THE DECIDABILITY OF DEPENDENCY IN INTUITIONISTIC PROPOSITIONAL LOGIC

L.A. Chagrova
Dick de Jongh

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Correction to page 6.

The D-formulae given are incorrect. The following simply replaces all of pages 6 and 7 but the bibliography. To get a generalization of Theorem 1.1 it suffices to ascribe D-formulae to the T-models \( (K_{n+1}^1)^+ \) and \( (K_{n+1}^1)^+ \). These could be found by applying a general method (Jankov 1968, de Jongh 1970), but the following formulae are nicer:

For \( (K_{n+1}^1)^+ \): \( g_{n+3}(q) \land (p \rightarrow f_{n+1}(q)) \rightarrow p := h_n(p, q) \),

for \( (K_{n+1}^1)^+ \): \( g_{n+3}(q) \land (p \leftrightarrow f_{n+1}(q)) \rightarrow p := k_n(p, q) \),

with some simpler degenerate cases for the lower numbers:

\[ \neg q \land (p \rightarrow q) \rightarrow p, \neg q \land (p \leftrightarrow q) \rightarrow p, \neg q \land \neg p \rightarrow p, \neg (q \land \neg p). \]

Here \( g_n(q) \) and \( f_n(q) \) are such that, for any 1-variable T-model \( L, L \vdash g_n(q) \) iff \( L \ll K_n \), \( L \vdash f_n(q) \) iff \( L \ll K_n \) or \( L \ll K_{n+1} \). The relevant properties of the \( g_n(q) \) and \( f_n(q) \) are:

\[ \vdash_{IPC} f_{n+1}(q) \rightarrow g_n(q) \lor g_{n+1}(q) \]

\[ \vdash_{IPC} g_{n+3}(q) \land (g_{n+2}(q) \rightarrow f_{n+1}(q)) \text{ and hence } \vdash g_{n+3}(q) \land (g_{n+2}(q) \rightarrow g_n(q) \lor g_{n+1}(q)). \]

3.1 Theorem. If for no \( n \in N \), \( \vdash_{IPC} g_{n+3}(B) \land ((A \rightarrow f_{n+1}(B)) \rightarrow A) \rightarrow A \) or

\[ \vdash_{IPC} g_{n+3}(B) \land (A \leftrightarrow f_{n+1}(B)) \rightarrow A, \text{ or } \vdash_{IPC} g_{n+3}(A) \land ((B \rightarrow f_{n+1}(A)) \rightarrow B) \rightarrow B \text{ or } \]

\[ \vdash_{IPC} g_{n+3}(A) \land (B \leftrightarrow f_{n+1}(A)) \rightarrow B, \text{ or one of the above degenerate cases is provable in } \]

\( IPC \) for \( A, B \), then \( A \) and \( B \) are independent over \( IPC \).

It is to be noted that just as Theorem 1.1 this theorem immediately applies to HA, since rooting the models is applicable in the case of HA by adjoining the standard model \( N \) to the new root (see Smoryński, 1973). That Theorem 3.1 is in a sense best possible can be demonstrated by showing that \( h_n(p, q) \) and \( k_n(p, q) \) are exactly provable. (Again this then applies to HA as well, now by the uniform version of the arithmetic completeness of \( IPC \) over HA, see Smoryński, 1973.)

3.2 Theorem. (a) The formula \( h_n(p, q) \) is exactly provable for

\( (g_{n+3}(q) \land ((p \rightarrow f_{n+1}(q)) \rightarrow p)) \land p \land q. \)

(b) The formula \( k_n(p, q) \) is exactly provable for

\( (g_{n+3}(q) \land (p \leftrightarrow f_{n+1}(q))) \land p \land q. \)

Proof. (a) Actually, we will show in general that \( C \land ((p \rightarrow D) \rightarrow p) \rightarrow p \) where \( C \) and \( D \)

do not contain \( p \) is exactly provable with the substitution \( (C \land ((p \rightarrow D) \rightarrow p)) \land p \) for \( p \)

and the identity for the other variables.
We first show that the required formula is actually provable. We apply the easily verified IPC-equivalence of \( A \rightarrow B \) to \(((A \rightarrow B) \rightarrow A) \rightarrow B\). Let us write \( p^* \) for \((C \land ((p \rightarrow D) \rightarrow p)) \lor p\). Then \( p^* \rightarrow D \) is equivalent to \((C \land ((p \rightarrow D) \rightarrow p) \rightarrow D) \lor (p \rightarrow D)\) and hence to \( p \rightarrow D\).

Therefore, \((p^* \rightarrow D) \rightarrow p^*\) is equivalent to \((p \rightarrow D) \rightarrow (C \land ((p \rightarrow D) \rightarrow p)) \lor p\) and hence implies \((p \rightarrow D) \rightarrow p\). Thus, \((C \land ((p^* \rightarrow D) \rightarrow p^*))\) implies \( p^*\).

Next we have to show that no stronger formulae are provable. For that it is sufficient to note that in any Kripke model validating \( C \land ((p \rightarrow D) \rightarrow p) \rightarrow p\) changing the valuation of \( p\) to that of \((C \land ((p \rightarrow D) \rightarrow p)) \lor p\) will leave all forcing relations as they are. This is obvious, because in any such Kripke model \((C \land ((p \rightarrow D) \rightarrow p)) \lor p\) is actually equivalent to \( p\).

(b) We will show in general that \( C \land (p \leftrightarrow D) \rightarrow p\) where \( C \) and \( D \) do not contain \( p\) is exactly provable with the substitution \((C \land (p \leftrightarrow D)) \lor p\) for \( p\) and the identity for the other variables. Obviously, the second part of the proof is the same as in (a), so it is sufficient to show that the relevant formula is provable. Let us write \( p^* \) for \((C \land (p \leftrightarrow D)) \lor p\). Then \( p^* \leftrightarrow D \) implies \( p \rightarrow D\) as well as \( D \rightarrow (C \land (p \leftrightarrow D)) \lor p\). The latter implies \( D \rightarrow p\). So, \((C \land (p^* \leftrightarrow D))\) implies \( p^*\).
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The D-formulae given are incorrect. The following simply replaces all of pages 6 and 7 but the bibliography. To get a generalization of Theorem 1.1 it suffices to ascribe D-formulae to the T-models \((K^1_n)^*\) and \((K^1_{n'} K^1_{n+1})^*\). These could be found by applying a general method (Jankov 1968, de Jongh 1970), but the following formulae are nicer:

For \((K^1_{n+3})^*\): \(g_{n+3}(q) \land ((p \rightarrow f_{n+1}(q)) \rightarrow p) \rightarrow p := h_n(p, q)\),
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3.1 Theorem. If for no \(n \in N\), \(\vdash IPC g_{n+3}(B) \land ((A \rightarrow f_{n+1}(B)) \rightarrow A) \rightarrow A\) or
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\[\vdash IPC g_{n+3}(A) \land (B \leftrightarrow f_{n+1}(A)) \rightarrow B, \text{ or one of the above degenerate cases is provable in IPC for } A, B\], then \(A\) and \(B\) are independent over IPC.

It is to be noted that just as Theorem 1.1 this theorem immediately applies to HA, since rooting the models is applicable in the case of HA by adjoining the standard model \(N\) to the new root (see Smoryński, 1973). That Theorem 3.1 is in a sense best possible can be demonstrated by showing that \(h_n(p, q)\) and \(k_n(p, q)\) are exactly provable. (Again this then applies to HA as well, now by the uniform version of the arithmetic completeness of IPC over HA, see Smoryński, 1973.)

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(b) The formula \(k_n(p, q)\) is exactly provable for
\((g_{n+3}(q) \land (p \leftarrow f_{n+1}(q))) \lor p\) and \(q\).

Proof. (a) Actually, we will show in general that \(C \land ((p \rightarrow D) \rightarrow p) \rightarrow p\) where \(C\) and \(D\) do not contain \(p\) is exactly provable with the substitution \((C \land ((p \rightarrow D) \rightarrow p)) \lor p\) for \(p\) and the identity for the other variables.
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Therefore, \((p^* \rightarrow D) \rightarrow p^*\) is equivalent to \((p \rightarrow D) \rightarrow (C \land ((p \rightarrow D) \rightarrow p)) \lor p\) and hence implies \((p \rightarrow D) \rightarrow p\). Thus, \(C \land ((p^* \rightarrow D) \rightarrow p^*)\) implies \(p^*\).

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3.1 Theorem. If for no \(n \in \mathbb{N}\), \(\vdash \text{IPC} g_{n+3}(B) \land ((A \rightarrow f_{n+1}(B)) \rightarrow A) \rightarrow A\) or
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\vdash \text{IPC} g_{n+3}(A) \land B \vdash f_{n+1}(A) \rightarrow B, \text{ or one of the above degenerate cases is provable in IPC for } A, B, \text{ then } A \text{ and } B \text{ are independent over IPC.}
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3.2 Theorem. (a) The formula \(h_n(p, q)\) is exactly provable for
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(g_{n+3}(q) \land (p \rightarrow f_{n+1}(q)) \rightarrow p) \lor p \text{ and } q.
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Therefore, $(p^* \to D) \to p^*$ is equivalent to $(p \to D) \to (C \land ((p \to D) \to p)) \lor p$ and hence implies $(p \to D) \to p$. Thus, $C \land ((p^* \to D) \to p^*)$ implies $p^*$.

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For $(K_n^1, K_{n+1}^1)^+$: $g_{n+3}(q) \land ((p \rightarrow f_{n+1}(q)) \rightarrow p) := h_n(p, q),$

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$\neg \neg q \land (p \rightarrow q) \rightarrow p$, $\neg \neg q \land (p \leftrightarrow q) \rightarrow p$, $\neg q \land \neg p \rightarrow p$, $\neg (\neg q \land \neg p)$.

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It is to be noted that just as Theorem 1.1 this theorem immediately applies to HA, since rooting the models is applicable in the case of HA by adjoining the standard model $\mathbb{N}$ to the new root (see Smoryński, 1973). That Theorem 3.1 is in a sense best possible can be demonstrated by showing that $h_n(p, q)$ and $k_n(p, q)$ are exactly provable. (Again this then applies to HA as well, now by the uniform version of the arithmetic completeness of IPC over HA, see Smoryński, 1973.)

3.2 Theorem. (a) The formula $h_n(p, q)$ is exactly provable for

$(g_{n+3}(q) \land ((p \rightarrow f_{n+1}(q)) \rightarrow p)) \land p$ and $q$.

(b) The formula $k_n(p, q)$ is exactly provable for

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Therefore, \( (p^* \rightarrow D) \rightarrow p^* \) is equivalent to \( (p \rightarrow D) \rightarrow (C \land ((p \rightarrow D) \rightarrow p)) \lor p \) and hence implies \( (p \rightarrow D) \rightarrow p \). Thus, \( C \land ((p^* \rightarrow D) \rightarrow p^*) \) implies \( p^* \).
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Proof. (a) Actually, we will show in general that \(C \land ((p \rightarrow D) \rightarrow p) \rightarrow p\) where \(C\) and \(D\) do not contain \(p\) is exactly provable with the substitution \((C \land ((p \rightarrow D) \rightarrow p)) \lor p\) for \(p\) and the identity for the other variables.
We first show that the required formula is actually provable. We apply the easily verified IPC-equivalence of \( A \rightarrow B \) to \( ((A \rightarrow B) \rightarrow A) \rightarrow B \). Let us write \( p^* \) for \( (C \wedge ((p \rightarrow D) \rightarrow p)) \vee p \). Then \( p^* \rightarrow D \) is equivalent to \((C \wedge ((p \rightarrow D) \rightarrow p) \rightarrow D) \wedge (p \rightarrow D)\) and hence to \( p \rightarrow D \).

Therefore, \((p^* \rightarrow D) \rightarrow p^*\) is equivalent to \((p \rightarrow D) \rightarrow (C \wedge ((p \rightarrow D) \rightarrow p)) \vee p\) and hence implies \( p \rightarrow D \rightarrow p \). Thus, \((p^* \rightarrow D) \rightarrow p^*\) implies \( p^* \).

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Therefore, $(p^* \rightarrow D) \rightarrow p^*$ is equivalent to $(p \rightarrow D) \rightarrow (C \wedge ((p \rightarrow D) \rightarrow p)) \vee p$ and hence implies $(p \rightarrow D) \rightarrow p$. Thus, $C \wedge ((p^* \rightarrow D) \rightarrow p^*)$ implies $p^*$.

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with some simpler degenerate cases for the lower numbers:
\(-q \land (p \to q) \to p, -q \land (p \leftrightarrow q) \to p, -q \land -p \to p, -(-q \land -p)\).

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INTUITIONISTIC PROPOSITIONAL LOGIC

L.A. Chagrova
Tver State University
Zhelyabova Str. 33
Tver, Russia 170013
Dick de Jongh
Department of Mathematics and Computer Science
University of Amsterdam
Abstract. A definition is given for formulae $A_1, \ldots, A_n$ in some theory $T$ which is formalized in a propositional calculus $S$ to be (in)dependent with respect to $S$. It is shown that, for intuitionistic propositional logic $IPC$, dependency (with respect to $IPC$ itself) is decidable. This is an almost immediate consequence of Pitts' uniform interpolation theorem for $IPC$. A reasonably simple infinite sequence of $IPC$-formulae $F_n(p, q)$ is given such that $IPC$-formulae $A$ and $B$ are dependent if and only if at least one of the $F_n(A, B)$ is provable.

1. Introduction. We denote the intuitionistic propositional calculus by $IPC$. Let us call formulae $A_1, \ldots, A_n$ of some intuitionistic theory $T$ $IPC$-dependent over $T$, or dependent over $T$ for short, if, for some $IPC$-formula $F(p_1, \ldots, p_n), \vdash_T F(A_1, \ldots, A_n)$, but $\not\vdash_{IPC} F(p_1, \ldots, p_n)$. Otherwise $A_1, \ldots, A_n$ are called independent. In de Jongh (1982) the behavior of formulae of one propositional variable in intuitionistic arithmetic $HA$ was discussed. The main result of that paper was that for arithmetic sentences $A$, if $\not\vdash_{HA} \neg\neg A \rightarrow A$ and $\not\vdash_{HA} \neg\neg A$, then $A$ is independent over $HA$ with respect to $IPC$. This result was generalized to formulae. We did not mention the fact that the result applies to the propositional calculus itself as well.

1.1 Theorem. If $\not\vdash_{IPC} \neg\neg A \rightarrow A$ and $\not\vdash_{IPC} \neg\neg A$, then $A$ is independent over $IPC$.

In fact, the proof in §2 of the article mentioned above applies immediately to this case. Naturally, for $IPC$ there is no immediate reason to look for a more constructive proof, as we did for $HA$ in the major part of that paper. A fortiori of course, the result implies that dependency is decidable for the one variable case: it can be checked whether an arbitrary formula $A$ is dependent by checking whether $\neg\neg A \rightarrow A$ or $\neg\neg A$ is provable. We call theorem 1.1 a minimal provability result: if anything non-trivial propositional is provable about $A$, $\neg\neg A \rightarrow A$ or $\neg\neg A$ is. The result leads to a characterization of the monadic propositional functions $F$ for which there exist $A$ such that exactly $\vdash F(A)$. This result holds for $HA$ as well as for $IPC$. To remind the reader of the definition for $n$ propositional variables:

1.2 Definition.

Exactly $\vdash F(A_1, \ldots, A_n)$ iff $\vdash F(A_1, \ldots, A_n)$ and, for all propositional $G$, $\vdash G(A_1, \ldots, A_n) \Rightarrow \vdash F(p_1, \ldots, p_n) \rightarrow G(p_1, \ldots, p_n)$.

This leads to the following classification of formulas of one propositional variable in $HA$ as well as in $IPC$. 

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1.3 Theorem. To each formula exactly one of the following cases applies (non-constructively, of course, in the case of HA),

(I) exactly ⊨ A

(II) exactly ⊨ ¬A

(III) exactly ⊨ ¬¬A

(IV) exactly ⊨ ¬¬A→A

(V) exactly ⊨ A→A (A is independent)

Examples exhibiting the five cases in IPC are respectively, (I) p→p, (II) p∧¬p, (III) ¬¬p→p, (IV) ¬p, (V) p.

In this note we will show that for n variables the decidability of dependency is a consequence of Pitts' uniform interpolation theorem (Pitts, 1992). Moreover, we will give an analogue for two propositional variables of theorem 1.1. A general analogue for theorem 1.3 seems much harder (see de Jongh-Visser, 1993, however, for some results). We will not go into that here except for remarking that in the case of arithmetic there are easy analogues of theorem 1.3 for restricted cases, e.g. if one restricts oneself to $\Pi^0_1$-sentences. In the monadic case we have for $\Pi^0_1$-sentences A, exactly ⊨ A or exactly ⊨ ¬A, or exactly ⊨ ¬¬A→A (i.e., in particular, an $\Pi^0_1$-sentence is never an independent one). In the binary case, if not ⊨ A, ⊨ ¬A, ⊨ B or ⊨ ¬B, then exactly

⊨ (¬¬A→A)∧(¬¬B→B) or

⊨ (A→B)∧(¬¬A→A)∧(¬¬B→B) or

⊨ (B→A)∧(¬¬A→A)∧(¬¬B→B) or

⊨ (A↔B)∧(¬¬A→A)∧(¬¬B→B). The only non-trivial relationship between $\Pi^0_1$-sentences is apparently the one of implication, as it is in classical arithmetic. We thank Albert Visser for many discussions on the subject.

2. Decidability of dependency over IPC. Pitts (1992) proved, among other things, that, for any IPC-formula $A(\overline{p}, \overline{r})$, there is a formula $\exists \overline{p} A(\overline{p}, \overline{r})$ such that, for any formula $B(\overline{p})$, $A(\overline{p}, \overline{r})\vdash_{IPC} B(\overline{p}) \iff \exists \overline{p} A(\overline{p}, \overline{r})\vdash_{IPC} B(\overline{p})$.

Consider the formulae $A_1, \ldots, A_n$ in the variables $\overline{r}$. From Pitts' Theorem it follows that

⊨ (A_1←p_1)∧\ldots∧(A_n←p_n)→B(\overline{p}) \iff \exists \overline{p} ((A_1←p_1)∧\ldots∧(A_n←p_n))→B(\overline{p})

On the other hand, ⊨ B(\overline{A}) \iff ⊨ (A_1←p_1)∧\ldots∧(A_n←p_n)→B(\overline{p}). Hence

$\exists \overline{p} ((A_1←p_1)∧\ldots∧(A_n←p_n))$ axiomatizes the propositional theory of $A_1, \ldots, A_n$. In consequence, $A_1, \ldots, A_n$ is independent \iff ⊨ $\exists \overline{p} ((A_1←p_1)∧\ldots∧(A_n←p_n))$, and the latter is decidable.
One may be of the opinion that A and B would more properly be defined to be dependent if, for some \( F, \vdash F(A, B) \) and, for no \( G, H \) such that \( \vdash G(A) \) and \( \vdash H(B) \), \( G(p), H(q) \vdash F(p, q) \). With this alternative definition e.g., \( \neg p \) and \( q \) would be independent, while under definition 1.2 they are dependent, since \( \neg \neg \neg \neg p \) is provable. (V. Shavrukov suggested this alternative to us.) It will be clear that decidability follows from the above proof for the alternative definition just as well. It seems to us that both definitions describe relevant concepts.

It is clear, of course, that, for any propositional logic \( S \) for which a uniform interpolation theorem holds, dependency is decidable for the logic itself. In fact the uniform interpolation theorem has been proved for the provability logic \( L \) by Shavrukov (1993), completely independently of the result by Pitts and by a completely different proof. Hence, dependency is decidable for \( L \). Unfortunately, for most modal logics not even a standard interpolation theorem holds (see e.g. Maksimova, 1982), so, for many logics a completely different method will have to be found if one wants to study the problem.

3. A minimal provability result for two variables. We first recall some facts about Kripke models for intuitionistic propositional logic.

(i) For each \( A \) of \( IPC \), if \( \not\vdash IPC A \), there is a finite tree-ordered Kripke model \( K=\langle W, \leq, \models \rangle \) such that \( K \not\models A \). (There is no essential reason to restrict oneself to tree-ordered Kripke models, but these are more easily described.)

(ii) We write \( w \uparrow \) for \( \{ w' \in W \mid w \leq w' \} \). The model \( K \) restricted to \( w \uparrow \) is called a generated submodel of \( K \).

(iii) A \( p \)-morphism from a Kripke model \( K=\langle W, \leq, \models \rangle \) to a Kripke model \( K'=\langle W', \leq, \models' \rangle \) is a surjection \( \phi: W \rightarrow W' \) such that:

\[(a) \ w \leq w' \Rightarrow \phi(w) \leq \phi(w') \]
\[(b) \ \phi(w) \leq \phi(w') \Rightarrow \exists w'' \geq w' (\phi(w'') = \phi(w)) \]
\[(c) \text{for all } w \in W \text{ and all propositional variables } p, \phi(w) \models p \iff w \uparrow p \]

It is easily shown then that \( (c) \) applies to all formulas.

(iv) A finite tree-ordered Kripke model is called irreducible if all its \( p \)-morhc images to tree-ordered Kripke models are isomorphic. We will call such a model here \( T \)-model for short.

(v) (Jankov 1968, de Jongh 1970) For each \( T \)-model \( K \) there are formulas \( C_K \) and \( D_K \) such that

\[(a) \ K \vdash C_K, K \not\vdash D_K \]
\[(b) \text{For each } T \text{-model } L \text{ such that } L \vdash C_K, L \text{ is isomorphic to a generated submodel of } K (L \leq K) \]
\[(c) \text{For each } T \text{-model } L \text{ such that } L \not\vdash D_K, K \leq L. \]
(vi) If we consider a Kripke model for the language consisting of the two propositional variables p and q, the values of p and q at the root of a model K are respectively i and j and the generated submodels corresponding to the immediate successors of the root are $L_1, \ldots, L_n$, then we denote K by

Each T-model with a domain of more than one element has such a form with $L_1, \ldots, L_n$ irreducible and none of the $L_j$ isomorphic to a generated submodel of any of the others. In case $n=1$, the root of $L_1$ has a forcing relation distinct from ij. All finite T-models can be obtained from the four irreducible p-q-models with one-element domains: 00, 11, 10, 11 by repeatedly adjoining roots with proper valuations to finite sets of $\prec$-incomparable T-models already obtained.

Let us now suppose that we have that $\not\prec D_K(A_1, A_2)$. Then in a sense, the model K is "available" for $A_1, A_2$, because any counter-model to $D_K(A_1, A_2)$ (and such a model has to exist) has to contain K in its valuations for $A_1, A_2$. Any counter-examples to formulas which can be given on K or its generated submodels then give rise to underviable formulas as well.

If we have a finite set of $D_L$'s which are not derivable for $A_1, A_2$, then we may also construct models by taking the set of the L's and adjoining a root below them. If it happens to be the case that the forcing on the root is automatically 00, then the model thus obtained is a model that gives rise to underviable formulas in its turn. This is so, if among the old roots at least one value 00 occurs or if both 10 and 01 occur. More exactly, if such a case applies and the model K arises in the construction from the models $L_1, \ldots, L_n$ and $D_{L_1}(A_1, A_2), \ldots, D_{L_n}(A_1, A_2)$, are not derivable in IPC, then neither is $D_K(A_1, A_2)$. In this case we will denote the newly obtained model by $(L_1, \ldots, L_n)^+$ and say that $(L_1, \ldots, L_n)^+$ has been obtained by rooting from $L_1, \ldots, L_n$.

Let us recall the one-variable case.

\[ K_0 = 1 \quad K_1 = 0 \]

\[ K_2 = \uparrow \]

\[ 0 \]
And, in general for any \( n \geq 0 \), \( K_{n+3} = \)

\[
\begin{array}{c}
K_{n+1} \\
\downarrow \\
W \\
\downarrow \\
K_n
\end{array}
\]

This means that all \( K_n \) can be constructed from \( X = \{K_0, K_1, K_2\} \) by the second method of rooting models. Also, \( K_1 \) is a generated submodel of \( K_2 \). The proof of theorem 1.1 is then actually contained in the above sketch, but then applied to the 1-variable case.

In the 2-variable case a set \( X \) of Kripke-models which is sufficient for the construction of all models is the set of all T-models with 00 only occurring at the root. All T-models with 00 occurring at the root can be obtained from \( X \) by repeatedly rooting models, and all other T-models are generated submodels of models in this set.

A simpler such set, however, is the following set \( X^* \):

Let us denote by \( K_{n,i}^1 \), \( K_n \) preceded by 1 everywhere, i.e. \( K_n \) with 0 replaced by 10 and 1 by 11. Similarly \( K_n^2 \) will denote \( K_n \) followed by 1 everywhere, i.e. \( K_n \) with 0 replaced by 01 and 1 by 11.

Now take \( X^* = \{ K_{n,i}^1 | n \in \mathbb{N}, i = 1, 2 \} \cup \{ K_n | n \in \mathbb{N}, i = 1 \} \cup \{ K_{n+1}^i | n \in \mathbb{N}, i = 1, 2 \} \)

To show that this set suffices it is sufficient to generate the original set \( X \) from \( X^* \) by taking generated submodels and rooting them. Take an arbitrary member

\[
L_i \quad L_n \quad i \quad j
\]

of \( X \). If a root of one of the \( L_i \) is 11, then that \( L_i = K_0^1 = K_1^1 \).

In general, any T-model with a root 11, 10 or 01 is a \( K_n^1 \). If among the \( L_i \) no root 01 occurs, then all the \( L_i \)'s are \( K_n^1 \)'s and we actually have one of the cases \( (K_n^1)^+ \) or \( (K_n^1, K_{n+1}^1)^+ \), similarly, if no root 10 occurs. If both 01 and 10 occur on the roots of the \( L_i \), then \( ij = 00 \) is forced and the model is obtained by rooting the \( L_i \), and the \( L_i \) themselves are generated submodels of models \( (K_n^1)^+ \) and \( (K_n^1, K_{n+1}^1)^+ \).
Now to get a generalization of Theorem 1.1 it suffices to ascribe D-formulae to the T-models \((K_n^1)^+\) and \((K_{n+1}^1)^+\). These could be found by applying a general method (Jankov 1968, de Jongh 1970), but the following formulae are nicer:

for \((K_n^1)^+\): \(g_{n+2}(q) \land ((p \rightarrow f_{n+2}(q)) \rightarrow p) \rightarrow p := h_n(p, q)\)

for \((K_{n+1}^1)^+\): \(g_{n+3}(q) \land ((p \rightarrow g_{n+1}(q)) \rightarrow p) \land ((p \rightarrow g_{n+2}(q)) \rightarrow p) \rightarrow p := k_n(p, q)\).

3.1 Theorem. If for no \(n \in \mathbb{N}\), \(\vdash_{IPC} g_{n+2}(B) \land ((A \rightarrow f_{n+2}(B)) \rightarrow A) \rightarrow A\) or

\(\vdash_{IPC} g_{n+3}(B) \land ((A \rightarrow g_{n+1}(B)) \rightarrow A) \land ((A \rightarrow g_{n+2}(B)) \rightarrow A) \rightarrow A\), then A and B are independent over IPC.

It is to be noted that just as Theorem 1.1 this theorem immediately applies to HA, since rooting the models is applicable in the case of HA as well by adjoining the standard model \(\mathbb{N}\) to the new root (see Smoryński, 1973). That this theorem is in a sense best possible can be demonstrated by showing that \(h_n(p, q)\) and \(k_n(p, q)\) are exactly provable. (Again this then applies to HA as well, now by the uniform version of the arithmetic completeness of IPC over HA, see Smoryński, 1973.)

3.2 Theorem. (a) The formula \(h_n(p, q)\) is exactly provable for

\((g_{n+2}(q) \land ((p \rightarrow f_{n+2}(q)) \rightarrow p)) \land p\) and q.

(b) The formula \(k_n(p, q)\) is exactly provable for

\((g_{n+3}(q) \land ((p \rightarrow g_{n+1}(q)) \rightarrow p) \land ((p \rightarrow g_{n+2}(q)) \rightarrow p)) \land p\) and q.

Proof. Actually, we will show in general that

(i) \(C \land ((p \rightarrow D) \rightarrow p) \rightarrow p\) where C and D do not contain p is exactly provable with the substitution \((C \land ((p \rightarrow D) \rightarrow p)) \land p\) for p and the identity for the other variables,

(ii) \(C \land ((p \rightarrow D) \rightarrow p) \land ((p \rightarrow E) \rightarrow p) \rightarrow p\) where C, D and E do not contain p is exactly provable with the substitution \((C \land ((p \rightarrow D) \rightarrow p) \land ((p \rightarrow E) \rightarrow p)) \land p\) for p and the identity for the other variables.

Of course, it suffices to prove (ii). We first show that the required formulae are actually provable. We apply the easily verified IPC-equivalence of \(A \rightarrow B\) to \(((A \rightarrow B) \rightarrow A) \rightarrow B\). Let us write \(p^*\) for \((C \land ((p \rightarrow D) \rightarrow p) \land ((p \rightarrow E) \rightarrow p)) \land p\).

\(p^* \rightarrow D\) is equivalent to \((C \land ((p \rightarrow E) \rightarrow p) \rightarrow (p \rightarrow D)) \land (p \rightarrow D)\) and hence to \(p \rightarrow D\).

\((p^* \rightarrow D) \rightarrow p^*\) is equivalent to

\((p \rightarrow D) \rightarrow ((C \land ((p \rightarrow D) \rightarrow p) \land ((p \rightarrow E) \rightarrow p)) \land p)\) and hence implies \((p \rightarrow D) \rightarrow p\).

Similarly \((p^* \rightarrow E) \rightarrow p^*\) implies \((p \rightarrow E) \rightarrow p\). That

\(C \land ((p^* \rightarrow D) \rightarrow p^* \land ((p^* \rightarrow E) \rightarrow p^*))\) implies \(p^*\) is now trivial.

Next we have to show that no stronger formulae are provable. For that it is sufficient to note that in any Kripke model validating
C ∧ ((p → D) → p) ∧ ((p → E) → p) → p changing the valuation of p to that of
(C ∧ ((p → D) → p) ∧ ((p → E) → p)) ∨ p will leave all forcing relations as they are. This is
obvious, because in any such Kripke model (C ∧ ((p → D) → p) ∧ ((p → E) → p)) ∨ p is
actually equivalent to p.

Bibliography

Jankov, V.A., 1968, Constructing a Sequence of Strongly Independent Super-
de Jongh, D.H.J., 1982, Formulas of one Propositional Variable in Intuitionistic
de Jongh, D.H.J., and A. Visser, 1993, Embeddings of Heyting Algebras, ILLC Pre-
publications, ML-93-14.
North-Holland, Amsterdam.
Maksimova L.L., 1982, Interpolation Properties of Superintuitionistic and Modal
sophica Fennica, pp. 70-78.
Pitts, A., 1992, On an Interpretation of Second Order Quantification in First Order
Intuitionistic Propositional Logic, JSL 57, 33-52.
Shavrukov, V., 1993, Subalgebras of Diagonalizable Algebras of Theories con-
taining Arithmetic, Dissertationes Mathematicae, Polska Akademia Nauk., Math-
ematical Institute.
Troelstra, A.S. (ed.) 1973, Metamathematical Investigations of Intuitionistic Arith-
Troelstra, A.S. and D. van Dalen (eds.), 1982, The L.E.J. Brouwer Centenary Sym-