Apportionment in Theory and Practice

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written by

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Abstract

Apportionment is the problem of translating an election outcome to a number of seats in fixed-size political house. Mathematically, the problem consists of translating a sequence of reals to a sequence of integers, while ensuring that the sum of the sequence sums to a pre-determined number. The problem arises because seats are indivisible, whereas an election outcome generally gives rise to fractional remainders. This thesis approaches the problem of apportionment from both a theoretical and a practical side. The theoretical part discusses all known apportionment methods and the problems these methods encounter; e.g., the Alabama paradox and quota violations. In the second, practical part this thesis investigates the apportionment system in the Netherlands. I answer the question to what extent the Dutch system suffers from the problems encountered with apportionment. This leads to the question whether alternative apportionment methods are more appropriate in the Dutch case.

Contents

1 An Introduction to Apportionment

1.1 A Cattle Conundrum

On his deathbed, a farmer divides his inheritance among his five sons. The four older sons are to receive 1/3, 1/4, 1/5 and 1/6 respectively. The youngest has to do with 1/20 of the inheritance. The old man possesses 16 cows. Assuming no changes may be made to the number of cows, and the cows will have to remain in one piece, how should the sons divide the livestock?

This general problem of fair division finds an interesting instance in the problem of *apportionment*, also known as *fair* or *proportional representation*. The central question of apportionment is how to divide the seats of a political body based on an election outcome in a fair manner. The cattle problem illustrates the matter. The sons are the parties in a political system competing for the seats in the house, or the cows in a herd. The relative share of each son amounts to the number of votes each party has received.

Mathematically, the problem consists of transforming an ordered set of non-negative real numbers (votes) into integers (seats), while ensuring the integers sum up to a predetermined number (house size).

The primary aim of apportionment is to find a seat distribution that is 'fair' in some sense or, more formally, that fits the vote distribution "as closely as possible." [33] Much of apportionment theory therefore focuses on quantifying the concept of closeness, defining 'distances' between seat distributions and minimizing these distances based on a set of criteria. The underlying political interpretation of 'fair' is that an apportionment approximates direct democracy – i.e., the principle of one-person, one-vote – as closely as possible [12].

The key problem of apportionment is that seats are indivisible, whereas voting outcomes generally give rise to fractional remainders. The problem manifests itself in Figure 1. Computing the exact shares of each son, note how four out of five sons are left with a fractional remainder. Hence, we need an algorithm that translates exact shares to whole numbers (cows), while ensuring that the total sums to a required number – in this case 16. Such algorithms are called *apportionment methods*. In this example two well-known apportionment methods are used, those once proposed by U.S. politicians Alexander Hamilton and Thomas Jefferson. How these methods yield these particular solutions is further explained in Chapter 2.3; for now we will just observe the results.

From this example, three observations stand out. Firstly, different methods can produce different results. In Figure 1 we see that the two methods differ on the apportionment of sons 1 and 5. Each apportionment technique has its merits, its imperfections and its own reasoning behind it. There are approximately ten different apportionment methods, with many more algorithms going under different names but being mathematically

Son	Fraction	Exact share	Hamilton's Method	Jefferson's Method
1	1/3	5.33	5	6
2	1/4	4	4	4
3	1/5	3.2	3	3
4	1/6	2.66	3	3
5	1/20	0.8	1	0
Total	1	16.00	16	16

Figure 1: Apportionment of the cattle using Hamilton's Method and Jefferson's Method.

equivalent.

The specific difference between the Hamilton and Jefferson apportionments highlights the second observation. It is puzzling that Jefferson's method gives the only cow of the fifth son to the already heavily endowed first son. Note, moreover, that the fifth son's exact share, or *quota*, of 0.8 is closer to 1, than the 5.33 of the eldest son is to 6.[1] This poses the question whether any apportionment method qualifies as more 'fair' than another. To answer this question, a set of seemingly reasonable criteria has emerged over the years. One of these criteria is that an apportionment method is unbiased towards the parties. This is not a given. Indeed, Jefferson's Method systematically favors larger parties. More criteria are discussed in Chapter 3.

It turns out that no apportionment method can ever be entirely fair. This third observation stems from a central impossibility result by Balinski and Young [7]. They prove that certain desirable criteria are mutually exclusive. A trade-off between criteria is not necessary, as a search for apportionment methods that minimize the risk of criteria violations is successful.

And the cows? It turns out that apportionment is a problem that is simply described, but quickly displays many difficulties.

1.2 The Importance of Apportionment

The theory of apportionment finds it roots in the 18^{th} century in the United States of America (U.S.A. or U.S.). When drafting the Constitution, the Founding Fathers of the U.S.A. failed to specify how the apportionment of seats in the House of Representatives should proceed. This omission started 200 years of heated debate – a debate that gave

[1] This is exactly the reason why Hamilton's Method awards the extra seat to the younger son. However, this reasoning, though intuitively appealing, is flawed and produces unwanted side-effects; see Section 3.2.

rise to the most important apportionment methods and the most important problems associated with their use. An excellent overview of this history can be found in [7], the key reference in the field of apportionment.

The debate in the U.S. House of Representatives highlights the importance of apportionment in several ways. A first, rather pragmatic reason is that apportionment directly affects the bread and butter of the politician: a state losing a seat in the House could mean a Congressman losing his job.

A more fundamental issue is, quite simply, that any formal apportionment algorithm is used at all. This was not always the case. In fact, in the 1793 apportionment – the very first apportionment for the House of Representatives – the statesman Alexander Hamilton proposed an apportionment based on political considerations rather than a mathematical algorithm. In particular, he proposed an apportionment that ensured enough seats for his own Federalist States. For this reason Hamilton's proposal was heavily attacked by his political opponent, the Republican Thomas Jefferson:

> "The bill does not say that it has given the residuary representatives *to the greatest fraction*; though in fact it has done so. It seems to have avoided establishing that into a rule, lest it might not suit on another occasion. Perhaps it may be found the next time more convenient to distribute them *among the smaller States*; at another time *among the larger States*; at other times to any other crochet which ingenuity may invent, and the combinations of the day give strength to carry." [7] (pp. 21–22)

In response to the criticism Hamilton quickly devised a method that happened to produce his initial apportionment to give residuary representatives to 'the greatest fraction.' Yet, Jefferson was not free of his own political machinations either. Even though he did use an algorithm, it conveniently turned out to strongly favor his Republican States. Nonetheless, his point was clear: the importance of apportionment is that it specifies a transparent and mathematically sound rule to elect a political body, and is not subject to 'the combinations of the day.'

There are additional reasons underlining the importance of apportionment. For instance, political 'realists' have sometimes downplayed the importance of the problem, because the contention usually revolves around only one or two seats – or one cow, as was seen in the previous section. But for smaller parties winning or losing one seat can be the difference between being represented or not. One extra seat can give absolute power to a party. Small differences for many districts, states or even countries (as in the European Union) can add up to large differences overall.

Also, apportionment is not limited to the political arena. As indicated, apportionment is an instance of the problem of fair division. Therefore the problem occurs wherever the indivisibility of resources forces us to translate a series of reals to integers, while these should sum up to some predetermined number. This may be the division of an inheritance or the allocation of a limited number of judges over several districts.

To sum up, apportionment is important for many reasons. But in the end, maybe the most important reason is that apportionment touches at the heart of democratic legitimacy.

1.3 Overview

The topic of apportionment knows both a theoretical (i.e., mathematical) side and a practical (i.e., political) side. This thesis will not attempt to separate them, after all they are often interlinked. Even though the aim of apportionment is to achieve exact proportional representation, "[one-person], one-vote is in fact a mathematical impossibility." [7] This automatically implies political decisions.

Following the distinction theoretical/practical, this thesis is divided into two main parts. The first part, consisting of Chapters 2 and 3, will set the theoretical framework. Chapter 2 will discuss the most commonly used apportionment methods. Chapter 3 pinpoints the problems these methods run into. These problems provide a natural bridge to a set of criteria for a satisfactory apportionment method. An impossibility result shows that certain desirable criteria are incompatible.

Regardless of its long history, apportionment continues to have political impact. For instance, in the U.S.A. debate over the apportionment method to be used for the House of Representatives continues unabated [18]. In the Netherlands, there are regular calls for imposing thresholds to the Second Chamber to keep out small parties in a fractured political landscape [26]. Therefore the second part of the thesis, consisting of Chapter 4, is distinctly practical in nature. This part takes the apportionment system of the Netherlands as a case study. The Netherlands uses two apportionment methods, those of Hamilton and Jefferson. The second part answers the question to what extent the problems known to be associated with these two methods occur. Comparing the historic overview of apportionments to alternative apportionments, this part subsequently answers the question whether the Dutch system should be changed.

Part I

Theoretical Framework

2 An Overview of Apportionment Methods

2.1 Introduction

I will start this thesis in earnest with laying down a theoretical foundation for apportionment. Starting with some basic definitions, I will concentrate primarily on describing the various existing apportionment methods. The contribution of this chapter is in unlocking the extensive, yet scattered literature on the topic by making an overview, as exhaustive as possible, of all apportionment methods that are currently known. Many seemingly different algorithms are often mathematically equivalent, and where this happens I have chosen for one consistent way of display. The discussion of these methods shows that each apportionment method has its own particular traits and flaws. We will look at these characteristics in more detail in Chapter 3.

2.2 Preliminary Definitions

Apportionment is assigning a number of seats to a political party based on a voting outcome. The problem is that a usually real-valued voting outcome has to be translated into an integer value (i.e., a number of seats). The problem is analogous to cases in several other fields; for instance, assigning seats to states or departments based on population numbers.[2]

More formally, let $\mathbf{v} = (v_1, v_2, ..., v_n)$ be the vector of valid votes won by the n parties, numbered $1, 2, ..., n$. Here $v = \sum_i v_i$ is the total number of valid votes cast. Let h be the total number of seats to be assigned in the house. The problem is to find, for each $h \geq 0$, an *apportionment* for $\mathbf{v}$: an n-tuple of non-negative integers $\mathbf{a} = (a_1, ..., a_n)$ whose sum is h.

A solution of the apportionment problem is therefore a function f which with every vector $\mathbf{v} = (v_1, v_2, ..., v_n)$ of non-negative integers and a non-negative integer h associates a vector $\mathbf{a} = (a_1, a_2, ..., a_n)$ of non-negative integers such that $\sum_i a_i = h$.

A regularly returning concept is that of *quota* q_i, i.e. the strictly proportional number of seats that party i is entitled to:

$$q_i = \frac{v_i}{v} \cdot h. \tag{1}$$

The vector of quotas for all parties i will be denoted as $\mathbf{q} = (q_1, q_2, ..., q_n)$.

[2] Considering the second part of this thesis, I will mainly focus on the example of voting. At some points historical examples concerning states and population figures illustrate a point particularly well. In those cases I have adhered to the original formulation. In general, the two formulations may be used interchangeably, although there are crucial political differences; see Section 3.7.

The lower quota $\lfloor q_i \rfloor$ is defined as the largest integer less than or equal to q_i; upper quota $\lceil q_i \rceil$ is defined as the smallest integer larger than or equal to q_i. The notation $[q_i]$ is used when a quota q_i is rounded, either up or down, depending on a specified criterion.

Finally, in legislatures that involve states and population numbers certain constraints may be imposed on the minimum number of seats that a state should receive. For instance, in the U.S. House of Representatives each state is assured at least one seat regardless of its size. Similarly, in France each *département* receives at least two representatives in the *Assemblée nationale*. Such *minimum requirements* are denoted as $\mathbf{r}_{min} = (r_1, r_2, ..., r_n)$ for all states 1 through n with r_i denoting the minimum number of seats state i is entitled to.

2.3 Apportionment Methods

Over the course of centuries of discussion, many apportionment methods have been proposed, tested and discarded. Confusingly, many of these methods have different names and guises, while being equivalent from a mathematical perspective. This section identifies which methods are mathematically identical and groups methods on the basis of commonalities.

Historically, there are roughly three types of approaches: (i) Hamilton-type methods, (ii) divisor methods, and (iii) a series of modern methods that combine elements from various techniques (e.g., power indices) and fields (e.g., social choice theory). Of the methods listed below some are no longer in use (or have never been), but I have included them for the sake of completeness.

Some apportionment methods do not always yield unique solutions; e.g., when two parties receive the same number of votes. Usually this is not a very relevant practical concern. I will assume that a suitable tie-breaking rule is defined, e.g. a random selection device.

As a running example throughout this thesis, consider the following five-party election outcome for a house of 26 seats (Figure 2). Note that each quota q_i equals $v_i/1,000$. For each of the nine apportionment methods discussed in this section I will show the solution it produces for this example. At the end of this section I have provided a full overview of all apportionments produced; see Figure 9.

2.3.1 Hamilton-type Methods

The class of Hamilton-type methods consists of two apportionment methods: Hamilton's method and Lowndes' method. Their intuitive and simple algorithm is their main appeal and as a result the basic algorithm – Hamilton's method – is still widely in use. However, in Chapter 3 it will become clear that these methods are heavily defective.

Party i	Votes v_i	Quota q_i
A	9,061	9.061
B	7,179	7.179
C	5,259	5.259
D	3,319	3.319
E	1,182	1.182
Total	26,000	26

Figure 2: Fictional election outcome for five parties and house size 26 (from [4]).

Hamilton's Method
First proposed in 1791 by the U.S. statesman Alexander Hamilton (1755 – 1804), this method is also known as the method of 'Largest Remainders,' 'Vinton's Method' and the 'Method of Roget.' It is still in use in, for instance, the Netherlands, Russia, Hong Kong and Namibia.

The algorithm is straightforward: give every state its lower quota and divide the remaining seats among those parties that have the largest remainder left.

Algorithm – Hamilton's Method:
1. Assign to each party the seats of its lower quota $\lfloor q_i \rfloor$.
2. For each party, compute the remaining fraction $q_i - \lfloor q_i \rfloor$.
3. Give the remaining seats to the parties with the largest remaining fractions.

Applied to Figure 2 this plays out as follows. In the first round, all states are given their lower quotas. That is, $a_A = 9$, $a_B = 7$, $a_C = 5$, $a_D = 3$ and $a_E = 1$, giving out a total of 25 seats. The remaining seat goes to the party with the highest residual fraction. In this case, party D ranks highest with a fraction of .319 and receives the remaining seat. The apportionment produced by Hamilton's method is $\mathbf{a} = \{9, 7, 6, 4, 1\}$.

Lowndes' Method
Proposed in 1822 by U.S. politician William Lowndes, this rule refines Hamilton's method by recognizing that the same fraction weighs heavier for a smaller party than for a larger party. For instance, a .4 remainder is quite substantial for a party that has only one seat, whereas it is of relatively less importance to a party that already has 45

seats. Therefore, Lowndes' method adjusts the remaining fractions according to party size.

The algorithm starts out as Hamilton's by giving each party the whole number in its quota. Subsequently, each party's vote total is divided by its lower quota to obtain the *adjusted remaining fraction*.[3] This measures the average constituency per seat awarded so far. The parties with the larger average constituencies per seat are less well represented and receive the remaining seats.

Algorithm – Lowndes's Method:

1. Assign to each party the seats of its lower quota $\lfloor q_i \rfloor$. If the lower quota for party i is 0, set $a_i = 1$.
2. For each party, compute the adjusted remaining fraction $m_i = v_i / \lfloor q_i \rfloor$, excluding all parties i for which $\lfloor q_i \rfloor = 0$.
3. Give the remaining seats to the parties with the largest adjusted remaining fractions.

Returning to the running example of page 11, each party gets assigned its lower quota leaving one remaining seat. The adjusted remaining fractions are: $m_A = 9{,}061 \ / \ 9 \approx 1{,}007$, $m_B \approx 1{,}026$, $m_C \approx 1{,}052$, $m_D \approx 1{,}106$ and $m_E \approx 1{,}182$. The remaining seat goes to party E, resulting in the apportionment $\mathbf{a} = \{9, 7, 6, 3, 2\}$.

Unfortunately, Lowndes' method gives a disproportional advantage to smaller parties. Balinski and Young provide an example in which a state with a quota of 24.917 does not receive an extra seat, but a state of 1.302 receives two ([7], pp. 25). For the formal explanation, consider two parties i and j with $v_i < v_j$ and one remaining seat. Observe that party i will receive the seat if $m_i > m_j$, that is if

$$\begin{aligned} m_i > m_j \quad &\Rightarrow \quad v_i / \lfloor q_i \rfloor > v_j / \lfloor q_j \rfloor \\ &\Rightarrow \quad v_i / v_j > \lfloor q_i \rfloor / \lfloor q_j \rfloor \\ &\Rightarrow \quad q_i / q_j > \lfloor q_i \rfloor / \lfloor q_j \rfloor \end{aligned}$$

The latter inequality is very likely to hold, since dropping the fraction for i has a much stronger effect than for j. More precisely, the decrease of the numerator when going from q_i to $\lfloor q_i \rfloor$ is much stronger than the decrease of the denominator.

Likely due to this undesirable bias for small parties, Lowndes' method has never been used and is not known under any other name. Note that E, the smallest party, re-

[3] Analogously, one can divide the exact quota by the lower quota, since the exact quota is precisely proportional to the vote total of a party.

ceiving the seat is consistent with the observation that Lowndes' method favors smaller parties.

2.3.2 Divisor Methods: Basic Explanation

Along with Hamilton's method, the most widely used apportionment methods come from the class of divisor methods. Divisor methods are characterized by the use of a *common* or *standard divisor* λ. For all parties i, λ divides the vote totals v_i and, after rounding, a seat assignment $\mathbf{a}$ is obtained.

To better understand divisor methods, remember that quotas for parties are computed through equation (1), which can be rewritten as

$$q_i = \frac{v_i}{(\frac{v}{h})}.$$

This clarifies the role of the part $(\frac{v}{h})$, which takes up the role of the standard divisor here. Divisor methods work by tinkering with the standard divisor until the resulting quotas sum to h when they are rounded according to some rule. The particular rule used characterizes each divisor method. Specifically, each method uses another cut-off point to round quotas up or down. This allows us to formulate a generic algorithm.

Generic Algorithm for Divisor Methods:

1. Initialize $\lambda = (\frac{v}{h})$.
2. Compute the exact quota $q_i = v_i/\lambda$ for each party.
3. Compute $[q_i]$: round q_i up if it exceeds a method-specific cut-off point; round down otherwise.
4. While $\sum_i [q_i] \neq h$:
   (a) If $\sum_i [q_i] > h$, increase λ; otherwise decrease λ.
   (b) For all parties i, compute the new $q_i = v_i/\lambda$.
   (c) Compute new $[q_i]$.

Any cut-off point can be used for rounding the q_i. The Marquis de Condorcet once proposed a threshold of .400, the literature also mentions the golden mean. Nonetheless, five classical divisor methods with five different thresholds have emerged as the only methods worth consideration. Huntington [19] proved that exactly these five divisor methods are *workable*. An apportionment method is workable if it produces *stable* apportionments. Intuitively, an apportionment is unstable if, after the apportionment, transfers of seats from one party to another still reduce a certain 'amount of inequality.' The five methods differ in how they measure the amount of inequality. I will go into more detail on stable apportionments and amounts of inequality in Section 2.3.3. For now it suffices to note that other methods using a common divisor are either mathematically equivalent to one of these five methods, or they do not produce stable apportionments.

Observe that Hamilton-type methods also follow the generic algorithm for divisor methods. After all, Hamilton's method takes the usual v/h as a common divisor, and as a threshold any real number x that contains the appropriate number of remaining fractions in the interval $[x, \lceil x \rceil]$. However, Hamilton's method misses a certain 'inner consistency'. Under any of the five classical divisor methods, a given fraction y is always rounded up or always rounded down. This contrasts with Hamilton-type methods where a fraction y is sometimes rounded up, but sometimes rounded down given the particular combination of the vote totals of other parties. This lack of consistency will prove to be a cause of many problems; see Chapter 3.

We will now turn to the five divisor methods.

Jefferson's Method

The oldest known divisor method was proposed in 1791 by U.S. statesman Thomas Jefferson (1743 – 1826). It goes under various names such as the 'Method of Greatest Divisors,' the 'Method of d'Hondt,' the 'Bader-Ofer Method' and the 'Procedure of Hagenbach-Bischoff.' It is still in use in many legislatures, among which The Netherlands, Israel, Japan and Venezuela.

The rule for rounding fractions is defined as follows:

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3. Compute $[q_i]$ by rounding down each q_i to $\lfloor q_i \rfloor$.

In Figure 3 I have applied Jefferson's method to the running example (page 11), illustrating the generic algorithm for divisor methods. The first divisor is $\lambda = v/h = 26,000/26 = 1,000$ (Step 1), which gives the exact quota q_i (Step 2). The method-specific rule for Jefferson rounds down all q_i (Step 3). This allots one seat too little, entering the loop at Step 4. The divisor λ needs to be adjusted downwards (Step 4(a)) and at $\lambda = 906.1$ the quota of party A is the first quota to jump to the next integer (Step 4(c)). Any divisor in the interval $[897.3, 906.1]$ will work.

Skipping ahead to the discussion on characteristics of methods in Chapter 3, observe the difference between the Jefferson apportionment and Hamilton-type apportionments. Whereas the latter methods give the remaining seat to the smallest parties D and E, Jefferson gives it to the largest party A. The explanation is that dropping fractions has a positive effect for larger parties. After Step 3 Jefferson's method will always hand out

Party i	v_i	Step 2: Compute q_i	Step 3: Rounding	Step 4 (b): New q_i	Step 4 (c): Rounding
A	9,061	9.061	9	10.07	10
B	7,179	7.179	7	7.98	7
C	5,259	5.259	5	5.84	5
D	3,319	3.319	3	3.69	3
E	1,182	1.182	1	1.31	1
Total	26,000	26.000	25	26.00	26

Figure 3: Apportioning with Jefferson's method.

too little seats. Consequently, the standard divisor will need to be adjusted downwards to increase the quotas. However, *ceteris paribus*, larger quotas increase more quickly than smaller ones.[4] In Section 3.3 we will see the effects of this *bias* towards larger states over longer periods of time.

Webster's Method

The rich history of apportionment in the U.S. House of Representatives produced each of the five classical divisor methods, among which this method by former Senator Daniel Webster (1782 – 1852). An expert orator, Webster persuaded with both rhetoric and simple logic. He suggested that when possible the most intuitive idea of all should be adopted: simply round the quotas to the nearest whole numbers. Webster's method is also known as 'Major Fractions,' the 'Method St. Lagüe,' 'Willcox's Method,' the 'Method of Odd Numbers' and the 'Procedure RE' ("Rounded off Exactly"). It is currently still in use in, for example, New Zealand, Norway and Nepal.

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3. Compute $[q_i]$ by taking the arithmetic mean as a threshold. That is, round up if $q_i \geq 1/2$; round down otherwise.

For Webster's and the remaining divisor methods I will not explain in detail how the results for the running example are obtained; the procedure should be clear. In Figure 4 an overview is given of all apportionment methods. For the running example of page

[4] Observe that $q_i = v_i \cdot (h/v) = v_i \cdot (1/\lambda)$. Thus, when lowering λ, the vote total v_i for party i leverages the factor $1/\lambda$.

11 Webster's method produces the apportionment $\mathbf{a} = \{9, 8, 5, 3, 1\}$.

Hill's Method

The third divisor method was proposed in the beginning of the 20^{th} century by Joseph A. Hill, an American statistician. It is better known as the method of 'Equal Proportions' or 'Huntington's Method' and is currently the method of choice in the U.S. House of Representatives. Hill's method uses another familiar threshold:

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3. Compute $[q_i]$ by taking the geometric mean as a threshold. That is, round up if $q_i \geq \sqrt{\lfloor q_i \rfloor \cdot \lceil q_i \rceil}$; round down otherwise.

Hill's method yields the apportionment $\mathbf{a} = \{9, 7, 6, 3, 1\}$.

Dean's Method

James Dean was a professor of astronomy and mathematics at the University of Vermont. At around the same time as Webster proposed his method, Dean formulated an algorithm that boils down to using the harmonic mean as a threshold for rounding the quotas. To my best knowledge, Dean's method has never been used and does not go by under any other names.

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3. Compute $[q_i]$ by taking the harmonic mean as a threshold. That is, round up if $q_i \geq \frac{\lfloor q_i \rfloor \cdot \lceil q_i \rceil}{(\lfloor q_i \rfloor + \lceil q_i \rceil)/2}$; round down otherwise.

Dean's apportionment is (in this case) identical to Hamilton's: $\mathbf{a} = \{9, 7, 5, 4, 1\}$.

Adams' Method

With the alternative name of method of 'Smallest Divisors,' this method is the mirror image of the Greatest Divisors rule, i.e. Jefferson's method. Instead of rounding down each fraction, this method rounds up each fraction.

3. Compute $[q_i]$ by rounding up q_i.

Exactly opposite to Jefferson's method, Adams's method favors the smaller parties. It should then not be surprising that the apportionment gives the remaining seat of the running example to the smallest party E: $\mathbf{a} = \{9, 7, 5, 3, 2\}$.

To sum up, Figure 4 provides an overview of the five apportionments produced by the five classical divisor methods for our running example of page 11.

Party i	v_i	q_i	Jefferson	Webster	Hill	Dean	Adams
A	9,061	9.061	10	9	9	9	9
B	7,179	7.179	7	8	7	7	7
C	5,259	5.259	5	5	6	5	5
D	3,319	3.319	3	3	3	4	3
E	1,182	1.182	1	1	1	1	2
Total	26,000	26	26	26	26	26	26

Figure 4: Overview of divisor methods.

Interestingly, five different methods produce five different apportionments. Chapter 3 will address the question which apportionment method, if any, is preferable to another. After all, taking one threshold seems as reasonable as another. However, we will first turn to the previously mentioned 'amounts of inequality' approach to deepen our understanding of apportionment methods. After this we turn to the remaining class of apportionment methods: the modern methods.

2.3.3 Divisor Methods: Amounts of Inequality

Given the indivisibility of seats in a legislative body, a perfectly proportional apportionment is hardly ever possible. As a result, some parties may be slightly advantaged, others slightly disadvantaged. A proper apportionment method, however, minimizes the inequality between the two groups. Daniel Webster had already noticed this when proposing his algorithm: "That which cannot be done perfectly must be done in a manner as near perfection as can be..." ([7], pp. 31). Hence, minimizing inequality between parties is a natural approach to apportionment. As Huntington puts it ([19], pp. 85):

> Between any two states there will practically always be a certain inequality which gives one of the states a slight advantage over the other. A transfer of one representative from the more favored state to the less favored state will ordinarily reverse the *sign* of this inequality, so that the more favored state now becomes the less favored, and *vice versa.* Whether such a transfer should be made or not depends on whether the *amount* of inequality between the two ... is less or greater than it was before; if ... reduced ..., it is obvious that the transfer should be made. The fundamental question therefore at once presents itself, as to how the "*amount of inequality*" between two states is to be measured.

This 'amounts of inequality' approach allows us to increase our understanding of divisor methods and apportionment in general. It should be noted that yet another way to describe divisor methods (as well as Hamilton's) method by using *minimal distances* is followed in, for instance, [33]. The difference is that the approach discussed here minimizes an amount of inequality between all pairs of parties. The approach in [33] minimizes the distance between the vectors **a** and **q**. Hamilton's method is an intuitive example of this approach: a Hamilton apportionment **a** solves $\min_{a_i} \sum(a_i - q_i)^2$.

Huntington leaves us with the question how to best measure the 'amount of inequality'. Some natural candidates automatically emerge. We could consider minimizing the difference between parties in terms of people represented per seat (the average constituency size v_i/a_i). We could also do it the other way around by minimizing the difference in per capita representation (a_i/v_i). Either option leaves open the choice whether to take the absolute or the relative difference. These considerations lead to three natural descriptions:

- Webster minimizes the absolute difference in per capita representation, i.e. $a_j/v_j - a_i/v_i$.
- Hill minimizes both the relative difference in constituency size as the relative difference in per capita representation, i.e. $v_i a_j / v_j a_i - 1$.
- Dean minimizes the absolute difference in constituency size, i.e. $v_i/a_i - v_j/a_j$.

The methods of Jefferson and Adams unfortunately do not have such natural explanations.

For the sake of completeness, let us briefly consider why the two relative measures coincide for Hill's method. If $x \geq y$, the *relative difference* between x and y is defined as the proportion by which x exceeds y: $(x-y)/y$. Given two parties i and j, we can express that party j is better off than party i by either $v_i/a_i \geq v_j/a_j$ (the relative difference in constituency size) or $a_j/v_j \geq a_i/v_i$ (the relative difference in per capita representation). Therefore:

$$\frac{v_i/a_i - v_j/a_j}{v_j/a_j} = \frac{a_j/v_j - a_i/v_i}{a_i/v_i} = \frac{v_i a_j}{v_j a_i} - 1.$$

The second column of Figure 5 lists the measures of inequality for the five classical divisor methods.

Method	Measure of Inequality (for $v_i/a_i \geq v_j/a_j$)	Rank Index $r(v_i, a_i)$
Jefferson	$a_j(v_i/v_j) - a_i$	$v_i/(a_i+1)$
Webster	$a_j/v_j - a_i/v_i$	$v_i/(a_i+\frac{1}{2})$
Hill	$v_i a_j/v_j a_i - 1$	$v_i/\sqrt{a_i(a_i+1)}$
Dean	$v_i/a_i - v_j/a_j$	$v_i/\left(\frac{2a_i(a_i+1)}{2a_i+1}\right)$
Adams	$a_j - a_i(v_j/v_i)$	v_i/a_i

Figure 5: Measures of Inequality (from [4], pp. 708).

Let us see how we can construct an actual apportionment method from these measures of inequality, taking the Hill measure as an example. Given a certain 'amount of inequality' we call an apportionment *stable* if, for each pair of parties, no transfer of seats reduces this amount. Recall that the five apportionment methods are the only methods to guarantee the existence of a stable apportionment. Our goal then is to minimize amount of inequality and hence produce stable apportionments.

Suppose we have a seat assignment $\mathbf{a}$ in which party j is better off than party i. The basic idea is that if the transfer of one seat from party j to party i reduces the amount of inequality, then the transfer should be made. So suppose we transfer a seat from j to i. The new apportionments (indicated by primes) for state i and j are then, respectively,

$$a'_i = a_i + 1 \quad \text{and} \quad a'_j = a_j - 1.$$

In case party j is still more advantaged than party i, then the transfer should obviously be made. In the other case the average constituency size is now in favor of state i, that is,

$$\frac{v_j}{a'_j} \geq \frac{v_i}{a'_i}.$$

Hill measures the inequality between i and j as follows

$$\frac{v_j a'_i}{v_i a'_j} - 1 = \frac{v_j(a_i+1)}{v_i(a_j-1)} - 1.$$

Recalling the definition of a stable apportionment, we check whether or not the transfer reduces the amount of inequality. If not, then no progress is made by transferring the seat and the apportionment as it stood was stable. In that case the following equation holds:

$$\begin{aligned} \frac{v_j(a_i+1)}{v_i(a_j-1)} - 1 &\geq \frac{v_i a_j}{v_j a_i} - 1, \text{ or,} \\ \frac{v_j^2}{(a_j-1)a_j} &\geq \frac{v_i^2}{a_i(a_i+1)}, \\ \frac{v_j}{\sqrt{(a_j-1)a_j}} &\geq \frac{v_i}{\sqrt{a_i(a_i+1)}}. \end{aligned} \tag{2}$$

The question is then how to construct an apportionment satisfying equation (2) for all pairs of parties i, j. An iterative approach is outlined in [4]. Seats are assigned one at a time, starting with $h = 0$ so that $a_k = 0$ for all states k. Each next seat is given to the party maximizing the right-hand side of equation (2). This expression $v_k/\sqrt{a_k(a_k+1)}$ is called the *rank index* (c.f., Figure 5). The rank index may be thought of as the amount of deviation of state k from the norm of equality either in average constituency size or per capita representation. The larger the deviation, the more eligible a party is for a seat in order to balance irregularities. Following from the measures of inequality, each divisor method has its own unique rank index.

Continuing with the apportionment method, for $h \geq 0$, an apportionment for $h + 1$ is obtained by assigning the additional seat to the party k that maximizes the rank index. This ensures that the apportionment satisfies equation (2), as each time the seat is given to the party that is 'most deserving' while ensuring that the apportionment does not become unstable. It is proved that the apportionment solutions so obtained at $h + 1$, and, continuing the process at $h+2$, $h+3$, ..., satisfy the stability criterion, and – except for ties – are unique [2].

To round off this 'amount of inequality' approach, a final interesting observation. Note that equation (2), given the final allocation **a**, implies the existence of a divisor λ satisfying:

$$\min_j \frac{v_j}{\sqrt{(a_j-1)a_j}} \geq \lambda \geq \max_i \frac{v_i}{\sqrt{a_i(a_i+1)}}. \tag{3}$$

Let us compute the used values for the running example using Hill's apportionment to get a better feeling for what this means; see Figure 6. The value minimizing the left-hand side of equation (3) and the value maximizing the right-hand side are printed in bold-face.

The minimum value for $v_i/\sqrt{a_j(a_j-1)}$ and the maximum value for $v_i/\sqrt{a_i(a_i+1)}$ are the thresholds at which the first party is to lose or gain a seat, respectively. It is not coincidental, of course, that $\sqrt{a_i(a_i+1)}$ is exactly the geometric mean, which is the threshold Hill's method uses for rounding given in the previous section; for the other rank indices similar observations hold. Party B is the first to exceed its geometric mean

Party i	v_i	a_i	$\frac{v_j}{\sqrt{(a_j-1)a_j}}$	$\frac{v_i}{\sqrt{a_i(a_i+1)}}$
A	9,061	9	1067.847	955.117
B	7,179	7	1107.751	**959.336**
C	5,259	6	**960.162**	811.449
D	3,319	3	1354.970	958.113
E	1,182	1	DIV/0	835.808

Figure 6: Finding λ range for Hill's method.

when decreasing λ. It should not be surprising that party C, the party to receive the last seat, is the first to lose it again when increasing λ.

In conclusion, we set out to understand this 'amount of inequality' approach to better understand divisor methods. Importantly, a seemingly random threshold such as the geometric mean actually has a more intuitive explanation as minimizing an amount of inequality measured in terms of the relative difference in constituency size.

It is worth noting that this alternative approach to divisor methods does not allow us to make an informed decision whether to choose any method over the other. As Balinski and Young noted ([4], pp. 709):

> (...) the essential problem with the [divisor] approach is: there is no *a priori* justification for choosing one test or measure of inequality over another.

In the next section we will investigate whether we *can* find such a justification in the sometimes fundamentally different approaches of the modern methods.

2.3.4 Modern Methods

With Hill's method in the early 20^{th} century the introduction of new apportionment methods came to a temporary halt – save for the re-invention of the same methods under different names. It took until 1974 for a genuinely new apportionment method to be proposed, the Quota Method [3]. Although the Quota Method is strongly influenced by the classical approach to apportionment, other recent methods are characterized by combining elements from more diverse fields (e.g., social choice theory, cooperative game theory) and techniques (e.g., power indices). In this section we will discuss three apportionment methods: the Quota Method by Balinski and Young, Minimax Apportionment by Gambarelli, and a more general approach that uses weighted voting to obtain the ideal of proportional representation.

The Quota Method

After having defined the formal framework for the apportionment problem, the economists Balinski and Young devised their own apportionment method: the Quota Method [3, 4]. Comparing the Quota Method to the preceding algorithms, a crucial distinction presents itself. Note that the previously discussed methods all started out with an objective to achieve. For instance, Hill's method was first designed with the aim of minimizing the relative difference in constituency size per seat. Only when the methods were put to practice their characteristics were analyzed. As one example, divisor methods do not necessarily stay within quota (i.e., they do not exclude the possibility of awarding more seats than the upper quota or less seats than lower quota; see Section 3.4). The Quota Method inverts this order. It first looks at the criteria that it wishes to satisfy, and constructs an algorithm that satisfies these. In the case of the Quota Method, the criteria concerned are *satisfying quota* and *respecting house monotonicity*. An overview of commonly used criteria can be found in Section 3.6. Minimax Apportionment, to be discussed hereafter, generalizes this approach to hold for arbitrary criteria.

The algorithm assigns seats one by one, much like when using amounts of inequality. For each seat, the party maximizing the rank index $v_i/(a_i + 1)$ receives the seat, where a_i is the number of seats already allotted to party i. Initially, $a_i = 0$ for all parties i. This is the same rank index as Jefferson's, but, crucially, the Quota Method puts restrictions on which parties are eligible for receiving the extra seat. A party is only eligible for a next seat if it will not exceed its intermediate upper quota.

Algorithm – Quota Method:

Define $\bar{\mathbf{a}}$ to be the *intermediate seat allocation*. Set $\bar{a}_i = 0$ for all parties i. Let $\bar{h}$ be the *intermediate house size*. Set $\bar{h} = 1$.

While $\bar{h} \leq h$:

1. Compute the *intermediate quotas* $\bar{q}_i = (v_i/v) \cdot \bar{h}$ for all parties i.
2. Eliminate all parties that exceed their intermediate *upper quota*, i.e. $\bar{a}_i + 1 > \lceil \bar{q}_i \rceil$.
3. Compute the rank index $v_i/(\bar{a}_i + 1)$ for each remaining party.
4. Assign the seat to the party maximizing $v_i/(\bar{a}_i + 1)$; i.e., $\bar{a}_i = \bar{a}_i + 1$.
5. Set the intermediate house $\bar{h} = \bar{h} + 1$.

We return to the running example (page 11) to apply the Quota Method. Figure 7 outlines the first four iterations of the algorithm.

For an intermediate house size $\bar{h} = 1$ we first compute the intermediate quotas. Since the intermediate upper quotas are all 1, all states are eligible to receive the first seat.

		$\bar{h}=1$		$\bar{h}=2$		$\bar{h}=3$		$\bar{h}=4$			$h=26$
Party i	v_i	$\bar{q}_i$	$\bar{a}_i$	$\bar{q}_i$	$\bar{a}_i$	$\bar{q}_i$	$\bar{a}_i$	$\bar{q}_i$	$\bar{a}_i$	...	a_i
A	9,061	.349	1	.697	1	1.046	1	1.394	2	...	10
B	7,179	.276	0	.552	1	.828	1	1.104	1	...	7
C	5,259	.202	0	.404	0	.607	1	.809	1	...	5
D	3,319	.128	0	.255	0	.383	0	.511	0	...	3
E	1,182	.045	0	.091	0	.136	0	.182	0	...	1
Total	26,000	1.000	1	2.000	2	3.000	3	4.000	4	...	26

Figure 7: The Quota Method applied to the running example.

The party that maximizes $v_i/(a_i+1)$ is in this case simply the party with the largest vote total, i.e. A.

After this first seat allotment, the situation changes for $\bar{h}=2$. Particularly, the intermediate upper quota for party A is still 1, but receiving an extra seat would put it at 2 and as such exceed its intermediate upper quota. As such, party A becomes ineligible for receiving the second seat. The measure $v_B/(\bar{a}_B+1) = 7,179/(0+1) = 7,179$ is maximal for the remaining parties i and therefore party B receives the second seat.

At $\bar{h}=3$ we note that A is now eligible again to receive a seat, since its intermediate upper quota equals 2 now. On the other hand, party B is not eligible on this round. It is, however, party C that maximizes $v_i/(a_i+1)$ and receives the third seat to be handed out.

For $\bar{h}=4$ we see that both A and B are eligible to receive the next seat, whereas C is not. Interestingly, party A receives the next seat, before parties D and E have even received their first seat.[5] Continuing these calculations, we obtain the apportionment listed in the right-most column.

Some final remarks on the Quota Method. By construction the algorithm satisfies upper quota; interestingly and not so obvious is that this method also satisfies lower quota [4]. Also, despite the similarities, the Quota Method is not a real divisor method, as it does not use one common standard divisor for all parties i. Lastly, this version of the Quota Method could also be called the 'Quota Jefferson Method', since it uses the same rank index. Other versions of divisor methods that stay within quota can be constructed by changing the rank index. In this way one could, for example, obtain a 'Quota Webster Method'. Note the consistency between the apportionment and the previous observation that Jefferson's method favors larger parties: the Quota Method assigns the only

[5] Compare $v_A/(\bar{a}_A+1) = 7,179/(1+1) = 3,589.5$, $v_D/(\bar{a}_D+1) = 3,319/(0+1) = 3,319$, $v_E/(\bar{a}_E+1) = 1,182/(0+1) = 1,182$.

remaining seat to the largest party A.

Minimax Apportionment

This method designed by Gambarelli [16, 17] is the most recent apportionment method. Similar to the Quota Method, it reverses the order of setting objective and formulating criteria. As Gambarelli describes: "an order of priority of the criteria is first established and then a 'customized' solution is developed for each individual case" [16]. In fact, Gambarelli generalizes the approach followed by the Quota Method to satisfy arbitrary criteria. The term 'minimax' refers to the fact that this method always returns the apportionment that minimizes the largest differences between individual parties as measured by the specified criteria.

Minimax Apportionment is extensive rather than complicated, so I will start with a general overview of the method; the pseudocode for this algorithm is given in [17]. The algorithm takes the full set of possible apportionments as its starting point. The criteria that are to be applied are then ordered by importance. The criterion deemed most important is applied to the set of apportionments, eliminating all those solutions that do not satisfy the criterion. This is repeated for all criteria. To avoid that the final set of solutions is empty, apportionments are formulated in terms of dominance rather than drastic exclusion. That is, apportionments are only excluded if other seat assignments exist that are (strictly) preferred.

Gambarelli uses the following four criteria:

1. *Quota*: No party receives more seats than its upper quota or fewer seats than its lower quota.
2. *Monotonicity*: No party can receive fewer seats than another party, when it actually has more votes.
3. *Normalization* ('N-criterion'): Evaluate percentage differences rather than absolute differences.
4. *Power Index* ('β-criterion'): Evaluate apportionments on their proportionality to the power index values that result from the voting outcome. A power index can be chosen as deemed suitable; Gambarelli works with the normalized Banzhaf power index.[6]

Gambarelli combines the Quota and Monotonicity criteria into a so-called 'F-criterion.' Applying the F-criterion always yields a non-empty set of seat distributions; e.g., any Hamilton apportionment suffices. Apportionments produced by divisor methods are not necessarily in the set as they do not always satisfy quota.

More formally, each criterion is described as a transform $t : \mathbb{R}^n_+ \to \overline{X}$, with $\overline{X}$ defined as

[6] A description of the Banzhaf power index can be found in Appendix B.

the simplex:

$$\overline{X} = \{(x_1, ..., x_n) \in \mathbb{R}^n_+ : \sum_{k=1}^{n} x_k = 1\}.$$

The algorithm then proceeds as follows. The initial set of all possible apportionments is denoted A^0; $C := (C_1, ..., C_n)$ is a sequence of criteria C_i ordered by importance. Applying C_{k+1} to a set of apportionments A^k produces the subset A^{k+1}. Let C_{k+1} be any criterion and $t_{C_{k+1}}$ its associated transform. The Minimax algorithm evaluates how well each of the remaining apportionments in A^k does on C_{k+1}. Thus, let $\mathbf{a} \in A^k$ be an apportionment and let a_i be the number of seats party i receives in $\mathbf{a}$.

The first step in the Minimax algorithm is to compute for each party i how a_i scores on C_{k+1} and how v_i scores on C_{k+1}. The difference between these values is called the *bonus* e_i of party i:

$$e_i(\mathbf{v}, \mathbf{a}) = t_C(a_i) - t_C(v_i).$$

As an example, consider the Power Index criterion. In this first step the algorithm would compute the (Banzhaf) power index value based on its number of seats a_i, as well as based on its actual number of votes v_i. If it gets more power under the apportionment $\mathbf{a}$ than it is entitled to based on v_i, the bonus is positive.

After computing the bonus for each party i in $\mathbf{a}$ under C_{k+1}, the Minimax method computes for all pairs of parties i, j the *complaint* of party j against party i:

$$c_{i,j}(\mathbf{v}, \mathbf{a}) = e_i(\mathbf{v}, \mathbf{a}) - e_j(\mathbf{v}, \mathbf{a}).$$

The complaints should be thought of as measuring the differences between parties given a criterion C_{k+1} and an apportionment $\mathbf{a}$. Complaints measure the differences between those parties that are well off and those that are disadvantaged.

The complaints give rise to a *complaint vector* $c(\mathbf{v}, \mathbf{a})$, whose components are the absolute values of the complaints listed in non-increasing order (i.e., the standard lexicographical order). A complaint vector is computed for each $\mathbf{a} \in A^k$. The next set of apportionments $A^{k+1} \subseteq A^k$ is obtained by preserving only the apportionments whose complaint vectors are (strictly) preferred to other complaint vectors. Note that the lexicographical ordering of the complaint vector minimizes the maximum complaint, hence the name Minimax Apportionment.

We will work through Minimax Apportionment using the running example of page 11 and the sequence of criteria $C := (F, \beta, N)$. We first eliminate all apportionments from A^0 that do not satisfy quota or monotonicity, leaving us with the set of apportionments A^1:

$$\begin{aligned} \mathbf{a}_1 &= (10, 7, 5, 3, 1), \\ \mathbf{a}_2 &= (9, 8, 5, 3, 1), \end{aligned}$$

$$\begin{aligned}\mathbf{a}_3 &= (9,7,6,3,1),\\ \mathbf{a}_4 &= (9,7,5,4,1),\\ \mathbf{a}_5 &= (9,7,5,3,2).\end{aligned}$$

The next criterion to be applied is the β-criterion, which is formalized as the transform $\beta : \mathbb{R}^n_+ \to \overline{X}$. To compute the bonuses for A^1-apportionments under the β-criterion, we first calculate the normalized Banzhaf indices as produced by the actual vote vector $\mathbf{v}$:

$$\beta(\mathbf{v}) = (\beta(v_A), \beta(v_B), \beta(v_C), \beta(v_D), \beta(v_E)) = (10/26, 6/26, 6/26, 2/26, 2/26).$$

For the five apportionments, where $\beta(\mathbf{a}_i) = (\beta(a_A), \beta(a_B), \beta(a_C), \beta(a_D), \beta(a_E))$:

$$\begin{aligned}\beta(\mathbf{a}_1) &= (11/25, 5/25, 5/25, 3/25, 1/25),\\ \beta(\mathbf{a}_2) &= (9/25, 7/25, 7/25, 1/25, 1/25),\\ \beta(\mathbf{a}_3) &= (9/25, 7/25, 7/25, 1/25, 1/25),\\ \beta(\mathbf{a}_4) &= (11/25, 5/25, 5/25, 3/25, 1/25),\\ \beta(\mathbf{a}_5) &= (10/26, 6/26, 6/26, 2/26, 2/26).\end{aligned}$$

Taking one apportionment $\mathbf{a}_i$ at a time, for each individual party j we compute the bonus $e_j(\mathbf{v}, \mathbf{a}_i) = \beta(a_j) - \beta(v_j)$, resulting in the following bonus vectors:

$$\begin{aligned}e(\mathbf{v}, \mathbf{a}_1) &\approx (0.055, -0.031, -0.031, 0.043, -0.037),\\ e(\mathbf{v}, \mathbf{a}_2) &\approx (-0.025, 0.049, 0.049, -0.037, -0.037),\\ e(\mathbf{v}, \mathbf{a}_3) &\approx (-0.025, 0.049, 0.049, -0.037, -0.037),\\ e(\mathbf{v}, \mathbf{a}_4) &\approx (0.055, -0.031, -0.031, 0.043, -0.037),\\ e(\mathbf{v}, \mathbf{a}_5) &= (0, 0, 0, 0, 0).\end{aligned}$$

For the sake of brevity, we will not compute the full complaint vectors. Instead, exploiting the property of the lexicographical order, we initially just look at the maximum complaint for each vector $c(\mathbf{v}, \mathbf{a}_i)$.[7]

$$\begin{aligned}\max(c(\mathbf{v}, \mathbf{a}_1)) &= e_B - e_D = |.055 + .037| = .092,\\ \max(c(\mathbf{v}, \mathbf{a}_2)) &= e_A - e_B = |.049 + .037| = .086,\\ \max(c(\mathbf{v}, \mathbf{a}_3)) &= e_A - e_B = |.049 + .037| = .086,\\ \max(c(\mathbf{v}, \mathbf{a}_4)) &= e_A - e_E = |.055 + .037| = .092,\\ \max(c(\mathbf{v}, \mathbf{a}_5)) &= 0.\end{aligned}$$

These maximum complaints allow us to uniquely define the ordering $\mathbf{a}_5 \succ \mathbf{a}_2 = \mathbf{a}_3 \succ \mathbf{a}_1 = \mathbf{a}_4$. This unique ordering preempts the need to apply the N-criterion. Minimax

[7] I abbreviate $e_i(\mathbf{v}, \mathbf{a})$ to e_i.

Apportionment awards the extra seat to party E. In general it may happen that two apportionments are tied on the first or even all elements. In this case the full complaint vectors do need to be calculated to see which apportionments proceed to the next round. In general, Minimax Apportionment does not guarantee a unique solution, since different apportionments may give rise to the same complaint vector, as can be seen in the running example. A tie-breaking rule or an extra criterion can be formulated to arrive at a unique apportionment.

As a final remark, Minimax Apportionment is the only apportionment method discussed in this thesis where the computational complexity of the algorithm becomes a serious concern. Firstly, the general framework starts out with the set of *all* apportionments ($\mathcal{O}(2^n)$) and performs a series of operations that run in constant time. Nonetheless, this accounts for a runtime of the general framework exponential in the number of parties. Secondly, the specific criteria may add to the complexity. For instance, computing Banzhaf power indices is NP-complete [27]. For the remaining apportionment methods it is relatively easy to see that they run in polynomial time. For instance, for divisor methods the interpretation using rank indices clearly shows a runtime of h iterations, each of which contains a number of operations that run in $\mathcal{O}(c)$.

Weighted Voting and Power Indices

The common denominator of the previous methods is that they approximate an ideal seat distribution. Nonetheless, an exactly proportional solution seems readily at hand in the form of weighted voting, in which a representative may get a vote with a weight different than 1. In particular, choosing the weight for each party equal to its quota could resolve the issue. “The great advantage of the weighted vote will be clear: the representation of the votes of the electorate in the house is perfect, as it should be, and independent of the procedure used for the determination of the actual seat distribution.” ([33], pp. 175)

The importance of weighted voting is illustrated by the running example of page 11. Consider Jefferson’s method again, which produced the apportionment

$$\mathbf{a} = (a_A, a_B, a_C, a_D, a_E) = (10, 7, 5, 3, 1).$$

Suppose the house considers a bill that requires a simple majority to pass. If the parties B, C and E are in favor, the bill would receive 13 votes, which is not a simple majority and so the bill does not pass. But if we sum the vote totals we see that $v_B + v_C + v_E = 7,179 + 5,259 + 1,182 = 13,620$, which *is* an absolute majority. This highlights the importance of perfectly proportional representation – especially for close decisions, as these are often the most contentious cases.

Weighted voting is a means to achieve exact proportional representation and is used in, for instance, the United Nations Security Council, the World Bank and the European Union Council [1]. Shareholders in a company usually get a vote proportional to the number of shares held. In the International Monetary Fund each Member State receives a weighted vote proportional to its annual monetary contribution.

Weighted voting can also be based on an election outcome [33]. Each party i deserves a weighted vote equal to its exact quota, $w_i = q_i$, which is spread over the representatives of the party. Specifically, start with any apportionment **a**. Each representative for party i receives the weighted vote $w'_i = q_i/a_i$.[8] If party i received less seats than its due share ($q_i > a_i$), then each of the a_i representatives of i will have a weighted vote slightly larger than 1. In the other case, the weighted vote per seat will be slightly smaller. "The voting weight should be (and can easily be) realized in practice as an automatic system. (...) The votes recorded by the representatives are summed automatically by the system, each with its proper weight."

Figure 8 applies weighted voting to the running example (page 11), taking Jefferson's apportionment as the initial apportionment. Note how the weight w'_A is slightly smaller than 1, which is caused by the fact that party A is the only party to be advantaged in the Jefferson apportionment.

Party i	v_i	$w_i = q_i$	a_i	$w'_i = q_i/a_i$	v_i/v	$\beta(w_i)$
A	9,061	9.061	10	.906	.349	.385
B	7,179	7.179	7	1.026	.276	.231
C	5,259	5.259	5	1.052	.202	.231
D	3,319	3.319	3	1.106	.128	.077
E	1,182	1.182	1	1.182	.045	.077
Total	26,000	26.000	26		1	1

Figure 8: Weighted voting as an apportionment method.

Despite the merits of this approach, already since [8] we know that "weighted voting doesn't work." Specifically, the influence that a representative wields in a committee, his *voting power*, is usually not proportional to the weight of the vote he may cast. In some situations a representative can have a vote, but no influence on the outcome of any decision. At other times a representative with a small vote can be as powerful as a representative with a large weighted vote.

Figure 8 also illustrates this phenomenon. The right-most column lists the Banzhaf power distribution β based on the weights w_i. For ease of reference I have also listed the

[8] Another interesting alternative is to give each party that attains the minimum requirement the same number of seats s and spread the weight w_i for party i over each of its s seats. This achieves proportional voting, prevents having to pick a 'random' apportionment **a** and additionally avoids the common situation where representatives of small parties have to carry more work, whereas in large parties this can be spread over multiple representatives.

normalized votes v_i/v. Since this measure is proportional to the quotas q_i ($v_i/v \propto q_i$) and thus the weights w_i, it should also be proportional to $\beta(w_i)$ – that is, if weighted voting accomplishes proportional representation. Obviously this is not the case. For instance, parties B and C have the same voting power, whereas their respective number of votes differs by almost 2,000. In other words, simply ensuring proportional vote weights does not guarantee a proportional power distribution.

Minimax Apportionment already showed how power indices can be used in an apportionment method; also see [21, 22, 28, 29]. A more general framework is outlined in [1, 25]. This framework iteratively approximates the vector $\mathbf{w} = (w_1, ..., w_n)$ of weights $w_i \in \mathbb{R}$ with $0 \leq w_i \leq 1$, so that under a certain power index P we obtain a pre-designed power distribution $\mathbf{d} = (d_1, ..., d_n)$:

$$\mathbf{d} = P(\mathbf{w}). \tag{4}$$

As a measure of best fit the algorithm minimizes the sum of squared errors $\sum_i (d_i - P(w_i))^2$.

Interestingly, equation (4) forms a class of apportionment methods using power indices. These methods aim to ensure that each party gets a number of seats so that its resulting *power distribution* minimizes a certain distance, rather than to ensure that the actual *apportionment* does so. For instance, the β-criterion from Minimax Apportionment selects the apportionment that minimizes the maximum difference between $d_i = (P(\mathbf{v}))_i$ and $(P(\mathbf{w}))_i = (P(\mathbf{a}))_i$ over all parties i. The algorithm from [1, 25] minimizes the overall error between $\mathbf{d}$ and $P(\mathbf{w})$ as measured by the sum of squared errors. In general, any distance measure between the two vectors $\mathbf{d}$ and $P(\mathbf{w})$ could be used.

Nonetheless, the use of power indices finds an important drawback in that not any desired $\mathbf{d}$ can be chosen. For instance, in a system with only two parties A and B there are only three possible power distributions: $\mathbf{p}_1 = (p_A, p_B) = (1, 0)$, $\mathbf{p}_2 = (0, 1)$ or $\mathbf{p}_3 = (1/2, 1/2)$. Suppose the threshold for passing decisions is set at 0.5. The first power distribution results if $v_A > v_B$, the second if $v_A < v_B$. Only if $v_A = v_B$, we end up in $\mathbf{p}_3$. The effect of increasing the threshold is that the number of election results that result in $\mathbf{p}_3$ increases. The number of possible power distributions does increase with the number of parties in the system, but never to the extent that it may accommodate the flexibility and dynamics of real-life elections. In particular, it is very difficult to construct power differences of small percentages, since one jumps from one level of power to the next.

The only natural choice for $\mathbf{d}$ is to choose it equal to $\mathbf{v}/v$. As a result, the use of power indices boils down to matching a vector $\mathbf{v}/v$ onto a limited set of power distributions $\mathbf{d}$ – a process eerily reminiscent of the apportionment problem. Unfortunately, the use of power indices does not allow for precisely proportional representation either. Weighted voting can be used if questions of power distributions are left aside.

2.4 In Sum: A Variety of Techniques

This concludes, to my knowledge, an exhaustive overview of apportionment methods. Figure 9 summarizes the apportionments produced by each of the nine methods discussed in this chapter.

Party i	v_i	q_i	Hm	L	J	W	H	D	A	Q	M
A	9,061	9.061	9	9	10	9	9	9	9	10	9
B	7,179	7.179	7	7	7	8	7	7	7	7	7
C	5,259	5.259	5	5	5	5	6	5	5	5	5
D	3,319	3.319	4	3	3	3	3	4	3	3	3
E	1,182	1.182	1	2	1	1	1	1	2	1	2
Total	26,000	26	26	26	26	26	26	26	26	26	26

Figure 9: Total overview all methods applied to the running example. *Legend*: Hm: Hamilton, L: Lowndes, J: Jefferson, W: Webster, H: Hill, D: Dean, A: Adams, Q: Quota Method, and M: Minimax Apportionment.

We have seen that different apportionment methods may yield different solutions. The correspondence in Figure 9 between Hamilton and Dean, or Lowndes and Adams, is coincidental. This is not always the case: some methods appear different at the surface, but are mathematically equivalent. This is the case, for instance, with the different ways of displaying divisor methods.

Which, if any, of these methods is preferred? Which is most fair? The problem is that there seems to be no *a priori* justification for any of the methods. To answer these questions it is necessary to look closer to the characteristics of apportionment methods. In the next chapter we will delve into this subject matter and analyze each method extensively to see which criteria of fairness they satisfy.

3 From Paradoxes to Criteria

3.1 Introduction

In Chapter 2 we saw a variety of apportionment methods producing almost as many different apportionments. The question arises: which one is correct? Or most fair? The main aim of this chapter is to gain more insight into these questions.

Several apportionment methods were first proposed and used, only to be discarded when problems arose with their use. Hamilton's method was once used in apportioning the U.S. House of Representatives, but then dispensed with because it displayed the so-called Alabama paradox – an instance of a particularly objectionable property. The problems and paradoxes found with apportionment are sorted into four categories:

1. Monotonicity violations;
2. Bias;
3. Quota violations;
4. Super- and subadditivity.

These properties violate what we would only consider to be natural behavior of a fair apportionment method. As such I take these problems as a starting point towards a comprehensive overview of the criteria a good apportionment method should satisfy.

There is no apportionment method that unites all desired behavior. A fundamental impossibility result implies a trade-off between monotonicity and staying within quota. As a result, each apportionment methods is subject to some problems, while avoiding others. When discussing each of the four problem categories, I will indicate which methods are susceptible to the problem and which are 'immune.'

In the final part of this chapter I will take stock on the theory of apportionment. This allows us to enter into a more normative decision process on apportionment methods.

3.2 Monotonicity Violations

The most compelling examples of problems with apportionment methods undoubtedly come from monotonicity violations. The literature holds many interpretations of the term 'monotonicity,' which I will discuss in turn. I will start with the most straightforward interpretation (also see page 24):

> Given an election outcome, for any pair of parties, the party entitled to fewer votes cannot win more seats than the party receiving more votes.

All apportionment methods discussed so far satisfy this criterion. Although this may seem obvious, the mathematics of apportionment are sometimes subtle, so I will verify this through somewhat informal proofs.

First consider Hamilton's method. Assume $v_i > v_j$, giving two relevant cases.

- *Case 1*: $\lfloor q_i \rfloor = \lfloor q_j \rfloor$ and $\lceil q_i \rceil = \lceil q_j \rceil$. Since $v_i > v_j$ implies $q_i > q_j$, it automatically follows that party i is always considered before j and can never receive less seats.
- *Case 2*: $\lfloor q_i \rfloor \geq \lceil q_j \rceil$. Since by lower quota[9] $a_i \geq \lfloor q_i \rfloor$ and by upper quota $a_j \leq \lceil q_j \rceil$, the result automatically follows.

For Lowndes' method this line of reasoning can be easily adapted.

For divisor methods again assume $v_i > v_j$, and a suitable divisor λ. Thus, $v_i/\lambda > v_j/\lambda$, which leads to the same two cases.

- *Case 1*: $\lfloor q_i \rfloor = \lfloor q_j \rfloor$ and $\lceil q_i \rceil = \lceil q_j \rceil$. Let x_i denote the threshold for party i, differing per divisor method, for which a quota is rounded up should it exceed it. In this case, $x_i = x_j$, leading to:
  1. $v_i/\lambda > x_i$ and $v_j/\lambda < x_j$ implies $a_i > a_j$,
  2. $v_i/\lambda > x_i$ and $v_j/\lambda > x_j$ implies $a_i = a_j$,
  3. $v_i/\lambda < x_i$ and $v_j/\lambda < x_j$ implies $a_i = a_j$.
- *Case 2*: $\lfloor q_i \rfloor \geq \lceil q_j \rceil$, so that $x_i > x_j$:
  1. $v_i/\lambda < x_i$ and $v_j/\lambda < x_j$ implies $a_i > a_j$,
  2. $v_i/\lambda < x_i$ and $v_j/\lambda > x_j$ implies $a_i \geq a_j$.

For the Quota Method, assume $v_i > v_j$ and, for the sake of a contradiction, suppose $a_j > a_i$. Then, on at least one iteration of the algorithm party j must have received a seat putting it exactly at one seat above party i. Let this be step k, and thus assume that after k: $\bar{a}_j = \bar{a}_i + 1$. I will show that this step can never have occurred. After iteration $k-1$, the rank indices for i and j equaled $v_i/(a_i+1)$ and $v_j/(a_j+1)$, respectively. Since j received the seat, $v_i/(a_i+1) < v_j/(a_j+1)$, so, by substitution, $v_i/a_j < v_j/(a_j+1)$. This is an obvious contradiction, since $v_i > v_j$ and $a_j < a_j + 1$.

Finally, Minimax Apportionment is monotone by construction of the F-criterion.

As indicated, in the context of apportionment the term 'monotonicity' is multi-interpretable. We now shift our attention to three monotonicity phenomena that manifest itself when data start changing in any of the three parameters: (i) the number of votes, (ii) the house size and (iii) the number of parties or states. Throughout history three paradoxes have arisen:

[9] In Section 3.4 it will become clear that Hamilton's method *satisfies quota*, i.e., for no party $a_i < \lfloor q_i \rfloor$ or $a_i > \lceil q_i \rceil$.

- The *population paradox* concerns the number of votes v_i for a party, or the population of a state. Over two subsequent elections, party i grows stronger than party j, yet loses a seat to j.
- The *Alabama paradox* pinpoints a problem with the house size h. When increasing h to $h + x$ while maintaining $\mathbf{v}$, the number of seats for party i decreases.
- The *new states paradox* occurs when changing the number of parties or states. Upon the entrance (or secession) of a party k, a party i loses a seat to a party j, while v_i and v_j are unchanged.

These three paradoxes were first observed in the context of the apportionment for the U.S. House of Representatives, which concerns assigning seats to states based on their populations. A a result, the examples given for these three paradoxes relate to states and populations, rather than parties and votes. I will discuss which methods suffer from which paradoxes in Section 3.2.4.

3.2.1 Population Paradox

The population paradox entails that a state i grows faster than state j, yet loses seats to j. An example was found by Balinski and Young ([7], pp. 42-43) of two states in the U.S. House of Representatives. Balinski and Young interpolated decennial census figures to obtain a discrepancy in the apportionment of 1900 and the 'virtual' apportionment of 1901.[10] At that point, Hamilton's method was used for assigning the 386 seats in the House of Representatives. In 1900, Virginia had a quota of 9.599 and Maine a quota of 3.595. Virginia's quota was rounded up to 10 seats, Maine's rounded down to 3 seats. Based on the interpolated figures, a year later Virginia would receive 9 seats and Maine 4. The problem: Virginia's population had grown much stronger than Maine's, yet Virginia lost a seat to Maine.

The origins of the population paradox are well-understood and already alluded to when discussing Jefferson's method. Given a percentage increase for all population figures, larger quotas increase faster in absolute terms, yet also decrease faster in the opposite case. In the beginning of the 20$^{\text{th}}$ century, the U.S.A. was growing at a faster pace than either Virginia or Maine. Therefore, both states' shares actually declined as compared to the total by 1901, to 9.509 and 3.548 respectively. Virginia, being the larger state, suffered most and lost a seat.

Another example of the population paradox can be found in a slightly adjusted version of the running example of the previous chapter (page 11). Figure 10 lists Hamilton apportionments for house sizes of 25, 26 and 27 seats.

[10] Apportionment in the House of Representatives is done every ten years following the decennial census.

		$h = 25$		$h = 26$		$h = 27$	
Party i	v_i	q_i	a_i	q_i	a_i	q_i	a_i
A	9,061	8.713	9	9.061	9	9.410	9
B	7,179	6.903	7	7.179	7	7.455	8
C	5,259	5.057	5	5.259	5	5.461	6
D	3,319	3.191	3	3.319	4	3.447	3
E	1,182	1.136	1	1.182	1	1.227	1
Total	26,000	25.000	25	26.000	26	27.000	27

Figure 10: Hamilton's method is susceptible to the three monotonicity paradoxes.

Consider $h = 27$ and a subsequent election in which all parties gain votes, though growth rates differ per party; see Figure 11, subsequent election results are indicated by *.

Party i	v_i	a_i	Growth rate	v_i^*	q_i^*	a_i^*
A	9,061	9	1.1	9967	9.315	9
B	7,179	8	1.1	7897	7.3806	7
C	5,259	6	1.15	6048	5.653	6
D	3,319	3	1.09	3618	3.3813	4
E	1,182	1	1.15	1359	1.270	1
Total	26,000	27	1.11	28,889	26.000	27

Figure 11: Hamilton's method displays the population paradox.

Note how party B grows stronger than party D, yet it would lose a seat to D under Hamilton's method. The total growth rate is larger than the growth rate of both B and D.

The population paradox tampers with the core principle that a change in votes or population figures should be correctly reflected in an apportionment. The possible consequences are clear: "A political party could give away some of its votes and thereby gain seats. In federal systems a state could deliberately undercount its population or encourage emigration to obtain an increase in its representation." [7]

3.2.2 Alabama Paradox

Until well into the 20$^{\text{th}}$ century, the U.S. House of Representatives was adjusted in size to accommodate new states while ensuring that no other state would lose any seats it previously held. Hamilton apportionments would be computed for several different house sizes until an apportionment was found that was politically acceptable.

In 1881 this practice gave rise to the Alabama paradox. Surprisingly, it turned out that Alabama was entitled to 8 seats within a house size of 299, but to only 7 seats with a size of 300. Another example is again hidden in the running example of Chapter 2; see Figure 10 with D being the affected party when going from $h = 26$ to $h = 27$.

The explanation of this paradox is very similar to that of the population paradox. As the house size increases, the quotas of all states increase by the same proportion, but not by the same absolute amounts. Returning to the U.S. example, Alabama's 1880 quota of 299 seats is 7.646, which gives it 8 seats under Hamilton. However, Alabama is unique in one respect: it is the state with the smallest remainder to receive an extra seat. Adding an extra seat to the house increases the quotas of all states with .33%, but in absolute terms, the remainders of larger states increase by more. As such, both Illinois' and Texas' remaining fractions surpass that of Alabama, causing it to lose its extra seat. In Figure 10 parties B and C, the larger parties, fulfill the role of Illinois and Texas.

3.2.3 New States Paradox

The new states paradox – upon the entrance (or secession) of a third state, a state loses a seat to a second state, while their respective populations remain identical – also finds an example in the history of the U.S. House of Representatives. In 1907 Oklahoma became a state and based on its population figure it was entitled to about five seats in the House, which it received. The total house size increased thereby from 386 to 391. While one expected that the rest of the Hamilton apportionment would remain unaffected, a recalculation in fact showed that New York would theoretically lose a seat to Maine.

This paradox occurs for much the same reason as the previous two paradoxes. Because of the entrance of a new state the total population increases. This weighs most heavy on the larger states, whose share drops heavier in absolute terms. Indeed, New York with 38 seats was considerably larger than Maine with only 3.

The new states paradox also occurs in our running example (p. 11). Consider Figure 12, in which party E decides to withdraw from the election. The relative shares of all parties increase, but more so for the larger parties B and C. These parties surpass D, which loses a seat as a result.

Party i	v_i	q_i	a_i	q_i^*	a_i^*
A	9,061	9.061	9	9.493	9
B	7,179	7.179	7	7.521	8
C	5,259	5.259	5	5.509	6
D	3,319	3.319	4	3.477	3
E	1,182	1.182	1	-	-
Total	26,000	26.000	26	26.000	26

Figure 12: The new states paradox also hides in the running example (p. 11).

3.2.4 In Sum: Monotonicity

All examples mentioned above concern Hamilton apportionments. An interesting result by Balinski and Young ([7], Theorem 4.3 and Corollary 4.3.1) provides a start for reviewing the susceptibility of the remaining methods for these paradoxes. The theorem states that *only* divisor methods avoid the population paradox. It is worthwhile to take a closer look at the behavior of Hamilton's method versus divisor methods to get a better grasp of the three monotonicity paradoxes.

As stated, the mechanism behind these paradoxes each time relates to the different effects of changing vote totals (population figures) on small and large parties (states). Whereas the relative change is equal, in absolute terms larger states are affected more heavily – either positive or negative. A possible consequence of this unequal effect is that the remaining fraction of a large party i overtakes the remaining fraction of a small party j when parameters change.

Nonetheless, this phenomenon occurs equally for Hamilton's method and divisor methods. The crucial difference is Hamilton's lack of 'inner consistency' (c.f., page 14), i.e., the threshold used for rounding that differs per situation. For Hamilton this means that if j received a remaining seat in the original situation, in the new situation it may lose it to i. That is, parties can lose seats to each other.

With divisor methods the remainder of party i can also overtake the remainder of j, but it can never result in j losing a seat to i. Consider this typical example of the Alabama paradox. Assume we go from h to $h+1$. For divisor methods, the divisor λ will need to be adjusted downwards so that an extra seat is handed out. This results in a percentage increase for all q_i and – just as with Hamilton – in a stronger absolute increase for large parties. Let i be a large party and j be a small party and assume that the remainder of i overtakes the remainder of j. Two scenarios can unfold. First, if j was below the threshold that the particular divisor method stipulates, then it may or may not exceed the threshold in the new situation (q_i'), but it will always attain minimally the seats it had. The interesting, second case is when the remainder of j's quota exceeded the threshold. The remainder of i may overtake that of j, but j will never fall below the threshold. In fact its quota will only increase. While party j may not receive the extra seat (this may or may not go to i), it will never lose it either.

As a result, the source of the monotonicity paradoxes concerns both the unequal effect of changes in parameters on small and large parties, as well as the differing threshold of Hamilton's method. These paradoxes violate the principle of proportionality: changes in one of the parameters – especially in population figures – should reflect proportionally in the apportionment. Any method that violates this does not satisfy the core principle of apportionment. "Population monotonicity says that as the conditions of a problem change – as populations shift, as the size of the house expands or contracts, and states join or secede – apportionments should respond accordingly, i.e., they should not move contrary to the relative changes of states' populations." [7]

Let us take a look at the remaining methods. As for the Quota Method, it is house monotone [4], but not population monotone.[11] Intuitively, the method avoids the new states paradox. Upon the entry of a new state k, for any pair of states i, j (with $i, j \neq k$), the comparisons of rank indices for i and j remain unaffected, particularly because v is not a factor in any of the equations.

For Minimax Apportionment we note that its susceptibility for these criteria is entirely dependent on the choice of criteria. If Minimax Apportionment is 'programmed' to yield the same solutions as a divisor method (e.g., by only allowing apportionments that minimize a certain amount of inequality), then it avoids all paradoxes. In all other case it will not escape so easily.

3.3 Bias

A second undesirable characteristic of some apportionment methods is a bias towards either smaller or larger parties. Lowndes' method, for instance, favors smaller parties, for Jefferson's method the opposite holds. Naturally, each apportionment solution favors some states over others. This is unavoidable with methods that are not purely proportional. Bias, however, is when a method manifests a systematic pattern of favoritism over the long run. Thus, an apportionment method is considered *biased* if it systematically favors either large or small parties, i.e., if the chance is better than 50% that the method will favor the large or the small respectively. A biased apportionment is (usually) unacceptable for the very reason that it attaches more value to one person's vote than another's.

It is impossible to give one example 'proving' the existence of a bias, as this only displays itself over the long term. The data from two centuries of apportionment in the U.S. House of Representatives serve to give insight; see Figure 13.

[11] By Balinski's and Young's impossibility result, see page 46.

Jefferson	Webster	Hill	Dean	Adams
-15.7	.3	3.4	5.2	18.3

Figure 13: Historical Average Percentage Biases in Favor of Small States ([7], Table 9.3).

Limiting the discussion to divisor methods for the moment, we see that Jefferson favors the large states, whereas Hill, Dean and Adams favor smaller states. Figure 13 suggests that Webster's method is the only unbiased divisor method; this suspicion is proved in ([7], Proposition 5.2). The intuitive explanation is that when using a common divisor to find the exact quota of a state, it is expected that its fractional part is above one-half just as often as it is below one-half. This holds equally for larger and smaller states. Using a threshold of exactly .5, Webster's method is the only one to appreciate this fact.

The fact that Hill's method, using relative difference, favors smaller states may come as a surprise. Nevertheless, the principle of relative difference gives more weight to the fractions of small states. After all, a difference of, say, .3, has more impact on a small state than a large state.

For the remaining methods, note that Hamilton's method is unbiased as remaining fractions are nearly randomly distributed regardless of state or party size [34]. Lowndes' method has the effect of overcompensating for a smaller party size and strongly favors smaller parties [7]. The Quota Method is observed to have a slight tendency to favor larger parties [5], which can be understood since it essentially uses the same rank index as Jefferson's method.

Minimax Apportionment is not biased *per se*; again it depends on the criteria chosen. The general framework is unbiased. Recall that the approach always minimizes the maximum complaint of party i against party j. The complaints in turn are computed through the differences in bonuses. There is no *a priori* reason to assume that the maximum bonus differences are 'caused' more often by the bonus difference of a small state and a large state, two medium ones, or any combination thereof. A case study of the Dutch Second Chamber strongly suggests that, indeed, Minimax Apportionment is unbiased; see Figure 22.

3.4 Quota Violations

The prevailing view on apportionment theory holds that any seat distribution should 'satisfy quota.' That is, no party should ever receive more seats than its exact quota rounded up or less seats than its quota rounded down. A specific instance is when q_i is an integer, in which case the apportionment method should generate $a_i = q_i$. Coincidentally, all examples given so far have satisfied this criterion, but this is not a mathematical certainty. In 1832 a proposed Jefferson apportionment violated quota; Senator Webster strongly objected:

> "The House is to consist of 240 members. Now, the precise portion of power, out of the whole mass presented by the number of 240, to which New York would be entitled according to her population, is 38.59; that is to say, she would be entitled to thirty-eight members, and would have a residuum or fraction; and even if a member were given her for that fraction, she would still have but thirty-nine. But the bill gives her forty ... for what is such a fortieth member given? Not for her absolute numbers, for her absolute numbers do not entitle her to thirty-nine. Not for the sake of apportioning her members to her numbers as near as may be because thirty-nine is a nearer apportionment of members to numbers than forty" [9]

Another, somewhat artificial example is given in Figure 14.

Party i	v_i	q_i	Jefferson	Hill
A	5,117	51.17	52	51
B	4,400	44.00	45	43
C	162	1.62	1	2
D	161	1.61	1	2
E	160	1.60	1	2
Total	10,000	100.00	100	100

Figure 14: Divisor methods violate quota (example from [9]).

In this example neither Jefferson's nor Hill's method awards party B its exact quota of 44 seats. Computing this for the remaining divisor methods shows that none of them does. A result by Balinski and Young shows that no divisor method satisfies quota [7], although clearly Jefferson's method satisfies lower quota and Adams' method satisfies upper quota. Most significantly, Hill's method can be *arbitrarily* far off quota – some compelling examples are given in [4].

As for the other apportionment methods, Hamilton's method does satisfy quota. Note that it can never violate lower quota, because each party gets assigned the whole number in its quota in the first step of the algorithm. To see that it also satisfies upper quota, consider the 'worst case' scenario in which all quotas are rounded down in step 1, whereas rounding up all quotas would actually yield $\sum_i a_i = h$. In other words, as many seats remain as there are parties. Since, by construction, no party can receive more than one seat before the remaining parties have received one (which depletes surplus seats), no party can receive more than 1 seat and thus exceed upper quota. Similar reasoning holds for Lowndes' method.

The Quota Method satisfies quota by construction. Minimax Apportionment is dependent on the criteria chosen, but satisfying quota is contained within the F-criterion.

3.4.1 Fairness of the Quota Requirement

To this day both researchers and politicians maintain that staying within quota is paramount for any apportionment method, if not the most important criterion. For long Balinski and Young were inclined to the same position: "The first principle is that any apportionment should satisfy quota." ([4], pp. 721) When proving the disturbing fact that Hill's method could produce 'arbitrary' solutions, they were exasperated: "Most seriously, [Hill's method] does not satisfy quota." ([4], pp. 707) But upon closer scrutiny, just how reasonable is this criterion?

It has been noted several times that changes in population figures or vote totals operate very differently on smaller parties versus larger parties. This also holds when forcing parties to stay within one seat of their quotas. One seat more or less for a large party has relatively little effect; for a small party the difference can be between representation or not. As a result, a few years later Balinski and Young reverse their position by 180 degrees: "In a word, staying within the quota is *not really compatible with the idea of proportionality at all*, since it allows a much greater leeway in the per capita representation of small states than it does for large states." ([7], pp. 80, italics added).

An additional piece of evidence against explicitly imposing the quota criterion is unveiled when investigating *how likely* quota violations are. Using Monte Carlo simulations of apportionments for the U.S. House of Representatives, Balinski and Young computed the following expected number of quota violations per 1,000 apportionment problems:

Jefferson	Webster	Hill	Dean	Adams
1,000	.61	2.86	15.40	1,000

Figure 15: Number of expected quota violations per 1,000 problems ([7], Table 10.3).

It turns out that for some methods the probability of going outside quota is negligible. Of all divisor methods, Webster's is the least likely to violate quota to the point that it can be considered to stay within quota for all practical purposes. To slightly lesser extent this also holds for Hill's method. Jefferson and Adams are predicted to violate quote always in case of the U.S. House of Representatives. Note that this does not mean that they always violate quota in other cases, nor even that they always will in the House.

In sum, although divisor methods do not necessarily stay within quota, the likelihood of such an event is often small. Moreover, questions can be raised on how sensible this criterion actually is.

3.4.2 Staying Near Quota

As a final consideration on quotas, of all divisor methods Webster's rule has an additional advantage: it always stays *near quota*. This criterion holds that for any pair of parties, it is impossible to take a seat from one party and give it to the other and simultaneously bring both of them nearer to their quotas. Webster's method is the only divisor method that stays near quota. Crucially, this is independent of the definition of 'nearness' in either absolute or relative terms [7]. Note that satisfying quota does not necessarily imply being near quota. A simple counter-example is the Jefferson apportionment for the example of page 11. Switching a seat from party A to party D would bring both parties nearer to quota.

I will show for the remaining apportionment methods whether they are near quota. Define the criterion of being near quota as follows: an apportionment method is *near quota* if for no apportionment that it produces there exist parties i and j such that

$$q_i - (a_i - 1) < a_i - q_i \text{ and } a_j + 1 - q_j < q_j - a_j.$$

First, Hamilton's method is indeed near quota. To prove this, note that switching a seat from party i to party j is only sensible when $a_i > q_i$ and $a_j < q_j$. As Hamilton's method satisfies quota, this implies $a_i = \lceil q_i \rceil$ and $a_j = \lfloor q_j \rfloor$. From here on, let $r(q_i) = q_i - \lfloor q_i \rfloor$ denote the fractional remainder of a quota q_i. Observe that when an apportionment method satisfies quota, a party k receiving a seat can only move closer to its quota when $a_k > 1/2$; when giving away a seat it can only move closer when $a_k > 1/2$. This leads to the following case distinction, where we move one seat from i to j:

1. $r(q_i) > 1/2$ implies that party i will move further away from its quota,
2. $r(q_i) < 1/2$ implies $r(q_j) < 1/2$, so that j will move away from its quota, and
3. $r(q_i) = 1/2$ implies no move for i, and j either stays the same (when $r(q_j) = 1/2$) or moves further away from its quota (when $r(q_j) < 1/2$).

By means of a counterexample, we show that the Quota Method, although satisfying quota, does not produce apportionments that are always near quota. Consider Figure 16, a two-party election for two seats:

Note that party A gets awarded the second seat as well, since on the second iteration its rank index $v_A/(\bar{a}_A + 1) = 1499/(1 + 1) = .7495 > v_B/(\bar{a}_B + 1) = 501/(0 + 1) = .501$. Nonetheless, giving one seat to B would bring both parties nearer to their quota.

Finally, Minimax Apportionment also does not necessarily stay near quota. Since the method starts with the entire set of possible apportionment, this also includes solutions that are within quota, but not near quota, not even after applying the F-criterion. The counterexample for the Quota Method also holds for Minimax Apportionment when applying only F.

		$\bar{h} = 1$		$\bar{h} = 2$	
Party i	v_i	$\bar{q}_i$	$\bar{a}_i$	$\bar{q}_i$	$\bar{a}_i$
A	1,499	.7495	1	1.499	2
B	501	.2505	0	.501	0
Total	2,000	1.000	1	2.000	2

Figure 16: The Quota Method is not always near quota.

3.5 Super- and Sub-Additivity

Lastly, we will explore the additivity of vote totals and the resulting seat distribution. A method is *superadditive* if a coalition of two parties gains at least a number of seats equal to the sum of the seats won by the individual parties. In this case the method is said to 'encourage coalitions'; subadditive methods are said to 'encourage schisms'. The distinction between super- and sub-additive methods gains importance when choosing apportionment methods either for federal systems or proportional representation systems; see Section 3.7.

Looking at divisor methods, Jefferson's is the only rule that encourages coalitions without exceptions ([7], Theorem 9.1). Adams' method can be characterized as the only divisor method that encourages schisms, whereas Webster's method is approximately 'coalition-neutral.' All divisor methods limit the number of seats awarded extra or taken away from a coalition to 1, a situation Balinski and Young call 'being stable' – not to be confused with stable apportionments; see page 13.

To gain some insight into the reason why Jefferson's method is superadditive, consider two parties A and B and a divisor λ that assigns exactly h seats. If A and B decide to form a coalition, their aggregate quota is simply the sum of their separate quotas: $q_{AB} = q_A + q_B$. Subsequently, two scenarios may occur. First, $r(q_{AB}) < 1$, in which case nothing happens. If, however, $r(q_{AB}) \geq 1$, Jefferson awards the coalition an extra seat. The total apportionment hands out one seat too much, though. Increasing λ to hand out fewer seats can cause the summed remainder of A and B to fall below 1 again, in which case the coalition is back at its original number of seats. If the remainder does not dip below 1, the coalition gained an extra seat.

As for the other apportionment methods, Hamilton and Lowndes are approximately coalition-neutral. The fractional remainder of a coalition of two parties will be randomly spread over the interval $[x, x + 1], x \in \mathbb{N}$. Moreover, the threshold used for rounding, which differs depending on the particulars of the election outcome, is also randomly distributed over the interval. As a result, the fractional remainders of coalitions are rounded up as often as they are rounded down. Hamilton and Lowndes follow the divisor methods in being stable ([7], Proposition 9.2).

The Quota Method is super-additive, since it copies Jefferson's behavior by using the same rank index. The Quota Method is also stable, since it stays within quota and the quota of a coalition is just the sum of the quotas of its members. Hence, the aggregate quota is still within quota and is therefore stable.

Lastly, for Minimax Apportionment super- or sub-additivity can be formulated as a criterion.

A practical example of superadditivity occurs when parties are allowed to connect their ballot lists, as in the Netherlands, where these are called 'list combinations.' Consider, for instance, Figure 17, which is the 1977 election outcome for the Dutch Second Chamber. Observe that party 7 is apportioned more seats than party 6, while it receives less votes. The Netherlands use Jefferson's method for elections of the Second Chamber (Appendix A). Here party 1 and party 7 entered into a list combination, meaning that their votes are first combined and a number of seats is computed as if they were one party. In a second round this number is divided over the constituting parties of the shared list using Hamilton's method.

Party i	Votes v_i	Seats a_i
1	2,813,793	53
2	2,655,391	49
3	1,492,689	28
4	452,423	8
5	177,010	3
6	143,481	2
7	140,910	3
8	79,421	1
9	77,972	1
10	69,914	1
11	59,487	1
Total	8,162,491	150

Figure 17: List combinations utilize Jefferson's super-additivity [33].

3.6 Criteria for Fair Apportionment

This section wraps up some of the results scattered over the previous pages. I will list which apportionment methods suffer from which problems (and, hence, violate which criteria). Based on these observations, in Section 3.7 I will evaluate what this means in terms of the 'fairness' of each apportionment method – the question we set out to answer.

First of all, let us summarize possible criteria for a 'fair' apportionment method.

- *Monotonicity*: For any pair of parties, the one entitled to fewer votes cannot win more seats than the other.
  - *Population Monotonicity*: Following an election, no party i grows stronger than a party j and loses seats to j.
  - *House Monotonicity*: When increasing h while maintaining $\mathbf{v}$, the number of seats for a party i can never decrease.
  - *New States Monotonicity*: Upon entry of a new party or secession of a present party, no changes in the relative seat distribution between other parties should occur.
- *Bias*: There is no systematic tendency to favor either large or small parties.
- *Quota*: No party can obtain more seats than are defined by its upper quota or fewer seats than it would win as stipulated by the lower quota.
  - *Near Quota*: No transfer of seats can bring both parties simultaneously closer to their quota.
- *Additivity*:
  - *Super-additivity*: A party formed by the union of two parties gains at least a number of seats equal to the sum of the seats won by the individual parties.
  - *Sub-additivity*: A party formed by the union of two parties gains at most the number of seats equal to the sum of the seats won by the individual parties.

In case of additivity the desired behavior (super- or sub-additivity) often depends on specific considerations regarding the political context in which the apportionment method is used.

Figures 18 and 19 provide a full overview of all apportionment methods discussed above and all criteria that can be found in the literature. For Minimax Apportionment I only assume that the standard F-criterion has been applied; a 'P' indicates 'possibly', i.e. the method could satisfy this criterion given the right transform. A '+' under bias indicates a tendency to favor small states, a '$\circ$' indicates being neutral. Where results were proved in this thesis I have indicated this with †.

Method	Monotonicity	Population M.	House M.	New States M.
Hamilton	$\surd^\dagger$	$\times$	$\times$	$\times$
Lowndes	$\surd^\dagger$	$\times$	$\times$	$\times$
Jefferson	$\surd^\dagger$	$\surd$	$\surd$	$\surd$
Webster	$\surd^\dagger$	$\surd$	$\surd$	$\surd$
Hill	$\surd^\dagger$	$\surd$	$\surd$	$\surd$
Dean	$\surd^\dagger$	$\surd$	$\surd$	$\surd$
Adams	$\surd^\dagger$	$\surd$	$\surd$	$\surd$
Quota	$\surd^\dagger$	$\times$	$\surd$	$\surd$
Minimax	$\surd^\dagger$	P	P	P

Figure 18: Overview of apportionment methods and their susceptibility to paradoxes (part 1).

Method	Bias	Upper Quota	Lower Quota	Near Quota	Additivity
Hamilton	$\circ$	$\surd$	$\surd$	$\surd^\dagger$	$\circ$
Lowndes	+	$\surd$	$\surd$	$\surd^\dagger$	$\circ$
Jefferson	-	$\times$	$\surd$	$\times$	+
Webster	$\circ$	$\times$	$\times$	$\surd$	$\circ$
Hill	+	$\times$	$\times$	$\times$	?
Dean	+	$\times$	$\times$	$\times$	?
Adams	+	$\surd$	$\times$	$\times$	-
Quota	-	$\surd$	$\surd$	$\times^\dagger$	+
Minimax	$\circ$	$\surd$	$\surd$	P	P

Figure 19: Overview of apportionment methods and their susceptibility to paradoxes (part 2).

3.7 An Impossibility Result: An Evaluation Of Apportionment Methods

If one conclusion is to be drawn from the previous tables it is this: no apportionment method satisfies all desirable criteria. This could have been expected. A central impossibility result by Balinski and Young reads ([7], pp. 79):

> There is no method that avoids the population paradox and always stays within quota.

The reasoning behind this proof is surprisingly simple. Balinski and Young first showed that the only apportionment methods that are free of the population paradox are divisor methods. But history had already given examples of each divisor method violating quota. The impossibility result follows easily from these two facts. No further (im)possibility result has been proved since.

The implications of the result pose a dilemma; maybe not so much about which method is perfect, but rather which criteria are most important in a trade-off. In order to choose an apportionment method all that is required is a preference over the set of criteria. This, of course, is the basic idea behind Minimax Apportionment. If the dilemma is between the population paradox and satisfying quota, we already noted that staying within quota is not always necessarily desirable due to its different effect on small and large parties. Additionally, certain divisor methods avoid the population paradox and have a virtually zero chance of violating quota, in particular Webster's method. As such, Webster's method is the only method that avoids all monotonicity paradoxes, is unbiased, is 'coalition-neutral', and for all practical purposes can be considered to stay within quota. "To conclude, while it is not possible to satisfy all of the principles all of the time, it is possible to satisfy all of them *almost* all of the time." ([7], pp. 83)

Still, an ordering over the set of criteria should be made with the particular political situation in mind in which the apportionment method is to operate. In particular, there is a difference between proportional representation (PR) systems, where the apportionment problem applies to votes and parties, and federal systems, in which apportionment involves population figures and states. Whereas the concepts and terms seem interchangeable, and in fact have been used as such throughout this thesis, there are essential differences.

The first in this regard is in the attitude towards smaller parties. In PR systems a concern often lies in avoiding small (extremist) splinter parties to gain representation. This is usually accomplished through voting thresholds. For instance, in the Netherlands a party needs at least enough votes as are necessary for one full seat ($v_i \geq \frac{v}{h}$) in the Second Chamber, otherwise a party will not receive any seats. Another method is to use an apportionment method that favors larger parties. On the other hand, in federal systems the concern is to give every district, no matter how small, some representation.

The second difference between the two situations concerns the flux in which PR systems operate. Whereas in federal systems the number of districts hardly ever changes, in PR systems parties emerge, disappear, merge and splinter. This may cause political instability. This makes a strong case for an apportionment method that 'is robust under a changing composition of parties, and that does not encourage fragmentation." That is, one that encourages coalition forming.

The observations above make a strong case to use Jefferson's method in countries where proportional representation systems are used. Jefferson's method encourages coalition forming, favors larger parties, and yet gives all (smaller) parties their due as it satisfies lower quota. Of course, it also avoids all monotonicity paradoxes. In pure PR systems, such as the Netherlands' Second Chamber or the Israeli Knesset, Jefferson might be the preferred option.

For federal systems, however, it seems safe to conclude that Webster's method is unambiguously best. Especially in light of the 'one-person, one-vote' principle, it is of paramount importance that methods are unbiased. Additionally, the essence of fair representation is that changes in population figures or vote totals accurately reflect in a seat distribution. A paradox-free method is therefore essential.

Some final remarks on two modern methods: the Quota Method and Minimax Apportionment. The Quota Method finds a severe drawback in being susceptible to the population paradox. It offers only one advantage over Webster's method, it stays within quota, but for all practical purposes so does Webster.

The overall evaluation for Minimax Apportionment is hard to compile, as many properties depend on which criteria are chosen in what order. Although its general framework is appealing in its flexibility, the algorithm also does not provide any clear advantages over either Jefferson's or Webster's method. Moreover, it is computationally expensive to execute.

These paragraphs clearly show that regardless of all mathematical and empirical arguments, the final say about apportionment is still distinctly political in nature. And on that note, we now turn to the second, practical part of this thesis where we focus on the Dutch political system.

Part II

Apportionment in Practice

4 Analyzing the Dutch Apportionment Procedure

4.1 Introduction

In the previous chapters we looked at apportionment from a rather theoretical point of view. Still, apportioning seats is a distinctly practical, and particularly political affair as well. To gain more insight into this practical aspect, in the second part of this thesis I will present the case of apportionment in the Dutch legislative system. The Netherlands uses two different apportionment methods: Hamilton's method for any body smaller than 19 seats (only municipalities) and Jefferson's method for every body of 19 seats or more (including large muncipalities, the Second Chamber and European Parliament); see Appendix A, articles P7 and P8. In the Netherlands these methods go under the name of Major Fractions (Hamilton) and Largest Averages (Jefferson). An important feature in the Dutch system is the imposition of minimum thresholds. In small municipalities the threshold is 75% of one full seat; i.e., a party may only receive a seat if $v_i > \frac{3}{4} \cdot v/h$ (Appendix A, article P7). In the remaining councils the threshold is set at one full seat (Appendix A, article P8).

Nevertheless, previously we observed that both Hamilton's and Jefferson's method violate several desirable criteria. Hamilton is susceptible to the three monotonicity paradoxes (Section 3.2); Jefferson is known to violate quota and be biased towards larger parties (Sections 3.3 and 3.4). The aim of this chapter is therefore to investigate the extent to which these criteria violations occur in the history of apportionment in the Netherlands. The case of small municipalities will illustrate the behavior of Hamilton's method; the Dutch Second Chamber will illustrate Jefferson's.

In both cases I will run simulations with alternative apportionment methods and put the results into the broader political context. Based on these results I will conclude this chapter by answering the question whether another apportionment method would be more appropriate in the Dutch situation.

4.2 Hamilton's Method for Small Councils

4.2.1 Problem Statement

The most notable flaws of Hamilton's method are the three monotonicity paradoxes (population, Alabama and new states). Apportionment in the PR system of the Netherlands is extremely dynamic with parties entering and leaving and voters' allegiance constantly shifting. In this context the population paradox is the most likely to occur and easy to identify.

The Dutch system is theoretically susceptible to the Alabama paradox, but instances are virtually impossible to detect. As the population of a municipality grows, the number of seats in the council grows by two seats for roughly every 5,000 citizens. From one election to the next, a municipality could gain two new seats, creating the right circumstances for the paradox. Note that seats are only added to a council before a new election, never during a council's term. However, over subsequent elections the votes for each party is a particularly volatile parameter. This makes it impossible to assess whether a party lost a seat due to the Alabama paradox, because it lost votes, or as the result of a change in any other parameter.

The same problem occurs with the new states paradox. Recall that this paradox occurs when a state or party enters or leaves the system – a relatively common phenomenon in a PR system. Again, since the number of votes for each party changes over subsequent elections, it is difficult to judge where a change in apportioned seats originates.

The population paradox occurs precisely because of changes in votes. Changes in other parameters can be compensated. The change in house size can be easily eliminated by only focusing on municipalities with constant house size over subsequent elections. The change in number of parties is a minor concern: Dutch voters are relatively faithful to a set of five to six political parties. Moreover, the effect can always be reconstructed afterwards by checking instances of the Population Paradox for effects of changing party numbers.

In sum, the question I will aim to answer is, if instances of the population paradox are found, is this reason enough to change the apportionment method in smaller councils in the Netherlands?

4.2.2 Experiment Setup

The experiment consisted of two parts: first, data acquisition, which was followed by the programming of routines that could sift through large quantities of data in search of the paradox.

As for the election data, only the results of the 2002 and 2006 elections were available digitally, and then only partially. The first step was therefore to digitalize the data of a number of elections enough to provide a representative sample. I chose to do five elections: 1982, 1986, 1990, 1994 and 1998. In 1982 the Netherlands still counted over 750 municipalities, which dropped to just over 500 in 1998 in a process of merging municipalities (in Dutch: 'gemeentelijke herindeling'). In each election roughly 15 parties participated, each represented by a (party_name, v_i) entry. I augmented the election outcome with meta-data to ensure the correctness of the data.[12] In total the data acquisition comprised the compilation of a 3250×20 matrix, or roughly 65,000 entries.

To find instances of the population paradox, I implemented several routines, which fall into two categories: apportionment methods and paradox-finding routines. I have used the Python programming language. The implementation of the apportionment methods

[12]For instance, the total number of valid votes in each municipality was added to compare to the sum of the vote totals of the individual parties.

is straightforward and simply follows the pseudo-code of Chapter 2. The routine to identify paradoxes took two election outcomes as input and consisted of two steps; see Appendix C for pseudocode of this main routine.

First, accounting for changes in other parameters, the routine selects all relevant municipalities. 'Relevant' entails that a municipality existed from one election to another, had less than 19 seats in both elections (so that Hamilton was used both times for the apportionment), and actually had the same number of seats in the body (so as to exclude interference by Alabama occurrences).

The second step of the algorithm performs the identification of paradoxes. The routine used the criterion that a party i grew stronger than party j, yet lost a seat to j. A corresponding case was formulated in case of negative growth. Growth of a party i was defined as the relative difference between the vote total of i in the first election and the vote total in the subsequent election.

The population paradox knows degrees of severity. The strongest case is the population paradox as described in Section 3.2 where party i loses a seat and party j wins a seat. However, this criterion does not exclude the possibility that party i neither gains nor loses, but party j wins even though it has a smaller growth rate. I will call the original formulation the 'strong population paradox' and the latter case the 'weak population paradox'. Observe how the weak version knows degrees in an of itself, as measured by the ratio `growth_rate_i / growth_rate_j`: the larger the ratio, the stronger the weak population paradox is.

4.2.3 Results & Analysis

In the period 1982 – 2006 I have not found an instance of the strong population paradox, yet I did find many instances of the weak paradox. Both observations are not entirely surprising. In a 2006 letter the then Dutch Minister of the Interior, Mr. Remkes, already noted that only "in exceptional cases a party can lose a seat to another party" as a result of the population paradox [32]. Although examples of the population paradox may be (readily) available in federal systems (Section 3.2), two particulars of PR systems make it less likely to find the strong paradox in Dutch municipalities.

First, population figures usually change less dramatically than voting outcomes. A vote moves faster from party to party, than a citizen does from state to state. For federal systems it is therefore more likely that a *ceteris paribus* assumption between subsequent apportionments is relatively accurate. On the contrary, for the strong population paradox to occur in PR systems it is necessary that (i) the total number of votes increases (decreases), (ii) a larger and a smaller party both receive more (less) seats, (iii) the larger party more so than the smaller on a percentage basis, and (iv) all other parties remain relatively unaffected. In a PR system the conditions (ii), (iii) and (iv) are likely to be more volatile and paradoxes less likely to occur.

The second reason is a characteristic of the small number of seats and parties in the Dutch municipalities where Hamilton is applied. As said, many criteria need to be satisfied for the population paradox to occur. In the US House of Representatives there are many more states (50) and seats (435), than parties ($\sim$ 6) and seats (max. 17) in the Dutch system. As a result, in small PR councils there is a much more limited set of combinations of parties, growth rates and apportionments to display the strong population paradox.

On the other hand, there are many instances of the weak paradox, another foreseeable fact. First let us look at some powerful examples of the weak population paradox. In the period 1982 – 2006 there were nine examples of the weak paradox where the ratio between growth rates of party A and B exceeded a threshold of 10. Three typical instances are:

1982 – 1986: Loppersum

- The votes for the PvdA party increase from 834 to 981 (+17.6%), yet the party stays at 4 seats.
- The votes for the VVD party increase from 339 to 340 (+0.3%), resulting in an increase from 1 seat to 2 seats.

1990 – 1994: Niedorp

- The votes for the CDA party increase from 1264 to 1286 (+1.7%), through which it holds on to 4 seats.
- The votes for the VVD party increase from 746 to 901 (+20.8%), but the party loses a seat going from 3 seats to 2.

1994 – 1998: Voerendaal

- The votes for the VVD party increase from 504 to 588 (+16.7%), yet the party is once again apportioned 1 seat.
- The votes for the D66 party increase from 1157 to 1165 (+0.7%), boosting its apportionment from 2 to 3 seats.

The source of the weak paradox is identical to the strong paradox. Consider the municipality Voerendaal, where the increase of 84 votes for the VVD party translated to a growth rate of 16.7% and an increased quota from $q_{VVD} \approx 0.97$ in 1994 to $q_{VVD} \approx 1.24$ in 1998. The D66 party saw a much smaller increase from 1157 to 1165 votes. These 8 votes on a larger total yielded a growth rate of only 0.69% and a quota increase from $q_{D66} \approx 2.29$ to $q_{D66} \approx 2.45$. Still, D66 gained a seat while the VVD remained stuck at 1. The underlying cause is the decrease of the total number of valid votes, from 7783

(1994) to 7126 (1998), a negative growth rate of -8.4%. As a result, the relative shares of both the VVD and D66 increased, and more so for the larger party D66. Hence, D66 stays ahead of VVD in terms of fractional remainders.

Although the source is the same, and the three examples above are very powerful, the weak paradox is certainly not the strong paradox. One could even question whether the weak paradox is a paradox at all. An obvious objection is that the 'disadvantaged' party of the most recent election may have been the advantaged party of the election before. Suppose A was such an advantaged party and received one of the remaining seats in step 3 of Hamilton's algorithm. Assume B was disadvantaged in not receiving such a seat, but having to be satisfied with its lower quota $\lfloor q_B \rfloor$. Party A may then grow stronger than B but not enough to reach the threshold to gain yet an extra seat. Party B on the other hand may already have been at the threshold and only needs a small increase to pass it and get an extra seat. The weak population paradox may be nothing more than setting the record straight.

This actually explains the weak paradox in Niedorp from 1990 to 1994. The party CDA wins marginally, going from 1264 votes to 1286 (+1.7%). The party VVD has a more substantial increase from 746 tot 901 (+20.8%). The total number of votes, however, increased even stronger, from 4187 to 5308 (+26.8%). The relative shares of both the CDA and the VVD thus decrease, as is reflected in the quotas: q_{CDA} from 4.53 to 3.63, q_{VVD} from 2.67 to 2.54. By the previous reasoning, the CDA, being the larger party, should be affected more by this decrease, the more so because they get less extra votes in absolute terms. However, looking at all the quotas, in 1990 the CDA was on the wrong end of the fractional remainders being the first to have received an extra seat should one more have been given out. In 1994 it was on the right end, being the last to receive an extra seat. The record was set straight.

As a result, there are many instances of the weak paradox by virtue of the process of approximation that lies at the heart of apportionment: some parties are slightly advantaged, others are not. This is adjusted when the next election sways the public opinion. The weak paradox is also not limited to Hamilton's method, divisor methods also display the phenomenon; see Figures 20 and 21. Observe how Webster displays the weak population paradox in Loppersum. Webster, Dean and Hill all display the weak population paradox in Voerendaal.

4.2.4 In Sum: Hamilton's Method for Small Councils

The recent history (1982 – 2006) of the Dutch municipal elections does not show any occurrence of the strong population paradox. The reasons are clear. First of all, the parameters of PR systems are dynamic; the population paradox needs some stability to occur and be detected. Second, small councils with few parties limit the occurrences of a strong population paradox even further.

There are, however, many instances of the weak paradox, but as a natural phenomenon

	v_i		Jefferson		Webster		Hill		Dean		Adams	
i	'82	'86	'82	'86	'82	'86	'82	'86	'82	'86	'82	'86
CDA	799	859	4	4	4	4	3	4	3	4	3	4
PvdA	834	981	4	5	4	4	4	4	4	4	4	4
VVD	339	340	1	1	1	2	2	2	2	2	2	2
GPV	183	207	1	1	1	1	1	1	1	1	1	1
O	173	0	1	0	1	0	1	0	1	0	1	0
Total	2328	2387	11	11	11	11	11	11	11	11	11	11

Figure 20: Loppersum 1982 – 1986.

	v_i		Jefferson		Webster		Hill		Dean		Adams	
i	'94	'98	'94	'98	'94	'98	'94	'98	'94	'98	'94	'98
CDA	2263	1583	4	3	4	3	4	3	4	3	4	3
PvdA	971	1333	2	3	2	3	2	3	2	3	2	3
VVD	504	588	1	1	1	1	1	1	1	1	1	2
D66	1157	1165	2	2	2	3	2	3	2	3	3	2
O	2888	2457	6	6	6	5	6	5	6	5	5	5
Total	7783	7126	15	15	15	15	15	15	15	15	15	15

Figure 21: Voerendaal 1994 – 1998.

in apportionment this is not alarming. Not only Hamilton's method displays the weak population paradox, divisor methods do so as well.

Still, the objection against using Hamilton's method holds: although the likelihood of the paradox occurring seems limited, the method does not accurately reflect changes in vote totals. On the basis of these results, however, there seems to be no compelling argument to change the apportionment method for small Dutch municipalities.

4.3 Jefferson's Method for the Second Chamber

4.3.1 Problem Statement

Section 3.7 made a case for Jefferson's method in PR systems. Still, the veracity of that claim may differ per political system. There are at least two question marks over the case for Jefferson.

The first question concerns Jefferson's bias towards larger parties. Unless there is a very good reason, there is no a priori reason that a method should be biased towards either smaller or larger parties. The reason for the Dutch PR system is that it would ensure more political stability. It represents a general shift of seats towards large, established parties, and is claimed to have two important consequences. First, a bias allows fewer small, extremist parties to enter the system and, second, coalition forming is easier because of the increased size of large parties. I will investigate these claims.

The second question revolves around Jefferson's quota violations. A specific issue here is the combination of quota violations and the bias of Jefferson's method. Since Jefferson cannot violate lower quota, only violations of upper quota can occur. But because Jefferson has a bias towards larger parties this means that the violations are more likely to occur for large parties. I will identify if and how many times quota violations have occurred.

Therefore in this section I will compare Jefferson's results to apportionments produced by two other methods: Webster, as this is the most suitable divisor method, and Minimax Apportionment, for its more unconventional nature. I will focus on these two questions in particular aiming to answer the question whether Jefferson's method is the most suitable apportionment method for the Dutch Second Chamber.

4.3.2 Experiment Setup

The experimental setup is similar to the one in the previous section in that it consists of two parts: data acquisition and simulations with several apportionment methods.

Contrary to the election data for municipalities, the apportionments for the Dutch Second Chamber are digitally available since 1946 [11]. I have looked at 50 years of election data, i.e., the period 1956 – 2006, enough for a representative sample. Also, in 1956 the size of the Dutch Second Chamber was increased from 100 to 150 seats, which accounts for a natural starting point. In total, this period entails 16 elections with an average of 21.3 parties participating per election.

The implemented routines need to answer the two questions concerning the consequences of bias and quota violations. I did so by computing for each election outcome the apportionment (i) for pure Jefferson, (ii) for Jefferson with minimum requirements, (iii) for Jefferson with minimum requirements and the use of list combinations (i.e., the actual apportionment), (iv) for pure Webster, (v) for Webster with minimum requirements, and (vi) for Minimax Apportionment with minimum requirements (both $FN\beta$ and $F\beta N$ criteria orders).

On the basis of these results I checked for bias and quota violations. For bias, I defined a large party as a party above the median, a small party as below the median. A bias is the probability that as a party you obtain more or less seats than entitled to by quota. I compensated for the imposition of minimum requirements – which are obviously against small parties – by evaluating only those parties that exceeded the minimum requirement. Also for quota violations I compensated for minimum requirements. Quotas were re-computed after eliminating the parties that did not reach the minimum requirement.

4.3.3 Results & Analysis

Bias. The period 1956 – 2006 confirms many of the theoretical results of Section 3.3. Figure 22 summarizes the most important findings.

	Jefferson			Webster		Minimax	
	(pure)	(minreq)	(real)	(pure)	(minreq)	$FN\beta$	$F\beta N$
Large $> q$	66.0	70.9	69.6	56.1	57.0	50.6	50.6
Small $> q$	22.3	12.7	21.5	63.3	40.5	48.1	45.6

Figure 22: Percentage Biases for Parties in Dutch Second Chamber (1956 – 2006); 'minimum requirements' abbreviated 'minreq'.

This figure displays the probability that a large or small party is apportioned more seats than its quota q. We see that Jefferson is biased towards larger parties without any significant differences between the three variations. A large party in the Dutch Second Chamber has a 19.6% bias working in its favor.

Webster is more neutral, with the important observation being that the probabilities of a large and small parties being advantaged are almost equal. The fact that $P(\text{small} > q)$ is fairly large is a direct consequence of taking away seats from large parties and distributing those to small one-seat parties. Whereas these parties i are normally eliminated because $q_i < 1$, Webster allows them representation. These are all parties that 'score' over quota, even though only by a small amount.[13] We see that this effect is indeed eliminated when imposing minimum requirements. Interestingly enough, Webster

[13]Note how in general $P(\text{large} > q) + P(\text{small} > q)$ do not sum to 1. This is because these probabilities hold for different sets of parties, i.e., the large and the small. However, $P(\text{large} > q) + P(\text{large} < q)$ *do* sum to 1.

with minimum requirements even seems to disadvantage small parties slightly, although the difference is expected to even out when taking a larger sample of election data.

Most interestingly, both criteria-orderings of Minimax Apportionment are almost completely neutral. This is not surprising: small and large parties are equally likely to minimize the maximum complaint – the defining decision criterion in Minimax Apportionment.

Bias was deliberately institutionalized for political stability, specifically because coalition forming could be easier because of larger large parties. An analysis of political history shows that Jefferson does not manifest a clear advantage over Webster in terms of encouraging coalitions. Coalitions[14] are always formed with a clear margin over the simple majority of 75 seats. Webster apportionments only shift a small number of seats: on average the difference between a Webster apportionment and a Jefferson apportionment is 3.88 seats on a total of 150. Additionally, the difference is virtually always one seat per party; in the period 1956–2006 there is only one occurrence of a two-seat difference. This means that an average coalition of three parties could typically lose three seats. This is not enough to lose the majority. In particular, Webster with minimum requirements would have lead to exactly the same coalitions.

Admitting Small Parties Pluralism is a laudable goal, but a country should remain governable and politically stable. In the Netherlands minimum requirements are imposed to keep out small and oftentimes extremist factions. The threshold of one seat is comparatively low; e.g., in Germany a 5% minimum requirement is set for elections for the Bundestag (Second Chamber) and Landestage (states). Figure 23 lists the average number of parties that would be admitted to the Second Chamber under each apportionment method.

Jefferson			Webster		Minimax	
(pure)	(minreq)	(real)	(pure)	(minreq)	$FN\beta$	$F\beta N$
10.7	10.3	10.3	12.9	10.3	10.3	10.3

Figure 23: Average number of parties in Dutch Second Chamber (1956 – 2006); 'minimum requirements' abbreviated 'minreq'.

The base level of 10.3 parties on average is determined by the minimum requirements. The pure forms of Jefferson and Webster admit extra parties in differing numbers. Without exception, however, the parties admitted extra by Jefferson (pure) and Webster (pure) are one-seat parties. Especially the pure form of Webster would allow many smaller parties: on average almost three parties extra in the Chamber as compared to

[14] 'Coalition' is used here in a very particular sense, which slightly differs from the notion in Section 3.5. A coalition here is the subset of parties that govern the country for a four-year term following an election.

the actual system in use today. In 1971, in an unusually fragmented election with many parties receiving enough votes, Webster would have admitted 19 parties in the Chamber, five more than any other method would have done. Nevertheless, the imposition of minimum requirements quickly resolves the issue.

The political upshot is that a proliferation of small parties is a realistic concern. Jefferson's bias for large parties largely resolves the issue. On the other hand, imposing minimum requirements achieves the same aim. It is unclear why both measures would have to be taken.

Quota Violations Figure 24 describes the extent to which quota violations occur under the different apportionment methods. Minimax Apportionment is not listed as these apportionments are trivially within quota.

Jefferson			Webster	
(pure)	(minreq)	(real)	(pure)	(minreq)
93.8	62.5	37.5	6.3	0

Figure 24: Percentage of Apportionments Subject to Quota Violations in Dutch Second Chamber (1956 – 2006); 'minimum requirements' abbreviated 'minreq'.

The main message is that Jefferson apportionments violate quota consistently, with or without minimum requirements. This is consistent with the simulations of Balinski and Young. Moreover, the violations are without exception in favor of the larger parties. Pure Webster shows one quota violation in the 1956 – 2006 history (an extra seat for a large party in 1986), but when imposing minimum requirements it does not. Webster with minimum requirements assigns the seats of small parties that were eliminated equally over smaller and larger parties. In general, the quota violations involve on average 1.7 seats per election – sometimes one party receives two seats over quota, more often the difference is spread over multiple parties.

There are two points of particular interest concerning quota violations: the role of minimum requirements and the role of list combinations. At first sight, the imposition of minimum requirements seems to bring both Jefferson's and Webster's apportionments more often within quota. Jefferson goes from 93.8% violations to 62.5%. Webster's case is less dramatic, but corroborates the claim.

A closer look at the data, in particular the role of list combinations, debunks the hypothesis. List combinations were not allowed in the Netherlands before 1978. The introduction of this possibility proved a watershed for Jefferson's quota violations. If we split up the results in the period before and after 1977, we see that 85.7% of the apportionments in the pre-1977 period violated quota and 0% afterwards. Minimum requirements were in place both before and after 1977.

List combinations employ the superadditivity of Jefferson's method and are used in the Netherlands almost exclusively by small parties. A list combination *de facto* creates a new large party that is a serious contender for an extra seat. Through Jefferson's bias these extra seats often go to large parties, frequently result in upper quota violations, but are also easy to migrate to parties that engage in a list combination.

These observations beg the question on list combinations. A 2010 recommendation by the Kiesraad[15] on this issue reads [23]:

> When introducing list combinations it was presumed that coalition-forming of existing parties would be encouraged and each vote would optimized. This presumption has turned out largely inaccurate – establishing list combinations is predominantly motivated by the possibility of gaining seats.

Additional arguments and counter-arguments for list combinations are given in [33]: "[The] combination possibility is not a completely satisfactory remedy for the bias of (Jefferson's method). Moreover, it is of an ad-hoc character, and it unnecessarily complicates the procedure as a whole." The latter part is undoubtedly true. Furthermore, the reason that it is not 'completely satisfactory' to remove bias lies in the fact that taking away a seat from a party that violates upper quota still leaves the party with an apportionment of upper quota or more.

Still, list combinations have a mitigating effect for quota violations – at least for Jefferson. In the overall picture, both Webster with minimum requirements, as well as Minimax Apportionment additionally stay within quota. The latter two methods, however, do not need the questionable addition in the form of list combinations.

4.3.4 In Sum: Jefferson's Method for the Second Chamber

The use of Jefferson in the Second Chamber led to two questions about its side-effects: Which effects warrant its bias towards large parties? And how prevalent are its (upper) quota violations? An analysis of Dutch Second Chamber apportionments in the period 1956 – 2006 shows that Jefferson's side-effects are mainly negative.

Jefferson's bias does prevent small parties from entering the system. However, imposing minimum requirements achieves precisely the same goal. Using both measures has the double effect of keeping out small parties, as well as advantaging large parties unduly.

The bias also does not significantly facilitate the process of coalition forming. In the Netherlands coalitions are formed with a clear margin over the simple majority of 75 seats. Webster apportionments only shift a small number of seats and never enough to lose the majority.

[15] The Kiesraad (Electoral Council) is the central electoral committee for the elections of the Lower House, the Upper House and the European Parliament. The Council determines official election results and advises on legislative questions concerning the electoral system.

As for quota violations, Jefferson's method, with or without minimum requirements, routinely violates quota and exclusively to the advantage of large parties. The introduction of list combinations changes this matter. By adding temporary large parties to the system, seats shift from large parties to combinations of small parties. Nonetheless, the tool of list combinations is complicated, of an ad-hoc character and its role may be achieved through different means.

These means are available in two alternative apportionment methods: Webster's method and Minimax Apportionment. Based on simulated apportionments with these methods, both seem attractive alternatives to Jefferson. Webster is approximately unbiased, but does allow many small parties into the system. This is resolved by imposing minimum requirements. Webster is virtually immune to quota violations – without the use of list combinations. Minimax Apportionment is also unbiased and avoids a proliferation of small parties through minimum requirements. It avoids quota violations by using the right criteria.

An overall picture emerges where Jefferson's negative side-effects are hard to justify and its positive qualities can be readily achieved by using other apportionment methods. Webster's method and Minimax Apportionment have corresponding behavior regarding bias and quota violations. Webster's method has the advantage that it is computationally much more effcient than Minimax Apportionment. On the other hand, the general framework provided by Minimax Apportionment allows for more customized apportionments.

5 Conclusion: Apportionment in Theory and Practice

Mathematical problems are often easy to state, but hard to solve. This thesis treated such a problem: translate a sequence of reals to a sequence of integers while ensuring that the sequence sums to a predetermined number. In the political arena this problem is known as the problem of *apportionment*. Based on an election outcome, how to divide the seats of a house over a number of parties, while ensuring that we reach exactly the house size? Getting the answer right here is important, because it may mean the difference between an absolute majority or not. The difference between representation or not. The difference between a conservative or a progressive vision.

But most of all, it is about political legitimacy. A well-functioning democracy sets itself apart from other government systems through transparency and accountability. Hence, it needs a mathematically sound apportionment method.

Over the course of history many solutions have been proposed under even more names. The U.S. statesmen Alexander Hamilton and Thomas Jefferson started debate on the matter when apportioning the very first U.S. House of Representatives in 1789. Their two methods gave rise to two broader classes: the Hamilton-type methods and divisor methods. More recently we became acquainted with the Quota Method, Minimax Apportionment, and avenues involving weighted voting and power indices.

Each apportionment method inevitably runs into problems. Violations of monotonicity criteria can lead to the Population Paradox, the Alabama Paradox and the New States Paradox. Some methods favor large states over small states, or vice versa. Other methods violate quota by giving a party more or less than what it is strictly entitled to. Regrettably, an impossibility result shows that there is no apportionment method that unites all possible behavior: there is no method that avoids the population paradox and always stays within quota.

In the second part of this thesis I have researched the apportionment system in the Netherlands by focusing on the question to what extent these problems have occurred. Historically, the Netherlands uses two systems: Hamilton's method for councils of 17 seats or less, Jefferson's method for all other councils. The system has two important additions in the form of imposing minimum requirements and allowing list combinations. If these problems are wide-spread, the question begs whether to look for other methods.

In a first experiment I have analyzed election data from small municipalities – the only councils of less than 19 seats – from 1982 to 2006. Focusing on Hamilton's susceptibility to the population paradox, I have not found any instances of the strong version of this paradox. The 'weak' version of the paradox did display itself frequently, but I showed that it has a common explanation in 'setting the record straight'.

In a second experiment I have looked at Jefferson's method. This method is biased towards larger parties, which is motivated under the pretext of achieving political stability. I have compared Jefferson's method to two other methods: Webster's method and Minimax Apportionment.

I have shown that Jefferson's bias indeed benefits large parties, but that its positive side-effects are either non-existent (there is no clear effect of encouraging coalition forming) or can be achieved through other means (small parties are kept out by minimum requirements). Finally, Jefferson avoids quota violations, but only for the dubious addition of list combinations.

On the other hand, Webster's method is unbiased and with the help of minimum requirements avoids a proliferation of small parties. Moreover, the method practically never violates quota – and it does so without list combinations. As the last method under consideration, Minimax Apportionment is unbiased in practice, the remaining aspects are dependent on the criteria chosen. Its main drawback may be its complicated nature and being computationally demanding.

In sum, for Dutch small municipalities there seems no reason to change the current apportionment method. For the Dutch Second Chamber, Webster is a good, unbiased alternative, especially in a streamlined version of the system without list combinations.

A Electoral Law (Kieswet)

As valid on August 4th, 2009. Lifted from http://wetten.overheid.nl/BWBR0004627/. The appendix only concerns the relevant passage concerning apportionment (Afdeling II > Hoofdstuk P > §2 > Artikelen P2 – P14):

Artikel P 2

1. Een stel gelijkluidende lijsten als bedoeld in artikel H 11, eerste lid, geldt voor de vaststelling van de uitslag van de verkiezing als één lijst.
2. Het centraal stembureau telt van deze gelijkluidende lijsten te zamen de stemcijfers en de aantallen op iedere kandidaat uitgebrachte stemmen.

Artikel P 3

Een lijstengroep als bedoeld in artikel H 11, tweede lid, geldt voor het bepalen van het aantal daaraan toe te wijzen zetels als één lijst met een stemcijfer gelijk aan de som van de stemcijfers van de lijsten waaruit de groep bestaat.

Artikel P 4

1. Een lijstencombinatie als bedoeld in artikel I 10 geldt voor het bepalen van het aantal daaraan toe te wijzen zetels als één lijst, met een stemcijfer gelijk aan de som van de stemcijfers van de lijsten waaruit die combinatie bestaat.
2. Een lijstencombinatie wordt slechts in aanmerking genomen, indien aan ten minste twee van de verbonden lijsten een zetel zou zijn toegewezen, indien geen lijstencombinaties zouden zijn gevormd. Verbonden lijsten die zelfstandig geen zetel zouden hebben verworven, worden geacht geen deel uit te maken van de lijstencombinatie.

Artikel P 5

1. Het centraal stembureau deelt de som van de stemcijfers van alle lijsten door het aantal te verdelen zetels.
2. Het aldus verkregen quotiënt wordt kiesdeler genoemd.

Artikel P 6

Zoveel maal als de kiesdeler is begrepen in het stemcijfer van een lijst wordt aan die lijst een zetel toegewezen.

Artikel P 7

1. De overblijvende zetels, die restzetels worden genoemd, worden, indien het aantal te verdelen zetels negentien of meer bedraagt, achtereenvolgens toegewezen aan de lijsten die na toewijzing van de zetel het grootste gemiddelde aantal stemmen per toegewezen zetel hebben. Indien gemiddelden gelijk zijn, beslist zo nodig het lot.
2. Indien het betreft de verkiezing van de leden van de Tweede Kamer, komen bij deze toewijzing niet in aanmerking lijsten waarvan het stemcijfer lager is dan de kiesdeler.

Artikel P 8

1. De restzetels worden, indien het aantal te verdelen zetels minder dan negentien bedraagt, achtereenvolgens toegewezen aan de lijsten waarvan de stemcijfers bij deling door de kiesdeler de grootste overschotten hebben. Hierbij worden lijsten die geen overschot hebben, geacht lijsten te zijn met het kleinste overschot. Indien overschotten gelijk zijn, beslist zo nodig het lot.
2. Bij deze toewijzing komen niet in aanmerking lijsten met een stemcijfer dat lager is dan 75% van de kiesdeler.
3. Wanneer alle lijsten die daarvoor in aanmerking komen een restzetel hebben ontvangen en er nog zetels te verdelen blijven, worden deze zetels toegewezen volgens het stelsel van de grootste gemiddelden als bedoeld in artikel P 7, eerste lid, met dien verstande, dat bij deze toewijzing aan geen van de lijsten meer dan één zetel wordt toegewezen.

Artikel P 9

Indien aan een lijst die de volstrekte meerderheid van de uitgebrachte geldige stemmen heeft verkregen, een aantal zetels is toegewezen, kleiner dan de volstrekte meerderheid van het aantal toe te wijzen zetels, wordt aan die lijst alsnog één zetel toegewezen en vervalt daartegenover één zetel, toegewezen aan de lijst die voor het kleinste gemiddelde of het kleinste overschot een zetel heeft verworven. Indien twee of meer lijsten voor hetzelfde kleinste gemiddelde of hetzelfde kleinste overschot een zetel hebben verworven, beslist het lot.

Artikel P 10

Indien bij de toepassing van de vorige bepalingen aan een lijst meer zetels zouden moeten worden toegewezen dan er kandidaten zijn, gaan de overblijvende zetel of zetels door voortgezette toepassing van die bepalingen over op één of meer van de overige lijsten, waarop kandidaten voorkomen aan wie geen zetel is toegewezen.

Artikel P 11

1. De verdeling van de aan een lijstencombinatie toegewezen zetels over de lijsten welke zijn gecombineerd, geschiedt als volgt.
2. Het centraal stembureau deelt het stemcijfer van de lijstencombinatie door het aantal aan de lijstencombinatie toegewezen zetels.
3. Het aldus verkregen quotiënt wordt combinatiekiesdeler genoemd.
4. Zoveel maal als de combinatiekiesdeler is begrepen in het stemcijfer van elk van de lijsten waaruit de combinatie bestaat, wordt aan die lijst een van de aan de combinatie toegewezen zetels toegewezen.
5. De restzetels worden achtereenvolgens toegewezen aan de lijsten van de combinatie waarvan de stemcijfers bij deling door de combinatiekiesdeler de grootste overschotten hebben. Hierbij worden lijsten die geen overschot hebben, geacht lijsten te zijn met het kleinste overschot. Indien overschotten gelijk zijn, beslist zo nodig het lot.

Artikel P 12

1. De verdeling van de aan een lijstengroep toegewezen zetels over de lijsten waaruit de groep bestaat, geschiedt als volgt.
2. Het centraal stembureau deelt het stemcijfer van de lijstengroep door het aantal aan de groep toegewezen zetels.
3. Het aldus verkregen quotiënt wordt groepskiesdeler genoemd.
4. Zoveel maal als de groepskiesdeler is begrepen in het stemcijfer van elk van de lijsten waaruit de groep bestaat, wordt aan die lijst een van de aan de groep toegewezen zetels toegewezen.
5. De restzetels worden achtereenvolgens toegewezen aan de lijsten van de groep waarvan de stemcijfers bij deling door de groepskiesdeler de grootste overschotten hebben. Hierbij worden lijsten die geen overschot hebben, geacht lijsten te zijn met het kleinste overschot. Indien overschotten gelijk zijn, beslist zo nodig het lot.

Artikel P 13

1. Indien bij de toepassing van artikel P 11 of artikel P 12 aan een lijst meer zetels zouden moeten worden toegewezen dan er kandidaten zijn, gaan de overblijvende zetel of zetels door voortgezette toepassing van dat artikel over op een van de andere lijsten van de combinatie, onderscheidenlijk van de groep, waarop kandidaten voorkomen aan wie geen zetel is toegewezen.
2. Zijn er na toepassing van het eerste lid nog zetels toe te wijzen, dan worden deze toegewezen volgens het stelsel van de grootste gemiddelden als bedoeld in artikel P 7, eerste lid.

Artikel P 14

De in de voorgaande artikelen bedoelde lotingen vinden plaats in de in artikel P 20 bedoelde zitting van het centraal stembureau.

B Banzhaf Power Index

The rationale behind voting power starts with the observation that the weight of a party or representative in a committee is not proportional to its voting power. For instance, in a two-party system where one party holds 51% of the votes, its voting power is 100% and not 51%. The other party, though holding 49% of the votes, has no influence at all. A power index is a measure to quantify voting power.

There is a sizable body of literature on voting power and different power indices. Arguably the most well-known index is named after its inventor John F. Banzhaf [8], though it was independently discovered by Coleman [13] and Penrose [31]. Other power indices are, for instance, the Shapley-Shubik power index [35] and Deegan-Packel power index [15].

The Banzhaf power index looks at the parties present in a system and considers which coalitions are 'winning' in the sense that the combined votes of all parties in the coalition exceed the threshold for passing a decision. The voting power of each party i is measured by its marginal contribution to all winning coalitions. If a coalition C is winning with the presence of i but losing without i, then party i is awarded a swing. The number of swings for a party indicates how crucial a party is: the more swings, the more powerful.

An example illustrates the matter. Consider a system with four parties A, B, C and D and suppose $\mathbf{a} = (a_A, a_B, a_C, a_D) = (4, 3, 2, 1)$. Decision making goes by simple majority, that is, there are 6 votes required to pass a decision. Figure 25 lists the winning coalitions and the swing parties for each coalition.

Winning coalition	Total votes	Swing parties
$ABCD$	10	-
ABC	9	A
ABD	8	A, B
ACD	7	A, C
BCD	6	B, C, D
AB	7	A, B
AC	6	A, C

Figure 25: An example of computing the Banzhaf power index.

In total there are 12 swings, which leads to the (normalized) Banzhaf power distribution $\beta = (\beta_A, \beta_B, \beta_C, \beta_D) = (5/12, 3/12, 3/12, 1/12)$.

C Pseudo-code For Population Paradox

This chapter describes the pseudocode for an important routine: finding the population paradox. For the computer scientist the pseudocode should be straightforward. One particular piece of notation may need explanation. Getting the value based on a key from a list of (key, value) pairs is short-handed to list[key].

Algorithm 1 FindPopulationParadox(outcome_year_1, outcome_year_2)

Input: Two subsequent election outcomes outcome_year_1 and outcome_year_2. Each election outcome is an ordered list of the form ((party_1, v_1), ..., (party_n, v_n)).

```
// Store here all instances of the paradox; initialize with an empty list.
paradoxes = ()

// The routine select_relevant selects municipalities that (i) exist in both years, (ii)
have less than 19 seats in both years, and (iii) have the same number of seats in both
years. The variable relevant_municipalities combines the outcomes for both years.
relevant_municipalities ← select_relevant(outcome_year_1, outcome_year_2)

// Run through all relevant municipalities, comparing years 1 and 2.
for all municipality in relevant_municipalities do
  // Compute the Hamilton apportionments for municipality.
  apportionment_year_1 ← hamilton(municipality)
  apportionment_year_2 ← hamilton(municipality)

  // Compute the growth rates of all parties in municipality.
  growth_rates ← compute_growth_rate(municipality)

  // Check all pairs of parties in municipality for instances of the population paradox.
  for all (party_i, party_j) in municipality do
    if growth_rates[party_i] > growth_rates[party_j] and
    apportionment_year_1[i] < apportionment_year_2[i] and
    apportionment_year_1[j] > apportionment_year_2[j] and
     then
      paradoxes.append([municipality, party_i, party_j])
    end if
  end for
end for

return  paradoxes
```

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