RANDOMNESS IN SET THEORY

Michiel van Lambalgen

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SEE INSIDE BACK COVER
RANDOMNESS IN SET THEORY

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Randomness in set theory

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0 INTRODUCTION  The fact that ZF set theory does not decide various easily expressible statements concerning the reals and sets of reals, leads one to search for additional axioms. Most axioms proposed so far use concepts which are formalizable in set theory and they express new, in some cases even plausible, properties of these concepts. Typical examples include large cardinal axioms, Martin's axiom, or the axiom of determinacy. But, as Kreisel pointed out, one might also think of a different way of introducing axioms, this time involving new primitives:

Let us try to expand the language of set theory, that is, add symbols for new primitive notions, and look for axioms in the wider language which are evident (for the notions given). They may imply set theoretic propositions, i.e. propositions in the language $L_E$ of set theory, which are not. Prima facie the case for this project is overwhelming. [Kreisel then goes on to discuss one such proposal.]

A more interesting, but also more problematic, expansion in the literature [Kruse 1967] concerns the primitive predicate of being a "random sequence". Myhill formulated axioms for the notion i.e. which, however, are not altogether plausible. As he himself observed, the intuitive notion of random sequence is certainly not extensional; specifically, if we think of a sequence as being produced by a die, it is random, but this does not ensure that some element of the sequence differs from, say, 1; also, it is not intuitively clear that there is a collection of (intensional) sequences such that (i) for each it is decided whether or not the sequence is random and (ii) the usual axioms of set theory are satisfied by the collection. So, once again, we have here an idea for an expansion rather than an effective use of an expansion ([1968], 100-1).

In the following pages we shall study various ways of introducing random sequences in set theory, and their consequences. As Kreisel remarks, it is a moot point whether the intuitive notion of randomness can be captured extensionally. We therefore pursue two different lines:

- an intensional approach, where one studies essentially only quantification over random sequences by means of a generalized quantifier;
- an extensional approach, where one adds a new predicate for random sequences.

The main conclusion will be that, once one accepts the axioms $Q$ for quantification over random sequences given in section 1, and moreover classical logic, the two approaches are essentially equivalent (theorem 2.4). The consequences of the axioms that have
been obtained so far, are all related to AC; e.g. in section 3 we show that Lebesgue measure has a translation invariant extension to all sets of reals (theorem 3.6).

To appreciate the potential significance of this result, recall that, in modern probability theory as codified by Kolmogorov, one proceeds by setting up first the apparatus of σ-algebras and σ-additive measures; afterwards the notions of independence and randomness are defined. The results of this article indicate that one can profitably reverse this procedure: we start with the notions of randomness and independence as true primitives, not even formalizable in set theory; it is then shown that one can dispense with the usual apparatus.

We now give a brief description of the contents of the paper. In section 1 we study the effect of adding a generalized quantifier Q with the intuitive meaning of "almost certain" to ZF. The most natural way to do so (allowing arbitrary sets as parameters in formulas with Q) leads to a refutation of AC. A strong restriction (allowing only real and ordinal parameters) still refutes \( V \neq L \). The extensional approach to randomness is studied in section 2. It so happens that this approach is strongly related to that of section 1 via the elimination of the quantifier Q given in van Lambalgen [1990]. In section 3 we introduce some further plausible axioms for randomness, which are analogous to the density and data axioms for the (intuitionistic) theory of lawless sequences. A consequence of the new axioms is the existence of a translation invariant extension of Lebesgue measure to all sets of reals, and furthermore the nonexistence of free ultrafilters on the powerset of \( \omega \). In section 4 we briefly indicate why, as a consequence of the preceding results, the addition of the theory of lawless sequences to intuitionistic set theory is no longer a conservative extension (as opposed to the situation in, say, intuitionistic second order arithmetic). The Epilogue, 5, contains some philosophical remarks. In an appendix, section 6, we compare our approach with Freiling's "Axioms of Symmetry" [1986]. Freiling shows how some simple axioms on throwing \( n \) random darts forces the cardinality of the continuum to be \( \geq R_n \). The main result here is, roughly, that the size of \( 2^{K_0} \) is related to the number of iterations of Q that one allows. In this section we also comment on axioms for randomness proposed by Myhill in 1963.

It is only fair to say that this work owes part of its inspiration to Freiling [1986] and Simms [1989]. In particular Freiling's emphasis on the fundamental importance of symmetry for randomness, and Simms suggestion that Fubini's theorem is related to extensions of measures have been instrumental. Furthermore I would like to thank Domenico Zambella for permission to include some of his results and Georg Kreisel for helpful correspondence.
1 QUANTIFICATION OVER RANDOM SEQUENCES. When we generate a random
sequence, there is prima facie nothing certain about the result, except for the finite
segment already generated. In this sense, random sequences behave as lawless
sequences and are indeed often adduced as examples of the latter. However, on an
intuitive level we are also almost certain, or practically certain, of many other properties:
that relative frequencies converge, that the same holds in suitably chosen subsequences
etc. The fact that practical certainty is not absolute certainty is what renders an
extensional theory of randomness difficult. But we have a different option: formalizing
the properties of "practically certain" itself and incorporating this notion in ZF.
So suppose we have some stochastic mechanism X that randomly produces infinite
binary sequences x. For X one could take, e.g., a fair coin. We add to the language of
set theory a generalized quantifier Q, where the intended interpretation of Qxφ(x) is: "if
x is randomly generated, it is practically certain that φ(x)". We shall state the axioms
first, and comment on them afterwards. Unless specified otherwise, we assume that the
variables bound by Q run over 2^ω. We first fix the logic to be used:
Q0 Axioms and inference rules of classical predicate logic.
The following five properties, expressing that Q is a nonprincipal filter, are fairly
immediate.
Q1 ¬Qx x≠x
Q2 Qy x≠y
Q3 Qxφ(...,x,..) → Qyφ(...,y,..) provided y is free for x in φ
Q4 Qxφ ∧ ∀x(φ → ψ) → Qxψ
Q5 Qxφ ∧ Qxψ → Qx(φ ∧ ψ)
So far, there is no difference between Q and the filter quantifiers studied in Kaufmann
[1983] and Kakuda [1989]. The crucial property is
Q6 QxQyψ ↔ QyQxψ.

COMMENTS ON THE AXIOMS
Q0 This choice of a logic is not altogether obvious, since we have expressly introduced
random sequences as objects about which we have partial information only. The force
of Q0 will become clear when we discuss Q4.
Q1 When read classically, this axiom states that there is some randomly generated x.
Without such a weak existence axiom, the enterprise would trivialize. But notice that
Q1 is still very weak; the following density axiom also seems to be justified: if a set A
has positive measure, then ¬Qx(x ∈ A → x ≠ x). This stronger axiom will be
considered in section 3.
Q2 This means that it is almost certain that randomly generated sequences differ from
some given sequence. This axiom is related to Kreisel's observation that "if we think of
a sequence as being produced by a die, it is random, but this does not ensure that some element of the sequence differs from, say, 1"; for y, take the sequence 111... Here, intensionality makes itself felt; we only require that it is \textit{almost} certain that a randomly produced x differs from y.

Q3 This is a syntactic expedient only.

Q5 states that the intersection of two almost certain events is again almost certain (though, intuitively, perhaps slightly less so).

Q4 expresses that Q is a monotone quantifier. That this is less innocent than may seem at first sight can be seen as follows. We show that $Q x \phi(x) \rightarrow \exists x \phi(x)$. Suppose $Q x \phi(x)$; assume for simplicity that $\phi$ does not contain other free variables. Since $\forall x (\phi(x) \rightarrow \exists y \phi(y))$, monotonicity implies $Q x \exists y \phi(y)$. Using classical logic and Q1 – 5 one can derive, when x does not occur free in $\exists y \phi(y)$: $Q x \exists y \phi(y) \rightarrow \exists y \phi(y)$.

For suppose $\neg \exists y \phi(y)$, then also $\forall x (x \neq y \rightarrow \neg \exists y \phi(y))$, whence by Q4 and Q2, $Q x \neg \exists y \phi(y)$. By Q5, $Q x (\neg \exists y \phi(y) \wedge \exists y \phi(y))$, which, again by Q4, contradicts Q1. Hence $\neg \exists y \phi(y)$, and so $\exists y \phi(y)$ by classical logic. To see why the existential import of Q is somewhat problematic, let us make a historical diversion. When Borel had proved (in [1909]) that almost all real numbers are absolutely normal (i.e. normal in every base), he noted that it would be of interest either to construct a concrete example of such a number or to show that none of the definable numbers is absolutely normal. He observed that, however paradoxical at first sight, the latter possibility did not contradict his result. In other words, the existential quantifier in $Q x \phi(x) \rightarrow \exists x \phi(x)$ need not have a constructive meaning.

Indeed, on an intensional view $Q x \exists y \phi(y) \rightarrow \exists y \phi(y)$ is controversial. It will be observed that, when x does not occur free in $\psi$, "$Q x$" in $Q x \psi$ can be read as a propositional modal operator, namely "it is practically certain that $\psi". The statement "$Q x \psi \rightarrow \psi$" (x not free in $\psi$) therefore expresses a collapse of modalities: $\psi$ is practically certain implies that $\psi$ is true. Hence, although each of Q1 – 5 seems justified on the intensional interpretation, together with classical logic they present a picture not unlike the extensional view. It should be observed here that the addition of Q1 – 5 to ZF is conservative so that no new existential statements become provable via $Q x \exists y \phi(y) \rightarrow \exists y \phi(y)$; however, this is no longer true when one adds Q6.

Q6 So far our axioms were concerned with randomness. The last axiom implicitly introduces a new primitive notion, namely, \textit{independence}. We think of the quantifiers Qx, Qy in a formula QxQy$\phi$ as referring to independent processes. Hence QxQy$\phi(x,y)$ means: "if x is randomly generated and y is randomly generated independently from x, then it is practically certain that $\phi(x,y)$". But if the processes generating x and y are independent, it shouldn't matter in which order we take them. (This intuition is also the motivation behind Freiling's "Axioms of Symmetry" [1986].)
The axiom system $Q_0 - 6$ will be denoted $Q$. We have seen just now that there is a
certain amount of idealization involved in rendering "almost certain" formally by means
of $Q$, in particular, the intuitive notion of "almost certain" may not be monotonic, or
satisfy classical logic. But if we adopt $Q$, we can use some work of Harvey Friedman
(see Steinhorn [1985a,b]), who introduced a quantifier with properties 1 - 6. We now
think of the formula $Q\phi(x)$ as meaning "$\{x \mid \phi(x)\}$ has Lebesgue measure 1". On this
interpretation, $Q_6$ bears some analogy to the Fubini theorem: for Fubini implies $Q_6$ if
we know in addition that $\{<x,y> \mid \phi(x,y)\}$ is measurable. Without this extra condition,
$Q_6$ expresses a very strong property, which is almost solely responsible for the peculiar
features of set theory with random sequences. Despite the fact that there is only an
analogy, we shall refer to $Q_6$ as the "Fubini property". Alternatively, $Q\phi x$ can be read topologically as "$\{x \mid \phi(x)\}$ is first category". In this
case, $Q_6$ corresponds (in the same sense as above) to the Kuratowski - Ulam theorem
(see Oxtoby [1980], Chapter 16). Friedman shows that the set $Q$ of axioms for $Q$ is
complete for both interpretations. However, since definable sets need not be
measurable, or have the Baire property, Friedman considers completeness with respect
to a class of nonstandard models, so called Borel structures.

Let $M$ be a model for a first order language $L$, with domain a subset of $2^0$ of positive
Lebesgue measure. On the measure theoretic interpretation, the satisfaction clause for $Q$
is:

$$M \models Q\phi(x,a_1,\ldots,a_n) \iff \lambda(\{ x \mid M \not\models \phi(x,a_1,\ldots,a_n) \}) = 0,$$
where $\lambda = (\emptyset,\emptyset)^{2^0}$
is Lebesgue measure on $2^0$.

The other logical constants have standard interpretation. A structure whose domain is a
subset of $2^0$ and all of whose relations and functions are Borel is called a Borel
structure (or model). A Borel structure for $L \cup \{Q\}$ is called totally Borel if any relation
defined using a formula from $L \cup \{Q\}$ (possibly with parameters) is Borel. Evidently
the axioms for $Q$ are valid on any totally Borel model.

Similarly, when $Q$ is interpreted in terms of category, we put

$$M \models Q\phi(x,a_1,\ldots,a_n) \iff \{ x \mid M \models \phi(x,a_1,\ldots,a_n) \} \text{ is comeagre;}$$
and we have definitions of Borel structure etc. analogous to the ones given before.

The main theorems in this subject are:

1.1 THEOREM (See Steinhorn [1985a,b]) Let $T$ be a consistent theory in $L$ with an
infinite model. Then $T$ has a totally Borel model with domain $2^0$.
1.2 THEOREM (See Steinhorn [1985a]) Let $T$ be a theory in $L \cup \{Q\}$. Then $T$ has a
totally Borel model whose domain is a full subset of $2^0$ iff $T$ is consistent in $Q$.
Mutatis mutandis, the same theorem holds for category.
We now study the effect of adding $Q$ to ZF. We shall first present a few quick consistency results using a theorem due to Shelah and Stern, in order not to overburden the exposition with the details of a forcing construction. In sections 2 and 3 we give a finer analysis.

1.3 THEOREM Assume there is a standard model of ZF. (I) Then there is a standard model of ZFC with the following properties:
(1) Any set of reals definable from ordinal and real parameters has the property of Baire (Shelah [1984], theorem 7.17).
(2) There is a sequence of $\mathbb{R}_1$ reals (Stern [1985], theorem 2(iii)).
(3) Any ordinal definable set of reals is Lebesgue measurable (Stern [1985], theorem 2(v)).
(II) As a consequence, there is a standard model of ZF + DC in which all sets of reals have the Baire property (Shelah [1984], theorem 7.17). The proofs show moreover that subsets of $(2^\omega)^n$, $n > 1$, have corresponding regularity properties.

We first show how not to add $Q$ to ZF. Generally, we have

1.4 LEMMA Let $T$ be any first order theory. Then $T + Q$ is conservative over $T$.
PROOFSKETCH This sketch is only intended to give the reader moral certainty; a more elementary proof can be found in van Lambalgen [1990] (theorem 2.1.3). Let $\mathcal{M}$ be a model of ZFC such that $T \in \mathcal{M}$. Let $\mathcal{N}$ be an extension of $\mathcal{M}$ in which every set of reals has the Baire property (theorem 1.3 (II)).
Let $\mathcal{A}$ be a totally Borel model of $T$ in $\mathcal{M}$, and let $\mathcal{A}^#$ be $\mathcal{A}$ as seen in $\mathcal{N}$. By absoluteness, $\mathcal{A}^#$ verifies $T$. Since in $\mathcal{N}$ all sets definable using $\forall$ and $Q$ have the Baire property, we can expand $\mathcal{A}^#$ to a model $<\mathcal{A}^#, Q>$ of the Friedman axioms. (Use the fact that $Q_6$ corresponds to the Kuratowski - Ulam theorem.)

In particular, ZFC + $Q$ is conservative over ZFC, even when the variables bound by $Q$ run over all sets. However, the theory ZFC + $Q$ is unnatural, since we do not allow $Q$ to occur in the schemata of ZF. This unnaturalness is reflected semantically: since no Borel relation can be a wellordering, the Borel models constructed in the above lemma are not transitive. We therefore consider only extensions of set theory in which $Q$ does occur in the separation and substitution schemata. It is useful to distinguish two extensions of this sort: ZF$Q$ and ZF$Q^0$. In ZF$Q$, $Q$ formulas may contain parameters for arbitrary sets; in ZF$Q^0$, only real and ordinal parameters are allowed in $Q$ formulas.
Hence in ZFQ we allow formulas "Qx(x ∈ A)" for arbitrary sets A in 2_0; in the weaker theory ZFQ^0, we still have formulas "Qx(x ∈ B)", with B a Borel subset of 2_0.

We emphasize that systems of this type (unlike the ones studied in section 2) are fairly weak in one respect: one cannot even express that random sequences are closed under nontrivial operations. (For a weaker notion of closure, see section 3.) Nevertheless, the axioms pose strong constraints on the set theoretic universe. We first give a simple result which shows that it is really the Fubini property Q6 that is responsible for the striking features of ZFQ that will be discussed later.

1.5 LEMMA ZFQ minus Q6 is conservative over ZF.
PROOF Define Qxφ explicitly as ∃y∀x(x ∈ y → φ), then Q1 - 5 are satisfied.

1.6 LEMMA If ZF is consistent, so is ZFQ + DC.
PROOF This follows from 1.3 (II).

1.7 LEMMA If ZF is consistent, so is ZFQ^0 + AC.
PROOF Use theorem 1.3 (I) (1). For an alternative proof, see lemma 3.4 below.

The main technical tool is the following additivity property of Q.

1.8 THEOREM (ZFQ) Let the family (A_α)_{α<κ} be given, where κ is an initial ordinal. ∀α < κQx(x ∈ A_α) → Qx∀α < κ(x ∈ A_α).
PROOF (The argument is adapted from van Benthem [1990]). We use induction on initial ordinals κ. Note that we need Q in the separation axiom to conclude from this that the result holds for all families (A_α)_{α<κ}. We may suppose the A_α are pairwise disjoint.

The case κ = 1 is trivial. So suppose for all λ < κ and all families (B_β)_{β<λ}, ∀α < λQx(x ∈ B_β) → Qx∀α < λ(x ∈ B_β). Define a relation S on 2^ω by S(x,y) ⇔ ∀α < κ[x ∈ A_α → (∃β > α(y ∈ A_β) ∨ ∀β < κ(y ∈ A_β))] ∨ [∀α < κ(x ∈ A_α) ∧ y ∈ 2^ω].

Put B := \bigcup_{α<κ} A_α.

We then have ∀xQyS(x,y). For if x ∈ B, then \{y \mid S(x,y)\} = 2^ω; and if x ∈ B, then \{y \mid S(x,y)\} = 2^ω - C, where C is a union of less than κ A_α's; now apply the induction hypothesis. Hence also QxQyS(x,y), whence by Q6, QyQxS(x,y). Consider the set \{x \mid S(x,y)\}. Obviously, if y ∈ B, then \{x \mid S(x,y)\} = B^c ∪ C, where C is a union of less than κ A_α's; and if y ∈ B, then \{x \mid S(x,y)\} = 2^ω, which is in Q. It follows from this observation that \{y \mid QxS(x,y)\} ⊇ B^c. If we have equality, then QxQyS(x,y) implies
Qx(x \not\in B). If not, then for some y \in B, \{x \mid S(x, y)\} = B^c \cup C, where C is a union of less than \kappa A_{\alpha}'s, and we can apply the induction hypothesis to obtain Qx(x \not\in C). But Qx(x \in B^c \lor x \in C) \land Qx(x \not\in C) implies Qx(x \in B^c).

In particular, we see that the continuum is not the union of \kappa Q - nullsets, for any initial ordinal \kappa.

1.9 COROLLARY (ZFQ) Suppose B \subseteq 2^\omega has a wellordering. Then Qx(x \not\in B).

PROOF Suppose B = \{x_\alpha \mid \alpha < \kappa\}. By Q2, Qx(x \not\in \{x_\alpha\}). Now apply the preceding theorem.

In other words, if a set can be "counted" at all, it must be small. Note that it is consistent with ZFQ that every wellorderable subset of 2^\omega is countable; this holds in Solovay's model where every set of reals is Lebesgue measurable. But, as we have seen in 1.3 (1) (2), the existence of a sequence of \kappa_1 reals (hence of a nonmeasurable set) is also consistent with ZFQ. Hence, if CH' denotes the aleph - free version of the continuum hypothesis, ZFQ is consistent with both CH' and \neg CH'.

1.10 COROLLARY (ZFQ) \neg AC.

PROOF If not, 2^\omega would have a wellordering, whence by the preceding corollary, Qx(x \not\in 2^\omega); but this contradicts Q1.

We thus see that it is the Fubini axiom Q6 that is responsible for eliminating some sets from models of ZFC.

By keeping track of definability, we obtain

1.11 COROLLARY (ZFQ') Let \phi(x, \alpha) be a formula which may contain additional ordinal and real parameters (but no other parameters). Then \forall \alpha < \kappa Qx\phi(x, \alpha) \to Qx\forall \alpha < \kappa \phi(x, \alpha). Moreover, if B = \text{Field}(\leq) for some definable wellordering \leq, then Qx(x \not\in B).

For the constructible hierarchy this means the following: if L(x) is the formula "x is constructible", we get Qx\neg L(x).

2 AN EXTENSIONAL STUDY OF RANDOMNESS We shall see presently how one can introduce a randomness predicate R and axioms governing R, such that ZFQ' is interpretable in the resulting theory. The moral of this construction is, that we may think of a formula "Qx\phi(x)" as meaning: "for all random x, \phi(x)"; in other words, that we can extensionalize the notion of randomness implicit in Q.
Our procedure is to eliminate $Q$ in terms of the $R$ predicate and $\forall$. In the first approximation, this elimination is a translation from $Q$-formulas to $R$-formulas which is the identity on formulas not containing $Q$, commutes with $\neg$, $\land$, $\lor$ and transforms $Qx\phi$ into $\forall x(R(x) \rightarrow \phi)$. However, we quickly run into trouble if $\phi$ is of the form $x \neq y$, say; for then $Qx(x \neq y)$ implies that $R$ is empty. Instead of $R(x)$ we therefore use a relation $R(x, \vec{y})$ with the intended interpretation "$x$ is independent of $\vec{y}$" or "$\vec{y}$ has no information about $x$". Here, $\vec{y}$ denotes a vector, of unspecified length, of variables; hence $R$ is a relation of indefinite arity. We always assume that the variable $x$ in $R(x, \vec{y})$ runs over $2^0$; the $\vec{y}$, however, may be interpreted by arbitrary sets. $\vec{y}$ may be empty, in which case we write $R(x, \emptyset) =: R(x)$. We may think of $x$ with $R(x)$ as random sequences. We are now in a position to define our translation formally; afterwards we shall specify the axioms which will turn the translation into an embedding.

2.1 DEFINITION Let $L$ be an arbitrary countable first-order language. We define a translation $*$ of $L(Q)$ into $L(R)$ as follows:

\[ (Qx\phi(x, \vec{y}))^* := \forall x(R(x, \vec{y}) \rightarrow \phi(x, \vec{y})^*) \]

We now give the axioms for the independence relation. Properties 1 - 5 are a slight weakening of the usual axioms for algebraic or linear independence. Observe that R5 corresponds to the Steinitz exchange principle.

R0. Axioms and inference rules for classical predicate logic

- R1. $\exists x R(x)$, $\forall \vec{y} \exists x R(x, \vec{y})$
- R2. $R(x, \vec{y}, \vec{z}) \rightarrow R(x, \vec{z})$
- R3. (a) $R(x, \vec{y}) \rightarrow R(x, \pi \vec{y})$ for any permutation $\pi$; (b) $R(x, y \vec{z}) \rightarrow R(x, yy \vec{z})$
- R4. $\neg R(x, x)$
- R5. $R(y, \vec{z}) \land R(x, y \vec{z}) \rightarrow R(y, x \vec{z})$.

The next axiom looks odd and we shall comment on it after the proof of the embedding theorem.

R6. Suppose $\phi(x, \vec{y})$ is in $L(R)$, and $z$ does not occur free in $\phi$. Then

\[ \forall x(R(x, z \vec{y}) \rightarrow \phi(x, \vec{y})) \rightarrow \forall x(R(x, \vec{y}) \rightarrow \phi(x, \vec{y})); \]

The system consisting of axioms R0 - 6 will be denoted $\mathcal{R}$.

Our results in this section will rely upon the possibility of interpreting $R$ by means of Solovay forcing. This was spelled out in van Lambalgen [1990], but since in the present paper we have opted for a stronger version of R6, we have to verify this anew.
Let $\mathcal{M}$ be a countable transitive model of $ZF + V = L$. In $\mathcal{M}$, consider $(2^{\omega})^\kappa$, where $\kappa \geq \omega_1$. We equip $(2^{\omega})^\kappa$ with the product topology and the product measure $\lambda^\kappa$ defined on the Borel $\sigma$-algebra $B((2^{\omega})^\kappa)$. Let $I$ denote the $\sigma$-ideal of $\lambda^\kappa$ nullsets, then the quotient algebra $B = B((2^{\omega})^\kappa)/I$ is a complete Boolean algebra. Let $\mathcal{G} \subseteq B$ be a generic ultrafilter; construct the generic extension $\mathcal{M}[\mathcal{G}]$. We shall refer to this extension as "generically adding $\kappa$ random reals".

For any sequence $\vec{y}$ of elements in $2^{\omega} \cap \mathcal{M}[\mathcal{G}]$, $\mathcal{M}[\vec{y}]$ is welldefined (via relative constructibility).

In $\mathcal{M}[\mathcal{G}]$, interpret $R(x, \vec{y})$ as

$$R(x, \vec{y}) \text{ iff for all Borel sets } B \text{ with code in } \mathcal{M}[\vec{y}], \text{ if } \lambda B = 1, \text{ then } x \in B.$$ 

If $R(x, \vec{y})$, we say that $x$ is (Solovay) random over $\mathcal{M}[\vec{y}]$.

2.2 LEMMA $R(x, \vec{y})$, interpreted as Solovay randomness, satisfies $\mathcal{R}$.

PROOF $R1$ holds because the set of reals random over $\mathcal{M}[\vec{y}]$ has outer measure 1 in $\mathcal{M}[\mathcal{G}]$. $R2 – 4$ are trivial and $R5$ was verified in van Lambalgen [1990], theorem 2.2.1.

To prove $R6$, we first observe that $R$ is in fact definable; hence it suffices to show that for $\phi$ in the language of $ZF$, if $z$ does not occur free in $\phi$,

$$\mathcal{M}[\mathcal{G}] \models \forall x (R(x, z, \vec{y}) \rightarrow \phi(x, \vec{y})) \rightarrow \forall x (R(x, \vec{y}) \rightarrow \phi(x, \vec{y})).$$

Suppose $\mathcal{M}[\mathcal{G}] \models \forall x (R(x, z, \vec{y}) \rightarrow \phi(x, \vec{y}))$. If $x$ is an element of $\mathcal{M}[\mathcal{G}]$, let $x$ denote its name. We claim that there exists a formula $\psi$ such that $\mathcal{M}[\mathcal{G}] \models \phi(x, \vec{y}) \iff \mathcal{M}[\vec{y}, x] \models \psi(x, \vec{y})$. By the homogeneity of $B$, we can write $\mathcal{M}[\mathcal{G}] = \mathcal{M}[\vec{y}, x][G']$, where $G' \subseteq B$ is a generic ultrafilter. If $\pi$ is any automorphism on $B$ such that the induced automorphism on $\mathcal{M}[\mathcal{G}]$ fixes $\vec{y}$ and $x$, then $\mathcal{M}[\vec{y}, x][\pi G'] \models \phi(x, \vec{y})$. It follows that $1_B \models \phi(x, \vec{y})$ and since forcing is expressible in the groundmodel, in this case $\mathcal{M}[\vec{y}, x]$, we have the required formula $\psi(x, \vec{y})$.

By the properties of Solovay forcing, there exists a Borel set $B$, with code in $\mathcal{M}[\vec{y}]$, such that for all $x$ random over $\mathcal{M}[\vec{y}]$, $x \in B \iff \mathcal{M}[\vec{y}, x] \models \psi(x, \vec{y})$.

By hypothesis, for all $x$ random over $\mathcal{M}[\vec{y}, z]$, $\mathcal{M}[\vec{y}, x] \models \psi(x, \vec{y})$. Since this set has outer measure 1, we must have $\lambda B = 1$. But then, because $B$ has code in $\mathcal{M}[\vec{y}]$, also for all $x$ random over $\mathcal{M}[\vec{y}]$, $\mathcal{M}[\vec{y}, x] \models \psi(x, \vec{y})$. This shows $\mathcal{M}[\mathcal{G}] \models \forall x (R(x, \vec{y}) \rightarrow \phi(x, \vec{y}))$. $\mathcal{M}[\mathcal{G}]$ is a transitive model of $ZF$.

2.3 THEOREM $^*$ is a faithful relative interpretation of $L(Q)$ into $L(R)$, i.e. for all $\phi$ in $L(Q)$: $Q \models \phi$ iff $R \models \phi^*$.

PROOF SKE LETCH For a weaker system $\mathcal{R}$ this was proved in van Lambalgen [1990]. We do the $\Rightarrow$ part again to show the reader which properties of $R$ correspond to quantifier properties. $R1$ ensures that $^*$ is correctly defined as a relative interpretation. The
remainder of the argument proceeds by a routine induction on the length of proofs in $Q$. That the axioms for $Q$ are derivable in $R$ can be seen as follows:

- \((-Qx \neq x)^* = \neg \forall x(R(x) \rightarrow x \neq x) \iff \exists x R(x), \) hence $Q_1$ corresponds to $R_1$ under *
- \((\forall x Q y \neq y)^* = \forall x \forall y (R(y,x) \rightarrow x \neq y) \iff \forall x \neg R(x,x) \) hence $Q_2$ corresponds to $R_4$ under *

- Q3 holds trivially
- \((Qx\phi(x,\bar{y}) \land \forall x(\phi(x,\bar{y}) \rightarrow \psi(x,\bar{z})) \rightarrow Qx\psi(x,\bar{z}))^* = \forall x(R(x,\bar{y}) \rightarrow \phi(x,\bar{y})^*) \land \forall x(\phi(x,\bar{y})^* \rightarrow \psi(x,\bar{z})^*) \rightarrow \forall x(R(x,\bar{z}) \rightarrow \psi(x,\bar{z})^*); \) the antecedent implies that $\forall x(R(x,\bar{y}\bar{z}) \rightarrow \psi(x,\bar{z})^*)$, hence by $R_6$ also $\forall x(R(x,\bar{z}) \rightarrow \psi(x,\bar{z})^*)^*

- \((Qx\psi(x,\bar{z}) \land Qx\psi(x,\bar{z})) \rightarrow Qx(\phi(x,\bar{y}) \land \psi(x,\bar{z}))^* = \forall x(R(x,\bar{y}) \rightarrow \phi(x,\bar{y})^*) \land \forall x(R(x,\bar{z}) \rightarrow \psi(x,\bar{z})^*) \rightarrow \forall x(R(x,\bar{y}\bar{z}) \rightarrow \phi(x,\bar{y})^* \land \psi(x,\bar{z})^*), \) hence $(Q5)^*$ can be derived in $R$ using $R_2$.

- \((QxQ\phi(x,y,\bar{z}) \leftrightarrow QyQx\phi(x,y,\bar{z}))^* \land Qx\forall y(R(x,\bar{z}) \land R(y,\bar{z}) \rightarrow \phi(x,y,\bar{z})^*) \leftrightarrow \forall y \forall x(R(x,\bar{y}) \land R(x,\bar{z}) \rightarrow \phi(x,y,\bar{z})^*), \) hence $(Q6)^*$ can be derived in $R$ using $R_5$ and $R_2$.

The induction step is almost trivial, since both in the case of $Q$ and $R$ the inference rules are those of classical predicate logic. To check the validity of the identity axioms, we have to verify that * commutes with substitution, i.e. that $(Qx\phi(x,y\bar{z}))^*[y = t(\bar{y})] \leftrightarrow (Qx\phi(x,y\bar{z})\land y = t(\bar{y}))^*$, but this can be shown using $R_2$ and $R_6$.

In van Lambalgen [1990] (theorem 2.2.1) we proved the converse direction by extending a totally Borel model (cf. theorem 1.2) for $Q \cup \{\neg \phi\}$ using forcing; this argument has to be changed slightly. Suppose $Q \cup \{\neg \phi\}$ is consistent. Let $M$ be a model of $ZF + V=L$, and let $A \in M$ be a totally Borel model for $Q \cup \{\neg \phi\}$. Let $M[G]$ be the extension for generically adding $\omega_1$ random reals. Now consider $A$ in $M[G]$ (by Borel absoluteness) and expand $A$ to $<A,R>$, where $R$ is interpreted as Solovay forcing. That $<A,R> \models R$ was verified in lemma 2.2. By induction one shows $A \models \psi(\bar{y}) \iff <A,R> \models \psi^*(\bar{y})$, which proves $<A,R> \models \neg \phi^*$. Hence if $Q \cup \{\neg \phi\}$ is consistent, so is $R \cup \{\neg \phi^*\}$. $\square$

The above proof makes clear that the odd-looking $R_6$ corresponds to that seemingly most innocent of all quantifier properties, monotonicity. To see what it means, let us first consider the contraposition:

$$\exists x (R(x,\bar{y}) \land \phi(x,\bar{y})) \rightarrow \exists x (R(x,z \bar{y}) \land \phi(x,\bar{y})).$$

In conjunction with the irreflexivity of $R$, this statement shows that if $\phi$ is satisfied by some random sequence, it is satisfied by infinitely many of them. Again, we see that random sequences have some indistinguishability properties. Moreover, in van Lambalgen [1990] it is shown that, as a consequence of $R_6$, random sequences are
transcendental in the modeltheoretic sense: \( R(a, \bar{b}) \) implies that \( a \) is not in the algebraic closure of \( \bar{b} \). As easy consequences we get

- \( \forall x \ ( R(x) \to x \neq a ) \), for constants \( a \) in \( L \); and more generally
- \( \forall x \ ( R(x, \bar{y}) \to x \neq t(\bar{y}) ) \), where \( t \) is a term in \( L \). In other words, if \( x \) is independent of \( \bar{y} \), then \( x \) cannot be obtained by applying a definable function to \( \bar{y} \). Hence the \( R \)-analogue of monotonicity for \( Q \) is a strong statement; perhaps this is connected to the reservations on monotonicity expressed in section 1. Further clarification of \( R_6 \), in terms of its relation to forcing, will be given after the proof of lemma 3.4.

We now study the effect of adding \( R \) to ZFC. As in the case of \( Q \), there are two possibilities. The system ZF\( R \) results from ZF by adding \( R \), where the parameters \( \bar{y} \) can be interpreted by arbitrary sets and where we allow \( R \) to occur in the schemata of ZF. The system \( ZF R^0 + AC \) results from ZFC by adding \( R \), with the following provisos

1. \( R \) may occur in the schemata
2. the variables in \( R \) are assumed to range over \( 2^\omega \)
3. the \( L(R) \) formulas occurring in \( R_6 \) may contain ordinal parameters.

It is easy to see that in this case the fundamental axiom is \( R_6 \):

2.4 LEMMA ZF\( R \) minus \( R_6 \) is conservative over ZF; similarly for ZF\( R^0 \).
PROOF If \( \bar{y} \) denotes the vector \( <y_1, ..., y_n> \), \#(x, \bar{y}) \) is defined as: \( x \neq y_1 \land ... \land x \neq y_n \).
Now interpret \( R \) as: \( R(x, \bar{y}) \leftrightarrow \#(x, \bar{y}) \).

Hence, for a reason that is not yet clear to me, the role of the Fubini axiom \( Q_6 \) in \( Q \) is taken over by the transcendentality axiom \( R_6 \) in \( R \).
The fundamental consistency result is given by:

2.5 LEMMA \( ZF R^0 + AC \) can be interpreted in the model obtained by generically adding \( \kappa \) random reals, for \( \kappa \geq \omega_1 \).
PROOF Interpret \( R \) as Solovay randomness and apply 2.2. Since \( R \) is definable, the schemata with the new predicate \( R \) are also satisfied.

COROLLARY 2.6 (a) If ZF is consistent, so is \( ZF Q^0 + AC \). (b) If ZF is consistent, so is \( ZF Q^+ + DC \).
PROOF (a) By theorem 2.3. (b) Consider again \( M[G] \); \( ZF Q^+ + DC \) holds in the inner model of \( M[G] \) of sets which are hereditarily definable from ordinal and real parameters.
We still have to prove one more consistency result, namely, if ZF is consistent, so is \( ZF+\text{DC} \). Before we turn to this question, and to the question posed in the introduction, to wit, what is the relation between the intensional, and the extensional approach to randomness?, let us first draw an easy consequence from \( ZF^0+\text{AC} \). We already know the next result (it follows from 1.11 and 2.5), but we give another proof here because it shows that \( R \) in the schemata of ZF makes life a little easier.

**Theorem 2.7 (ZF^0+AC)** There is no definable wellordering (not even with real and ordinal parameters).

**Proof** Suppose \( x < y \) is a formula that defines a wellordering. Suppose for the present that \( x < y \) does not contain additional real parameters (ordinal parameters are allowed). Define \( S(x,y) := R(y,x) \land \forall z(R(z,x) \rightarrow y \leq z) \). We claim that \( \forall x \exists y S(x,y) \). Uniqueness is obvious. If for some \( x \), \( \forall y (R(y,x) \rightarrow \exists z(R(z,x) \land z < y)) \), then by using AC and recursion one can construct an infinite descending \( < \)-chain (here, it is essential that \( R \) occurs in the schemata); this proves the claim.

But since \( S \) defines a function, we also have \( \forall x \exists y S(x,y) \rightarrow \forall x y (R(y,x) \rightarrow \neg S(x,y)) \). For by R4, \( \forall x z (S(x,z) \rightarrow \forall y (R(y,zx) \rightarrow \neg S(x,y))) \), whence by R6, \( \forall x z (S(x,z) \rightarrow \forall y (R(y,x) \rightarrow \neg S(x,y))) \). \( \forall x \exists y S(x,y) \) now implies \( \forall x y (R(y,x) \rightarrow \neg S(x,y)) \), but this is a contradiction. (Observe that we haven't used R5 in the proof.) If there are other parameters in \( x < y \), simply add these to \( R \) in the definition of \( S \). \( \Box \)

We now examine the relation between \( ZF^R \) and \( ZF^Q \) and their \( ^0 \)-versions. These results are in part due to Domenico Zambella.

2.8 **Theorem** (i) \( ZF^R \) has an inner model satisfying \( ZF^Q \). (ii) \( ZF^Q \) has an inner model satisfying \( ZF^R \) (as a consequence \( ZF^R \) is consistent).

**Proof** (i) follows from theorem 2.3 *. In the other direction, it suffices to find an explicit definition of an \( R \), satisfying R1–5 and the weak form of R6, in \( ZF^Q \). Suppose first that we have a formula \( \theta(x,\alpha,\vec{y}) \) enumerating (in \( \alpha \) ) all sets of reals which are definable (in \( L(Q!) \)) from \( \vec{y} \) and ordinal parameters. Define

\[
R(x,\vec{y}) \leftrightarrow \forall \alpha(Qx\theta(x,\alpha,\vec{y}) \rightarrow \theta(x,\alpha,\vec{y})).
\]

Observe that we must have \( \forall \vec{y} Qx R(x,\vec{y}) \); for this statement is equivalent to \( \forall \vec{y} Qx \forall \alpha(Qx\theta(x,\alpha,\vec{y}) \rightarrow \theta(x,\alpha,\vec{y})) \), which follows from the tautology \( \forall \vec{y} \forall \alpha(Qx\theta(x,\alpha,\vec{y}) \rightarrow Qx\theta(x,\alpha,\vec{y})) \) by theorem 1.8. Hence we have established R1. R2 and R3 are trivial. For R4 observe that \( R(x,y) \) implies \( Qx(x \neq y) \rightarrow x \neq y \), which implies \( x \neq y \) by Q2. The verification of R5 is somewhat more involved:
We have to show \( R(y, z) \land R(x, y z) \rightarrow R(y, x z) \). Suppose \( \neg R(y, x z) \). Then for some \( \phi(y, x z) \), \( \neg (Qy \phi(y, x z) \rightarrow \phi(y, x z)) \). By \( R(y, z) \), \( \neg Qy \neg (Qy \phi(y, x z) \rightarrow \phi(y, x z)) \), and by \( R(y, z) \), \( \neg Qy \neg \neg (Qy \phi(y, x z) \rightarrow \phi(y, x z)) \). We apply Q6 and obtain \( \neg Qy \neg \neg (Qy \phi(y, x z) \rightarrow \phi(y, x z)) \), whence also \( \neg Qy \neg (Qy \phi(y, x z) \land Qy \neg \phi(y, x z)) \). From this we get \( \neg Qx(x = x) \), but this conflicts with Q2 and Q4. Lastly, R6 holds trivially:

For suppose \( \forall x (R(x, z \overline{\gamma}) \rightarrow \phi(x, \overline{\gamma})) \), then by definition \( \forall x [\forall \alpha (Qx \theta(x, \alpha, z \overline{\gamma}) \rightarrow \theta(x, \alpha, z \overline{\gamma})) \rightarrow \phi(x, \overline{\gamma})] \). We have to show \( \forall x [\forall \alpha (Qx \theta(x, \alpha, \overline{\gamma}) \rightarrow \theta(x, \alpha, \overline{\gamma})) \rightarrow \phi(x, \overline{\gamma})] \). Suppose for some \( x \) and \( \alpha \), \( Qx \theta(x, \alpha, \overline{\gamma}) \rightarrow \theta(x, \alpha, \overline{\gamma}) \), then also \( Qx (\theta(x, \alpha, \overline{\gamma}) \land z = z) \rightarrow (\theta(x, \alpha, \overline{\gamma}) \land z = z) \), hence \( \phi(x, \overline{\gamma}) \). This proves \( \forall x (R(x, \overline{\gamma}) \rightarrow \phi(x, \overline{\gamma})) \).

It remains to construct the formula \( \theta(x, \alpha, \overline{\gamma}) \). The obvious way to do this, is to copy the construction of the formula defining ordinal definability. To this end, we must extend the reflection theorem to formulas in the language \( L(Q) \), i.e. we have to show:

\[
\text{if } \phi \text{ is a formula in } L(Q), \text{ then } \forall \alpha \exists \beta > \alpha (\phi \iff \phi^\beta). 
\]

We show how to amend Kunen's proof ([1980], 137). We first observe that the quantifier \( Q \) is represented as a set in \( ZFQ \); for by the separation axiom we have

\[
\exists B \forall A (A \in B \iff A \subseteq 2^\omega \land Qx(x \in A)).
\]

We shall call this \( B \). Q. We can therefore rewrite a formula \( Qx \phi(x) \) as \( \exists u \in Q \forall x (x \in u \iff \phi(x)) \), so we have a reduction to the case of first order formulas. ✗

2.9 COROLLARY (i) \( ZFQ^0 \) has an inner model satisfying \( ZFQ^\alpha \). (ii) \( ZFQ^\alpha \) has an inner model satisfying \( ZFQ^0 \).

PROOF SKETCH Only the argument for (ii) needs modification. We can no longer conclude immediately from separation that \( Q \) is represented as a set in \( ZFQ^0 \). We therefore use "bootstrapping": let \( \theta_0(x, \alpha, \overline{\gamma}) \) denote the formula representing all sets ordinal definable (in \( \{e, =\} \)) from \( \overline{\gamma} \). Using \( \theta_0 \), we can define (in \( ZFQ^\alpha \)) a set \( Q^1 \) by

\[
A \in Q^1 \iff \exists \alpha (\forall x (x \in A \iff \theta_0(x, \alpha, \overline{\gamma})) \land Qz \theta_0(z, \alpha, \overline{\gamma})).
\]

As in 2.8, we can use \( Q^1 \) to show that the reflection theorem holds for \( L(Q) \) formulas involving one \( Q \) only. This absoluteness can then be used to construct a formula enumerating all sets (of reals) which are definable from ordinals and reals using formulas which may contain one \( Q \); etc. ✗

Hence there is not much to choose between the extensional approach and the approach which speaks about random sequences only implicitly. Of course, this holds true only provided one accepts \( Q \); and as we have seen, one may argue that both monotonicity
(Q4) and classical logic not justified on a more intensional way of looking at randomness. In any case it is clear that the Fubini property, in the guise of theorem 1.8, plays a major role in the equivalence.

3 DATA AXIOMS So far our axioms have made explicit only very general properties of randomness; since the axioms also allow an interpretation in terms of category, essential features of randomness are missing. Even worse, the axioms do not force simple properties like density of random sequences:

3.1 LEMMA ZFQ has models in which the random sequences are not dense.

PROOF Let [0] be the cylinder set of all sequences starting with 0. We may suppose \( \neg \exists x (x \in [0]) \), otherwise we are done. Define a new quantifier \( Q^{[0]} \) by \( Q^{[0]} \phi \leftrightarrow Q x (x \in [0] \rightarrow \phi) \). Then ZFQ is also satisfied when we interpret \( Q \) as \( Q^{[0]} \). Hence in the resulting model there are no random sequences in \([1]\). \( \Box \)

To formulate additonal axioms, we take our cue from the intuitionistic theory of lawless sequences. A lawless sequence (cf. also the next paragraph) is a process of choosing infinitely many 0's and 1's such that at any stage only finitely many values are known and no restrictions are imposed upon future choices. Two characteristic axioms are:

Density: every finite binary word is the initial segment of a lawless sequence

and

Open Data: if we know \( A(\alpha) \) for lawless \( \alpha \), this can only be due to our knowledge of an initial segment \( \alpha(n) \) of \( \alpha \); hence for all \( \beta \), if \( \beta(n) = \alpha(n) \), then \( A(\beta) \).

We will now investigate the analogues of these axioms for random sequences. By now we have four formal systems, and the formulation of the data and density axioms differs for each of them. To spare the reader, we look only at the two most important cases: ZFQ and ZFQ0.

We first consider ZFQ. As a density axiom we propose

Density: let \( A \) be a Borel set such that \( \lambda A > 0 \), then \( \neg \exists x \neg (x \in A) \).

In other words, a set of positive probability contains a random sequence. As a kind of converse, we have

Inner Data: \( Q x (\phi (x) \rightarrow \exists A (A \text{ Borel} \land \lambda A > 0 \land x \in A \land Q z (z \in A \rightarrow \phi (z, y))) \).

The justification of this principle runs as follows. We think of random sequences as incomplete objects, i.e. objects about which we have only partial information; typically
their mode of generation and some initial segment. (In this respect, random sequences are analogous to lawless sequences.) Suppose we know that \( \phi(x) \) holds for some random \( x \). Since we cannot know \( x \) in its entirety, our knowledge of the truth of \( \phi(x) \) must be based on our knowledge of the process generating \( x \). But it seems that such a process must be of the following type: a randomness preserving (Borel) function \( f: 2^{\omega} \rightarrow 2^{\omega} \) applied to a sequence generated by coin tossing (or some isomorphic process).

Now \( f \) preserves randomness if \( \lambda f^{-1} \ll \lambda \), i.e. if \( \lambda f^{-1} \) is absolutely continuous with respect to \( \lambda \). Hence we have justified

\[
Qx(\phi(x) \rightarrow \exists f(2^{\omega} \rightarrow 2^{\omega} \text{ Borel } \land \lambda f^{-1} \ll \lambda \land x \in \text{ ran}(f) \land Qx\phi(f(x)))
\]

but this is equivalent to Inner Data by general nonsense.

It is clear that, by resorting to a specimen of intuitionistic reasoning in the above justification, we have given a more or less constructive meaning to the dual quantifier \( \neg Q \neg \). For a thoroughgoing classical analysis of randomness, see the remarks on Myhill's work in 6.1.

We next consider \( \mathsf{ZFP}^0 \). One could obtain a R version of Inner Data by restricting the formulas \( \phi(x) \) to have only real and ordinal parameters, and applying \( * \) from definition 2.1. However, this would not yield a statement that is true for our preferred interpretation of R, Solovay randomness. The true R - analogue is obtained by explicitly taking parameters into consideration. Suppose \( \phi(x) = \phi(x, \langle y \rangle) \), where all real parameters are exhibited. Observe that, by applying Q2, Q4 and Q5, ID is equivalent to

\[
Qx(\phi(x, \langle y \rangle) \rightarrow \exists A(A \text{ Borel } \land \lambda A > 0 \land x \in A \land Qz(z \in A \land \#(z, \langle y \rangle) \rightarrow \phi(z, \langle y \rangle))).
\]

If we translate this version of Inner Data in the R language, using \( * \), we get

\[
\forall x(R(x, \langle y \rangle) \land \phi(x, \langle y \rangle) \rightarrow \exists A(\lambda A > 0 \land x \in A \land \forall z(R(z, \langle y \rangle) \land \#(z, \langle y \rangle) \land z \in A \rightarrow \phi(z, \langle y \rangle))).
\]

but by R4 this is equivalent to

\[
\forall x(R(x, \langle y \rangle) \land \phi(x, \langle y \rangle) \rightarrow \exists A(\lambda A > 0 \land x \in A \land \forall z(R(z, \langle y \rangle) \land \#(z, \langle y \rangle) \land z \in A \rightarrow \phi(z, \langle y \rangle))).
\]

As we shall see in the proof of 3.2, this statement expresses a truth about Solovay forcing. In fact, I do not know of any other way of verifying Inner Data than by way of the parametrized version. The reader may also wish to compare this formulation of Inner Data with the parametrized form of the axiom of Open Data, given in 4.

Henceforth we abbreviate Inner Data to ID, and Density to D; we use these abbreviations indiscriminately for the Q and R versions. When we add both D and ID to a theory we indicate this by IDD.
3.2 LEMMA \( ZF^0 + AC + IDD \) holds in the model obtained by generically adding \( \kappa \geq \omega_1 \) random reals to a ground model satisfying \( V = L \).

PROOF Let \( M[G] \) be the generic extension constructed in theorem 2.6. It suffices to show that Density and ID hold in \( M[G] \). Density follows from the definition of Solovay randomness. Moreover, ID was already implicitly verified in the proof of R6. For we showed that, given a formula \( \phi(x, \vec{y}) \), there exists a Borel set \( A \) with code in \( M[\vec{y}] \), such that for all \( x \) random over \( M[\vec{y}] \), \( x \in A \) iff \( M[G] \models \phi(x, \vec{y}) \). If \( \phi(x, \vec{y}) \) is verified for some random \( x \), \( \lambda A > 0 \). Hence we have shown

\[
\forall x (R(x, \vec{y}) \land \phi(x, \vec{y}) \rightarrow \exists A (\lambda A > 0 \land x \in A \land \forall z (R(z, \vec{y}) \land z \in A \rightarrow \phi(z, \vec{y}))).
\]

Exactly as in section 2 we obtain

3.3 COROLLARY \( ZF^0 + DC + IDD \) is consistent.

A consequence of ID that is useful in applications is given in

3.4 LEMMA \( (ZFQ + DC + ID) \) For any formula \( \phi(x) \) there is a Borel set \( A \) such that \( Qx(x \in A \leftrightarrow \phi(x)) \).

PROOF We may assume \( \neg Qx \neg \phi(x) \), for otherwise we can take \( A = \emptyset \). Consider the set \( \{ A \mid \lambda A > 0 \land Qx(x \in A \rightarrow \phi(x)) \} \) which exists in \( ZFQ \). Let \( A_0 \) be an element of maximal measure. \( A_0 \) exists because \( s := \sup \{ \lambda A \mid \lambda A > 0 \land Qx(x \in A \rightarrow \phi(x)) \} \leq 1 \); choose \( (DC!) A_1, ..., A_n, ... \) such that \( \lambda A_n \rightarrow s \), and put \( A_0 := \bigcup A_n \). Then we must have \( Qx(x \in A_0 \leftrightarrow \phi(x)) \), for otherwise \( \neg Qx \neg (\phi(x) \land x \notin A_0) \) and ID shows that \( A_0 \) wasn't maximal after all.

The reader will have noticed that the proofs of R6 and ID in the random real extension are very similar. It may therefore be instructive to observe that ID can be derived from two weaker axioms, with the help of R6. The axioms are

(1) Let \( B \) be a Borel set with Borel code \( u \), such that \( \lambda B = 1 \). Then \( \forall x (R(x, u) \rightarrow x \in B) \).

(2) For any formula \( \phi(x, \vec{y}) \), there exists a Borel set \( B \) such that \( \forall x (R(x, \vec{y}) \rightarrow (\phi(x, \vec{y}) \leftrightarrow x \in B)) \).

Suppose \( R(x, \vec{y}) \land \phi(x, \vec{y}) \). To prove ID, it suffices to show that for the \( B \) given by (2), \( \lambda B > 0 \). Suppose \( \lambda B = 0 \). Let \( B \) have Borel code \( u \). By (1), \( \forall x (R(x, u) \rightarrow x \notin B) \), whence by R2, \( \forall x (R(x, u \vec{y}) \rightarrow x \notin B) \). By (2), \( \forall x (R(x, u \vec{y}) \rightarrow \neg \phi(x, \vec{y})) \). We may now apply R6 to obtain \( \forall x (R(x, \vec{y}) \rightarrow \neg \phi(x, \vec{y})) \), a contradiction.

Conversely, in \( ZF^0 \) (minus R6) + DC, ID implies (1), (2) and R6. (2) follows from 3.4. To prove (1), let \( B \) be a Borel set with code \( u \) such that \( \lambda B = 1 \) and suppose that

17
for some $x$, $R(x, y) \land x \in B$. By ID, there exists $A$ such $\lambda A > 0$ and $\forall z (R(z, y) \land z \in A \rightarrow z \in B)$. We can rewrite this as $\forall z (R(z, y) \rightarrow (z \in B \lor z \in A))$; but since $\lambda(B^c \cup A^c) < 1$, we have a contradiction with D. Lastly, we prove R6. Suppose $\forall x (R(x, z, y) \rightarrow \phi(x, y))$ and $\exists x (R(x, y) \land \neg \phi(x, y))$. ID gives us a Borel set $A$ such that $\lambda A > 0$ and $\forall x (R(x, y) \land x \in A \rightarrow \neg \phi(x, y))$. By hypothesis, $\forall x (R(x, y) \land x \in A \rightarrow \neg R(x, z, y))$, which is equivalent to $\forall x (R(x, y) \land R(x, z, y) \rightarrow x \in A)$. By R2 we obtain $\forall x (R(x, z, y) \rightarrow x \in A)$, which conflicts with D.

The presence of (1) of course explains why we had to use Solovay forcing.

We have already noted that the Q language is not strong enough to express the closure of the universe of random sequences under suitable operations. E.g. if $f$ is measure preserving, there is no way to express "for all random $x$, $f(x)$ is random". A weaker type of closure is given by

3.5 LEMMA (ZFQ+IDD) Let $f : 2^\omega \rightarrow 2^\omega$ be a bijective measure preserving Borel function. Then $Q_x\phi(x) \leftrightarrow Q_x\phi(f(x))$.

PROOF Suppose $Q_x\phi(x) \land \neg Q_x\phi(f(x))$. By Inner Data, we obtain a Borel set $A$ such that $\lambda A > 0$ and $Q_x(x \in A \leftrightarrow \neg \phi(f(x)))$. Then $f[A]$, the image of $A$ under $f$, is Borel and we have $Q_x(x \in f[A] \leftrightarrow \neg \phi(x))$; but since $Q_x\phi(x)$, we get $Q_x(x \in f[A]$, whence $\lambda f[A] = 0$; a contradiction. For the other direction, consider $f^{-1}$ (which is also Borel and measure preserving).

We are now ready for the main result of this section.

3.6 THEOREM (ZFQ+ DC + IDD) There exists a translation invariant extension $\mu$ of Lebesgue measure to all of $\mathcal{P}(2^\omega)$. This $\mu$ can be computed as follows: every $B \in \mathcal{P}(2^\omega)$ can be written as $A \Delta N$, where $A$ is Borel and $N$ is a $\mu$ - nullset, and $\mu A \Delta N = \lambda A$.

PROOF The idea that Fubini's theorem is connected to extensions of Lebesgue measure is suggested by Simms [1989]. Let $\mathcal{A}$ denote the Borel $\sigma$ - algebra on $2^\omega$. Let $\mathcal{N}$ denote $\{B \in \mathcal{P}(2^\omega) \mid Q_x(x \notin B)\}$. By theorem 1.8, $\mathcal{N}$ is closed under countable unions. By Q4, $\mathcal{N}$ is closed under subsets. Hence $\mathcal{N}$ is a $\sigma$ - ideal. Let $B = \{A \Delta N \mid A \in \mathcal{A}, N \in \mathcal{N}\}$. It is easy to see that $B$ is a $\sigma$ - algebra. We would like to define $\mu$ on $B$ such that $\mu A \Delta N = \lambda A$. To make $\mu$ welldefined, we should have $A_1 \Delta N_1 = A_2 \Delta N_2$ implies $\lambda A_1 = \lambda A_2$. But $A_1 \Delta N_1 = A_2 \Delta N_2$ implies $A_1 \Delta A_2 = N_1 \Delta N_2$ and since $N_1 \Delta N_2 \in \mathcal{N}$ we get $Q_x(x \notin A_1 \Delta A_2)$. By Inner Density, $\lambda A_1 \Delta A_2 = 0$. Hence $\mu$ is welldefined and is easily shown to be $\sigma$ - additive. Moreover, $N \in \mathcal{N}$ implies $\mu N = 0$.
We now show that $B = \varphi(2^{\omega})$. Choose $B \in \varphi(2^{\omega})$. Apply lemma 3.4 to get a Borel
set $A$ such that $Qx(x \in A \iff x \in B)$, i.e. $Qx(x \not\in A \Delta B)$. Hence $A \Delta B \in \mathcal{A}$ (We may now
write $B = A \Delta (A \Delta B)$, which shows that $B \in \mathcal{B}.
That $\mu$ is translation invariant follows from lemma 3.5.

3.7 Lemma (ZFQ+ DC + IDD) $\mu$ is $\kappa$-additive, for any ordinal $\kappa$.
Proof Suppose $\{A_\alpha\}_{\alpha < \kappa}$ are pairwise disjoint subsets of $2^{\omega}$. At most countably
many of them can have positive $\mu$ measure; say these are $\{A_n\}_{n < \omega}$. Then
\[
\sum_{\alpha < \kappa} \mu A_\alpha = \sum_{n < \omega} \mu A_n + \mu \cup \{A_\alpha | \omega \leq \alpha < \kappa\}; \text{ by the additivity property 1.8, the last term}
equals 0.

The meaning of these results can perhaps best be explained by referring to two rivalling
traditions in probability theory. In the approach universally used today (associated with
the name of Kolmogorov), a probability is taken to be a $\sigma$-additive measure on a $\sigma$-
algebra. Other notions that are of interest to a probabilist, foremost among them
independence, are defined in terms of measure. But there is an older tradition, now
nearly extinct, in which rather randomness and independence are taken as primitive.
This approach goes back to von Mises, who tried to capture the independence inherent
in random sequences by means of axioms about subsequence selection. Although these
axioms themselves are fairly unwieldy, it still seems to me that the general idea of
treating fundamental probabilistic notions as primitives has some potential. For
instance, we have seen just now that the whole business about $\sigma$-algebras actually
becomes superfluous. This may not be immediately apparent, because we formulated D
and ID in terms of Borel sets and Lebesgue measure; but a glance at the axioms will
show that we could have used closed sets instead, and their Lebesgue measure is easily
computed.

It may also be of interest to note that von Mises' axioms in their original form are
derivable in ZFQ+ DC + IDD. First some notation:

Put $\text{LLN}(x) := \forall \varepsilon \exists n_0 \forall n \geq n_0 \left| \frac{1}{n} \sum_{k=1}^{n} x_k - \frac{1}{2} \right| < \varepsilon$ and define a partial operation $/: 2^{\omega} \times 2^{\omega}$
$\to 2^{\omega}$ by: $(\chi/y)_n = x_m$ if $m$ is the index of the $n^{th}$ 1 in $y$ and undefined if there is no
such index. (The subsequence $\chi/y$ of $x$ is infinite if $y$ has infinitely many 1's.)
In van Lambalgen [1990] we used the systems $\mathcal{Q}$ and $\mathcal{R}$ to formalize von Mises' notion
of randomness. We added for instance the following axioms:

$$Qx\text{LLN}(x) \quad ("a \text{ random sequence satisfies the law of large numbers")$$

and
QxQyLLN(\(\lambda/y\)) ("If one selects a subsequence from a random sequence 
x by means of a random sequence y which is independent of x, that 
subsequence satisfies the law of large numbers").

It is easy to see that these properties are derivable in ZFQ + DC + IDD. QxLLN(x) 
follows from ID and the fact that \(\lambda\{x | LLN(x)\} = 1\). To prove QxQyLLN(\(\lambda/y\)), we first 
observe that \(\lambda: 2^\omega \to 2^\omega\) is measure preserving. Hence by 3.5, \(\forall y \forall x \forall \lambda\)QxLLN(\(\lambda/y\)). By 
applying monotonicity and Fubini we obtain QxQyLLN(\(\lambda/y\)).

The next consequence of ZFQ + DC + IDD is also related to AC:

3.8 THEOREM (ZFQ + DC + IDD) There are no ultrafilters on \(\mathcal{P}(\omega)\).

PROOF Let \(U\) be an ultrafilter on \(\mathcal{P}(\omega)\). We identify \(U\) with a subset of \(2^\omega\). By 
theorem 3.6, \(U\) must be \(\mu\)-measurable. Hence \(U\) must be of the form \(A \Delta N\), where \(A\) 
is Borel and \(N \in \mathcal{A}\). We claim that \(\lambda A = 0\) or \(1\). This follows if we can show that \(A\) is 
a tailset. So let \(\tau\) be a transformation on \(2^\omega\) that changes a fixed finite number of 
coordinates of \(x \in 2^\omega\). It suffices to show that \(x \in A\) implies \(\tau x \in A\). Observe that, 
by lemma 3.3, each \(M \in \mathcal{A}\) is a tailset. If \(x \in A \Delta U\), then \(x \in N\) and hence \(\tau x \in N\); but 
since for all \(y, y \in U\) implies \(\tau y \in U\), we must have \(\tau x \in A\). Suppose now that \(x \in A \cap U^c\). Again, 
generally \(y \in U^c\) implies \(\tau y \in U^c\). If \(x \in N\), \(x \in A \cap N\), \(x \in A \cap N\), 
whence \(x \in U\). So we must have \(x \in N\) and hence \(\tau x \in N\). If \(\tau x \in A\), then \(\tau x \in A \cap N\), 
\(\tau x \in A \cap N\) and it follows that \(\tau x \in U\), a contradiction. This shows that \(\tau x \in A\), i.e. 
that \(A\) is a tailset. By the \(0 - 1\) law, \(A\) has measure 0 or 1. By definition of \(\mu\), \(\mu U\) 
eq 0 or 1. But \(U^c\) is obtained from \(U\) by the measure preserving transformation of 
exchanging 0's and 1's, and we get a contradiction from lemma 3.5.\(\Box\)

4 EXCURSION INTO INTUITIONISM: LAWLESS SEQUENCES We indicate here, as a 
consequence of the preceding results, why the addition of the axioms LS for lawless 
sequences to intuitionistic set theory ZFI is not conservative. ZFI (called T in Powell 
[1975]) is the following set of axioms, formulated in intuitionistic logic:

1. (Extensionality) \(x = y \land x \in w \rightarrow y \in w\)
2. (\(\epsilon\)-induction) \(\forall x(\forall u \in x \phi(u) \rightarrow \phi(x)) \rightarrow \forall x \phi(x)\)
3. (Separation) \(\exists y \forall z (z \in y \leftrightarrow z \in x \land \phi)\)
4. (Pairs) \(\forall y (u \in y \land v \in y)\)
5. (Union) \(\exists y \forall z (\exists u \in x (z \in u) \rightarrow z \in y)\)
6. (Replacement) \(\exists y \forall v (\exists u \in x (\phi(v) \land \forall w (\phi(w) \rightarrow v = w)) \rightarrow v \in y)\)
7. (Power set) \(\exists y \forall z (\forall u (u \in z \rightarrow u \in x) \rightarrow z \in y)\)
8. (Double complement) \(\forall y \forall z (\neg z \in x \rightarrow z \in y)\)
9. (Infinity) \( \exists y (\exists z (z \in y) \land \forall x \in y \exists z \in y (x \in z)) \)

Here, \( \neg \phi \) is defined as \( \phi \to \bot \), where \( \bot \) is some absurdity; e.g. \( \bot = \forall x y (x \in y) \).

Furthermore, equality is defined as \( x = y \iff \forall z (z \in x \iff z \in y) \).

We now turn to lawless sequences. Lawless sequences are processes of assigning (in our case) 0 or 1 to the arguments 1, 2, 3, ..., such that (1) at any stage only finitely many values are known and (2) at no stage the possibility for choosing values is restricted.

We think of ZF\(^1\) with variables for lawless sequences as a two-sorted theory: one sort for sets, one sort for lawless sequences \( \alpha, \beta, ... \in 2^\omega \). We first introduce some notational conventions. We use \( w \) as a variable over finite binary sequences. \([w]\) is the set of infinite binary sequences which have initial segment \( w \).

\( \# (\alpha, \beta_1, ..., \beta_n) \) is defined as \( \alpha \neq \beta_1 \land ... \land \alpha \neq \beta_n \); and the quantifier \( \forall \gamma B (\gamma, \beta_1, ..., \beta_n) \) is defined as \( \forall \gamma (\# (\gamma, \beta_1, ..., \beta_n) \to B (\gamma, \beta_1, ..., \beta_n)) \).

To show nonconservativity we need only axioms which are an immediate consequence of the intuitive explanation of lawlessness given above (and not the more problematic continuity principles relying on the bar theorem). The axioms are as follows:

**LS1** \( \forall w \exists \alpha (\alpha \in [w]) \)

**LS2** \( \alpha = \beta \lor \alpha \neq \beta \)

**LS3** \( A (\alpha, \beta_1, ..., \beta_n) \land \# (\alpha, \beta_1, ..., \beta_n) \rightarrow \exists w (\alpha \in [w] \land \forall \gamma \in [w] A (\gamma, \beta_1, ..., \beta_n)) \)

We add a few words of explanation; for the full motivation the reader is referred to Troelstra [1977] or Troelstra and van Dalen [1988].

LS1 says that we are allowed to specify an initial segment in advance. LS2 is based on the fact that extensional and intensional identity coincide for lawless sequences. Let \( \equiv \) denote intensional identity: \( \alpha \equiv \beta \) means that \( \alpha \) and \( \beta \) are given to us as the same process. Then obviously \( \alpha \equiv \beta \lor \neg \alpha \equiv \beta \) and it is easy to show that for extensional identity \( = \), \( \alpha = \beta \leftrightarrow \alpha \equiv \beta \).

LS3 is clearly inherent in the meaning of lawlessness, especially if we consider the parameterfree version: \( A (\alpha) \rightarrow \exists w (\alpha \in [w] \land \forall \gamma \in [w] A (\gamma)) \). For if we have a proof of \( A (\alpha) \), this can only be on the basis of a finite segment of \( \alpha \). In the general case of formulas with parameters, we have to add some provisos (like the condition \( \# (\alpha, \beta_1, ..., \beta_n) \)) to avoid inconsistency. The formula \( A (\alpha, \beta_1, ..., \beta_n) \) may contain quantifiers over arbitrary sets, but the parameters \( \beta_1, ..., \beta_n \) are assumed to be lawless.

(Troelstra remarks: "LS3 seems also to be justified for predicates containing ... parameters for non-lawlike objects not constructed from lawless sequences ... , provided extensionality holds w.r.t. all function and set parameters."
(Troelstra [1977], p. 29) In other words, one would need an independence relation more general than \( \neq \) to formulate LS3.)

Call the resulting two sorted system ZF\(^1\)L\( \text{S} \). We then have
4.1 THEOREM \( \text{ZF}^I \text{LS} \) is not conservative over \( \text{ZF}^I \).

PROOF We first indicate how to embed a slight variant of \( \text{ZFQ}_0 \) into \( \text{ZF}^I \text{LS} \). For reasons that will become clear below we have to consider the Q - closures of the axioms Q1 – 6, instead of the universal closures. Call this system weak - \( \text{ZFQ}_0 \). Inspection of the proof of corollary 1.11 shows that it takes place inside weak - \( \text{ZFQ}_0 \).

Powell [1975] constructs an interpretation * of \( \text{ZF} \) into \( \text{ZF}^I \) as follows. Put \( \sim z := \{ x \mid \exists x \in z \} \) and define by recursion a function s which satisfies \( s(x) = \sim \{ s(u) \mid u \in x \} \).

We can now define * by recursion:

\[
\begin{align*}
(\phi \in y)^* &= s(x) \in s(y) \\
(\neg \phi)^* &= \neg \phi^* := \phi \rightarrow \bot \\
(\phi \lor \psi)^* &= \neg (\neg \phi^* \land \neg \psi^*) \\
(\phi \land \psi)^* &= \phi^* \land \psi^* \\
(\phi \rightarrow \psi)^* &= \phi^* \rightarrow \psi^* \\
(\forall x \phi)^* &= \forall x u (u = s(x) \rightarrow \phi^*(u)) \\
(\exists x \phi)^* &= \neg (\forall x \neg \phi)^*.
\end{align*}
\]

Powell shows \( \text{ZF} \vdash \phi \iff \text{ZF}^I \vdash \phi^* \).

We extend this interpretation with the following clause for Q: \( (Q x \phi(x,y_1,...,y_n))^* := \forall \alpha (\#(x,y_1,...,y_n) \rightarrow \phi^*(\alpha,s(y_1),...,s(y_n))) \). If one of the parameters \( y_1,...,y_n \) is in fact a lawless sequence \( \beta \), we put \( s(\beta) = \beta \).

All Q axioms except Q4 are trivially true under this interpretation. To verify Q4, one uses the axioms of density and open data, and it is here that our convention on weak – \( \text{ZFQ}_0 \) becomes important, for now we may assume that all parameters are lawless. So suppose we have \( \forall \alpha (\alpha \neq \beta \rightarrow \phi^*(\alpha,\beta)) \) and \( \forall \alpha (\phi^*(\alpha,\beta) \rightarrow \psi^*(\alpha,\gamma)) \). Then also \( \forall \alpha (\alpha \neq \beta, \gamma \rightarrow \psi^*(\alpha,\gamma)) \) and we have to show \( \forall \alpha (\alpha \neq \gamma \rightarrow \psi^*(\alpha,\gamma)) \).

If \( \beta = \gamma \) we are done. So (LS1) suppose \( \alpha \neq \gamma, \beta \neq \gamma \). By applying LS3 to \( \beta \) in \( \forall \alpha (\alpha \neq \beta, \gamma \rightarrow \psi^*(\alpha,\gamma)) \) we get

\[
\exists w (\beta \in [w] \land \forall \beta' \in [w] (\beta' \neq \gamma \rightarrow \forall \alpha (\alpha \neq \beta', \gamma \rightarrow \psi^*(\alpha,\gamma)))).
\]

By LS2 we can take \( \beta' \in [w] \) such that \( \alpha \neq \beta' \) and \( \gamma \neq \beta' \) and \( \psi^*(\alpha,\gamma) \). This completes the proof of the interpretation of Q4.

Suppose \( \text{ZF}^I \text{LS} \) were conservative over \( \text{ZF}^I \). Hence if weak - \( \text{ZFQ}_0 \vdash \phi \), then \( \text{ZF}^I \text{LS} \vdash \phi^* \). By assumption \( \text{ZF}^I \vdash \phi^* \) and since * is an embedding, \( \text{ZF} \vdash \phi \). This would show that weak - \( \text{ZFQ}_0 \) is conservative over \( \text{ZF} \), quod non.

\[\square\]

4.2 REMARK We could have interpreted full \( \text{ZFQ}_0 \) in \( \text{ZF}^I \text{LS} \) if in the latter system LS3 were formulated with an abstract independence relation instead of \( \neq \) (cf. Troelstra's remark above).
This result shows that, in contrast to the situation in second order intuitionistic arithmetic, when added to set theory lawless sequences can no longer be considered a "figure of speech" (Troelstra and van Dalen [1988], p. 644). In other words, quantifiers over lawless sequences can no longer be interpreted by means of expressions not containing those quantifiers.

5 EPILOGUE Gödel writes in a letter to Tarski, commenting on the failure of his "square axioms" for the continuum:

My confidence that \( 2^{\aleph_0} = \aleph_2 \) has of course somewhat been shaken. But it still seems plausible to me. One reason is that I don't believe in any kind of irrationality such as, e.g. random sequences in any absolute sense. (Gödel [1990], 175)

The preceding considerations seem to show that randomness is related to structure or complexity rather than cardinality. It is true that Freiling [1986] has claimed that suitable randomness axioms can force \( 2^{\aleph_0} \geq \aleph_{n+1} \), for any \( n \). But an examination of his proofs (see the next section) makes clear that what he really shows is that the continuum cannot have a wellordering of length \( \omega_n \), for any \( n \). We would rather interpret his results as indicating that the continuum cannot be wellordered at all. This is what ZFQ implies and, as can be seen below, ZFQ is a natural extrapolation of Freiling's axioms. Of course, what we mean is really a conditional assertion: \( \text{if} \) one believes in the reality of randomness and \( \text{if} \) one believes that ZFQ is a correct description of this reality, then the axiom of choice is false. For different notions of set AC may still be true.

To us, the interesting feature of the randomness axioms is rather their strong connection with forcing. Fairly straightforward hypotheses on random sequences turn out to describe essential properties of a particular type of forcing extension; as such they are a "poor man's Martin's axiom". But, unlike Martin's axiom, the axioms can also lay claim to some intuitive justification.

6 APPENDIX: RELATED APPROACHES Here we discuss two other attempts to add axioms for randomness to set theory, those of Myhill and Freiling.

6.1 MYHILL'S AXIOMS These were alluded to in the quotation from Kreisel in section 0. They are reported in paragraph 10 of Kruse [1967].

Again, let \( 2^{\text{bo}} \) be the space of infinite binary sequences and let \( \lambda \) denote Lebesgue measure on this space, i.e. \( (\{\text{true}, \text{false}\})^{\text{bo}} \). Let \( R(x) \) be a predicate which should intuitively be interpreted as "\( x \) is random". Myhill tries to formalize the intuition that random sequences should satisfy "all" properties of probability 1, and no other properties. His axioms (M) are
(M1) \( \lambda \{ x \in 2^\omega \mid R(x) \} = 1 \)

(M2) If \( \phi(x) \) is a formula in one free variable, then \( \lambda \{ x \in 2^\omega \mid \phi(x) \} = 1 \) implies \( \forall x (R(x) \rightarrow \phi(x)) \).

The restriction that \( \phi \) contain no parameter beside \( x \) is obviously necessary, since otherwise we could take the formula \( x \neq y \). The formulation of (M2) contains a deliberate ambiguity, however: is \( R \) allowed to occur in \( \phi \) or not? If not, then \( M \) is obviously conservative over ZFC. On the other hand, if we do allow \( R \) in \( \phi \), then \( R \) need no longer be explicitly definable. Myhill comments:

\[
\text{[This] would accord with a prejudice of mine which I derived from Feller, i.e. that randomness is an intensional notion, not definable in the usual mathematical terms. The "circularity" of the schema above with \( R \) allowed to appear in \( \phi \) is quite justified if we are convinced that \( R \) belongs to a new order of ideas, entirely outside the set theoretic order.}
\]

An immediate consequence is that \( M \) is no longer conservative over ZFC, for it implies the by now familiar consequence that there is no definable wellordering of the continuum: if there were such a wellordering, then we could define the least random element, in contradiction with (M2). It is clear that this proof works only when \( R \) is also allowed to occur in the schemata of ZF. Note that, on the liberal interpretation of (M2), the consistency of ZFC + \( M \) (with \( R \) allowed in the schemata of ZF) is not immediate.

6.1.1 LEMMA ZFC + \( M \) is consistent.

PROOFSKETCH Use the construction of theorem 1.3 (I), starting from a model \( \mathcal{M} \) of ZF + \( V = L \). Interpret \( R(x) \) as: \( x \) is (Solovay) random over \( \mathcal{M} \). In the generic extension of \( \mathcal{M} \) verifying 1.3 (I), the random reals have measure 1. Furthermore, \( R \) is definable, hence to verify (M2) it suffices to show that a definable set of reals with measure 1 contains all random reals. This is established in Stern [1985], section 4.7. □

In order to facilitate comparison between \( M \) and the axioms introduced in the previous sections, let us reformulate \( M \) in terms of axioms of density and data. (M1) is equivalent to the axiom of Outer Density:

If \( \{ x \mid \phi(x) \} \) has positive outer measure, then \( \exists x (R(x) \land \phi(x)) \).

and (M2) is equivalent to Outer Data:

If \( \exists x (R(x) \land \phi(x)) \), then \( \{ x \mid \phi(x) \} \) has positive outer measure.

It then becomes clear that Myhill's approach is eminently classical. If \( \{ x \mid \phi(x) \} \) has positive outer measure, we do not have enough information to construct a random \( x \) which almost certainly verifies \( \phi \). By the same token, knowledge of the truth of \( \exists x (R(x) \land \phi(x)) \) does not allow us to infer that we must have been able to construct
(with practical certainty) a random \( x \) such that \( \phi(x) \); hence we obtain only the weaker conclusion that \( \{ x \mid \phi(x) \} \) has positive outer measure.

One final remark: in view of the extensional theory of randomness outlined in section 2, the following modification of Myhill's axioms seems reasonable: we have a relation \( R(x, y) \) such that

(M0) \( R(x, y) \) satisfies \( R \)

(M1) \( \forall y \exists x \in 2^\omega | R(x, y) \) = 1

(M2) If \( \lambda(\{ x \in 2^\omega | \phi(x, y) \}) = 1 \), then \( \forall x (R(x, y) \rightarrow \phi(x, y)) \).

Is it possible to prove the consistency of this theory short of using an inaccessible?

6.2 FREILING AND THE CONTINUUM HYPOTHESIS

We will next investigate the relation of Freiling's axioms of symmetry [1986] to the axioms introduced here. Freiling writes:

Suppose we were to throw a random dart at the real number line and ask whether the dart landed on a rational number. The outcome is, of course, predictable. We could say in advance that the dart will (with probability one) land on an irrational number. Furthermore, let us agree that the reason does not depend on any particular property of the set of real numbers except that it is countable and its members are determined before we make our throw.

Now suppose we were to throw two darts and ask whether the second dart was a rational multiple of the first one. The answer would likewise be no, since by the time we throw the second dart there are only countably many points which it has to miss, and membership in this countable set is predetermined by the first dart.

Suppose then that we have a function \( f: \mathbb{R} \rightarrow \mathbb{R} \), (i.e. \( f \) assigns to each real a countable set of reals). The second dart will not be in the countable set assigned to the first dart. Now by the symmetry of the situation (the real line does not know which dart was thrown first or second), we could also say that the first dart will not be in the set assigned to the second. This leads us to the following natural proposition:

\( A_{\mathbb{R}_0} \forall f: \mathbb{R} \rightarrow \mathbb{R} \exists x (x \not= f(y) \land y \not= f(x)) \),

the intuition being that \( x \) and \( y \) could be found by independently throwing two random darts.

He then proceeds to prove that \( A_{\mathbb{R}_0} \) is equivalent to \( \neg \text{CH} \). We intend to show here that Freiling's intuitive motivation for \( A_{\mathbb{R}_0} \) is entirely captured by \( Q_0 \) in the following sense: a suitable fragment of \( ZFQ \) (basically one allows two iterations of \( Q \) only) suffices to derive \( A_{\mathbb{R}_0} \) and, conversely, that fragment can be interpreted in \( \text{ZFC} + A_{\mathbb{R}_0} \).

Although the resulting fragment is admittedly adhoc, we shall try to motivate it by referring back to the proof of 1.8. It will be observed that, in that proof, we needed only statements of the form \( Q_0(\langle x, y, v \rangle \in U) \) or \( Q_0Q_0(\langle x, y \rangle \in V) \). Accordingly, we can define a class of "elementary statements" as follows.

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6.2.1 DEFINITION The class of n – elementary statements is defined by
(1) \(<x_1 , ... , x_n> \in U\), where \(U \subseteq (2^\omega)^n\), is n – elementary
(2) n – elementary statements are closed under \(\rightarrow, \neg, \wedge, \vee, \forall, \exists\)
(3) if \(\phi\) is elementary and not of the form \(\forall x\psi\) or \(\exists x\psi\), then \(Qx\phi\) is n – elementary.

\(ZF\mathcal{E}_n\) is obtained from \(ZFQ\) by applying the following restrictions
(1) the \(Q\) axioms are formulated for n – elementary statements only
(2) we allow only n iterations of \(Q\)
(3) in n – elementary statements, we allow only those \(U \subseteq (2^\omega)^n\) for which \(\{ x \mid <x_1 , ... , x_{n-1}, x> \in U\}\) is countable for all \(<x_1 , ... , x_{n-1}>\).

We apologize for the lack of elegance, both of the preceding definitions and of the proof of the main result, 6.2.7. We include this material only because it shows that Freiling's intuitions fit squarely in the framework of the preceding sections. Furthermore, it may be of interest to see that \(Q\) can also have a cardinality interpretation and that the size of the continuum is related to the number of iterations of \(Q\).

6.2.2 THEOREM \((ZF\mathcal{E}_2)\) \(A\mathcal{K}_0\)
PROOF Suppose we are given \(f: 2^\omega \rightarrow (2^\omega)^{\mathcal{K}_0}\). It suffices to show \(QxQy(x \notin f(y) \wedge y \notin f(x))\), for generally \(Qx\phi\) implies \(\exists x\phi\). By Q6 and Q5, we only have to prove \(QxQy(y \notin f(x))\). However, this follows from \(\forall xQy(y \notin f(x))\), which is a consequence of 1.8.

Clearly, Freiling's symmetry principle ("the real line does not know the order of the darts") corresponds to Q6.

6.2.3 COROLLARY \((ZF\mathcal{E}_2)\) \(\neg CH\)
PROOF If \(CH\), there is a wellordering \(\leq\) of \(2^\omega\) of length \(\mathcal{K}_1\). Now consider \(f(y) := \{ x \mid x \leq y\}\). Then \(\forall x\forall y(x \in f(y) \vee y \in f(x))\); but since each \(f(y)\) is countable we get a contradiction from \(A\mathcal{K}_0\).

Actually something stronger holds. Let \((2^\omega)_n\) denote the set of n - element subsets of \(x\). Since \((2^\omega)_n\) can be coded into \(2^\omega\), we can also consider functions \(f: (2^\omega)_n \rightarrow (2^\omega)^{\mathcal{K}_0}\). This means that we are able to formulate and prove a generalization of \(A\mathcal{K}_0\) in our set up. First a

6.2.4 DEFINITION Let \(f: (2^\omega)^n \rightarrow (2^\omega)^{\mathcal{K}_0}\). A set \(X \subseteq 2^\omega\) is called \(f\) - incomparable if for any n distinct elements \(x_1 , ... , x_n\) of \(X\), \(x_n \notin f(\{ x_1 , ... , x_{n-1}\})\).

The generalization of \(A\mathcal{K}_0\) can then be formulated as follows:
For all $n$, $A_{\aleph_0}^n$.

**Proof** Along the same lines as for $n = 2$. To show how Q6 is applied we do the case $n = 3$. Choose $f$. By 1.8 $\forall x \forall y Qz(z \notin f(x, y))$, hence by monotonicity $QxQyQz(z \notin f((x, y)))$. By Q6, we get $QzQyQx(z \notin f((x, y)))$, whence $QxQyQz(x \notin f((z, y)))$ by Q3; and similarly $QxQyQz(y \notin f((x, z)))$. Now apply Q5.

Freiling proves

**Theorem (ZFC)** $A_{\aleph_0}^n$ is equivalent to $2^\aleph_0 \geq \aleph_n$.

We will now establish an equivalence between Freiling’s approach and ours.

**Theorem ZF$E_2^+$ AC is interpretable in ZFC + $A_{\aleph_0}$**.

**Proof** The obvious interpretation of Q would be to put A in Q if A is co-countable, but this runs afoul of Q6. We therefore have to give a contextual definition of Q, depending on an additional parameter. Indeed, it seems impossible to give a uniform interpretation of Q in the absence of strong hypotheses like, e.g., "all sets of reals have the Baire property".

Let $\mathcal{A}$ be the $\sigma$-algebra of countable (i.e. finite or countably infinite) and co-countable sets. Define $\nu : \mathcal{A} \to \{0, 1\}$ by $\nu A = 1$ iff A is co-countable. $\nu$ is $\sigma$-additive. We show how to extend $\nu \times \nu$ on $2^\omega \times 2^\omega$ to a measure $\mu$ on an extension of $\mathcal{A} \times \mathcal{A}$ that will serve to interpret Q. Expressions like "full", "null" or "almost all" refer to $\nu$.

We first need a lemma which shows that Sierpinski’s counterexample to a Fubini theorem without joint measurability condition does not exist under $A_{\aleph_0}$.

**Lemma (ZFC + $A_{\aleph_0}$)** There is no set that is null for almost all horizontal sections and full for almost all vertical sections.

**Proof** See Freiling [1986], p.197.

Let $\mathcal{N}_x$ denote $\{ A \subseteq 2^\omega | \forall x \in 2^\omega A_x \text{ is null } \}$ and similarly $\mathcal{N}_y := \{ A \subseteq 2^\omega | \forall y \in 2^\omega A_y \text{ is null } \}$. Obviously $\mathcal{N}_x$ and $\mathcal{N}_y$ are $\sigma$–ideals.

**Lemma (ZFC + $A_{\aleph_0}$)** Let $A$ be $\nu \times \nu$ measurable and suppose $A \subseteq \bigcup B_i$, where $\{B_i\}$ is a countable family contained in $\mathcal{N}_x \cup \mathcal{N}_y$. Then $\nu \times \nu A = 0$. 

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PROOF (Cf. Simms [1989], lemma 2) Suppose not, then we would have \( \nu \times \nu A = 1 \).
We may suppose \( A \subseteq X \cup Y \), where \( X \in \mathcal{K}_x \) and \( Y \in \mathcal{K}_y \). Hence for all \( x \), \( A_x \subseteq X_x \cup Y_x \). By Fubini's theorem applied to \( \nu \times \nu \), for almost all \( x \), \( A_x \) is full.
Since for all \( x \), \( X_x \) is null, it follows that \( Y_x \) must be full for almost all \( x \), which is impossible by the previous lemma.

We can now define the extension \( \mu \) of \( \nu \times \nu \). Let \( \mathcal{B} \) be the \( \sigma \)-algebra generated by \( \mathcal{A} \cup (\mathcal{K}_x \cup \mathcal{K}_y) \). It is easy to show that \( \mathcal{B} = \{ B \mid B = A \Delta N, A \in \mathcal{A}, N \in \mathcal{K}_x \cup \mathcal{K}_y \} \).
Define \( \mu \) on \( \mathcal{B} \) by \( \mu(A \Delta N) = \nu \times \nu A \). \( \mu \) is a welldefined \( \sigma \)-additive measure because \( A_1 \Delta N_1 = A_2 \Delta N_2 \) implies \( A_1 \Delta A_2 = N_1 \Delta N_2 \); now apply the previous lemma. Obviously \( \mu \) satisfies \( \mu N = 0 \) for all \( N \in \mathcal{K}_x \cup \mathcal{K}_y \). We can explicitly define a disintegration \( \{ \mu_x \mid x \in 2^{\omega} \} \) of \( \mu \) as follows: put \( \mu_x(A_x \Delta N_x) := \nu A_x \). Then \( \mu_x \) is a \( \sigma \)-additive measure that extends \( \nu \), for fixed \( B \in \mathcal{B} \) the function \( x \mapsto \mu_x B_x \) is \( \nu \)-measurable and we have

\( (*) \quad \mu(A \Delta N) = \nu \times \nu A = \int \nu A_x d\nu(x) = \int \mu_x A_x d\nu(x) = \int \mu_x (A_x \Delta N_x) d\nu(x). \)

This property will serve to validate Q6. We interpret Q in ZFC + A,\( _{\alpha 0} \) as follows:

1. \( Q_x \psi(x) \Leftrightarrow \nu_1 \{ x \mid \psi(x) \} = 1 \)
2. \( Q_x \phi(x,y) \Leftrightarrow \mu_x \{ y \mid \phi(x,y) \} = 1 \)

for 2 - elementary \( \phi \) and \( \psi \). This interpretation is correct because, if \( \phi \) is 2 - elementary, either \( \forall x \{ y \mid \phi(x,y) \} \) is null or \( \forall x \{ y \mid \neg \phi(x,y) \} \) is null; whence by construction of \( \mu_x \), \( \{ <x,y> \mid \phi(x,y) \} \) is \( \mu \)-measurable.

We now show that the Q axioms are valid under this interpretation. Q1 - 3,5 hold trivially.

We prove Q6 by means of (*):

\[ Q_x Q_y \phi(x,y) \Leftrightarrow \nu_1 \{ x \mid \mu_x \{ y \mid \phi(x,y) \} = 1 \} = 1 \Leftrightarrow \int \mu_x \{ y \mid \phi(x,y) \} d\nu(x) = 1 \Leftrightarrow \mu_1 \{ <x,y> \mid \phi(x,y) \} = 1 \Leftrightarrow \int \mu_y \{ x \mid \phi(x,y) \} d\nu(y) = 1 \Leftrightarrow \nu \{ y \mid \mu_y \{ x \mid \phi(x,y) \} \phi(x,y) \} = 1 \Leftrightarrow Q_y Q_x \phi(x,y). \]

The validity of Q4 is connected to some simple symmetry properties. Suppose \( Q_x \phi(x,y) \) and \( \forall x \phi(x,y) \rightarrow y(x,z) \). Then \( \mu_y \{ x \mid \phi(x,y) \} = 1 \), hence \( \mu_y \{ x \mid \psi(x,z) \} = 1 \).

Let \( \tau : 2^{\omega} \rightarrow 2^{\omega} \) be an automorphism that maps \( y \) to \( z \) and let \( \tau = \text{id} \times \pi \) be the induced automorphism on \( 2^{\omega} \). Obviously \( \mu \) is invariant under \( \tau \). Hence if we put \( C := \{ <x,z> \mid \psi(x,z) \} \) we have \( \mu C = (\mu \tau^{-1}) C \) which implies \( \mu_z C_z = (\mu \tau^{-1}) z C_z = \mu_y C_z = 1. \)

6.2.10 COROLLARY ZFQ \( + \) AC is interpretable in ZFC + \( 2^{\mathfrak{K}_0} = \mathfrak{K}_2 \).

Analogous results hold for \( n \) iterations of Q, \( n \geq 3 \).

The main philosophical difference between Freiling's approach and ours, is that ZFQ is a purely qualitative theory. Unlike Freiling, we remain agnostic about which sets are small (hence will almost certainly be missed by a random dart), we formulate properties
of small sets *per se*. (It is only after adding ID that we commit ourselves.) What Freiling proves in each case is that the continuum cannot have a wellordering of length $\omega_n$, only by applying AC can we conclude from this that $2^{\aleph_0} \geq \aleph_{n+1}$. It seems to us that his arguments may be taken as well as evidence that AC is false for a universe containing random sequences. Indeed, ZFQ, which contains Q6, the natural extrapolation of Freiling's symmetry principle, forces AC to be false. But we have seen that, although the continuum has no wellordering at all, the aleph-free version of CH can still be true in the presence of ZFQ (even with ID added). We conclude from this that randomness is related to the structure, rather than to the cardinality of the continuum.

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