SAHLQVIST'S THEOREM FOR
BOOLEAN ALGEBRAS WITH OPERATORS

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BOOLEAN ALGEBRAS WITH OPERATORS

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Sahlqvist's Theorem for Boolean Algebras with Operators

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1 Introduction

The aim of this note is to explain how a well-known result from Modal Logic, Sahlqvist's Theorem, can be applied in the theory of Boolean Algebras with Operators to obtain a large class of identities, called Sahlqvist identities, that are preserved under canonical embedding algebras. These identities can be specified as follows. Let $\sigma = \{f_i : i \in I\}$ be a set of (normal) additive operations. Let an untied term over $\sigma$ be a term that is either (i) negative (i.e., in which every variable occurs in the scope of an odd number of complementation signs only), or (ii) of the form $g_1(g_2 \ldots (g_n(x)) \ldots)$, where the $g_i$s are duals of unary elements of $\sigma$ (i.e., $g_i$ is defined by $g_i(x) = -f_i(-x)$ for some unary operator in $\sigma$), or (iii) obtained from terms of type (i) or (ii) by applying $+, \cdot$ and elements of $\sigma$ only. Then, an equality is called a Sahlqvist equality if it is of the form $s = 1$, where $s$ is obtained from complemented untied terms $-u$ by applying duals of elements of $\sigma$ to terms that have no variables in common, and $\cdot$ only.

Examples of Sahlqvist identities are abundant in algebraic logic—in fact, all axioms for both relation and cylindric algebras can be brought in a Sahlqvist form. For instance, Johan van Benthem observed that the axiom $x^2; -(x; y) \leq -y$ in relation algebra has a Sahlqvist equivalent $-[x^2; -(x; y)) \cdot y)] = 1$.

To prove that such Sahlqvist equalities are indeed preserved under canonical embedding algebras we will not have to prove any really new results, but we will merely have to order some known results in an appropriate way.

As this note is aimed primarily at algebraists, we assume that the reader is familiar with basic algebraic notions and facts; for algebraic details not explained in this note we refer the reader to [2]. We will be somewhat more explicit concerning the modal logical results and definitions we will need; most of them will be presented in §2. After that, in §3, we describe the modal counterparts of the above Sahlqvist equalities, and partially prove a Sahlqvist Theorem, which says that Sahlqvist formulas are both canonical and first order. From this the preservation of Sahlqvist equalities under canonical embedding algebras is easily derived. Finally, §4, which is essentially a part of the second author's dissertation [9], contains a detailed demonstration of the usefulness of the Sahlqvist Theorem. It shows that by the Sahlqvist Theorem the equivalence of two equations may be proved or disproved by reasoning on modal frames (or atom structures) rather than by manipulating these

*This author was supported by the Netherlands Organisation for Scientific Research (NWO).
equations themselves; as an example illustrating this method Henkin’s equation in cylindric algebras is proved to be equivalent to an equality of a simpler form.

The reader is advised to skip §2 upon a first reading, and only to return to it later on to look up a definition.

We would like to thank Johan van Benthem for stressing the importance of T. Sahlqvist’s Theorem, and Andréka Hajnal, István Németi and Ildikó Sain for encouraging us to write this note.

2 Preliminaries

A Boolean algebra with operators (BAO) is an algebra \( \mathfrak{B} \) of type \( \{+,\cdot,-,0,1\} \cup \{ f_i : i \in I \} \) such that \((B,+,-,0,1)\) is a Boolean algebra, and the operators \( \{ f_i : i \in I \} \) are (finitely) additive (join preserving) in every argument; a BAO is called normal if for every \( f_i \), \( f_i(\overline{a}) = 0 \) whenever one of the terms \( x_j = 0 \).

Let us quickly move on to the Stone Representations of BAO’s, the so-called general frames. First, a modal similarity type is a pair \( S = (O, \rho) \), where \( O = \{ \forall_i : i \in I \} \) is a set of modal operators, and \( \rho \) is a rank function for \( O \). As variables ranging over modal operators we use \( \forall, \forall_1, \ldots \); for monadic modal operators we use \( \Diamond, \Diamond_1, \ldots \). For \( \forall_i \in S \) its dual operator \( \forall_i^c \) is defined as \( \forall_i^c(\phi_1, \ldots, \phi_{\rho(i)}) \equiv \neg \forall_i(\neg \phi_1, \ldots, \neg \phi_{\rho(i)}) \); the dual of a monadic operator \( \Diamond_i \) is denoted \( \Box_i \). A modal language is a pair \( M = (S, Q) \), where \( S \) is a modal similarity type, and \( Q \) is a set whose elements are called proposition letters. From the modal and Boolean constants, and the proposition letters, the modal formulas are built up in the obvious way, using \( \neg, \wedge, \text{ and the operators in } S \). When no confusion arises we write \( M(S) \) or even \( M \) rather than \( M(S, Q) \).

A general frame \( \mathfrak{F} \) of similarity type \( S \) is a tuple \((W, \{ R_i : i \in I \}, \mathcal{W}) \) where \( W \neq \emptyset \), \( R_i \subseteq W^{\rho(i)+1} \), and \( \mathcal{W} \subseteq \text{Sb}(W) \) contains \( \emptyset \), and is closed under \( \cdot, \neg \), and the operators \( \{ f_{R_i} : i \in I \} \), where \( f_{R_i} : \text{Sb}(W)^{\rho(i)} \rightarrow \text{Sb}(W) \) is defined by

\[
(1) \quad f_{R_i}(Y_1, \ldots, Y_{\rho(i)}) = \{ x_0 : \exists x_1 \ldots x_{\rho(i)}(R_i(x_0, x_1, \ldots, x_{\rho(i)})) \wedge \bigwedge_{1 \leq j \leq \rho(i)} (x_i \in Y_i) \}.
\]

For future use we also define \( g_{R_i} : \text{Sb}(W)^{\rho(i)} \rightarrow \text{Sb}(W) \), by putting \( g_{R_i}(Y_1, \ldots, Y_{\rho(i)}) = -f_{R_i}(-Y_1, \ldots, -Y_{\rho(i)}) \). A Kripke frame or atom structure of similarity type \( S \) is a tuple \((W, \{ R_i : i \in I \})\), with \( W \) and \( \{ R_i : i \in I \} \) as before. A general frame \( \mathfrak{F} \) defines a Kripke frame \( \mathfrak{F}^\# \) via the forgetful functor \( (\cdot)^\# : (W, \{ R_i : i \in I \}, \mathcal{W}) \mapsto (W, \{ R_i : i \in I \}) \). A Kripke frame \( \mathfrak{F} \) defines the general frame \( \mathfrak{F}^\# \) via \( (\cdot^\# : (W, \{ R_i : i \in I \}) \mapsto (W, \{ R_i : i \in I \}, \text{Sb}(W)) \).

Given a general frame \( \mathfrak{F}^+ = (W, \{ R_i : i \in I \}, \mathcal{W}) \) its complex algebra is the BAO \( \mathfrak{F}^+ = (W, \cup, \cap, 0, W, -, \{ f_{R_i} : i \in I \}) \), where \( f_{R_i} : \text{Sb}(W)^{\rho(i)} \rightarrow \text{Sb}(W) \) is defined as in (1).

Given a BAO \( \mathfrak{B} \) with operators \( \{ f_i : i \in I \} \), the general frame \( \mathfrak{B}^+ \) is the tuple \((X_{\mathfrak{B}}, \{ R_{f_i} : i \in I \}, \mathcal{W}) \), where \( X_{\mathfrak{B}} \) is the set of ultrafilters on \( \mathfrak{B} \), \( R_{f_i} \subseteq X_{\mathfrak{B}}^{\rho(i)+1} \) is de-

\[\text{Algebraists may be accustomed to seeing the argument places reversed in the definition of the function} \]

\( f_{R_i}(Y_1, \ldots, Y_{\rho(i)}) = \{ x_0 : \exists x_1 \ldots x_{\rho(i)}(R_i(x_0, x_1, \ldots, x_{\rho(i)})) \wedge \bigwedge_{1 \leq j \leq \rho(i)} (x_i \in Y_i) \} \) in (1). Being modal logicians we like to think that the modal notation is the more elegant one.
defined by

\[ R_f(a_0, a_1, \ldots, a_{\rho(i)}) \text{ iff } \forall j (1 \leq j \leq \rho(i) \rightarrow x_j \in a_j) \text{ implies } f_i(x_{\rho(i)} + i) \in a_0, \]

and \( W \subseteq Sb(X_B) \) is \( \{ x : x \in B \} \) for \( B = \{ a \in X_B : x \in a \} \). The canonical structure \( CS_B \) of \( B \) is the structure \( (B, \#) \). By definition the complex algebra of the canonical structure of \( B \) is called the canonical embedding algebra of \( B \) : \( CS_B = (CS_B)^+ \). By a canonical variety we mean one that is closed under canonical embedding algebras.

A valuation on a general frame \( \mathcal{G} \) is a function taking propositional letters to elements of \( W \); a valuation on a Kripke frame \( \mathcal{G} \) is a valuation on \( \mathcal{G}^\# \). In algebraic terms: a valuation is an assignment to the variables of elements of the ‘subcomplex’ algebra \( W \). Truth of a modal formula in a model \( (\mathcal{G}, V) \) is then defined as follows: \( (\mathcal{G}, V), w_0 \models p \) iff \( w_0 \in V(p) \); \( (\mathcal{G}, V), w_0 \models \neg \phi \) iff \( (\mathcal{G}, V), w_0 \not\models \phi \); \( (\mathcal{G}, V), w_0 \models \phi \land \psi \) iff \( (\mathcal{G}, V), w_0 \models \phi \land (\mathcal{G}, V), w_0 \models \psi \); and \( (\mathcal{G}, V), w_0 \models \forall_i(\phi_1, \ldots, \phi_{\rho(i)})(w_1, \ldots, w_{\rho(i)}) \) iff \( \forall w : (\mathcal{G}, V), w_1, \ldots, w_{\rho(i)} \models R_i(w_0, w_1, \ldots, w_{\rho(i)}) \land \forall 1 \leq j \leq \rho(i)(\mathcal{G}, V), w_j \models \phi_j \). We write \( (\mathcal{G}, V), w \models \phi \) for: for all \( w \in W, (\mathcal{G}, V), w \models \phi \); \( \mathcal{G}, w \models \phi \) is short for: for all valuations \( V \) on \( \mathcal{G} \), \( (\mathcal{G}, V), w \models \phi \); and \( \mathcal{G}, w \models \phi \) is short for: for all \( w \in W \), \( \mathcal{G}, w \models \phi \).

A modal formula \( \phi \) in \( n \) proposition letters induces an \( n \)-ary polynomial \( h_\phi(x_1, \ldots, x_n) \) which may be defined as follows:

\[
\begin{align*}
h_{p_i}(x_1, \ldots, x_n) & \equiv x_j \\
h_{\neg \phi}(x_1, \ldots, x_n) & \equiv -h_\phi(x_1, \ldots, x_n) \\
h_{\phi \land \psi}(x_1, \ldots, x_n) & \equiv h_\phi(x_1, \ldots, x_n) \cdot h_\psi(x_1, \ldots, x_n) \\
h_{\forall_i(\phi_1, \ldots, \phi_{\rho(i)})(x_1, \ldots, x_n)} & \equiv f_{R_i}(h_\phi(x_1, \ldots, x_n), \ldots, h_{\phi_{\rho(i)}}(x_1, \ldots, x_n)).
\end{align*}
\]

And conversely, each polynomial in a similarity type of BAO’s is of the form \( h_\phi \) for some modal formula \( \phi \) in a modal language of the appropriate type. This identification of formulas and terms is made explicit in the following proposition.

**Proposition 2.1** Let \( S \) be a modal similarity type. Let \( \mathcal{G} \) be a general frame of type \( S \). Let \( \phi \) be a formula in \( M(S) \). Then \( \mathcal{G}, w \models \phi \) iff \( \mathcal{G}^\# \models h_\phi = 1 \).

A (normal) modal logic in a language \( M(S) \) is a subset \( \Lambda \) of the set of formulas in \( M(S) \) that contains as axioms all propositional tautologies (PL), as well as

\[
(DB) \quad \forall i (P_1, \ldots, P_{j-1}, P, P_{j+1}, \ldots, P_{\rho(i)}(x_1, \ldots, x_n)) \land \forall i (P_1, \ldots, P_{j-1}, P', P_{j+1}, \ldots, P_{\rho(i)}(x_1, \ldots, x_n)) \land
\]

and that is closed under the following derivation rules:

\[
(MP) \quad \text{if } \phi, \phi \rightarrow \psi \in \Lambda \text{ then } \psi \in \Lambda
\]

\[
(UG) \quad \text{if } \phi \in \Lambda \text{ then } \neg \forall i (\phi_1, \ldots, \phi_{j-1}, \neg \phi, \phi_{j+1}, \ldots, \phi_{\rho(i)}) \in \Lambda
\]

\[
(SUB) \quad \text{if } \phi \in \Lambda \text{ then all substitution instances of } \phi \text{ are in } \Lambda.
\]

For a logic \( \Lambda \) a canonical general frame for \( \Lambda \) is defined by \( \mathcal{G}_\Lambda(\alpha) = (\mathcal{A}_\Lambda(\alpha))^+ \), where \( \mathcal{A}_\Lambda(\alpha) \) is the free algebra (on \( \alpha \) generators) of the variety \( V_\Lambda \), where \( \mathcal{A} \in V_\Lambda \) iff \( \mathcal{A} \models h_\phi = 1 \), for all \( \phi \in \Lambda \). For a class of general or Kripke frames \( K \), let \( \text{Th}(K) = \{ \phi : \text{for all } \mathcal{G}, \mathcal{G} \models \phi \} \). We call a logic \( \Lambda \) sound with respect to a class of general or Kripke frames \( K \) if \( \Lambda \subseteq \text{Th}(K) \), and complete with respect to \( K \) if \( \text{Th}(K) \subseteq \Lambda \). A logic \( \Lambda \) is called canonical if \( \mathcal{G}_\Lambda(\alpha))^+ \models \Lambda \), for every canonical general frame \( \mathcal{G}_\Lambda(\alpha) \).

\[3\] In [3] the canonical embedding algebra of \( B \) is called the Stone extension of \( B \); these, in turn, form a special case of the arbitrary extensions dealt with in [5], of a kind called perfect extension.
$L_0(S)$ is the first order language of type $S$; it has relation symbols $R_i$ ($i \in I$) of arity $\rho(i) + 1$. $L_1(S)$ is $L_0(S)$ extended with unary predicate symbols $P_j$ corresponding to the proposition letters of our modal language. $L_2(S)$ is the language of monadic second order logic with relation symbols $R_i$ ($i \in I$) of arity $\rho(i) + 1$, and variables $P_j$s ranging over sets. A modal formula $\phi$ locally corresponds to an formula $\alpha(x)$ if for all Kripke frames $\mathfrak{F}$ of the appropriate type, $\mathfrak{F}, w \models \phi$ iff $\mathfrak{F}, w \models \alpha[w]$. A modal formula $\phi$ corresponds to a sentence $\alpha$ if for all Kripke frames $\mathfrak{F}$ of the appropriate type, $\mathfrak{F} \models \phi$ iff $\mathfrak{F} \models \alpha$. When interpreted on frames modal formulas correspond to $L_2(S)$-formulas (cf. [1]).

3 A Sahlqvist theorem

To describe the modal counterparts of the earlier Sahlqvist equalities we need the following definition.

**Definition 3.1** Let $S$ be a modal similarity type. Positive and negative occurrences of a proposition letter $p$ are defined as usual by: (i) $p$ occurs positively in $p$, (ii) a positive (negative) occurrence of $p$ in $\phi$ is negative (positive) occurrence of $\neg p$ in $\phi \rightarrow \psi$, and a positive (negative) one in $\phi \lor \psi, \phi \land \psi, \forall_i (\phi_1, \ldots, \phi_i, \ldots, \phi_{\rho(i)}) , \exists_i (\phi_1, \ldots, \phi_i, \ldots, \phi_{\rho(i)})$ ($\forall_i \in S$). A formula $\phi$ in $M(S)$ is positive (negative) if every proposition letter occurs only positively (negatively) in $\phi$. $\phi$ is monotone in the proposition letter $p$ if for every model $(\mathfrak{F}, V)$ and every valuation $V'$ on $\mathfrak{F}$ with $V(p) \subseteq V'(p)$ and otherwise the same as $V, (\mathfrak{F}, V), w \models \varphi$ implies $(\mathfrak{F}, V'), w \models \varphi$.

Note that in a positive formula negations of modal or Boolean constants are allowed. Also, if $\phi$ is positive then $\phi$ is monotone in all proposition letters.

**Definition 3.2** Fix a modal similarity type $S$. A formula $\phi$ in $M(S)$ is a Sahlqvist antecedent if it is built up from formulas that are either negative or of the form $\Box_{i_1} \ldots \Box_{i_n} p$, using only $\lor, \land$ and $\forall_i$, where $\Box_{i_1}, \ldots, \Box_{i_n}, \forall_i \in S$.

Define the set of Sahlqvist formulas in $M(S)$ as being the smallest set $X$ such that if $\phi$ is a Sahlqvist antecedent, and $\psi$ is a positive formula, then $\phi \rightarrow \psi \in X$; if $\sigma_1, \sigma_2 \in X$ then $\sigma_1 \land \sigma_2 \in X$; and if $\sigma_1, \ldots, \sigma_{\rho(i)} \in X$ have no proposition letters in common, then $\forall_i (\sigma_1, \ldots, \sigma_{\rho(i)}) \in X$.

For a modal similarity type $S$ that contains only unary operators several definitions exist of what it is for a formula in $M(S)$ to be a Sahlqvist formula; however, all are equivalent to (or are covered by) the restriction of 3.2 to such similarity types.

We believe that the generalization to arbitrary similarity types is in fact ours. One may wonder whether this is the obvious generalization from the `unary case', e.g., why are boxes (i.e., duals of unary normal, additive operations) allowed in Sahlqvist antecedents, while for $n \geq 2$ duals of $n$-ary operations in $S$ are not? The reason why we are interested in Sahlqvist formulas is that they may be shown to be complete and to define certain first order properties of the underlying relations in (generalized) frames. A look at the kind of formulas forbidden in Sahlqvist antecedents in the unary case in order to guarantee these properties, shows that they typically include combinations of the form $\Box(\ldots \lor \ldots)$, or, in first order terms, $\forall(\forall \ldots \lor \ldots)$, But these are precisely the combinations that pop up when we have $n$-ary boxes ($n \geq 2$) around! (In fact, if $\lor$ is a binary modal operator, and $\land$ is its dual, then $(p \land p) \land p \rightarrow (p \lor p) \lor p$ may already be shown to be non-elementary.)
Before proving an important property of Sahlqvist formulas we recall that for a binary relation $R$, $\bar{R} = \{(y, x) : Rx \}$, To each modal formula $\phi$ we associate a set operator $F^\phi$ as follows. Let $P_1, \ldots, P_k$ be sets and let $\bar{P}$ abbreviate $P_1, \ldots, P_k$. $F^P_j = P_j$ $(1 \leq j \leq k)$, while $F^\phi(\bar{P}) = (F^\phi(\bar{P}))^c$, and $F^\phi \land \psi(\bar{P}) = F^\phi(\bar{P}) \land F^\psi(\bar{P})$. $F^{\forall_i(\phi_1, \ldots, \phi_k(i))}(\bar{P}) = f_{R_i}(F^\phi(\bar{P}), \ldots, F^{\phi_k(i)}(\bar{P}))$, while $F^{\forall_i(\phi_1, \ldots, \phi_k(i))}(\bar{P}) = g_{R_i}(F^\phi(\bar{P}), \ldots, F^{\phi_k(i)}(\bar{P}))$. We assume that the set operator corresponding to Boolean or modal constants is provided by the context in which these constants occur.

**Theorem 3.3** Let $S$ be a modal similarity type. Let $\chi$ be a Sahlqvist formula in $M(S)$. Then $\chi$ corresponds to an $L_0(S)$-sentence $\alpha_\chi$ effectively obtainable from $\chi$.

**Proof.** This is more or less similar to the proof of [7, Theorem 8] (cf. also [1, Theorem 9.10]). Assume that $\chi$ has the form $\phi \rightarrow \psi$.

Let $p_1, \ldots, p_k$ be the proposition letters occurring in $\chi$. Having $\mathcal{G} = (W, \{ R_i : i \in I \}) \models \chi$ means having $\mathcal{G} \models \forall \bar{P} \forall x (x \in F^\chi(\bar{P}))$. By assumption the latter formula has the form

$$\forall \bar{P} \forall x (x \in F^\phi(\bar{P}) \rightarrow x \in F^\psi(\bar{P})),$$

where $\phi$ is a Sahlqvist antecedent, and $\psi$ is a positive formula. Next, using such equivalences as

$$\forall \ldots \left( (\Phi \land x \in F^\psi(\bar{P})) \rightarrow \Psi \right) \leftrightarrow \bigwedge_{j=1,2} \forall \ldots \left( (\Phi \land x \in F^{\psi_j}(\bar{P})) \rightarrow \Psi \right),$$

$$\forall \ldots \left( (\Phi \land x \in F^{\forall_i(\phi_1, \ldots, \phi_k(i))}(\bar{P})) \rightarrow \Psi \right) \leftrightarrow \forall \ldots \forall y_1 \ldots y_{\rho(i)} \left( (\Phi \land R_i x y_1 \ldots y_{\rho(i)} \land \bigwedge_j (y_j \in F^{\phi_j}(\bar{P}))) \rightarrow \Psi \right),$$

and

$$\forall \ldots \left( (\Phi \land x \in F^\nu(\bar{P})) \rightarrow \Psi \right) \leftrightarrow \forall \ldots \left( \Phi \rightarrow (\Psi \lor x \in F^{-\nu}(\bar{P})) \right),$$

(2) can be rewritten as a conjunction of formulas of the form

$$\forall \bar{P} \forall x \forall y \forall z \left( (\Phi \land \bigwedge_{j=1}^{m_j} (y_{ij} \in g_{R_{n_{ij}}} \ldots g_{R_{1_{ij}}}(P_j))) \rightarrow \bigvee_{j=1}^{h} (z_j \in F^{\psi_j}(\bar{P})) \right),$$

where $\Phi$ is a quantifier free $L_0$-formula ordering its variables in a certain way, and where all the $\psi_j$s are monotone. If a predicate variable $P$ occurs only in the consequent $\bigvee_{j=1}^{h} (z_j \in F^{\psi_j}(\bar{P}))$ in (6), then, by the monotonicity of the $\psi_j$s, it can be replaced by $\bot$, and the quantifier binding $P$ may be deleted. Thus we may assume that every predicate letter occurs in the consequent of (6) only if it occurs in the antecedent of (6).

By an easy argument we have that $\land_{i=1}^{m_j} (y_{ij} \in g_{R_{n_{ij}}} \ldots g_{R_{1_{ij}}}(P_j))$ if and only if we have $\bigcup_{l=1}^{m_j} f_{R_{1_{ij}}} \ldots f_{R_{n_{ij}}} \{ (y_{ij}) \} \subseteq P_j$. Thus by universal instantiation (6) implies the first order formula

$$\forall x \forall y \forall z \left( \Phi \rightarrow \bigvee_{j=1}^{h} (z_j \in F^{\psi_j}(\bigcup_{l=1}^{m_1} f_{R_{1_{i1}}} \ldots f_{R_{n_{i1}}}(\{ y_{i1} \}), \ldots, \bigcup_{l=1}^{m_k} f_{R_{1_{ik}}} \ldots f_{R_{n_{ik}}}(\{ y_{ik} \})) \right),$$

But, conversely, by the monotonicity of the functions $F^{\psi_j}$ (7) implies (6), and we are done.
To prove the general case one may argue inductively. If the Sahlqvist formulas \( \chi_1, \chi_2 \) have been shown to correspond to \( \alpha_1, \alpha_2 \), respectively, then \( \chi_1 \land \chi_2 \) corresponds to \( \alpha_1 \land \alpha_2 \); and if \( \chi_1, \ldots, \chi_{\rho(i)} \) are Sahlqvist formulas that have no proposition letters in common, and that have been shown to correspond to \( \forall x \alpha_1, \ldots, \forall x \alpha_{\rho(i)} \), then \( \psi(\chi_1, \ldots, \chi_{\rho(i)}) \) corresponds to \( \forall x y (R_1 x y_1 \ldots y_{\rho(i)} \rightarrow \alpha_1(y_1) \lor \ldots \lor \alpha_{\rho(i)}(y_{\rho(i)})) \).

Two remarks are in order. First, in the above result we may in fact replace ‘corresponds’ by ‘locally corresponds’. But given the algebraic application we have in mind the global version is more natural. Second, although the algorithm in the above general proof may seem somewhat intractable or even obscure, in particular examples it is quite manageable, as is witnessed in §4.

**Theorem 3.4** Let \( S \) be a modal similarity type. For \( j \in J \), let \( \chi_j \) be Sahlqvist formulas in \( M(S) \). Let \( \Lambda \) be the modal logic axiomatized by \( \{ \chi_j : j \in J \} \). Then \( \Lambda \) is canonical. Hence \( \chi \) is complete with respect to the class of Kripke frames defined by \( \{ \alpha_{\chi_j} : j \in J \} \).

**Proof.** The case where \( S \) contains only unary modal operators is [7, Theorem 19]. To prove the general case one may use the same arguments together with the canonical frame construction for modal logics of arbitrary similarity type as found in [9, Chapter 2]. (An alternative proof of the unary case may be found in [8].) \( \square \)

We leave it to the reader to check that every Sahlqvist formula induces a Sahlqvist identity, and conversely.

**Theorem 3.5** Let \( \Sigma \) be a set of Sahlqvist equalities. Let \( V_\Sigma \) be the variety defined by \( \Sigma \). Then \( V_\Sigma \) is canonical.

**Proof.** Let \( \widehat{\Sigma} \) be the set of modal translations of the elements of \( \Sigma \). So \( \widehat{\Sigma} \) is a set of Sahlqvist formulas. Now, to prove the theorem, let \( \mathfrak{B} \in V_\Sigma \). Let \( \mathfrak{A}_\Sigma(\{B\}) \) be the free \( \Sigma \)-algebra on \( \{B\} \) generators. Then \( \mathfrak{A}_\Sigma(\{B\}) \cong \mathfrak{B}, \) and hence \( \text{em} \mathfrak{A}_\Sigma(\{B\}) \rightarrow \text{em} \mathfrak{B}, \) by [2, Corollary 3.2.5(6)]. So we are done once we have shown that \( \text{em} \mathfrak{A}_\Sigma(\{B\}) \in V_\Sigma. \)

\[ \begin{array}{ccc}
\mathfrak{B} & \leftarrow & \mathfrak{A}_\Sigma(\{B\}) \\
\downarrow & & \downarrow \\
\text{em} \mathfrak{B} & \leftarrow & \text{em} \mathfrak{A}_\Sigma(\{B\})
\end{array} \]

Figure 1.

Since \( \mathfrak{A}_\Sigma(\{B\}) \models \Sigma, \mathfrak{A}_\Sigma(\{B\})_+ \models \widehat{\Sigma}. \) So by 3.4 \( \text{em} \mathfrak{A}_\Sigma(\{B\}) = (\mathfrak{A}_\Sigma(\{B\})_+)_\# \models \Sigma. \) But then \( \text{em} \mathfrak{A}_\Sigma(\{B\}) = ((\mathfrak{A}_\Sigma(\{B\})_+)_\#)^\# \models \Sigma, \) i.e. \( \text{em} \mathfrak{A}_\Sigma(\{B\}) \in V_\Sigma. \) \( \square \)

**Remark 3.6** In [5] Jónsson and Tarski also describe a class of equalities that are preserved under canonical embedding algebras. The class they define contains all equalities \( h_1 = h_2, \) where both \( h_1 \) and \( h_2 \) are symbols for functions that are either additive, or obtained from additive ones by using composition only. Obviously, all Jónsson and Tarski equalities may be seen as (a conjunction of two) Sahlqvist equalities; but conversely, not every Sahlqvist
equality is a Jónsson and Tarski equality. (As an example, $\Diamond \Box p \rightarrow \Box \Diamond p$ is a Sahlqvist formula, and hence its algebraic counterpart is a Sahlqvist equality; it is not a Jónsson and Tarski equality, however.) Hence, the class of Sahlqvist equalities forms a strict superset of the class of Jónsson and Tarski equalities.

It should be noted that unlike our result the Jónsson and Tarski result applies also to non-normal (but additive) BAO’s. In a paper by Henkin [3], one can also find a description of a class of equalities whose validity is preserved under canonical embedding algebras; however, the BAO’s considered there need not even be additive.

4 An example: simplifying Henkin’s equation

As an application of theorems 3.3 and 3.5 we show that, in order to prove that two Sahlqvist equations are equivalent over a canonical variety $V$, it suffices to show the equivalence (in $\mathbf{At} V$) of their first order translations. This means that reasoning can be done in the Kripke frames, which is usually much easier than manipulating algebraic equations.

Proposition 4.1 Let $V$ be a canonical variety, and $\eta_1$ and $\eta_2$ two Sahlqvist equations with first order correspondents $\alpha_1$ and $\alpha_2$. Then

$$\mathbf{At} V \models \alpha_1 \leftrightarrow \alpha_2 \iff V \models \eta_1 \leftrightarrow \eta_2.$$ 

Proof. From left to right: let $A$ be an algebra in $V$ with $A \models \eta_1$. By the fact that $\eta_1$ is a Sahlqvist equation, $\eta_1$ holds in $\mathbf{em} A = (\mathbf{cs} A)^+$. This gives $\mathbf{cs} A \models \alpha_i$, so by assumption $\mathbf{cs} A \models \alpha_j$. But then again $\mathbf{em} A \models \eta_j$, so $\eta_j$ holds in $A \leq \mathbf{em} A$.

From right to left: let $\mathfrak{F}$ be a frame in $\mathbf{At} V$ with $\mathfrak{F} \models \alpha_i$. Then $\mathfrak{F}^+ \models \eta_i \Rightarrow \mathfrak{F}^+ \models \eta_j \Rightarrow \mathfrak{F} \models \alpha_j$. $\square$

We assume familiarity with the notion of a cylindric algebra (cf. [6], [4]), but we modify some notation and definitions. Without loss of generality we may confine ourselves to the two-dimensional case. The algebraic language $\mathcal{L}_2$ has a constant $d_{01}$ and two unary operators $c_0$ and $c_1$, which we write as $\Diamond_0$ and $\Diamond_1$, respectively, if we want to stress the modal aspects of the subject. A cylindric-type frame is a quadruple $\mathfrak{F} = (W, \sim_0, \sim_1, D)$ with $\sim_i$ a binary accessibility relation (for $\Diamond_i$) on $W$, and $D$ the subset of $W$ where $d_{01}$ holds. In the following table we list the modal versions of the axioms governing the variety of cylindric algebras, together with their first order equivalents ($i \in \{0, 1\}$):

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(C1_i)$</td>
<td>$p \rightarrow \Diamond_i p$</td>
</tr>
<tr>
<td>$(C2_i)$</td>
<td>$p \rightarrow \Box_i \Diamond_i p$</td>
</tr>
<tr>
<td>$(C3_i)$</td>
<td>$\Diamond_i p \rightarrow \Diamond_i \Diamond_i p$</td>
</tr>
<tr>
<td>$(C4_i)$</td>
<td>$\Diamond_i \Diamond_j p \rightarrow \Diamond_j \Diamond_i p$</td>
</tr>
<tr>
<td>$(C5_i)$</td>
<td>$\Diamond_i d_{01}$</td>
</tr>
<tr>
<td>$(C6_i)$</td>
<td>$\Diamond_i (d_{01} \land p) \rightarrow \Box_i (d_{01} \rightarrow p)$</td>
</tr>
<tr>
<td>$(N1_i)$</td>
<td>$\forall x \ x \sim_i x$</td>
</tr>
<tr>
<td>$(N2_i)$</td>
<td>$\forall x y (x \sim_i y \rightarrow y \sim_i x)$</td>
</tr>
<tr>
<td>$(N3_i)$</td>
<td>$\forall x y z ((x \sim_i y \land y \sim_i z) \rightarrow x \sim_i z)$</td>
</tr>
<tr>
<td>$(N4_i)$</td>
<td>$\forall x y z ((x \sim_i y \land y \sim_j z) \rightarrow \exists u (x \sim_j u \land u \sim_i z))$</td>
</tr>
<tr>
<td>$(N5_i)$</td>
<td>$\forall x \exists y (x \sim_i y \land Dy)$</td>
</tr>
<tr>
<td>$(N6_i)$</td>
<td>$\forall x y z ((x \sim_i y \land x \sim_i z \land Dy \land Dz) \rightarrow y = z)$</td>
</tr>
</tbody>
</table>

We define $C1 = C1_0 \land C1_1$, etc. A cylindric algebra is an algebra $A = (A, +, -, c_0, c_1, d_{01})$ such that $(A, +, -)$ is a Boolean Algebra, $c_0$ and $c_1$ are normal and additive, and $C1, \ldots, C6$ are valid in $A$. The variety of cylindric algebras is denoted by CA.
A cylindric frame is a cylindric type frame $\mathfrak{F}$ such that $N_1, \ldots, N_6$ are valid in $\mathfrak{F}$. So a frame $\mathfrak{F} = (W, \sim_0, \sim_1, D)$ is cylindric iff $\sim_0$ and $\sim_1$ are equivalence relations ($N1, N2$ and $N3$ for respectively reflexivity, symmetry and transitivity), every $\sim_i$-equivalence class contains precisely one 'diagonal' element in $D$ ($N5$ for existence, $N6$ for unicity), and $\sim_0$ and $\sim_1$ permute ($N4$). Below these facts may be used without notice.

The following proposition is immediate by the Sahlqvist form of $C1, \ldots, C_6$, and theorems 3.3 and 3.4.

**Proposition 4.2** (i) $\mathfrak{F}$ is a cylindric frame iff $\mathfrak{F}^+$ is a cylindric algebra.

(ii) $\mathcal{CA}$ is a canonical variety.

Besides the axioms $C1, \ldots, C_6$ governing the variety of cylindric algebras, additional equations play an important rôle, especially *Henkin's equation*  

(\eta)  

\[ c_0(x \cdot y \cdot c_1(x \cdot y)) \leq c_1(-d_{01} \cdot c_0 x). \]

For example, it can be shown that adding $\eta$ to $C1, \ldots, C_6$, one obtains a complete equational axiom system for the set of equations valid in the variety of representable cylindric algebras. (This is only true in the two-dimensional case; in the higher dimensional case the rôle of $\eta$, though important, is not decisive.) One might wonder why the authors of [4] decided against giving $\eta$ the status of a CA-axiom. One of the reasons may have been that $\eta$ is less transparent than the other seven. In the remainder of this section we will show that $\eta$ has a simpler equivalent (over the variety $\mathcal{CA}$), and that the equivalence is very easy to prove using the Sahlqvist form of the equations.

So let us define the intended simplification of Henkin's equation:

(\eta')  

\[ d_{01} \cdot c_0(-x \cdot c_1 x) \leq c_1(d_{01} \cdot c_0 x). \]

Clearly both $\eta$ and $\eta'$ are Sahlqvist equations. Let us compute their first order equivalents.

**Definition 4.3** Let $\alpha, \alpha'$ be the formulas

(\alpha) \quad \forall u \forall v \forall w \left( (u \sim_0 v \sim_1 w \land v \neq w) \rightarrow \exists x (\neg Dx \land u \sim_1 x \land (x \sim_0 v \lor x \sim_0 w)) \right)

(\alpha') \quad \forall u \forall v \forall w \left( (Du \land u \sim_0 v \sim_1 w \land v \neq w) \rightarrow \exists x (\neg Dx \land u \sim_1 x \sim_0 w) \right).

The following pictures explain the meaning of $\alpha$ and $\alpha'$ for cylindric frames:

![Figure 2: $\alpha$](image1)

![Figure 3: $\alpha'$](image2)
Proposition 4.4 Let $\mathfrak{F}$ be a frame of the appropriate type. Then $\mathfrak{F} \models \alpha \iff \mathfrak{F}^+ \models \eta$ and $\mathfrak{F} \models \alpha' \iff \mathfrak{F}^+ \models \eta'$.

Proof. For $\eta$, we will spell out the algorithm of theorem 3.3 to find its first order correspondent. First consider its modal variant

\[(\chi) \quad \Diamond_0 (p \land \neg q \land \Diamond_1 (p \land q)) \rightarrow \Diamond_1 (\neg d_{01} \land \Diamond_0 p) .\]

Let $\phi$ and $\psi$ be respectively the antecedent $\Diamond_0 (p \land \neg q \land \Diamond_1 (p \land q))$ and the consequent $\Diamond_1 (\neg d_{01} \land \Diamond_0 p)$ of this formula. Clearly $\chi$ is a Sahlqvist formula, as $\phi$ is a Sahlqvist antecedent and $\psi$ is positive.

Now let $\mathfrak{F} = (W, \sim_0, \sim_1, D)$ be a Kripke frame for the language, then $\mathfrak{F} \models \chi$ iff

\[(8) \quad \mathfrak{F} \models \forall x \forall P \forall Q (x \in \mathfrak{F}^x (P, Q)) .\]

Now the formula $x \in F^x (P, Q)$ is by definition equivalent to

\[(9) \quad x \in F^\phi (P, Q) \rightarrow x \in F^\psi (P, Q) .\]

Step by step we will rewrite (9), abbreviating $u \in P$ by $Pu$. Starting with the antecedent of (9), we obtain

$$\exists y_1 (x \sim_0 y_1 \land y_1 \in F^{\forall \sim_0 \land \forall \sim_1 (p \land q)} (P, Q)) \rightarrow x \in F^\psi (P, Q) ,$$

or better

$$\forall y_1 (x \sim_0 y_1 \land y_1 \in F^{\forall \sim_0 \land \forall \sim_1 (p \land q)} (P, Q)) \rightarrow x \in F^\psi (P, Q) ,$$

yielding the effect of (4). Then we get

$$\forall y_1 (x \sim_0 y_1 \land P y_1 \land \neg Q y_1 \land y_1 \in F^{\forall \sim_1 (p \land q)} (P, Q)) \rightarrow x \in F^\psi (P, Q) ,$$

and (5) gives

$$\forall y_1 (x \sim_0 y_1 \land P y_1 \land y_1 \in F^{\forall \sim_1 (p \land q)} (P, Q)) \rightarrow (x \in F^\psi (P, Q) \lor Q y_1) .$$

Using (4), we obtain

\[(10) \quad \forall y_1 \forall y_2 (x \sim_0 y_1 \land P y_1 \land y_1 \sim_1 y_2 \land P y_2 \land Q y_2) \rightarrow (x \in F^\psi (P, Q) \lor Q y_1) .\]

So we have $\mathfrak{F} \models \chi$ iff the following formula holds in $\mathfrak{F}$:

$$\forall x \forall P \forall Q \forall y_1 \forall y_2 (x \sim_0 y_1 \land y_1 \sim_1 y_2 \land P y_1 \land P y_2 \land Q y_2) \rightarrow (x \in F^\psi (P, Q) \lor Q y_1) .$$

Comparing this formula with (6), we observe that for both $y_1$ and $y_2$ the sequence $g_{R_n i j} \cdots g_{R_1 i j}$ of (6) is empty, so the universal instantiation mentioned just above (7) simply means replacing $Pu$ by $u \in \{y_1, y_2\}$ (or better, by $(u = y_1 \lor u = y_2)$), and $Qu$ by $(u = y_2)$.

So (10) is equivalent to the following instance of (7), viz.

$$\forall x \forall y_1 \forall y_2 (x \sim_0 y_1 \land y_1 \sim_1 y_2) \rightarrow (x \in F^\psi (\{y_1, y_2\} \lor (y_1 = y_2)) ,$$

9
which really means

\[ \forall x \forall y_1 \forall y_2 \left( (x \sim_0 y_1 \land y_1 \sim_1 y_2) \rightarrow (y_1 = y_2 \lor \exists z_1 (x \sim_1 z_1 \land z_1 \sim_0 z_2 \land (z_2 = y_1 \lor z_2 = y_2))) \right) \]

Transporting \((y_1 = y_2)\) back to the antecedent, and after some straightforward formula manipulation, we finally obtain

\[ \forall x \forall y_1 \forall y_2 \left( (x \sim_0 y_1 \land y_1 \sim_1 y_2 \land y_1 \neq y_2) \rightarrow \exists z_1 (x \sim_1 z_1 \land \neg D z_1 \land (z_1 \sim_0 y_1 \lor z_1 \sim_0 y_2)) \right) \]

which is what we were after. \(\dagger\)

**Proposition 4.5** Let \(\mathfrak{A}\) be a cylindric algebra. Then \(\mathfrak{A} \models \eta \iff \mathfrak{A} \models \eta'\).

**Proof.** By the previous two propositions it is sufficient to show that for a cylindric frame \(\mathfrak{F}, \mathfrak{F} \models \alpha \iff \mathfrak{F} \models \alpha'\).

\((\Rightarrow)\) Assume that \(\mathfrak{F} \models \alpha'\). To prove that \(\mathfrak{F} \models \alpha\), let \(u, v\) and \(w\) be worlds in \(\mathfrak{F}\) with \(u \sim_0 v \sim_1 w\) and \(v \neq w\). We have to find an \(x\) with \(x \notin D\), \(u \sim_1 x\) such that \(x\) is in the 0-equivalence class with \(v\) or with \(w\). Distinguish the following cases:

Case 1: \(u \in D\).
Then \(\mathfrak{F} \models \alpha'\) immediately gives us the desired \(x\), with \(x \sim_0 w\).

Case 2: \(u \not\in D\).
Then \(\mathfrak{F}\) itself is the desired \(x\), as \(u \sim_0 v\) and \(u \sim_1 w\).

\((\Leftarrow)\) For the other direction, we assume that \(\mathfrak{F} \models \alpha\), we consider arbitrary \(u, v\) and \(w\) in \(\mathfrak{F}\) with \(u \not\in D\), \(u \sim_0 v \sim_1 w\) and \(v \neq w\), and set ourselves the task to find an \(x\) with \(x \not\in D\) and \(u \sim_1 x \sim_0 w\), viz. Figure 3.

Since \(\mathfrak{F} \models \alpha'\), there is a \(y \notin D\) with \(u \sim_1 y\) and \(y \sim_0 v\) or \(y \sim_0 w\). Distinguish

Case 1: \(y \sim_0 w\).
This means we are finished immediately: take \(x = y\).

Case 2: \(y \sim_0 v\).
Since \(\mathfrak{F} \models \mathcal{N}4\), there is a \(z\) in \(\mathfrak{F}\) with \(u \sim_1 z \sim_0 w\), as in Figure 4:
Distinguish

Case 2.1: $z \notin D$.
Again we are finished: take $x = z$.

Case 2.2: $z \in D$.
This implies $z = u$ because $\mathfrak{B} \models N6$, so we have the situation depicted in Figure 5. We now have $w \sim_0 z = u \sim_0 v \sim_0 y$, so $y \sim_0 w$ after all, and we are back in case 1: take $x = y$.

References


