, \gamma) \rightarrow \gamma \leq \psi \beta);$ 
2) $A(\beta) \vdash C(\beta, \gamma) \leftrightarrow \forall n D(\beta, \gamma, n)$, where $D$ is a prime formula.

Then there is a $p$-functor $\Phi$ such that:
$EL + KL + MP \vdash A(\beta) \rightarrow \Phi \beta \downarrow \& \forall \gamma (\gamma \in V_{\Phi \beta} \leftrightarrow C(\beta, \gamma))$.

Proof. If $D$ is the prime formula from 2), there is a prime formula $D'(\beta, \sigma)$ such that $D(\beta, \gamma, n)$ is equivalent to $D'(\beta, \gamma, n)$. Now let $\chi$ be defined as follows: $\chi(\sigma) = 0$ if $D'(\beta, \sigma)$; $\chi(\sigma) = 1$ else.

Now put $\Phi := \Lambda \beta, j(\chi, \psi)$, where $\psi$ is the functor from condition 1).

Lemma 3.3. There is a $p$-functor $\beta_3$ such that
$EL + KL + MP \vdash \forall \alpha (\beta_3 \alpha \downarrow \& V_{\beta_3 \alpha} = u \in V_{\alpha} V_\gamma)$.

Proof. We apply sublemma 3.1.

$\epsilon \in u \in V_{\alpha} V_\gamma \rightarrow \epsilon \leq \max \{j_2 \gamma \leq j_2 \alpha\}$ is easy to see. Furthermore, the formula $\beta \in \Sigma_{\gamma} V_{\alpha} V_\gamma$ is equivalent to $\exists \gamma \leq j_2 \alpha \forall n (j_1 \alpha(\gamma, n) = 0 \& \beta(n) \leq j_2 \gamma(n) \& j_1 \gamma(\beta, n) = 0)$ which is, modulo KL and MP, equivalent to $\forall \alpha \exists \gamma (\alpha(\gamma(n) = 0 \& \forall k < n (\alpha_k \leq j_2 \alpha(k) \& (\beta_k) \leq j_2 (\sigma_k) \& (\beta_k < n \rightarrow j_1 (\sigma_{k+1}) = 0)])$ which is a formula of the form required in condition 2) of the sublemma.

Lemma 3.4. There is a $p$-functor $\Phi$ such that
$EL + KL + MP \vdash \forall \phi, \beta (\forall \alpha (\alpha \in V_\beta \rightarrow \phi(\alpha) \downarrow \rightarrow \Phi(\phi, \beta) \downarrow \&$ $\forall \alpha (\alpha \in V_{\Phi(\phi, \beta)} \leftrightarrow \exists \gamma (\gamma \in V_\beta \& \alpha = \phi \gamma)]].$

In other words: $V_\beta \subseteq \text{dom}(\phi) \rightarrow \phi(V_\beta) = V_{\Phi(\phi, \beta)}$.

In the following, for $p$-functors $\phi$, we will abbreviate $\phi^*$ for the $p$-functor $\Lambda \beta, \Phi(\phi, \beta)$. 

Proof. Again, we check the conditions of sublemma 3.1.
1) Suppose \( \forall \alpha (\alpha \in V_\beta \rightarrow \phi |\alpha|^\downarrow) \). So:

\[ \forall x \forall z \leq j_2 \beta (\forall n (j_1 \beta (\overline{\alpha n})=0) \rightarrow (\phi |\alpha|(x))^\downarrow) \]
which is equivalent to

\[ \forall x \forall z \leq j_2 \beta \rightarrow \exists n (j_1 \beta (\overline{\alpha n})=0 \vee \phi (\langle x \rangle * \overline{\alpha n}) \neq 0) \]
which by MP is equivalent to

\[ \forall x \forall z \leq j_2 \beta \exists n (j_1 \beta (\overline{\alpha n})=0 \vee \phi (\langle x \rangle * \overline{\alpha n}) \neq 0) \]
which in turn, by FAN, is equivalent to

\[ \forall x \exists n \forall z \leq n (j_1 \beta (\overline{\alpha z}) \neq 0 \vee \phi (\langle x \rangle * \overline{\alpha z}) \neq 0) \].

Note that the part following \( \forall x \exists n \) is actually quantifier-free, so define \( \psi \) by

\[ \psi = \lambda x . \mu n [\forall z \leq n (j_1 \beta (\overline{\alpha z}) \neq 0 \vee \phi (\langle x \rangle * \overline{\alpha z}) \neq 0)] \]

Let \( \Phi (x, z) \) be \( (\phi |\alpha|(x), \text{if } \phi (\langle x \rangle * \overline{\alpha z}) \neq 0 \text{ (and otherwise, for example, undefined). Now put} \)

\[ \eta (x) = \max \{ \Phi (x, z) | z \leq \psi (x) \text{ and } z \text{ witnesses } (\phi |\alpha|(x))^\downarrow \}; \]

\[ = 0 \text{ if this set is empty} \]
then \( \chi := \lambda x . \eta (x) \) is the required upper bound.

2) Now let \( \phi [V_\beta] \) be, modulo \( V_\beta \subseteq \text{dom}(\phi) \), equivalent to a \( \Pi^0_1 \)-formula: for, \( \exists \delta (n \leq j_2 \beta (n) \wedge j_1 \beta (\overline{\delta n})=0 \wedge \exists z (\forall k < z \phi (\langle n \rangle * \overline{\delta k})=0 \wedge \phi (\langle n \rangle * \overline{\delta z})=\gamma (n)+1)) \),
which, modulo \( \forall \alpha (\alpha \in V_\beta \rightarrow \phi |\alpha|^\downarrow) \), is equivalent to

\[ \exists \delta (n \leq j_2 \beta (n) \wedge j_1 \beta (\overline{\delta n})=0 \wedge \forall z (\forall k < z \phi (\langle n \rangle * \overline{\delta k})=0 \wedge \phi (\langle n \rangle * \overline{\delta z})>0 \rightarrow \phi (\langle n \rangle * \overline{\delta z})=\gamma (n)+1)) \],
and this is, in view of the boundedness of \( \delta \), in \( \text{EL}+\text{KL}+\text{MP} \) equivalent to a \( \Pi^0_1 \)-formula, by the kind of derivation we have seen before.

Lemma 3.5. For every formula \( A \) in the language of \( \text{EL} \) there is a \( p \)-functor \( \chi_A \), which may contain free variables occurring in \( A \), such that

\[ \text{EL}+\text{KL}+\text{MP}+\forall \beta [V_\beta \neq \emptyset \wedge \forall \alpha \in V_\beta (\alpha \in A) \rightarrow \chi_A |\beta|^\downarrow \wedge \chi_A |\beta|^\Delta \).

Proof. \( \chi_A \) is defined by induction on the logical complexity of \( A \):

1) \( \chi_A = [1] \) if \( A \), \( \chi_A = [0] \) if \( \neg A \), for \( A \) atomic.

Remember that \( [0]|\alpha|^\downarrow \) for every \( \alpha \).

2) \( \chi_A = \Delta \beta . \mu \lambda \beta \chi_B |j_2 \beta | \beta \chi_C |j_2 \beta \beta \) if \( A \equiv B \& C \).
For suppose $V_\beta \not= \emptyset$, $\forall \alpha \in V_\beta (\alpha \in B \land C)$, then
$j_1[[V_\beta]_t]_1 = V_{j_1[\beta]} (\leq B)$, so $\chi_B[j_1[\beta]]$ is a subset of $\chi_B[j_1[\beta]] \subseteq B$; analogously for $C$.

3) $\chi_A \equiv$
$\Lambda \gamma. (\chi_C[(\psi_\gamma \ast \beta)])$, where $\psi_\gamma$ is such that $\forall \alpha. \psi_\gamma \ast \alpha = \alpha \gamma$, if $A \equiv B \rightarrow C$.

For suppose $V_\beta \not= \emptyset$, $\forall \alpha \in V_\beta (\alpha \in B \rightarrow C)$, and $\gamma \in B$, then
$\psi_\gamma[[V_\beta]_t]_1 = V_{\psi_\gamma \ast \beta} \not= \emptyset$ and $\forall \delta \in V_{\psi_\gamma \ast \beta} \subseteq C$, so $\chi_C[(\psi_\gamma \ast \beta)]$ and $\in C$.

4) $\chi_A \equiv$
$\Lambda \gamma. (\chi_A[(\psi_\gamma \ast \beta)])$, where $\psi_\gamma$ is such that
$\forall \alpha. \psi_\gamma \ast \alpha = \alpha [\gamma(0)]$, if $A \equiv \forall x A(x)$.

For suppose $V_\beta \not= \emptyset$, $\forall \alpha \in V_\beta (\alpha \in \forall x A(x))$, $\gamma$ arbitrary, then
$\psi_\gamma[[V_\beta]_t]_1 = V_{\psi_\gamma \ast \beta} \not= \emptyset$, $\forall \delta \in V_{\psi_\gamma \ast \beta} \subseteq A(x)[\gamma(0)/x]$, so $\chi_A[(\psi_\gamma \ast \beta)]$ and $\in A(x)[\gamma(0)]$.

5) $\chi_A \equiv$
the functor $\beta_\gamma$ from lemma 3, if $A \equiv \exists x B(x)$ or $\exists \alpha B(\alpha)$.

6) $\chi_A \equiv$
$\Lambda \gamma. (\chi_B[\alpha][\gamma/\alpha]_1[(\psi_\gamma \ast \beta)])$, where $\psi_\gamma$ is such that $\forall \alpha. \psi_\gamma \ast \alpha = \alpha \gamma$, if $A \equiv \forall x B\alpha$.

For if $V_\beta \not= \emptyset$, $\forall \delta \in V_\beta (\delta \in \forall x B\alpha)$, $\gamma$ arbitrary, then
$\psi_\gamma[[V_\beta]_t]_1 = V_{\psi_\gamma \ast \beta} \not= \emptyset$, $\forall \delta \in V_{\psi_\gamma \ast \beta} \subseteq B\alpha[\gamma/\alpha]$ so
$\chi_B[\alpha][\gamma/\alpha]_1[(\psi_\gamma \ast \beta)] \subseteq B\alpha[\gamma/\alpha]$, etc.

**Lemma 3.6.** For every formula $A$ in the language of EL such that
EL $\vdash A$ there is a $p$-functor $\psi_A$ such that EL+KL+MP $\vdash \psi_A$ and $\psi_A \in A; \psi_A$ may contain variables occurring free in $A$.

**Proof.** This goes by induction on proofs in EL+KL+MP. Since our realizability differs only in the existential clauses from Kleene’s, we only have to check the lemma for those rules and axioms of two-sorted predicate calculus that concern existential formulas, as well as for QF-AC_oo.

It is clear that
$\Lambda \alpha. \beta_2 j_1[[t], \alpha] \in A(t) \rightarrow \exists x A(x)$,
$\Lambda \alpha. \beta_2 j_1[[\phi], \alpha] \in A(\phi) \rightarrow \exists \alpha A(\alpha)$, for $\beta_2$ from lemma 3.2.

Now suppose $\alpha \in A(y) \rightarrow C$, $y$ possibly in $\alpha$, not in $C$.

Then $\Lambda \gamma. \chi_C[(\psi_\gamma \ast y)] \subseteq \exists x A(x) \rightarrow C$, where $\chi_C$ from lemma 3.5 and $\psi$ such that $\psi[\beta = \alpha[\beta(0)/y]] \subseteq j_2 \beta$.

For suppose $y \in \exists x A(x)$, so $V_y \not= \emptyset$ and $\forall \beta \in V_y (j_2 \beta \subseteq A(j_1 \beta(0)))$. Then for $\beta \in V_y$ we
have that $\psi|\beta \subseteq C$, so $\forall \delta \in \psi|V^1_\beta = V_{\psi|\beta}(\delta \subseteq C)$, so $\chi_C(\psi*|\gamma) \subseteq C$.

Completely analogous for $\langle A(\phi) \rightarrow C \rangle \rightarrow \langle \exists \alpha A(\alpha) \rightarrow C \rangle$.

The following sublemma will be useful for the proof that $\text{QF-AC}_{00}$ is realised.

**Sublemma 3.2.** There is a functor $\chi$ such that

$\text{EL+KL+MP} \vdash \forall \varepsilon \left[ (\forall n \ (e[n] \downarrow \& V_{e[n]} \neq \emptyset) \rightarrow \chi|_e \downarrow \& V_{\chi|_e} \neq \emptyset \& \forall \gamma \in V_{\chi|_e} \forall n (\gamma[n] \downarrow \& \gamma[n] \in V_{e[n]}) \right]$.  

**Proof.** To apply sublemma 3.1, we construct a bounded primitive recursive condition for sequences $\sigma$ which says that $\sigma$ is "for the time being" an initial segment of a $\gamma$ such that $\forall n (\gamma[n] \downarrow \& \gamma[n] \in V_{e[n]}$).

Let $\sigma[n]$ denote the maximal $\tau$ such that $\gamma[n] \downarrow \& \gamma[n] \in V_{e[n]}$ for all $\gamma$ with $\gamma[n] \downarrow$ and $\gamma \in \sigma$.

(This is clearly primitive recursive in $n$ and $\sigma$).

We formulate our condition $A(e,\sigma)$ in 4 stages:

1) $\forall n < \omega (\sigma[n] \subseteq \sigma)$;

2) $\forall n < \sigma \forall i < \omega (e[n]_i) \rightarrow (\sigma[n]_i \leq j(\langle e[n] \rangle[n]))$ (so if $\gamma \in \sigma$ then for the time being $\gamma[n] \leq j(\langle e[n] \rangle[n])$);

3) $\forall n < \omega \forall i < \omega (\sigma[n]) \rightarrow (\langle e[n] \rangle[n])_i \leq \omega (\sigma[n])$ (This will ensure that $\forall m A(e,\gamma_m) \rightarrow \gamma[n] \downarrow$);

4) $\forall n < \omega \forall \tau \geq \sigma[n] (\tau \leq \omega (\langle e[n] \rangle[n]) \rightarrow j_1(\langle e[n] \rangle[n]) \leq 0)$ (So $\chi|_e \in V_{e[n]}$ if $\forall m A(e,\gamma_m)$).

Now let (sublemma 3.1) $\delta$ be such that $\forall n (\forall n \in V_\delta \leftrightarrow \forall n A(e,\gamma(n))$; and put $\chi \equiv \Lambda e. \delta$. Now if $\forall n (e[n] \downarrow \& V_{e[n]} \neq \emptyset)$, then there are arbitrarily long sequences $\sigma$ with $A(e,\sigma)$; with KL we conclude $V_{\chi|_e} \neq \emptyset$.

**QF-AC}_{00}.** Let $F \equiv \forall x \exists y A(x,y)$ be an instance of $\text{QF-AC}_{00}$ and suppose $\delta$ realizes the premiss. Then:

$\forall n \delta[n] \downarrow \& V_{\delta[n]} \neq \emptyset \& \forall \gamma \in V_{\delta[n]} (j_2(\gamma) \in A(n,j_1(\gamma)))$.

Let $\psi$ such that $\psi|\gamma \equiv j_2(\langle j_1(\gamma) \rangle[n])$. Then for all $n$: $V_{\psi|\delta[n]} = \psi|\delta[n] \neq \emptyset$ (Lemma 3.4), and $\forall \gamma \in V_{\psi|\delta[n]} j_2(\gamma) \in A(n,j_1(\gamma(n)))$. Apply sublemma 3.2 to find a $\chi$ such that $\forall n \in V_{\chi|\delta} \forall n (\gamma[n] \downarrow \& \gamma[n] \in V_{\psi|\delta[n]})$.

then this $\chi$ realizes the conclusion of $F$. 
We now get some lemmas that are analogous to lemmas 1.1 and following.

**Lemma 3.7.** Define a realizability \( r' \) by the clauses:

1. \( r' A \equiv \forall \alpha \neq \emptyset \wedge \forall \beta \in \alpha \) for \( A \) atomic;
2. \( r' A \& B \equiv \forall \alpha \neq \emptyset \wedge \forall \beta \in \alpha \) \( j_1 \beta \ r' A \wedge j_2 \beta \ r' B \);
3. \( r' A \rightarrow B \equiv \forall \alpha \neq \emptyset \wedge \forall \beta \in \alpha \) \( \forall \chi (\gamma \ r' A \rightarrow \beta \gamma \downarrow \wedge \beta \gamma \ r' B) \);
4. \( r' \forall x A x \equiv \forall \alpha \neq \emptyset \wedge \forall \beta \in \alpha \) \( \forall n (\beta[n] \downarrow \wedge \beta[n] \ r' A n) \);
5. \( r' \exists x A x \equiv \forall \alpha \neq \emptyset \wedge \forall \beta \in \alpha \) \( (j_2 \beta \ r' A (j_1 \beta)) \);
6. \( r' \forall \beta A (\beta) \equiv \forall \alpha \neq \emptyset \wedge \forall \beta \in \alpha \) \( \forall \gamma (\beta \gamma \downarrow \wedge \beta \gamma \ r' A (\gamma)) \);
7. \( r' \exists \beta A (\beta) \equiv \forall \alpha \neq \emptyset \wedge \forall \beta \in \alpha \) \( (j_2 \beta \ r' A (j_1 \beta)) \).

Then for all formulas in the language of EL there are \( p \)-functors \( \phi_A \) and \( \psi_A \) such that:

\[
\begin{align*}
\text{EL+KL+MP} \models & \ \forall \alpha (\alpha \rightarrow \phi_A (\alpha \downarrow \& \forall \alpha \ r' A)) \\
\text{EL+KL+MP} \models & \ \forall \alpha (r' A \rightarrow \psi_A (\alpha \downarrow \& \forall \alpha \ A (\alpha))).
\end{align*}
\]

**Proof.** For those who are not yet asleep, we give the definitions.

i) \( \phi_{s = s} = \lambda \alpha. \beta_2 \downarrow \lambda \alpha [\downarrow] \)

\( \psi_{s = s} = \lambda \alpha [\downarrow] \)

ii) \( \phi_{s \& B} = \lambda \alpha. \beta_2 \downarrow (\phi_A \downarrow j_1 \alpha, \phi_B \downarrow j_2 \alpha) \)

\( \psi_{s \& B} = \lambda \alpha. \beta_2 \downarrow (\psi_A \downarrow j_1 \alpha, \psi_B \downarrow (\psi_B \downarrow j_2 \alpha) \downarrow \alpha) \)

iii) \( \phi_{s \rightarrow B} = \lambda \alpha. \beta_2 \downarrow (\lambda \gamma \phi_B \downarrow (\lambda \alpha \psi_A \downarrow \gamma) \downarrow \alpha) \)

\( \psi_{s \rightarrow B} = \lambda \alpha. \beta_2 \downarrow (\lambda \gamma \chi_A \downarrow \downarrow \lambda \gamma \psi_B \downarrow (\lambda \alpha \psi_A \downarrow \gamma) \downarrow \alpha) \)

iv) \( \phi_{\exists x A x} = \lambda \alpha. \beta_2 \downarrow (\lambda \alpha \xi_A \downarrow \downarrow \lambda \alpha \psi_A \downarrow \downarrow \alpha) \)

\( \psi_{\exists x A x} = \lambda \alpha. \beta_2 \downarrow (\lambda \alpha \xi_A \downarrow \downarrow \lambda \alpha \psi_A \downarrow \downarrow \alpha) \)

v) \( \phi_{\exists x A x} = \lambda \alpha. \beta_2 \downarrow (\lambda \alpha \xi_A \downarrow \downarrow \lambda \alpha \psi_A \downarrow \downarrow \alpha) \)

\( \psi_{\exists x A x} = \lambda \alpha. \beta_2 \downarrow (\lambda \alpha \xi_A \downarrow \downarrow \lambda \alpha \psi_A \downarrow \downarrow \alpha) \)

vi) \( \phi_{\exists y A y} = \lambda \alpha. \beta_2 \downarrow (\lambda \alpha \xi_A \downarrow \downarrow \lambda \alpha \psi_A \downarrow \downarrow \alpha) \)

\( \psi_{\exists y A y} = \lambda \alpha. \beta_2 \downarrow (\lambda \alpha \xi_A \downarrow \downarrow \lambda \alpha \psi_A \downarrow \downarrow \alpha) \)

vii) \( \phi_{\exists y A y} = \lambda \alpha. \beta_2 \downarrow (\lambda \alpha \xi_A \downarrow \downarrow \lambda \alpha \psi_A \downarrow \downarrow \alpha) \)

\( \psi_{\exists y A y} = \lambda \alpha. \beta_2 \downarrow (\lambda \alpha \xi_A \downarrow \downarrow \lambda \alpha \psi_A \downarrow \downarrow \alpha) \)

We hope that it is clear by now how to transpose the rest of \( \Phi 1 \) to the case of EL; therefore we state the following lemmas without proof.

**Definition.** The class \( \Gamma \) of \( \Sigma^1_2 \)-negative formulas is the smallest
satisfying:
i) Formulas of form $\exists \alpha A(\alpha)$ are in $\Gamma$, with a quantifier-free;
ii) Formulas of form $\exists \alpha \leq \beta \forall n A(\alpha, n)$ are in $\Gamma$, with a quantifier-free;
iii) $\Gamma$ is closed under $\neg, \land, \forall x, \forall \alpha$.

**Lemma 3.8.** For every $\Sigma^1_2$-negative formula $A(a)$ with free variables $a$ there is a p-functor $\xi_a$ such that

$$\text{EL+KL+MP} \vdash \exists \alpha (\alpha \not= \emptyset \land \forall \zeta \in V_{\forall \alpha \beta} B(\alpha, \zeta)),$$

with the restriction that $A$ must be $\Sigma^1_2$-negative.

**Corollary 3.9.** EL+KL+MP is sound for $r'$.

**Definition.** Let GC$_L$ be the following schema:

$$\forall \alpha (A \alpha \rightarrow \exists \beta B(\alpha, \beta)) \rightarrow \exists \gamma \forall \alpha (A \alpha \rightarrow \forall \zeta \in V_{\forall \alpha \beta} B(\alpha, \zeta)),$$

with the restriction that $A$ must be $\Sigma^1_2$-negative.

**Lemma 3.10.** GC$_L$ is $r'$-realizable.

**Theorem 3.11.**

i) $\text{EL+KL+MP+GC}_L \vdash \exists \alpha (\alpha \not= \emptyset)$

ii) $\text{EL+KL+MP} \vdash \exists \alpha (\alpha \not= \emptyset) \iff \text{EL+KL+MP+GC}_L \vdash A$

As a minor application of $r'$-realizability we have that GC$_L$, so a fortiori not GC, is not sufficient to prove GC, the principle of Generalized Continuity:

$$\forall \alpha (A \alpha \rightarrow \exists \beta B(\alpha, \beta)) \rightarrow \exists \gamma \forall \alpha (A \alpha \rightarrow \forall \zeta \in V_{\forall \alpha \beta} B(\alpha, \zeta)),$$

which is considered in Troelstra 1973 and is proven there to axiomatize Kleene's realizability based on partial continuous application.

We can do better, for the weakest well-known continuity principle without uniqueness-condition in the premiss, the schema WC-N:

$$\text{WC-N} \quad \forall \alpha \exists n A(\alpha, n) \rightarrow \forall \alpha \exists n \exists m \forall \beta \in \omega n A(\beta, m),$$

(weak continuity for numbers), is already incompatible with KL:

**Proposition 3.12.** WC-N and KL are incompatible w.r.t. EL.

**Proof.** Define a functor $\Gamma$ as follows:
\[\Gamma(\langle\rangle) = 0\]

\[\Gamma(\langle\sigma\rangle \star n) = 0\text{\ if } 1\text{th}(\sigma) > 1\text{th}(n)\]

\[= 1\text{\ if } 1\text{th}(\sigma) \leq 1\text{th}(n) \& \forall i < 1\text{th}(n) (n_i \neq 0 \& n_i \neq 1)\]

\[= 1\text{\ if } 1\text{th}(\sigma) \leq 1\text{th}(n) \& \exists i < 1\text{th}(n) (n_i = 0 \& \forall j < i n_j \neq 1) \& \forall i < 1\text{th}(\sigma) \sigma_i = 0\]

\[= 2\text{\ if } 1\text{th}(\sigma) \leq 1\text{th}(n) \& \exists i < 1\text{th}(n) (n_i = 0 \& \forall j < i n_j \neq 1) \& \exists i < 1\text{th}(\sigma) \sigma_i \neq 0\]

\[= 1\text{\ if } 1\text{th}(\sigma) \leq 1\text{th}(n) \& \exists i < 1\text{th}(n) (n_i = 1 \& \forall j < i n_j \neq 0) \& \forall i < 1\text{th}(\sigma) \sigma_i = 1\]

\[= 2\text{\ if } 1\text{th}(\sigma) \leq 1\text{th}(n) \& \exists i < 1\text{th}(n) (n_i = 1 \& \forall j < i n_j \neq 0) \& \exists i < 1\text{th}(\sigma) \sigma_i \neq 1.\]

Then

\[(\Gamma|\alpha)(\sigma) = \Gamma(\langle\sigma\rangle \star \alpha(1\text{th}(\sigma))) - 1,\]

is always defined.

Let \(\gamma\) be such that

\[\forall \alpha \gamma \alpha = \{\Gamma|\alpha, [1]\}.\]

Then we have:

\[\forall \alpha \forall n \exists i (\forall i < 1\text{th}(\sigma) \sigma_i = 1 \& (\Gamma|\alpha)(\sigma) = 0 \& 1\text{th}(\sigma) = n \& \forall \tau \supset \sigma (\Gamma|\alpha)(\tau) = 0 \& \& \forall i < 1\text{th}(\sigma) \sigma_i = \sigma_i),\]

so with KL we conclude:

\[\forall \alpha \exists \beta (\forall n, \beta n \leq 1 \& \forall n, m \beta n = \beta m \& \forall n (\Gamma|\alpha)(\beta n) = 0),\]

in other words:

\[\forall \alpha \exists n (n \leq 1 \& [n] \in \mathcal{V}_{|\alpha}).\]

Furthermore

\[\forall \alpha (\forall n \alpha n > 1 \rightarrow \forall \beta (\forall n, \beta n \leq 1 \rightarrow \beta \in \mathcal{V}_{|\alpha}) \& \exists n (\alpha n = 0 \& \forall m \leq n \alpha m \neq 1) \rightarrow \mathcal{V}_{|\alpha} = \{0\} \& \exists n (\alpha n = 1 \& \forall m \leq n \alpha m \neq 0) \rightarrow \mathcal{V}_{|\alpha} = \{1\}] \text{ holds.}\]

Now we cannot have:

\[(\ast) \quad \forall \alpha \exists n \exists m \forall \beta \exists \bar{m}(m \leq 1 \& [n] \in \mathcal{V}_{|\beta}).\]

For suppose so; let \(n\) and \(m\) satisfy \((\ast)\) for \(\alpha = [2]\).

Then if \(n = 0\) and \(\beta < 2\star m \star [1]\) we would have \([0] \in \mathcal{V}_{|\beta};\) if \(n = 1, \beta < 2\star m \star [0]\)

then \([1] \in \mathcal{V}_{|\beta},\]

which is a contradiction in both cases.

\[\S 4. \text{ A topos for Lifschitz' realizability.}\]

A further generalization of Lifschitz' realizability for HA can be obtained with the machinery of tripos theory, developed in Hyland, Johnstone & Pitts 1980, to be abbreviated HJP 1980 hereafter. They describe some ways of defining triposes, and how to associate a topos with each tripos. In
Hyland 1982 the topos associated with the tripos constructed out of the partial combinatory structure $\langle\mathbb{N},\{,\}\rangle$, the "effective topos", is described and it is shown that for the natural number object in this topos, exactly those sentences hold that are realized in Kleene's sense by some natural number. We will show that a similar tripos can be defined for Lifschitz' realizability, leading to the "Lifschitz topos" giving an extension of Lifschitz' realizability to all finite types.

First of all let $S$ be $\{e \in \mathbb{N} \mid V_e \neq \varnothing\}$. We define an application $\iota$ on $S$ by putting $e[f] \downarrow$ iff $\forall h \in V_e(h)(f) \downarrow$, and then $e[f]$ to be a code for $\{(h)(f) \mid h \in V_e\}$ (Note that such a code can be obtained recursively in $e$ and $f$). Let $\Sigma$ consist of those $H \subseteq S$ that satisfy i) $e \in H$, $V_e = V_{e'[f]} \Rightarrow e' \in H$, and ii) $e, f \in H$, $V_e = V_e \cup V_f \Rightarrow g \in H$.

We define an implication $\rightarrow: \Sigma \times \Sigma \rightarrow \Sigma$ by $F \rightarrow G = \{e \mid \forall f \in F \ e[f] \downarrow \land e \in G\}$. One checks immediately that this is well-defined. Now we define for each set $X$ a preorder $(\rho X, \iota_X)$ by $\rho X = \Sigma^X$, $\phi \vdash \chi \Phi$ iff $\cap(\phi(x) \rightarrow \psi(x)) | x \in X$ is nonempty; for functions $f: I \rightarrow J$ let $\rho f: \rho J \rightarrow \rho I$ be composition with $f$, and $\forall f: \rho I \rightarrow \rho J$ defined by $\forall f(\psi) = \lambda j. \cap(I f(i) = j \rightarrow \psi(i)) | i \in I$, where $I f(i) = j \Rightarrow S$ if $f(i) = j$, and $\varnothing$ otherwise. So $\forall f(\psi)(j) = \{e \mid \forall i \in I \forall h \in S (f(i) = j \Rightarrow e \in h \downarrow \land e \in h \psi(i))\}$.

As generic element $\sigma$ we take $id_2 \in \rho S$. Now the verification (with the help of Theorem 1.4 of HJP 1980) that this defines a tripos does not give any problem; the only difference with a tripos constructed out of the partial combinatory structure $\langle S, \iota \rangle$ is that we do not take the full powerset of $S$.

**Proposition 4.1.** Conjunction and disjunction in $\rho I$ can be defined as follows:

1. $\phi \land \psi \equiv \lambda i. \{e \in S \mid \forall h \in V_e(j_1 h = \phi(i) \land j_2 h = \psi(i))\}$.
2. $\phi \lor \psi \equiv \lambda i. \{e \in S \mid \forall h \in V_e(j_1 h = \phi(i) \lor j_1 h \neq 0 \Rightarrow \phi(i)) \land j_1 h = 0 \Rightarrow \psi(i))\}$.

Moreover, for any function $f: I \rightarrow J$, existential quantification along $f$ can be defined by:

1. $\exists f \phi \equiv \lambda j. \{e \in S \mid \forall h \in V_e \exists i e(f(i) = j \land h = \phi(i))\}$.

**Proof.** Apply the definitions given in Theorem 1.4 of HJP 1980. According to these,

1. $\phi \land \psi = \lambda i. \{e \in \Sigma f(\phi \rightarrow (\psi \rightarrow \rightarrow \Rightarrow e[\downarrow] \land e \in G))\}$. Suppose $e \in \phi \land \psi(i)$, let $G$ be $\phi \land \psi(i)$. Let $b$ be a total recursive function such that $b(a)$ codes $(a)$. Put
f = \lambda s.\lambda t. b(j(s,t)))$, then $f \in \phi(i) \rightarrow (\psi(i) \rightarrow G)$, so $\text{el} f \downarrow \& \text{el} f \in G = \phi \& \psi(i)$, so

\[ \lambda c. \text{el} f \in (\phi \& \psi(i) \rightarrow \phi \& \psi(i)) \text{ for all } i. \]

Conversely, if $c \in \phi \& \psi(i)$, then $f \in (\phi \& \psi(i) \rightarrow G)$, then $\forall h \in V_c (f[j_1,h][j_2] \downarrow \& (f[j_1,h][j_2] \in G)$ so $\psi$ is such that $\psi(e)$ codes

\[ \cup (V_{(f[j_1,h][j_2] \downarrow h \in V_c}), \text{ then } \psi \in \phi \& \psi(i) \rightarrow \phi \& \psi(i) \text{ for all } i. \]

ii) $\phi \& \psi = \lambda i. (\text{el} \forall G \in \Sigma \exists f(f(e \phi(i) \rightarrow G \& \psi(i) \rightarrow G) \Rightarrow \text{el} f \downarrow \& \text{el} f \in G)).$ Now if $c \in \phi \& \psi(i)$, let $G$ be $\phi \& \psi(i)$; $h = b(\lambda s. j(0,s))$, $h_2 = b(\lambda s. j(1,s))$; then $h \in \phi(i) \rightarrow G$, $h_2 \in \psi(i) \rightarrow G$, so $f = b(j(h_1,h_2)) \in (\phi(i) \rightarrow G \& \psi(i) \rightarrow G)$, so $\text{el} f \downarrow \& \text{el} f \in G = \phi \& \psi(i)$.

In the other direction, if $c \in (\phi \& \psi(i) \rightarrow G \& \psi(i) \rightarrow G)$, let $\Phi$ be such that $\Phi(h,g) = (j_1,h)(j_2,g)$ if $j_1 = 0, (j_2,h)(j_2,g)$ if $j_1 = 0$; then $\forall h \in V_f \forall g \in V_c \Phi(h,g) \downarrow$ and $\forall h \in V_f \forall g \in V_c \Phi(h,g) \in G$, so $\{\Phi(h,g)\} \in V_c$, $h \in V_f \in G$, and this can be coded recursively in $c$ and $f$.

iii) It is enough to show that $\exists f$ is left adjoint to $\phi f$. Suppose $\exists f \phi I \psi$, so let $c \in \cap (\exists f \phi(j) \rightarrow \psi(j) \in J)$, $f \in \phi(i)$; then $b(f) \in \exists f \phi(f(i))$, so $\text{el} b(f) \downarrow \& \text{el} b(f) \in \psi(f(i))$. So $\lambda f. \text{el} b(f) \in (\phi(i) \rightarrow \psi(f(i)) \in J)$, so $\phi f \psi$.

In the other direction, suppose $\phi f \psi$, $c \in \cap (\phi(i) \rightarrow \psi(f(i)) \in J)$, let $f \in \exists f \phi(j)$. Then $\forall h \in V_f \exists i \in (f(i) = j \& h \in \phi(i))$, so $\forall h \in V_f \text{el} h \downarrow \& \text{el} h \in \psi(j)$. But then, because $\psi(j) \in \Sigma$, we must have (a code for) $\cup (V_{\text{el} h} \in V_f \psi(j) \in J)$; so $\lambda f. (\cup (V_{\text{el} h} \in V_f \in \cap (\exists f \phi(j) \rightarrow \psi(j) \in J)$, i.e. $\exists f \phi I \psi$.

**Proposition 4.2.** The coproduct in the topos of $\wp$-sets may be defined as follows: $(X,=) \cup (Y,=) \in (X \cup Y,=_{X \cup Y})$ with

\[ [W = _{X \cup Y}] = \{c \in \forall h \in V_c \in \{j_1 = 0 \& j_2 \in [W = _X Z]\} \} \text{ if } W, Z \in X \]

\[ \{c \in \forall h \in V_c \in \{j_1 = 0 \& j_2 \in [W = _Y Z]\} \} \text{ if } W, Z \in Y \]

\[ \emptyset \text{ else.} \]

**Proof.** Straightforward verification.

**Proposition 4.3.** The object $(N,=)$ with $= \in \wp$-sets defined by:

\[ n = m \equiv \{c \in V_c = \{n\} \cap \{m\}\} \]

is a natural number object in $\wp$-sets.

It is now a matter of calculation to show the equivalence of Lifschitz’ realizability with the internal logic of $\mathbb{N}$ in $\wp$-sets.
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