ON THE INDEPENDENT AXIOMATIZABILITY OF MODAL AND INTERMEDIATE LOGICS

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§0. This paper gives a solution to an old problem connected with the efforts to describe the lattices of all normal modal and intermediate logics. The problem is as follows:

*Does every normal modal or intermediate logic have an independent set of axioms?*

For intermediate logics it was formulated by A. Tsytkin in Logic Notebook [1986, Problem 148].

A way to the negative solution to this problem is opened by the following observation of Kleyman [1983], which is presented here in a form suitable for our purpose:

**Lemma 1** Suppose a logic $L_1$ has an independent axiomatization. Then, for every finitely axiomatizable logic $L_2 \subset L_1$, the interval of logics $[L_2, L_1] = \{ L : L_2 \subseteq L \subseteq L_1 \}$ contains an immediate predecessor of $L_1$, that is a logic $L \subset L_1$ which has no extension lying properly between $L$ and $L_1$.

**Proof.** If $L_1$ is finitely axiomatizable then the existence of an immediate predecessor of $L_1$ in $[L_2, L_1]$ follows from Zorn’s Lemma.

Suppose now that $L_1$ has an infinite independent set of axioms $\{ \varphi_i : i \in \omega \}$. Since $L_2$ is a finitely axiomatizable sublogic of $L_1$, there is $n < \omega$ such that $L_2$ is contained in the logic with the axioms $\varphi_0, \ldots, \varphi_n$. Let $L_3$ be the logic with the axioms $\varphi_0, \ldots, \varphi_n, \varphi_{n+2}, \varphi_{n+3}, \ldots$. Since the set of $L_1$’s axioms is independent, $L_2 \subset L_3 \subset L_1$ and $\varphi_{n+1} \notin L_3$. And now again Zorn’s Lemma provides us with an immediate predecessor of $L_1$ in the interval $[L_3, L_1]$.

Thus, to prove that there is a logic without an independent axiomatization it suffices to produce a finitely axiomatizable logic $L_2$ and its proper extension $L_1$ having no immediate predecessor in the interval $[L_2, L_1]$.

A lattice (e.g. the lattice of extensions of a given logic) is called *strongly coatomically* if each its interval $[L_2, L_1]$ with $L_2 \subset L_1$ contains an immediate predecessor of $L_1$. Blok
[1980] proved that the lattice of normal modal logics is not strongly coatomic (more exactly, he showed that the dual lattice of varieties of modal algebras is not strongly atomic). However, it seems unlikely that in the interval \([L_2, L_1]\), constructed by Blok and containing no immediate predecessor of \(L_1\), the logic \(L_2\) is finitely axiomatizable; in any case its semantic definition involves the set of squares of natural numbers which can hardly be described by a finite set of axioms.

We will strengthen appropriately Blok's result to construct logics without independent axiomatizations lying above \(K4, S4, Grz\) and intuitionistic logic, answering incidentally his question concerning the strong coatomicity of the lattices of intermediate logics and modal logics containing \(S4\).

\(\S\, 1.\) We use standard notions and notations in the realm of non-classical logic. Here we mention only those of them that have variants.

We denote by \(\Box^+\varphi\), \(\Diamond^+\varphi\), \(\Box^n\varphi\) and \(\Diamond^n\varphi\) the formulas \(\varphi \land \Box \varphi\), \(\varphi \lor \Diamond \varphi\), \(\Box \ldots \Box \varphi\) and \(\Diamond \ldots \Diamond \varphi\), respectively; \(\varphi(\psi/p)\) means the result of replacement of all occurrences of the variable \(p\) in \(\varphi\) with \(\psi\).

All modal logics in this paper, except those in the final \(\S\), are assumed to be normal, i.e. containing \(K\) and closed under modus ponens, substitution and necessitation \(\varphi/\Box \varphi\). The smallest normal modal logic to contain a logic \(L\) and a set of formulas \(\Gamma\) is denoted by \(L \oplus \Gamma\). Intermediate logics are consistent extensions of intuitionistic logic \(Int\) closed under modus ponens and substitution. \(L + \Gamma\) means the closure of the set \(L \cup \Gamma\) under the latter two rules.

Let \(L\) be a logic and \(\Gamma, \Delta\) sets of formulas in the language of \(L\). \(\Gamma\) is said to be an independent set of axioms for \(L\) over \(\Delta\) if, for every \(\Sigma \subseteq \Gamma\), \(L\) is the closure of \(\Sigma \cup \Delta\) under the postulated inference rules of \(L\) iff \(\Sigma = \Gamma\). For instance, we can say about independent axiomatization of an intermediate logic over \(Int\) or that of a modal logic over \(K\). If \(\Gamma\) is an independent set of axioms for \(L\) over \(\Delta = \emptyset\) then \(\Gamma\) is called an (absolutely) independent set of axioms for \(L\). A logic \(L\) is independently axiomatizable (over \(\Delta\)) if there is an independent set of axioms for \(L\) (over \(\Delta\)).

It is clear that the following lemma holds.

**Lemma 2** If a logic \(L\) is independently axiomatizable over a finitely axiomatizable logic then \(L\) is absolutely independently axiomatizable.

As to our semantic apparatus, we use here differentiated general frames. Recall that a general frame \(\langle W, R \rangle\), where \(\bar{W} = \langle W, R \rangle\) is a Kripke frame and \(P\) a set of possible values in \(\bar{W}\), is differentiated if, for every two distinct points \(x, y \in W\), there is a set \(X \subseteq P\) such that \(x \in X\) and \(y \notin X\). For more information on general frames consult Goldblatt [1976], from which it follows in particular that every normal modal logic is characterized by a class of rooted differentiated general frames.

All our frames are assumed to be transitive. We will define them by drawing diagrams (directed graphs) in which reflexive and irreflexive points are denoted by \(\circ\) and \(\bullet\), respectively, and, for distinct points \(x\) and \(y\), \(x R y\) means that there is a directed path from \(x\)
to \( y \). We write \( x \mathcal{R} y \) if \( x \mathcal{R} y \) or \( x = y \). So \( \mathfrak{F} = (W, \mathcal{R}) \) is rooted if there is \( x \in W \) such that \( x \mathcal{R} y \) for every \( y \in W \); in this case \( x \) is called a root of \( \mathfrak{F} \).

§2. First we give a solution to the independent axiomatizability problem for modal logics containing \( K4 \). Though afterwards stronger results will be obtained, we prefer to begin with logics above \( K4 \) because in this case our construction is more transparent.

We require a number of modal formulas:

\[
\begin{align*}
\alpha &= p \land \neg \Box p, \quad \alpha' &= \alpha(\Box p/p), \quad \alpha'' = \alpha'(\Box p/p) = \alpha(\Box^2 p/p), \\
\alpha_i &= \alpha(\Box^i T/p), \quad \alpha_{i+1} = \alpha'(\Box^i T/p), \quad \alpha_{i+2} = \alpha''(\Box^i T/p), \\
\beta &= \Box \alpha \land \neg \Box^+ \alpha', \quad \beta' = \beta(\Box p/p), \\
\beta_i &= \beta(\Box^i T/p) = \Box \alpha_i \land \neg \Box^+ \alpha_{i+1}, \\
\beta_{i+1} &= \beta'(\Box^i T/p) = \Box \alpha_{i+1} \land \neg \Box^+ \alpha_{i+2}, \\
\gamma &= \Box \beta' \land \Box \alpha'' \land \neg \Box \beta, \quad \gamma' = \gamma(\Box p/p), \\
\gamma_{i+1} &= \gamma(\Box^i T/p) = \Box \alpha_{i+1} \land \Box \alpha_{i+2} \land \neg \Box \beta_i, \\
\gamma_{i+2} &= \gamma'(\Box^i T/p) = \Box \beta_{i+2} \land \Box \alpha_{i+3} \land \neg \Box \beta_{i+1} \quad (i \geq 0).
\end{align*}
\]

Define \( L_2 \) as

\[
L_2 = K4 \oplus \{ ax_1, ax_2, ax_3, ax_4, ax_5. \psi : \psi \in \{ \alpha, \beta, \gamma \} \},
\]

where

\[
\begin{align*}
ax_1 &= \alpha_0 \lor \Box^+ \alpha_1, \quad ax_2 = \gamma \rightarrow \Box \gamma, \quad ax_3 = \gamma \rightarrow \Box \gamma', \\
ax_4 &= \Box \beta' \land \Box \alpha'' \rightarrow \Box \gamma, \quad ax_5. \psi = \Box^+(q \rightarrow \neg \psi) \lor \Box^+(q \rightarrow \neg \psi).
\end{align*}
\]

It is not hard to verify that \( L_2 \) is consistent. Indeed, all its axioms are valid in the frame shown in Fig. 1 with empty \( V \).

Our first goal is to characterize the constitution of rooted differentiated frames for \( L_2 \). To this end we require the following substitution instances of its axioms:

\[
\begin{align*}
av_2.i &= \gamma_i \rightarrow \Box \gamma_i = ax_2(\Box^i T/p), \\
av_3.i &= \gamma_i \rightarrow \Box \gamma_{i+1} = ax_3(\Box^i T/p), \\
av_4.i &= \Box \beta_i \land \Box \alpha_{i+1} \rightarrow \Box \gamma_i = ax_4(\Box^i T/p) \quad (i \geq 1), \\
av_5.\alpha_i &= \Box^+(q \rightarrow \neg \alpha_i) \lor \Box^+(q \rightarrow \neg \alpha_i) = ax_5.\alpha(\Box^i T/p), \\
av_5.\beta_i &= \Box^+(q \rightarrow \neg \beta_i) \lor \Box^+(q \rightarrow \neg \beta_i) = ax_5.\beta(\Box^i T/p), \\
av_5.\gamma_{i+1} &= \Box^+(q \rightarrow \neg \gamma_{i+1}) \lor \Box^+(q \rightarrow \neg \gamma_{i+1}) = ax_5.\gamma(\Box^i T/p), \quad (i \geq 0).
\end{align*}
\]

For each \( n \geq 1 \), by \( \mathfrak{F}(n, V) \) we denote the rooted subframe of the frame in Fig. 1 generated by \( c_n \); \( \mathfrak{F}(1, V) \) is that frame itself. Here \( V \) is a (possibly empty) set of points which see all \( \alpha_i \)'s and are seen from all \( c_i \)'s (as it follows from the diagram, \( b_i \)'s do not see
points in $V$ and are not seen from them); the accessibility relation between points in $V$ is of no concern to us.

Observe that the points $a_i$, $b_{i+1}$, $c_{i+1}$, for $i \geq 0$, are characterized in $\mathcal{G}(1, V)$ by the formulas $\alpha_i$, $\beta_{i+1}$, $\gamma_{i+1}$, respectively, in the sense that under any valuation in $\mathcal{G}(1, V)$ we have:

$$
\{x : x \models \alpha_i \} = \{a_i\}, \quad \{x : x \models \beta_{i+1} \} = \{b_{i+1}\}, \quad \{x : x \models \gamma_{i+1} \} = \{c_{i+1}\}.
$$

And the points in $V$ are exactly those points in $\mathcal{G}(1, V)$ at which all $\Diamond \alpha_i$'s are true and all $\Diamond \beta_{i+1}$'s are false, for $i \geq 0$.

**Lemma 3** Suppose $\langle \mathcal{G}, P \rangle$ is a rooted differentiated frame for $L_2$. Then $\mathcal{G}$ is (isomorphic to) a rooted generated subframe of a frame of the form $\mathcal{G}(1, V)$, for some $V$, and $\{a_i\}, \{b_{i+1}\}, \{c_{i+1}\}$ are in $P$, for all $i \geq 0$.

**Proof.** Let $r$ be the root of $\mathcal{G}$. As it was done above, we classify the points in $\mathcal{G}$ according to which of the formulas $\alpha_i$, $\beta_i$ and $\gamma_i$ are true at them.

Say that a point $x$ in $\mathcal{G}$ is of type $a_i$ (respectively, $b_{i+1}$, $c_{i+1}$) if $\alpha_i$ (respectively, $\beta_{i+1}$, $\gamma_{i+1}$) is true at $x$; $x$ is of type $a_{i\omega}$ if $x \models \Diamond \alpha_i$ and $x \nvdash \Diamond \beta_j$, for all $i \geq 0$, $j \geq 1$.

Since $\langle \mathcal{G}, P \rangle \models ax5.\alpha_i$, $\mathcal{G}$ contains at most one point of type $a_i$, for each $i \geq 0$. Indeed, suppose there are two distinct points $x$, $y$ of type $a_i$. Since $\langle \mathcal{G}, P \rangle$ is differentiated, there is $X \in P$ such that $x \in X$ and $y \not\in X$. Define a valuation $\mathcal{V}$ in $\mathcal{G}$ by taking $\mathcal{V}(q) = X$. Then $r \nvdash ax5.\alpha_i$, which is a contradiction. Likewise, for each $i \geq 1$, there are at most one point of type $b_i$ and one point of type $c_i$.

By the definition of $\alpha_i$, each point $x$ of type $a_i$, if any, is irreflexive and must see a point of type $a_j$, for every $j < i$, and every point accessible from $x$ is of type $a_j$, for some $j < i$. Therefore, in view of their uniqueness, the points of type $a_i$, $i \geq 0$, form a descending chain in $\mathcal{G}$.
By $ax3.i$, each point of type $c_i$ for $i \geq 1$, if any, sees a point of type $c_j$, for every $j > i$, and, by the definition of $\gamma_i$, a point of type $a_j$, for every $j \geq 0$; besides, by $ax2.i$ and the uniqueness of points of type $c_i$, every such point is reflexive.

If some point $x$ in $\mathfrak{s}$ sees a point of type $a_i$ and neither sees a point of type $a_{i+1}$ nor is of type $a_{i+1}$ itself then, by the definition of $\beta_i$, $x$ is of type $b_i$. Besides, by $ax1$, $ax4.i$ and the properties of points of types $c_j$ and $a_j$ established above, every point accessible from $x$ is of one of the types $a_0, \ldots, a_i, b_i$. It follows in particular that $x$ is reflexive. For if $x$ is irreflexive then either it sees only points of types $a_0, \ldots, a_i$ and so is of type $a_{i+1}$ itself, contrary to our assumption, or sees a point of type $b_i$, contrary to the uniqueness of such a point.

It should be clear from the arguments above that each point in $\mathfrak{s}$ is of at most one type. We show now that each point in $\mathfrak{s}$ is of some type indeed.

Let $x$ be an arbitrary point in $\mathfrak{s}$. By $ax1$, among the points $y$ such that $x \not\rightarrow y$ there is at least one point of type $a_i$, for some $i \geq 0$. If $x$ sees only finitely many points of type $a_i$, $i \geq 0$, then, as was established above, $x$ is either of type $a_i$ or of type $b_i$, for some $i$. If $x$ sees points of type $a_i$ for all $i \geq 0$ then we have the following alternatives. First, $x$ sees no point of type $b_j$, for $j \geq 1$, which means that $x$ is of type $a_\omega$. Second, $x$ sees a point of type $b_j$, for some $j \geq 1$, and no point of type $b_k$, for $0 < k < j$, which means that $x$ of type $c_j$. We have exhausted all the possibilities, and so each point in $\mathfrak{s}$, in particular $r$, is of some unique type.

The isomorphism we are after is quite clear now: we map every point of type $a_i$ (respectively, $b_{i+1}$, $c_{i+1}$) to $a_i$ (respectively, $b_{i+1}$, $c_{i+1}$). The uniqueness of points of types $a_i$, $b_{i+1}$ and $c_{i+1}$ guarantees that $P$ satisfies the desirable condition. ♦

Now we are in a position to define $L_1$. Let $C_1$ be the class of all differentiated frames for $L_2$ whose underlying Kripke frames have the form shown in Fig. 2. Since $\mathfrak{s}(1, \emptyset) \models L_2$ and the frame in Fig. 2 with empty $V$ is a generated subframe of $\mathfrak{s}(1, \emptyset)$, $C_1 \neq \emptyset$. We define $L_1$ as the logic characterized by the class $C_1$, i.e. put

$$L_1 = \{ \varphi : \forall \mathfrak{s} \in C_1 \mathfrak{s} \models \varphi \}.$$

Observe that $L_2 \subseteq L_1$; moreover, this inclusion is proper, since $\neg \gamma_1 \in L_1 - L_2$.

**Lemma 4** $L_1$ has no immediate predecessor in the interval $[L_2, L_1]$. 

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Proof. Suppose otherwise. Let $L$ be an immediate predecessor of $L_1$ containing $L_2$. Since $L \subseteq L_1$, there exists a rooted differentiated frame $\langle g, Q \rangle$ such that $\langle g, Q \rangle \models L$ and $\langle g, Q \rangle \not\models L_1$. On the other hand, since $L_2 \subseteq L$, we have $\langle g, Q \rangle \models L_2$ and so, by Lemma 3, $\langle g, Q \rangle$ is of the form $\langle g(n, V), P \rangle$, for some $n \geq 1$, $V$ and $P$. Then $\neg \gamma_n \not\models L$; for, as we know, $c_n \models \gamma_n$.

Let $C'$ be the class of frames containing all the frames in $C_1$ and also the subframe of $\langle g(n, V), P \rangle$ generated by $c_{n+1}$, and let $L'$ be the logic characterized by $C'$. By the definition, $L \subseteq L' \subseteq L_1$. Moreover, $\langle g(n + 1, U), Q \rangle \models \neg \gamma_n$, for every $U$ and $Q$, from which $\neg \gamma_n \in L'$, and $c_{n+1} \models \gamma_{n+1}$, from which $\neg \gamma_{n+1} \not\models L'$, while $\neg \gamma_{n+1} \in L_1$. Therefore, $L \subseteq L' \subset L_1$, contrary to $L$ being an immediate predecessor of $L_1$. \( \square \)

As a consequence of Lemmas 1 and 4 and the fact that $L_2$ is finitely axiomatizable we obtain our main result:

**Theorem 5** $L_1$ has no independent axiomatization.

Remark. It is worth noting that $L_1$ is recursively axiomatizable. Indeed, using Lemma 3 one can readily prove that

$$L_1 = L_2 \oplus \{ \neg \gamma_i : i \geq 1 \}.$$  

§3. Now we show how to modify the construction above in order to obtain much stronger logics without independent axiomatizations. First we consider intermediate logics.

The construction in §2 was based upon the frame in Fig. 1 containing the descending chain $a_0, a_1, \ldots$ of irreflexive points. We replace it with "Fine's ladder" consisting of the pairs of reflexive points $a_0^1, a_0^2, a_1^1, a_1^2, \ldots$; see Fig. 3 where the points $a_{-1}^1$ and $a_{-1}^2$ play an auxiliary role (cf. Fine [1974, p.26]).
Since in the case under consideration variable free formulas are not expressive enough — there are only two of them (up to equivalence, of course), namely, \( \bot \) and \( \top \) — we shall use as a "starting formula" the following one:

\[
\delta = (p \rightarrow q \lor \neg q) \lor (\neg p \rightarrow q \lor \neg q).
\]

It is not hard to see that a rooted Kripke frame \( \mathcal{F} \) refutes \( \delta \) iff it contains a (not necessarily generated) subframe of the form shown in Fig. 4, with \( a \) and \( b \) having no common successors in \( \mathcal{F} \). Since the frame in Fig. 3 contains only one (modulo interchanging superscripts) subframe of that sort, without loss of generality we may assume that under any valuation refuting \( \delta \) in the frame we have:

\[
a_0^1 \models p, \ a_0^1 \not\models q \lor \neg q, \ a_{-1}^1 \models q,
\]

\[
a_0^2 \models \neg p, \ a_0^2 \not\models q \lor \neg q, \ a_{-1}^2 \models q.
\]

Now, taking the formulas

\[
\alpha_{-1}^1 = p \land q \rightarrow \bot, \ \alpha_{-1}^2 = \neg p \land q \rightarrow \bot,
\]

\[
\alpha_0^1 = p \rightarrow q \lor \neg q, \ \alpha_0^2 = \neg p \rightarrow q \lor \neg q,
\]

\[
\alpha_{i+1}^1 = \alpha_i^2 \lor \alpha_{-i-1}^2, \ \alpha_{i+1}^2 = \alpha_i^1 \lor \alpha_{-i-1}^1,
\]

\[
\beta_i = \alpha_{i+1}^1 \land \alpha_{i+1}^2 \rightarrow \alpha_i^1 \lor \alpha_i^2,
\]

\[
\gamma_{i+1} = \beta_i \rightarrow \beta_{i+1} \lor \alpha_{i+2}^1 \lor \alpha_{i+2}^2 \ (i \geq 0)
\]

we obtain, under a valuation refuting \( \delta \), a classification of points in the frame in Fig. 3 similar to that in \( \S 2 \):

\[
\{ x : x \not\models \alpha_i^1 \} = \{ a_i^1 \}, \ \{ x : x \not\models \alpha_i^2 \} = \{ a_i^2 \} \ (i \geq -1),
\]

\[
\{ x : x \not\models \beta_i \} = \begin{cases} \{ b_i \} & \text{if } i \geq 1 \\ \emptyset & \text{if } i = 0 \end{cases}, \ \{ x : x \not\models \gamma_{i+1} \} = \{ c_{i+1} \} \ (i \geq 0).
\]

Here \( x \not\models \varphi \rightarrow \psi \) means \( x \models \varphi \) and \( x \not\models \psi \).

\( L_2 \) can be defined by adding to \( \textbf{Int} \) the following axioms:

\[
\beta_0, \ \zeta_2 \rightarrow \zeta_1 \lor \delta, \ \zeta_1 \rightarrow \eta_1 \lor \xi_2 \lor \xi_2^\prime \lor \delta,
\]

\[
\begin{array}{c}
\text{c} \\
\downarrow \\
\text{a}
\end{array}
\quad
\begin{array}{c}
\text{d} \\
\downarrow \\
\text{b}
\end{array}
\]

Figure 4:
\[ \phi(\xi_0) \lor \xi_1 \lor \xi'_1, \ \phi(\xi_1) \lor \xi_1', \ \phi(\xi_2) \lor \xi_1', \ \phi(\eta_1), \ \phi(\xi_i), \]

where
\[ \xi_{-3} = r_1, \ \xi'_{-3} = r_2, \ \xi_{-2} = s_1, \ \xi'_{-2} = s_2, \]
\[ \xi_n = \xi_{n-1} \lor \xi_{n-2} \lor \xi'_{n-2}, \ \xi'_n = \xi_{n-1} \lor \xi_{n-2} \lor \xi_{n-2} \ (n \geq -1), \]
\[ \eta_n = \xi_{n+1} \land \xi'_{n+1} \rightarrow \eta_n \lor \xi'_n \ (n \geq 0), \]
\[ \zeta_n = \eta_{n-1} \rightarrow \eta_n \lor \xi_{n+1} \lor \xi'_{n+1} \ (n \geq 1) \]

and \( \phi(\varphi \rightarrow \psi) \) is an abbreviation for \( (t \land \varphi \rightarrow \psi) \lor (\varphi \rightarrow t \lor \psi) \lor \delta \). The meaning and purpose of the axioms above are analogous to those of the axioms in §2; namely, the first axiom is similar to \( ax1 \), the second one to \( ax3 \), the third to \( ax4 \), the forth, fifth and sixth axioms play the same role as \( ax5.\alpha \), the seventh is like \( ax5.\beta \) and the eighth is like \( ax5.\gamma \).

By using these axioms one can prove an analog of Lemma 3 which looks like this: if a rooted differentiated frame \( (\mathfrak{A}, P) \) for \( L_2 \) refutes \( \delta \) then \( \mathfrak{A} \) is isomorphic to a generated subframe of a frame of the form shown in Fig. 3, with the sets generated by each of the points \( a_j, b_k, c_k \), for \( i \in \{1, 2\}, j \geq -1, k \geq 1 \), belonging to \( P \). Now, by defining \( L_1 \) as the intermediate logic characterized by the class of all differentiated frames validating \( \delta \) and all differentiated frames for \( L_2 \) whose underlying Kripke frames have the form shown in Fig. 3, but with the points \( c_i \)'s removed, we obtain an analog of Lemma 4 for intermediate logics. Thus we arrive at

**Theorem 6** There is an intermediate logic without an independent axiomatization.

Lemma 4 (for intermediate logics) provides us with an interval \([L_2, L_1]\) of intermediate logics in which \( L_1 \) has no immediate predecessors. This result and the Blok–Esakia Theorem, according to which the lattices of varieties of pseudo–Boolean (alias Heyting) algebras and Grzegorczyk algebras are isomorphic, give a solution to the Blok’s [1980] problem:

**Theorem 7** (i) The lattice of varieties of pseudo–Boolean algebras is not strongly atomic.

(ii) The lattice of varieties of topological Boolean (and even Grzegorczyk) algebras is not strongly atomic.

§4. Now we consider the correlation between the independent axiomatizability of intermediate logics and normal modal logics above \( S_4 \). We remind the reader that there is a lattice homomorphism \( \rho \) from the lattice of normal extensions of \( S_4 \) onto the lattice of extensions of \( \text{Int} \) which is defined as follows: for every normal logic \( M \supset S_4 \),

\[ \rho M = \{ \varphi : T \varphi \in M \} \]

where \( T \) is the Gödel translation prefixing \( \square \) to every subformula of an intuitionistic formula. The logic \( M \) is called a modal companion of \( \rho M \). The set of all modal companions of an intermediate logic \( L = \text{Int} + \{ \varphi_i : i \in I \} \) forms the interval of logics \([\tau L, \sigma L] \), where

\[ \tau L = S_4 \oplus \{ T \varphi_i : i \in I \}, \]

\[ \sigma L = S_4 \oplus \{ T \varphi_i : i \in I \}, \]
\[ \sigma L = \tau L \oplus \text{Grz} = \tau L \oplus \Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p, \]

with \( \tau \) being an isomorphism between the intervals \([\text{Int}, \text{Cl}]\) and \([\text{S4}, \text{S5}]\) and \( \sigma \) the Blok–Esakia isomorphism between the lattices of extensions of \text{Int} and normal extensions of \text{Grz} mentioned at the end of §3. For more information on modal companions of intermediate logics and references consult Chagrov and Zakharyaschev [1992].

It follows immediately from these facts and Lemma 4 for intermediate logics that in the intervals \([\tau L_2, \tau L_1]\) and \([\sigma L_2, \sigma L_1]\), where \( L_1 \) and \( L_2 \) are the intermediate logics constructed in §3, the modal logics \( \tau L_1 \) and \( \sigma L_1 \) have no immediate predecessors, respectively. Thus we obtain

**Theorem 8** There are a normal modal logic in the interval \([\text{S4}, \text{S5}]\) and a normal logic containing \text{Grz} without independent axiomatizations.

**Remark.** It is not hard to modify the proof of Theorems 6 and 8 to construct a normal extension of the Gödel–Löb provability logic \text{GL} without an independent axiomatization.

Another consequence of the properties of \( \tau \) and \( \sigma \) mentioned above is

**Theorem 9** For every intermediate logic \( L \), the following conditions are equivalent:

- \( L \) is independently axiomatizable over \text{Int};
- \( \tau L \) is independently axiomatizable over \text{S4};
- \( \sigma L \) is independently axiomatizable over \text{Grz}.

The maps \( \rho \), \( \tau \) and \( \sigma \) can be characterized with the help of the apparatus of the modal and intuitionistic canonical formulas, which are denoted here by \( \alpha(\exists, \mathcal{D}, \bot) \) and \( \beta(\exists, \mathcal{D}, \bot) \), respectively; for a brief exposition and further references consult Zakharyaschev [1993]. Namely, a normal logic \( M \supseteq \text{S4} \) is a modal companion of an intermediate logic

\[ L = \text{Int} + \{\beta(\exists_i, \mathcal{D}_i, \bot) : i \in I\} \]

iff \( M \) can be represented in the form

\[ M = \text{S4} \oplus \{\alpha(\exists_i, \mathcal{D}_i, \bot) : i \in I\} \oplus \{\alpha(\emptyset_j, \emptyset_j, \bot) : j \in J\}, \]

where each \( \emptyset_j \), for \( j \in J \), contains at least one proper cluster; in particular,

\[ \tau L = \text{S4} \oplus \{\alpha(\exists_i, \mathcal{D}_i, \bot) : i \in I\}, \]

\[ \sigma L = \text{S4} \oplus \{\alpha(\exists_i, \mathcal{D}_i, \bot) : i \in I\} \oplus \alpha(\emptyset, \emptyset). \]

Here \( \emptyset \) is the two point cluster.

**Theorem 10** If an intermediate logic \( L \) has an infinite independent axiomatization over \text{Int} then every logic in the interval \([\tau L, \sigma L]\) is independently axiomatizable (over \text{S4}).
Proof. Suppose \( L = \textbf{Int} + \{ \varphi_i : i \in \omega \} \) with independent axioms \( \varphi_i \). According to the characterization above, every logic \( M \in [\tau L, \sigma L] \) can be represented as
\[
M = \textbf{S4} \oplus \{ T\varphi_i : i \in \omega \} \oplus \{ \alpha(\mathfrak{g}_i, \mathcal{D}_i, \bot) : i \in \omega \},
\]
where each \( \mathfrak{g}_i \), for \( i \in \omega \), contains a proper cluster. Therefore,
\[
M = \textbf{S4} \oplus \{ T\varphi_i \land \alpha(\mathfrak{g}_i, \mathcal{D}_i, \bot) : i \in \omega \}.
\]
The latter axiomatization is independent over \( \textbf{S4} \), for otherwise we would have, for some \( i \in \omega \)
\[
T\varphi_i \in M' = \textbf{S4} \oplus \{ T\varphi_j \land \alpha(\mathfrak{g}_j, \mathcal{D}_j, \bot) : j \in \omega, j \neq i \},
\]
and hence
\[
\varphi_i \in \rho M' = \textbf{Int} + \{ \varphi_j : j \in \omega, j \neq i \},
\]
which is a contradiction. By Lemma 2, \( M \) is absolutely independently axiomatizable. \( \text{\dag} \)

That \( L \) in Theorem 10 is infinitely independently axiomatizable over \( \textbf{Int} \) is essential. For, as is shown by the following theorem, \( \textbf{Int} \) itself has a modal companion without an independent axiomatization.

**Theorem 11** The interval \([\tau \textbf{Int}, \sigma \textbf{Int}] = [\textbf{S4}, \textbf{Grz}]\) contains a logic without an independent axiomatization.

Proof (a sketch). We point out how to change the proof of Theorem 5 in order to obtain a logic we need.

As a ”starting formula” \( \delta \), we take a modal formula which is refuted in a rooted Kripke frame \( \mathfrak{g} \) iff \( \mathfrak{g} \) contains a subframe shown in Fig. 4, \( a \) and \( b \) have no common successors in \( \mathfrak{g} \) and \( d \) (or \( c \)) is contained either in a proper cluster or in an infinite strictly ascending chain. Besides, in the frame in Fig. 3 we replace \( \alpha_{2^{-1}} \) with the two point cluster.

Then we construct a finite number of axioms for \( L_2 \) in such a way that Lemma 3 holds for every rooted differentiated frame for \( L_2 \) refuting \( \delta \). And \( L_1 \) is defined as the logic characterized by the class of all differentiated (reflexive) frames validating \( \delta \) and all differentiated frames for \( L_2 \) of the form shown in Fig. 3 with \( \alpha_{2^{-1}} \) replaced by the two point cluster and the points \( c_i, i \geq 1 \), removed. This class contains all the finite partially ordered frames (since all of them validate \( \delta \)) which means that \( \rho L_1 = \textbf{Int} \). The fact that \( L_1 \) has no independent axiomatization is proved in the same way as in §2 and §3. \( \text{\dag} \)

That the property of independent axiomatizability is not in general preserved while passing from an intermediate logic to its arbitrary modal companion can hardly be regarded as a great surprise. Many other properties (such as the decidability, finite model property, Kripke completeness, etc.) behave in this respect in the same way. What is rather unexpected is that unlike the other ”good” properties of logics (at least those known to us) the independent axiomatizability is not in general preserved under the map \( \rho \).
Theorem 12 There is an independently axiomatizable normal modal logic \( M \supset S_4 \) such that \( \rho M \) does not have an independent axiomatization.

Proof. We are going to construct an independently axiomatizable modal logic \( M \) such that \( \rho M = L_1 \), where \( L_1 \) is the intermediate logic without an independent axiomatization constructed in the proof of Theorem 6. By the definition of \( L_1 \), each subframe \( \mathfrak{b}_i \) of the frame in Fig. 3 generated by \( b_i \), for \( i \in \omega \), validates \( L_1 \), and so each frame \( \mathfrak{z}_i \), which is obtained from \( \mathfrak{b}_i \) by replacing \( b_i \) with the two point cluster is a frame for \( \tau L_1 \). For \( i \in \omega \), we denote by \( \beta_i^* \) the formula

\[
T(\alpha_{i+1}^* \land \alpha_{i+2}^*) \rightarrow T(\alpha_i^* \lor \alpha_{i+2}^*) \lor (\Delta(\square(r \rightarrow \square r) \rightarrow r)) \rightarrow r),
\]

where \( \alpha_i^* \)'s are taken from the proof of Theorem 6. It is not hard to verify that \( \mathfrak{z}_i \not\models \beta_i^* \) and \( \mathfrak{z}_j \models \beta_i^* \), for every \( j \neq i \). Therefore, the set \( \{ \beta_i^* : i \in \omega \} \) is independent over \( \tau L_1 \).

Let \( \{ \varphi_i : i \in \omega \} \) be a set of axioms for \( L_1 \) over \( \text{Int} \). Then, by defining \( M \) as

\[
S_4 \oplus \{ T(\varphi_i) : i \in \omega \} \oplus \{ \beta_i^* : i \in \omega \},
\]

we clearly have \( \tau L \subset M \subset \sigma L \), with

\[
S_4 \oplus \{ T(\varphi_i) \land \beta_i^* : i \in \omega \}
\]

being an independent axiomatization of \( M \). \( \dashv \)

Remark. It may be of interest that it is impossible to extract an independent set of axioms for \( M \) from the axiomatization (1). By using the logic \( L_1 \) constructed in the proof of Theorem 6, it is not difficult to construct an intermediate logic with the same property.

§5. We conclude the paper with some questions to which we could not find answers.

The first three questions concern the difference between absolutely independent axiomatizability and independent axiomatizability over a finitely axiomatizable logic.

- Is an absolutely independently axiomatizable logic \( L_1 \) containing a finitely axiomatizable logic \( L_2 \) is independently axiomatizable over \( L_2 \)?

- Does the conversion of Lemma 1 hold?

- Do Theorems 9 and 10 hold for the case of absolutely independent axiomatizability?

Our forth question is connected with that there are two ways of axiomatizing modal logics, namely, with the rule of necessitation and without it. The results above establish the existence of modal logics having no independent axiomatizations only of the former kind. In the proof of Theorem 5 the rule of necessitation was used together with the formulas \( ax3.i \), which can be rewritten as \( \square \neg \gamma_i+1 \rightarrow \neg \gamma_i \), to ensure that \( \neg \gamma_i \) is in an extension of \( L_2 \) whenever \( \neg \gamma_j \) belongs to it, for some \( j > i \). Without this rule the set \( \{ \neg \gamma_i : i \geq 1 \} \) is independent over \( L_2 \), and it is not hard to show that \( L_1 = L_2 + \{ \square^+ \neg \gamma_i : i \geq 1 \} \). In the proof of Theorem 8 we used the Blok–Esakia isomorphism between the lattices of intermediate logics and normal extensions of \( Grz \), with the condition of normality being essential here (for details see Chagrov and Zakharyaschev [1992]).
• Do there exist modal logics having no independent axiomatizations without the postulated rule of necessitation?

One can show, using the mystical part $V$ of the frames in Fig. 1 and 3 that all the logics without independent axiomatizations above have rooted frames of infinite width and depth. Besides, the frames in Fig. 1 and 3 are closely related to the frame which was used by Fine [1974] for constructing an incomplete modal logic. So our three final questions are:

• Do there exist Kripke complete (modal or intermediate) logics without an independent axiomatizations?

• Do there exist (modal or intermediate) logics without an independent axiomatizations but with the finite model property?

• Do there exist (modal or intermediate) logics of finite width or finite depth without an independent axiomatizations?

(As to the last question, our conjecture is that such logics do not exist.)

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