Context Dependence of Epistemic Operators in Dynamic Evidence Logic

MSc Thesis (*Afstudeerscriptie*)

written by

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Abstract

In this thesis it is argued, supported by the relevant alternatives theory, that epistemic operators are context dependent. They are context dependent, in the sense that the context filters out certain irrelevant alternatives from the information state of an agent. As a consequence, rational knowledge and rational beliefs are evaluated on only a subset of the information state of the agent. This filtering process of the context origins from the fact that an agent is not in all circumstances capable of overseeing every possible alternative. The alternatives an agent is capable of overseeing are determined by the context.

Furthermore, a concrete proposal of an epistemic logic that incorporates this context dependence of epistemic operators will be given. This proposal is an extension of the evidence logic of Johan van Benthem and Eric Pacuit, which basically is an epistemic logic where beliefs are formed on the basis of evidence in the agent's information state. In this proposal, called contextual evidence logic, beliefs are formed on the basis of the relevant evidence in the agents information state. Again, the relevance of the evidence is determined by the context.

Moreover, some characteristics will be exploited, the contextual evidence logic will be made dynamic in order to incorporate dynamic context changes. Finally, natural examples, of contextual influence on rational beliefs and how the contextual evidence logic will deal with this, will be provided.

Key words: dynamic epistemic logic, context, epistemic operators, relevant alternatives theory, evidence

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Chapter 1

Overview

A fundamental problem of epistemic logics is that they assume logical omniscience of the agents. That is, if q is logically implied by p and the agent knows p, the agent is assumed to know q as well.

In this thesis we approach one fallacy of logical omniscient agents with respect to human behavior. This fallacy is due to the fact that in epistemic logics an agent is assumed to know a proposition, only if she can rule out every contradicting alternative. However, as will be argued in chapter 3, humans can know a proposition without ruling out every contradicting alternative. It is enough for us to rule out only the relevant contradicting alternatives, in order to claim knowledge of that proposition.

Throughout this thesis we will be working with the assumption that epistemic operators are based on underlying evidence. This assumption will provide us with a rich logic, which is able to incorporate more information provided by the context than in normal epistemic logics as we will see in chapter 4. In this chapter we also propose a static contextual evidence logic that incorporates that agents can know, already on the basis of the relevant alternatives in the context.

However, due to an insight of Lewis ([12]), discussed in chapter 3, we learned that the relevant alternatives can change by actions that change the context. Therefore, chapter 5 provides a discussion on how different actions on the context influence the belief state of an agent.

In chapter 6 we provide some examples to convince the reader of how adding context dependence of epistemic operators to evidence logic, gives the logic the expressibility to model natural situations, which we are not able to model without taking the context dependence of epistemic operators into account.

Finally, we will discus the achievements in this thesis and give some (of the many imaginable) ideas for future work on this topic.

Chapter 2

The Logical setting

The argumentation in this thesis will mainly be guided by dynamic epistemic and doxastic logics. These logics operate on models which can be viewed as information states of agents. Agent's information states capture which propositions are *known* or *believed* by the agent. Dynamic actions have been defined on the static models, to model how certain speech acts or other actions can influence the information state of an agent by changing the model. Hereby, these logics respect the fact that information exchange processes are dynamic and, especially, that interaction changes an agents information state. See van Benthem, [19], and references in their for further readings.

In this thesis we will use two systems in particular in order to arrive at a contextual dependent epistemic logic. The first one, *Epistemic logic* ([3]), is probably the most basic. In this logic two worlds are connected in the agent's information state, if she can not distinguish between the two worlds. The second one is *Evidence Logic* ([21]), which implies a partial ordering on the set of worlds by the evidence the agent has gathered. Belief in the evidence logic is totally dependent and determined by the evidence available to the agent.

We begin this chapter with some preliminary notations, which will be maintained in the remaining of this thesis. Afterwards the basics of epistemic logic will be briefly stated. Finally, we end this chapter with an introduction to evidence logic.

2.1 Notations

In this section we will introduce some basic definitions which will be taken for granted in the remaining chapters.

We will often use the truth set of a formula, of which the definition is just as to be expected.

Definition 1 (Truth set)**.** Given some model $\mathcal{M}$ and a set of worlds W defined by this model, we have that the truth set of some formula φ evaluated in $\mathcal{M}$ equals the set of all worlds in $\mathcal{M}$ that model φ:

$$[\![\varphi]\!]_{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \models \varphi\}$$

We will write $[\![\varphi]\!]$ for $[\![\varphi]\!]_{\mathcal{M}}$ when the model $\mathcal{M}$ is clear from the context.

Furthermore, we will need a somewhat less straightforward notion of the truth set of a set of formulas. The truth set of a set of formulas will be taken to be a new set containing the truth sets of all these formulas.

Definition 2 (Truth set of a set of formulas)**.** Given some model $\mathcal{M}$, we have that the truth set of a set of formulas Φ evaluated in $\mathcal{M}$ equals the set of all the truth sets of $\varphi \in \Phi$:

$$[\![\{\varphi_1, ..., \varphi_n\}]\!]_{\mathcal{M}} = \{[\![\varphi_1]\!]_{\mathcal{M}}, ..., [\![\varphi_n]\!]_{\mathcal{M}}\}$$

Just as before, we will write $[\![\Phi]\!]$ for $[\![\Phi]\!]_{\mathcal{M}}$ when the model $\mathcal{M}$ is clear from the context.

The information that is maintained by keeping a set of sets instead of combining the information into the union of the truth sets will play a key role in this thesis. However, in some cases we will find ourselves in situations where the combined information is the only relevant information. Therefore, we want to be able to take the union of a sets of sets.

Definition 3 (Union of a set of sets)**.** Let $\mathbb{X} = \{X_1, ..., X_n\}$ be a set of sets. The union of $\mathbb{X}$ equals the union of all the sets in $\mathbb{X}$:

$$\bigcup \mathbb{X} = \bigcup_{X \in \mathbb{X}} X$$

The same holds for the intersection of a set of sets.

Definition 4 (Intersection of a set of sets)**.** Let $\mathbb{X} = \{X_1, ..., X_n\}$ be a set of sets. The intersection of $\mathbb{X}$ equals the intersection of all the sets in $\mathbb{X}$:

$$\bigcup \mathbb{X} = \bigcap_{X \in \mathbb{X}} X$$

Finally, we also want to be able to restrict a set of sets with respect to some other set.

Definition 5 (Restriction of a set of sets)**.** Let $\mathbb{X} = \{X_1, ..., X_n\}$ be a set of sets and Y a set. The restriction of $\mathbb{X}$ to Y equals the set of the sets in $\mathbb{X}$ intersected with Y:

$$\mathbb{X}\restriction_Y = \{X_1 \cap Y, ..., X_n \cap Y\}$$

2.2 Epistemic Logic

We will shortly recall the basic epistemic logic. For more extensive readings on epistemic logic we refer to [19] and further literature in there.

The model of epistemic logic is defined as follows.

Definition 6 (Epistemic model)**.** Given a fixed set of propositional variables At and a set of agents Ag. An epistemic model is a tuple $\mathcal{M} = \langle W, \{\sim_i \mid i \in Ag\}, V\rangle$, where

- W is a non-empty set of worlds,
- $\sim_i \subseteq W \times W$ is an epistemic accessibility relation,
- $V : At \to \wp(W)$ is a valuation function.

As explained in the introduction of this chapter, we have that agent i is unable to distinguish between two worlds $w, v \in W$, if they are connected trough her epistemic accessibility relation $w \sim_i v$. That an agent is unable to distinguish between the actual world w and some other world w', means that the agent does not know whether w or w' is the actual world.

Next, we define a language on the epistemic model, which will give us the tools to make the above idea explicit.

Definition 7 (Epistemic language)**.** Given a fixed set of propositional variables At and a group of agents Ag, with $i \in Ag$. Let $\mathcal{L}^K$ be the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_i\varphi$$

The truth definitions are as follows.

Definition 8 (Truth)**.** Let Ag be a set of agents and $\mathcal{M} = \langle W, \{\sim_i \mid i \in Ag\}, V\rangle$ be an evidence model. Truth of a formula $\varphi \in \mathcal{L}^K$ is defined inductively as follows:

$$\begin{array}{lll}
\mathcal{M}, w \models p & \textit{iff} & w \in V(p) \quad \text{(with } p \in At) \\
\mathcal{M}, w \models \neg\varphi & \textit{iff} & \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \varphi \wedge \psi & \textit{iff} & \mathcal{M}, w \models \varphi \ \&\ \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models K_i\varphi & \textit{iff} & \text{for all } t \in W \ \textit{s.t.}\ s \sim_i t \text{ we have that } \mathcal{M}, t \models \varphi
\end{array}$$

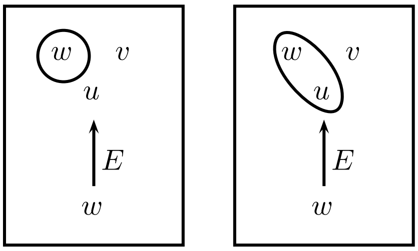
The truth conditions of the connectives are as to be expected. The knowledge operator states that agent i knows φ if all worlds that she can not distinguish from the actual world make true φ[1]. Namely, if all worlds, that are indistinguishable from the actual world for the agent, agree on the truth value of φ, then there is no doubt whatsoever about φ being true or false. Therefore, dependent on whether the worlds all agree that φ is true or they all agree that φ is false, the agent knows φ or the agent knows $\neg\varphi$.

2.3 Evidence Logic

In the previous section we saw that the epistemic model models the information states of a group of agents by connecting two worlds for some agent, if the agent can not distinguish between these worlds and the actual world. However, as we normally assume our agents to be rational, there should be underlying reasons that an agent is able to distinguish between certain worlds and not between others. For example, by looking out of the window we can see that it is not raining outside. Therefore, we are now able to distinguish the actual world, where it is not raining, from all the possible worlds where it is raining. Thus, we now know that it is not raining. All possible sorts of (rational) reasons, that make it possible for agents to distinguish between the actual worlds and other possible worlds, will be referred to as evidence. Johan van Benthem and Eric Pacuit ([21]) propose to represent such a piece of evidence an agent receives as the set of all the worlds where the piece of evidence provides evidence for.

Example 1. For examples of evidence, see figure 4.3.2, where 'Bush is on television' is true at w, 'Janet Jackson is on television' is true at v and 'Will Ferrell is on television' is true at u.

[1]Note that she can for sure not distinguish between the actual world and the actual world.



(a) There is evidence for ‘Bush is on television’.
(b) There is evidence for ‘There is a man on television’.

Figure 2.1: example of evidence representation.

The evidence model is a based on neighborhood models. In neighborhood models we have that the modal relation is between worlds and sets of worlds, instead of between worlds and worlds, like in classical modal logics. In the evidence model this means that the evidence relation is a relation between a world and the pieces of evidence that are available from world. For motivation and mathematical properties of neighborhood models see [13] and [10] and references in there.

Before we will come to the more powerful evidence logic about beliefs as proposed by van Benthem and Pacuit ([21]), we will first describe how the evidence logic will look like according to us, when the evidence is so reliable that it provides actual knowledge to the agent instead of only believes.

Definition 9 (Evidence model for knowledge)**.** Given a fixed set of propositional variables At and a group of agents. An evidence model for knowledge is a tuple $\mathcal{M} = \langle W, \{E_i \mid i \in Ag\}, V\rangle$, where

- W is a non-empty set of worlds,
- $E_i \subseteq \{(w, X) \mid w \in W \;\&\; X \in \wp(W) \;\&\; w \in X\}$ is an evidence relation,
- $V : At \to \wp(W)$ is a valuation function.

By assumption, we cannot know something which is actually untrue. Thus, when our knowledge is based on evidence, the evidence can only give truthful information. Therefore, we have that all the evidence sets for an arbitrary world must contain that world[2].

[2] We do not need all the evidence sets for an arbitrary world to contain that world, in order to get only truthful evidence. It is enough to ensure that every piece of evidence in the actual world contains the actual world. However, we chose this stronger definition since, then, the actual world does not need to be specified in the model.

The language will be the same as the language for the epistemic model $\mathcal{L}^K$.

Definition 10 (Evidence language for knowledge)**.** Given a fixed set of propositional variables At and a group of agents Ag, with $i \in Ag$. Let $\mathcal{L}^{EK}$ be the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_i\varphi$$

The truth definitions for the logical connectives are also the same as for the epistemic logic (see definition 8). The only difference is the truth definition for the knowledge operator K_i.

Definition 11 (Truth)**.** Let $M = \langle W, E, V \rangle$ be an evidence model for knowledge. Truth of a formula $\varphi \in \mathcal{L}^{EK}$ is inductively defined by definition 8 for the logical connectives and by the following truth definition for the knowledge operator:

$$\mathcal{M}, w \models K_i\varphi \quad \textit{iff} \quad \bigcap E_i(w) \subseteq [\![\varphi]\!]$$

Since every piece of evidence provides only truthful information to the agent, we have that the agent only can not distinguish between worlds that are in the intersection of all the evidence pieces. For all the other worlds not in the intersection, there is at least one piece of information which distinguishes this world from the actual world. Thus, in the same way as above we have that the agent knows φ if all the worlds that are not distinguishable by the evidence from the actual world agree on φ being true.

Note, that since every evidence is taken to provide only truthful information it cannot be that the intersection of all the evidence pieces is empty.

As we already noted above, the evidence model for knowledge as described here follows easily from the evidence model proposed by van Benthem and Pacuit in [21]. However, van Benthem and Pacuit do not work with this strong reliable evidence, which is needed for knowledge. The evidence in their model only has to be strong enough for an agent to form beliefs. Thus, beliefs of rational agents are based on evidence, just as knowledge of rational agents is based on evidence.

The consequence of working with evidence that only has to be strong enough for an agent to form beliefs based on this evidence, is that pieces of evidence can contradict each other. Namely, if agents are confronted with evidence which contains untrue information, it is no longer guaranteed that there is one world which is contained by every piece of evidence. In this way, it becomes possible that pieces of evidence contradict each other.

The evidence model, which capture the evidence pieces that underly the agent's beliefs is defined as follows.

Definition 12 (Evidence model)**.** Given a fixed set of propositional variables At and a group of agents Ag. An evidence model is a tuple $\mathcal{M} = \langle W, \{E_i \mid i \in Ag\}, V \rangle$, where

- W is a non-empty set of worlds,
- $E_i \subseteq W \times \wp(W)$ is an evidence relation,
- $V : At \to \wp(W)$ is a valuation function.

An evidence model for beliefs, which we will abbreviate with 'evidence model', looks similar to an evidence model for knowledge, except for the fact that the evidence relation of an evidence model is not restricted to pieces of evidence which contain truthful information. In the evidence model every possible set of worlds can represent a piece of evidence in every situation.

The evidence language which is used on this evidence model is as follows.

Definition 13 (Evidence language)**.** Given a fixed set of propositional variables At and a group of agents Ag, with $i \in Ag$. Let $\mathcal{L}^E$ be the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_i\varphi \mid B_i\varphi$$

The truth definitions for the logical connectives are as usual (see definition 8). We will only state the definitions for the evidence modality ($\Box_i$) and the belief operator (B_i). However, in order to give the truth definition, we need the notion of maximal finite intersection property ([21]).

Definition 14 (Finite intersection property)**.** A set of sets of worlds $\mathbb{X} \subseteq \wp(W)$ has the finite intersection property (f.i.p.) if for all finite subsets $\mathbb{Y} \subseteq X$ we have that $\bigcap \mathbb{Y} \neq \emptyset$.

The set $\mathbb{X}$ has the maximal finite intersection property (max f.i.p.) if $\mathbb{X}$ has the f.i.p. and there is no $\mathbb{Y}'$ such that $\mathbb{X} \subset \mathbb{Y}'$ and $\mathbb{Y}'$ has the f.i.p.

Now we can define the truth definitions for the evidence modality and the belief operator.

Definition 15 (Truth)**.** Let $M = \langle W, E, V\rangle$ be an evidence model. Truth of a formula $\varphi \in \mathcal{L}^E$ is inductively defined by definition 8 for the logical connectives and by the following truth definitions for the evidence modality and knowledge operator:

$$\mathcal{M}, w \models \Box_i\varphi \quad \textit{iff} \quad \exists X{\in}E_i(w) \ s.t. \ X \subseteq [\![\varphi]\!]$$
$$\mathcal{M}, w \models B_i\varphi \quad \textit{iff} \quad \text{for each max f.i.p. } \mathbb{X} \subseteq E_i(w) \text{ we have that } \bigcap \mathbb{X} \subseteq [\![\varphi]\!]$$

Thus, we have $\Box_i\varphi$ if and only if i has a piece of evidence in the model which supports the formula φ. From this definition it follows that $\Box_i$ satisfies upward monotonicity, thus from $\varphi \to \psi$ we can infer $\Box_i\varphi \to \Box_i\psi$. Although the pieces of evidence in the evidence model do not satisfy upward monotonicity, which means it is not the case that for $X, X' \subseteq W$ such that $X \subseteq X'$ we have that wEX implies wEX', we do have upward monotonicity for the evidence operator in the language.

Note as well that by definition 15 we do not have that $(\Box_i\varphi \wedge \Box_i\psi) \to \Box_i(\varphi \wedge \psi)$. Thus, if an agent has evidence that there is a man on television and the evidence that either Bush or Janet Jackson is on television, then the agent does not have evidence that Bush is on television. However, when only the above mentioned evidence is provided, the set of the two pieces of evidence above has the max f.i.p. in this example. Furthermore, it would be the only set that has the max f.i.p. and it contains all the worlds in which Bush is on television, thus the agent in this example would believe that Bush is on television.

Consequently, the evidence modality only satisfies the principles of the classical modal logic. However, when we adopt the assumption of van Benthem and Pacuit that an agent knows her space and therefore the whole universe W is an evidence set for every agent[3], the belief operator satisfies the modal logic KD. That is, the belief operator satisfies $B_i(\varphi \to \psi) \to (B_i\varphi \to B_i\psi)$, since if all φ worlds in the intersection of the max f.i.p. sets are also ψ worlds and there are only φ worlds in these intersections of the max f.i.p. sets, then all these worlds are ψ worlds. The belief operator satisfies $B_i\varphi \to \neg B\neg\varphi$. Since, if all worlds in the intersections of the max f.i.p. sets satisfy φ and by the assumption that there is at least one max f.i.p. (by definition with a non empty intersection), the worlds in the intersection satisfy φ and not $\neg\varphi$. Thus, we have $\neg B\neg\varphi$. Finally, the belief operator satisfies the rule of necessitation. If φ is true in every world than it must be true for every world in the intersections of all max f.i.p. sets, and therefore

[3] Actually it is enough to suppose that every agent has at least one evidence set, in order for the belief operator to satisfy the axiom in KD.

the agent will belief φ[4].

2.3.1 Public Announcements

In chapter 5, we will discuss the dynamics we propose on the extended evidence logic with respect to public announcements. Therefore, we briefly recall some basics of public announcements ([3]) as dicussed by van Benthem and Pacuit ([21]).

Definition 16 (Updated model for public announcements)**.** Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model, Ag a group of agents and $\varphi \subseteq \mathcal{L}^E$ a context. The model updated with the public announcement φ equals $\mathcal{M}^{!\varphi} = \langle W^{!\varphi}, \{E_i^{!\varphi} \mid i \in Ag\}, V^{!\varphi} \rangle$. With:

- $W^{!\varphi} = [\![\varphi]\!]$,
- $E^{!\varphi}(w) = \{Y \cap [\![\varphi]\!] \mid \exists Y \in E(w) \ s.t. \ Y \cap [\![\varphi]\!] \neq \emptyset\}$ for all $w \in W$,
- $V^{!\varphi}(p) = V(p) \cap [\![\varphi]\!]$ for all $p \in A$t.

In order to be able to state the reduction axiom for public announcements, we have to extend the language with conditional evidence.

Definition 17 (Evidence language for public announcements)**.** Given a fixed set of propositional variables At and a group of agents Ag, with $\in Ag$. Let $\mathcal{L}^{PE}$ be the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_i \varphi \mid \Box_i^{\varphi} \psi$$

The truth definitions are the same as the truth definitions for the evidence language, only we need to define the truth definition for conditional evidence.

Definition 18 (Truth)**.** Let $M = \langle W, E, V \rangle$ be an evidence model. Truth of a formula $\varphi \in \mathcal{L}^{PE}$ is inductively defined by definition 15 for the logical connectives and the evidence modality and by the following formula for the conditional evidence modality:

$$\mathcal{M}, w \models \Box_i^{\psi} \varphi \quad \textit{iff} \quad \exists X {\in} E_i(w) \ s.t. \ \emptyset \neq (X \cap [\![\psi]\!]) \subseteq [\![\varphi]\!]$$

[4] Note that for the rule of necessitation we do not need the assumption that every agent has at least one evidence set. If an agent has no evidence sets, there are no max f.i.p. sets, thus every max f.i.p. set satisfies the requirement for belief. Therefore, $B - i\varphi \to \neg B_i \neg\varphi$ is the only axiom that does require this assumption.

We extend the language with a corresponding action, $[!\varphi]\psi$, for public announcements. The truth condition for this action is straightforward.

$$\mathcal{M}, w \models [!\varphi]\psi \quad \textit{iff} \quad \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}^{!\varphi}, w \models \psi \tag{2.1}$$

We then have that the reduction axioms look as follows (see [21] for proofs).

(PA1)	$[!\varphi]p$	$\leftrightarrow$	$(\varphi \rightarrow p)$
(PA2)	$[!\varphi](\psi \wedge \chi)$	$\leftrightarrow$	$([!\varphi]\psi \wedge [!\varphi]\chi)$
(PA3)	$[!\varphi]\neg\psi$	$\leftrightarrow$	$(\varphi \rightarrow \neg[!\varphi]\psi)$
(PA4)	$[!\varphi]\Box_i\psi$	$\leftrightarrow$	$(\varphi \rightarrow \Box_i^{\varphi}[!\varphi]\psi)$
(PA6)	$[!\varphi]\Box_i^{\psi}\chi$	$\leftrightarrow$	$(\varphi \rightarrow \Box_i^{\varphi \wedge [!\varphi]\chi}[!\varphi]\psi)$

Chapter 3

Context Dependence of Epistemic Operators

In this chapter we will argue that epistemic operators like knowledge and beliefs are context dependent. We start with an argumentation that knowledge and beliefs are dependent on relevant alternatives, based on the relevant alternatives theory. Subsequently, we will argue that the context determines which alternatives are relevant and which are not. These two arguments in the end lead to the conclusion that knowledge and beliefs are dependent on the context.

3.1 The relevant alternatives theory for knowledge

Consider the following example:

Zebra You are in the zoo and looking at the zebras. Do you know that the animals you are looking at are zebras? Probably you do, since they look like zebras and the zoo has information available, telling you that they are zebras. However, what would you say when I ask you whether you know that they are not mules cleverly disguised as zebras? The evidence that leads you to think that they are zebras, provides evidence for the hypothesis that the animals are mules cleverly disguised as zebras as well. (example by Fred Dretske, [5, p.1015])

This example is introduced by Fred Dretske in order to motivate the relevant alternative theory (RAT). He claims with this example that an agent who knows p, does not necessarily have to know $\neg q$, even though q

and p are incompatible. That is:

$$\text{If } p \text{ implies } \neg q\text{, then } Kp \text{ does } \textit{not} \text{ impy } K\neg q.$$

Namely, in the example, you know that the animals you are looking at are zebras. However, this does not imply that you know that the animals you are looking at are not mules cleverly disguised as zebras. Nevertheless, it is definitely the case that, if the animals you are looking at are zebras, they are not mules cleverly disguised as zebras.

Dretske explains his claim that knowledge is not closed under implication, by the relevant alternative theory. He argues that: "To know that x is A is to know that x is A within a framework of relevant alternatives, B, C and D." ([5, p.1022]). Therefore, an irrelevant alternative E can exists, such that E is incompatible with A and the agent does not know that x is not E, and nonetheless the agent knows that x is A within the framework B, C and D.

David Lewis ([12]) agrees with Dretske ([5, 4]) that knowledge is relative to the relevant alternatives: "S knows that P iff S's evidence eliminates every possibility in which not-P - Psst! - except for those possibilities that we are properly ignoring." ([12, p.566]). However, a crucial difference is that, within the relative alternative theory, Lewis upholds that knowledge is closed under implication. According to him, it is the context (or circumstance of evaluation) that determines the set of relevant alternatives instead of the proposition itself.

Explained in the zebra example this means that at first you know that you are looking at zebras, within the relevant alternatives. However, the moment that I ask you whether you know that they are not mules cleverly disguised as zebras, this alternative becomes a relevant alternative after the question as well. Therefore, if you do not know whether the animals you are looking at are mules cleverly disguised as zebras, you start doubting whether the animals you are looking at are zebras, as well. As Jonathan Vogel ([22]) does, one can argue that you have perfectly good reasons to know that the zebras in the zoo are not mules cleverly disguised as zebras, since the zoo has no reason to put mules there instead of zebras and it would take a lot of effort to do so (more reasons can be found). You could find these reasons strong enough to assume knowledge that the animals are not mules cleverly disguised as zebras. This implies that you know as well, that the animals you are looking at are zebras. However, you could also judge the reasons to be unconvincing[1]. In that case you do not know whether the animals are mules cleverly disguised as zebras and also you would not know whether you are looking at zebras. Thus, although you knew they were zebras in the beginning, a context switch has taken place which makes you doubt this knowledge. Namely, after the context switch the context provides more alternatives that may not be ruled out (strong enough) by your evidence.

As shown above, an immediate consequence of making the relative alternatives context dependent instead of proposition dependent, is that this view can handle the critique that you know that the zebras are not mules cleverly disguised as zebras. Furthermore, this view of the RAT can be used to explain that when the stakes are higher, then more alternatives become relevant. Namely, the context is such that there is more at stake and therefore less is known in such a context (see [2]). To illustrate this, think of a mother who wants to know whether her child has a fever. In a normal situation (context), she might use her hand to feel if the child's forehead is warm, to know whether the child has fever or not. However, suppose her child just came out of surgery and a fever can have huge consequences for the child's health, she would want to rule out the alternative that the child's forehead feels cold while he has fever. So in the second case, there is more at stake then in the first case. Therefore, in the first case we have that the alternative that the child's forehead is cold even though he has a fever is irrelevant. However, when more is at stake this alternative becomes relevant and thus, has to be ruled out by evidence.

Finally, we want to show that the view by Lewis also explains question dependence of knowledge as discussed by Schaffer ([16]). Schaffer argues that knowing the answer p to a question Q is not the same as knowing p. To illustrate we will use an example from his article, which we will refer to later in this thesis. Suppose you are watching a speech of Bush on television. Probably, you will be able to answer the question 'Is Bush or Janet Jackson on television?'. However, you might not be able to answer another question with the same answer, 'Is Bush or Will Ferell[2] on television?', even though you were able to answer the first question. In other words, 'knowing whether Bush or Janet Jackson is on television' is not the same as 'knowing whether Bush or Will Ferell is on television', even though, assuming the answer to both questions is Bush, they both imply the knowledge that Bush is on television.

[1]The fact that there is no consensus on whether these reasons are convincing enough, supports the view that knowledge is dependent on the circumstance of evaluation.

[2]Will Ferell is a Bush impersonator

When we look at the observation of Schaffer that knowing the answer to a question is dependent on the alternatives provided by the question, we can see that the question modifies the relevant alternatives available to the agent. The question: 'Is Bush or Janet Jackson on television?' provides only the alternatives that Bush is on television and that Janet Jackson is on television as relevant alternatives. A different question: 'Is Bush or Will Ferell on television?' makes other alternatives available; that Bush is on television and that Will Ferell is on television. For the first question we have that the evidence that it is a man you are watching, is enough to eliminate the possibility that Janet Jackson is on television. Thus, according to knowledge defined by Lewis (see above), we have that you know that Bush is on television. However, the alternative that Will Ferell is on television may not have been ruled out by any evidence you received by watching television. Therefore, you might not know the answer to the second question.

3.2 The relevant alternatives theory for beliefs

Although the emphasis of the discussion until now has been on knowledge, in the first place the claim of Dretske was not only about knowledge. Namely, his claim was about all epistemic operators in general, where he gave the following examples of epistemic operators ([5, p. 1009]):

1. S knows that ...
2. S sees or can see that ...
3. S has reasons (or a reason) to believe that ...
4. There is evidence to suggest that ...
5. S can prove that ...
6. S learned (discovered, found out) that ...
7. In relation to our evidence it is probable that ...

For reasons of convenience we will only provide the arguments for the relative alternatives theory for knowledge and beliefs in this thesis. Even though the title aims at the suggestion that the RAT is of influence for all epistemic operators, we have to leave this discussion to another occasion.

Let us first make a note of our use of 'beliefs', which will actually also hold for knowledge. In the remainder of the thesis we will speak only of beliefs and knowledge that are formed based on evidence. One could be mistaken by thinking that we thereby make the claim that one can only form beliefs and knowledge on the basis of evidence. However, it is just the case that in this thesis we are only concerned with those beliefs and knowledge that are indeed based on evidence. Thus, further claims about beliefs and knowledge are claims about beliefs and knowledge based on evidence and not about any other imaginable form.

Actually, a lot of the examples that have been used in the literature in order to motivate RAT for knowledge, can also be used in order to motivate RAT for beliefs. For example, if you walk past a red wall, you are inclined to believe that this wall is red. Namely, the evidence that you see a red wall supports the belief that the wall is red. However, this same evidence also supports the belief that the wall is white and cleverly illuminated to look red (the example comes from Dretske, [5, p. 1015]). This last alternative is not relevant in a default context and therefore not considered by the agent in forming her beliefs. The same as with knowledge, we have that it is impossible to have the beliefs that we have, if we must have evidence that discriminate between these beliefs and all other possible alternatives. Just as with beliefs, for some fixed set of pieces of evidence on which a belief is based one can always come up with an alternative that is supported by this same set of pieces of evidence, but is not believed since it is an irrelevant alternative. Only, the moment this alternative is made available by the context, we start considering it and judge whether we have evidence discarding our believe in this alternative, or that we have to explore new evidence in order to distinguish between these two alternatives.

An important difference between beliefs and knowledge, however, is that in order for an ascriber to ascribe beliefs to a subject, it is not required that the ascriber does not know that the belief is false in the actual world of the subject. On the contrary, an ascriber cannot ascribe knowledge to a subject when she knows that it is false in the actual world of the subject.

3.3 Context dependence of relative alternatives

As mentioned above, the relevant alternatives theory gives a plausible theory of knowledge, which can explain some peculiarities of knowledge behavior in everyday use. However, if knowledge would be dependent on the relevant alternatives and independent of all the irrelevant alternatives, these relevant alternatives have to be triggered in some way.

This is exactly the point where Dretske and Lewis disagree, as we discussed above. According to Dretske the relevant alternatives are dependent on the proposition at hand. Namely, if the proposition is that we are looking at zebras, then the alternative that we are looking at mules cleverly disguised as zebras is an irrelevant alternative in order to know the proposition. However, if the proposition is that we are looking at mules cleverly disguised as zebras, this is of course a relevant alternative in order to know the proposition. On the other hand, Lewis argues that when we say that we know that we look at zebras we are saying this in a certain context. However, the moment someone brings the alternative to your attention that you might be looking at mules cleverly disguised as zebras, the context changes. Namely, by mentioning the alternative it becomes a relevant alternative in the context.

By explaining the relevance of the alternative that we are looking at mules cleverly disguised as zebras in the second context, we used Lewis's rule of attention. Lewis discussed several rules which aim at describing which alternatives can be irrelevant alternatives, and therefore can be properly ignored and which are mostly relevant alternatives, and therefore should not be ignored. In this thesis we will leave the discussion about which context triggers which relevant alternatives for what it is and will only focus on the context which creates and removes relevant alternatives. However, in the examples we will make assumptions about which context triggers which relevant alternatives, just as we applied the rule of attention above. Nonetheless, without agreement on these triggers, we can still agree that the context causes this changes in relevance of the alternatives.

Thus, it is the context that determines which alternatives are relevant and which are not. Although we can influence the context, we cannot choose it to be one way or another. Take for example the court room. Sometimes a piece of evidence is obtained unjustified and the court is ordering the jury not to take this piece of evidence into account. Although we can assume the jury will try not to let the evidence influence their decision, after it is mentioned it is impossible to just simply ignore it. This is how it works with knowledge as well. Once some alternative is brought to ones attention, one cannot simply ignore its existence. For instance, in the zebra example above we have that once the alternative that it might not be zebras but mules cleverly disguised as zebras comes to our attention, we need to search for evidence that rules out this alternative. After the alternative has been brought to our attention we can no longer just simply ignore it.

3.4 Extra power

The idea that knowledge and beliefs are dependent on the relevant alternatives of the context, creates a lot of extra reasoning power to us. Where we could barely know anything if we have to test knowledge against every imaginable alternative, the context dependence gives us a pretty good working heuristic for knowledge.

It seems to us as though most of the hardest problems in artificial intelligence are due to the inability of computers to ignore the irrelevant alternatives in the context (see for a concrete example the frame problem [15]).

Furthermore (also related to problems in AI), imagine how hard it would be to plan a vacation, if we would need to check every possible scenario. For example, in a default context we just ignore the alternative that the plane will not fly that day. We ignore the possibility that our wallet may get stolen or that we will loose our passport. Imagine how much more tiresome it would be, if not impossible, to plan a vacation when we have to consider every possible scenario.

All the examples above may lead you to think that relevant alternatives are always very implausible alternatives, one would want to rule out anyway, even if they would become relevant due to a context change. However irrelevant alternatives are not always implausible, these kind of examples are just very natural to explain RAT. Consider for example the reason why nowadays, when you are trying to book your vacation, you can always immediately opt for a cancellation insurance as well. For many people, the option of getting the insurance was irrelevant at first, however, now that the option is made available by the context some of them do get the insurance. The same story holds true for all kinds of commercials, that make use of the fact that the commercials make you consider to buy stuff you would not have considered, if the context did not make the possibility available. Moreover, when playing chess you might overlook that a certain move will lead to your defeat, since the next move of your opponent was simply not available in your context. However, any move of your opponent that will lead to your defeat is normally assigned the highest probability.

Thus, although there are some disadvantages of having knowledge and beliefs depending on the context, we can also emphasize that we are reasoning impressively well in every day live, due to our ability to focus on the relevant alternatives in the (everyday) context.

Chapter 4

Towards a Model of Context Dependence Epistemic Operators.

In the previous chapter we argued for context dependence of epistemic operators. In this chapter we will show how we can extend already existing logics (discussed in chapter 2) to incorporate that epistemic operators are context dependent. Thus, this chapter proposes a concrete model for the above philosophical observations.

We start out with some arguments on how to represent the context on which notions, like knowledge and belief, should become dependent. Subsequently, we will extend the discussed epistemic logic based on the ideas discussed in the previous chapter. However, extending the epistemic logic will confront us with the fact that the framework is not expressible enough to incorporate all the information coming from the context, we want to make knowledge dependent on. This motivates our switch towards the more expressible evidence logic.

4.1 The Representation of the Context

In the previous chapter we have seen that an agent's judgments are dependent on the relevant alternatives. The relevant alternatives are the only available alternatives that an agent considers in a certain context. Thus, the context determines the alternatives the agent considers. In order to make this context dependence concrete by incorporating it into models for information states of agents, we first need to make concrete how the context will be represented.

In his paper [18], Stalnaker proposes a context set for the setting in which a speech act takes place. According to Stalnaker the context set is a set containing those worlds that are compatible with the information that is assumed to be common knowledge by the participants. Thus, according to Stalnaker the context set can be represented as a set of worlds. Following Stalnaker we can represent the context we are dealing with in this thesis as a set of worlds as well, in which the relevant alternatives are inside this set and the non-relevant alternatives are not[1].

However, we will argue here that the context set as proposed by Stalnaker is not expressive enough. In order to do so, we will use literature on questions combined with the conversational maxims of Grice. Since an exhaustive amount of literature on the subject exists, we will focus our arguments on natural language use, especially those concerning questions. However, we are still making the general claim that the more expressive context set we will propose here is also useful outside of natural language.

In his paper [9], Hamblin postulates three general principles regarding questions:

1. An answer to a question is a sentence, or a statement
2. The possible answers to a question form an exhaustive set of mutually exclusive possibilities.
3. To know the meaning of a question is to know what counts as an answer to that question.

In the case that our context set is determined by (the meaning of) a question, we have, by the above principle 3, that our context set is determined by the set of all statements that count as an answer to that question. By representing the context set as a set of possible worlds, one states that the context defined by the question equals the logical space defined by the question. Since the set of possible answers is exhaustive according to principle 2, this logical space of the question equals the union of all the possible answers to the question. However, if we will only take the logical space defined by the question into account, we will lose all information about the partition the question proposed on this space. To see why loosing information about

[1]This is exactly what Schaffer proposes in his paper ([16]) in order to represent the relevant answers to a question on which a particular use of 'knowledge' is dependent.

the partition proposed by the question is problematic for the context, we will need the maxims proposed by Grice.

Grice ([8]) proposed a set of maxims, which can be understood as general heuristics for participants in a conversation. Under normal circumstances, audience can draw semantic inferences from the speakers words by assuming the speaker will comply to these maxims. It concerns the following maxims.

- Maxim of Quantity
- Maxim of Quality
- Maxim of Relevance
- Maxim of Manner

The key point for here is that, since the audience expects the speaker to comply to this maxims. Therefore, a speaker should have access to the information required to comply to these maxims in order to participate in a conversation. To comply to the maximum of quantity, second in the list above, the speakers contribution has to be as informative as required and not more informative than required. Thus, in order to participate in a conversation a participant should have access to the required level of information.

In the case the contribution of the speaker is a reply to a question, the required level of information is determined by the partition on the logical space proposed by the question. Thus, in order to comply to the maxims of Grice in a conversation, one should be able to represent the partition proposed by the question in the context.

If we use a set of worlds as the context set, as proposed by Stalnaker, the context would imply which alternatives are relevant and which are not. This enables us to comply to the maxim of relevance, third in the list. However, it lacks the expressiveness to represent the partition proposed by the question. Thus, if the context set is defined as a set of worlds, a speaker cannot access the required level of information, determined by the context.

The more expressive context set we will propose here, follows easily from the above observations. The context set in this thesis will be a set of sets of worlds. In the case the context is determined by a question, this will exactly be the set of all possible answers, which is in accordance with Hamblin's third principle. In accordance with Hamblin's first principle, the answers are represented as the set of worlds that are compatible with it.

By Hamblin's second principle, we have that the set of all possible answers is an exhaustive set. This means precisely that the union of all the possible answers gives the logical space defined by the question. Thus, we will not lose any information when representing the context as a set of sets of worlds.

When the context is not (or only partly) determined by a question, we still have that the level of information that is required is determined by the discourse in a conversation. Therefore, in these cases, when the context is a set of sets of worlds, the speaker is also able to comply to the maxim of quality.

However, when we are returning to the original situation, namely epistemic operators in a context, the same observations hold. The context is making certain possibilities unavailable to us (the complement of the logical space defined by the context), but furthermore it defines the level of information we have access to in that context. More precisely, the context rules out certain possibilities and makes other possibilities available, where possibilities are interpreted as sets of worlds. Therefore a context is a set of set of worlds, namely the set of possibilities that are made available in the context.

4.2 Contextual Epistemic Logic

As we have seen in the previous chapter, epistemic operators are dependent on the relevant alternatives, which are determined by the context. As a consequence, knowledge can never exist outside a context. Thus, 'knowing' always means 'knowing' in a certain context.

However, the classical analysis of knowledge ('knowing that') is of the form $K_i\varphi$, which translates to: agent i knows that φ ([16, 1, 7]). This translation of knowledge is independent of the context and, therefore, cannot be a good translation of knowledge according to the above observations. In order to give a better translation for the epistemic operator knowledge, we will propose a contextual epistemic logic.

We will extend the epistemic model defined in chapter 2 with the observation that agents are not always able to consider every option. That is, they cannot always consider every possible world in their information state. Since one can ask a question, such that the set of answers restrict the available options, but also since the agent is simply not aware of all possible options at all times. For example, an agent might know that some shop is open, since she knows the opening hours and the actual time. However, she might not know that the shop is closed due to a just committed robbery. Since the agent does not know this, she must think that it might be a possibility. However, since in a normal situation, the context provides no reason to consider that the shop is closed due to a just committed robbery, this possible world is ruled out by this default context. If the agent, on the other hand, heard somewhere that a serious robbery had taken place in the shop, the context provides good reasons for the agent to take the possibility in consideration that the shop might be closed due to this robbery.

From the above example we can see that the context operates as a filter; the context filters out certain possible worlds and makes others salient. The contextual epistemic model will, based on these observations, resemble the epistemic model, only extended with an extra contextual filter.

Definition 19 (Contextual epistemic model)**.** Given a fixed set of propositional variables At and a group of agents Ag. An epistemic model is a tuple $\mathcal{M} = \langle W, \{\sim_i \mid i \in Ag\}, \{\mathbb{C}_i \mid i \in Ag\}, V\rangle$, where

- W is a non-empty set of worlds,
- $\sim_i \subseteq W \times W$ is an epistemic accessibility relation,
- $\mathbb{C}_i \subseteq \wp(W)$ is a contextual filter,
- $V : At \to \wp(W)$ is a valuation function.

The agents still have their own personal accessibility relation, which defines which worlds are indistinguishable from one another for that agent. Subsequently, agents now have their own personal contextual filter, which defines the relevant alternatives for the agents in their own personal context. We chose to personalize the contextual filter for each agent. This does not mean that they necessarily differ from one another, since the backgrounds of the agents might be different and the background can influence the contextual filter of the agent. In the earlier discussed shop example it could be the case that for two agents in a conversation, one of the agents heard from the serious robbery and the other agent did not. In this case they operate in different contexts, even though they participate in the same conversation.

In the following we will sometimes write $\sim$ for $\{\sim_i \mid i \in Ag\}$ and $\mathbb{C}$ for $\{\mathbb{C}_i \mid i \in Ag\}$.

The language on the contextual epistemic model is the same as the language we had on the epistemic model (see definition 7).

Definition 20 (Contextual epistemic language)**.** Given a fixed set of propositional variables At and a set of agents Ag, with $i \in Ag$. Let $\mathcal{L}^{CK}$ be the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_i\varphi$$

Furthermore, the truth definitions for the contextual evidence language are similar to the truth definitions for the epistemic language, except for the truth definition of the knowledge operator. In this contextual case, the knowledge operator will dependent on the contextual filter which filters out all worlds that are inaccessible for the agent in the current context.

Definition 21 (Truth)**.** Let $M = \langle W, \sim, \mathbb{C}, V\rangle$ be a contextual epistemic model. Truth of a formula $\varphi \in \mathcal{L}^{CK}$ is inductively defined by definition 8 for the logical connectives and by the following truth definition for the knowledge operator:

$$\mathcal{M}, w \models K_i\varphi \quad \textit{iff} \quad \text{for all } t \in W \cap \bigcup \mathbb{C}_i \textit{ s.t. } s \sim_i t \textit{ we have that } \mathcal{M}, t \models \varphi$$

By this definition of context dependent knowledge, we have that agent i knows that φ if all the possible worlds, that are relative alternatives in the context and the agent cannot distinguish from the actual world, agree on φ being true. Therefore, we restrict the set of worlds that we want to agree on φ being true, to the set of worlds that are not irrelevant in the current context. So before, in the epistemic logic knowledge required that all the worlds that the agent cannot distinguish from the actual world agree on φ being true. On the other hand, this weaker notion of knowledge only requires that the indistinguishable worlds, which are *relevant alternatives* in the context, agree on φ. As a logical consequence we have that if $\mathcal{M}^{\mathbb{C}} = \langle W, \sim, \mathbb{C}, V\rangle$ is a contextual epistemic model and $\mathcal{M} = \langle W, \sim, V\rangle$ its related epistemic model, then $\mathcal{M}, w \models K_i\varphi$ implies that $\mathcal{M}^{\mathbb{C}}, w \models K_i\varphi$, but the implication the other way around does not hold.

In general, if for two contexts $\mathbb{C}$ and $\mathbb{C}'$, we have that $\bigcup \mathbb{C} \subseteq \bigcup \mathbb{C}'$, then $\langle W, \sim, \mathbb{C}', V\rangle, w \models K_i\varphi$ implies that $\langle W, \sim, \mathbb{C}, V\rangle, w \models K_i\varphi$. Since, if some set of indistinguishable worlds that are relevant alternatives in the context agree on φ being true, then definitely a subset of these indistinguishable worlds that are now relevant will agree on φ being true. The implication here is a generalization of the implication above, by the observation that $\langle W, \sim, \mathbb{C}, V\rangle = \langle W, \sim, V\rangle$, whenever $\mathbb{C}$ is such that $\bigcup \mathbb{C} = W$.

However, we face a problem with this definition. As we argued in chapter 4.1, the context has more influence on the knowledge operator then only ruling out the worlds that are irrelevant in the current context. Namely, the context also defines a partition on the logical space it defines, which, as we argued, is also of importance for the knowledge operator. Nevertheless, in the truth definition of the knowledge operator above, the partition of $\mathbb{C}$ does not influence the knowledge operator in any way. It is only the union of the possibilities given by the context, which is exactly the logical space it defines, that influence the knowledge operator. As a consequence we can just as well define the context set to be a set of worlds, as Stalnaker ([18]) proposed, which will give the same results. Nonetheless, the good news is that this problem can be solved by switching to a more expressive logic, evidence logic, as we will see in the next section.

4.3 Contextual Evidence Logic

Since epistemic logic combined with context dependent knowledge turned out to totally ignore the partition given on the relevant worlds, we will propose a contextual extension of evidence logic that will take into account the relevant worlds and the proposed partition on these relevant worlds. We will first work out this proposal for contextual evidence logic and afterwards give some characteristics of its logical behavior.

4.3.1 Adding a context component

In chapter 2 we discussed two different kinds of evidence logics; one for knowledge and one for beliefs. The evidence model for knowledge was actually a restriction of the evidence model for beliefs by van Benthem and Pacuit ([21]). Therefore, we will only extend the evidence model for beliefs to a contextual evidence logic. All the next proposals are easily adaptable to the restricted evidence logic of knowledge. For convenience we will just refer to the evidence logic of beliefs as the evidence logic throughout this paper.

In the contextual epistemic logic we already saw that the context could be included in the model and operates as a filter on the possible worlds, thereby filtering out all the irrelevant alternatives. In the contextual evidence logic, we will use this same technique in order to make the epistemic operator context dependent.

Definition 22 (Contextual evidence model)**.** Given a fixed set of propositional variables At and a group of agents Ag. A contextual evidence model is a tuple $\mathcal{M} = \langle W, \{E_i \mid i \in Ag\}, \{\mathbb{C}_i \mid i \in Ag\}, V\rangle$, where

- W is a non-empty set of worlds,
- $E_i \subseteq W \times \wp(W)$ is an evidence relation,
- $\mathbb{C}_i \subseteq \wp(W)$ is a contextual filter,
- $V : At \to \wp(W)$ is a valuation function.

From now on we will write E for $\{E_i \mid i \in Ag\}$ and $\mathbb{C}$ for $\{\mathbb{C}_i \mid i \in Ag\}$.

The language for contextual epistemic logic will be the same as the language for evidence logic (definition 13).

Definition 23 (Contextual evidence language)**.** Given a fixed set of propositional variables At and a group of agents Ag, with $i \in Ag$. Let $\mathcal{L}^{CE}$ be the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_i\varphi \mid B_i\varphi$$

However, the truth conditions for the evidence modality and the operator will be different, since they will now both depend on the context as well.

Definition 24 (Truth)**.** Let $M = \langle W, E, \mathbb{C}, V\rangle$ be a contextual evidence model. Truth of a formula $\varphi \in \mathcal{L}^{CE}$ is inductively defined by definition 8 for the logical connectives and by the following truth definitions for the evidence modality and the belief operator:

$$\begin{array}{lll} \mathcal{M}, w \models \Box_i\varphi & \textit{iff} & [\![\varphi]\!] \in \mathbb{C}_i \;\&\; \exists X \in E_i(w) \; s.t. \; \emptyset \neq (X \cap \bigcup \mathbb{C}_i) \subseteq [\![\varphi]\!] \\ \mathcal{M}, w \models B_i\varphi & \textit{iff} & [\![\varphi]\!] \in \mathbb{C}_i \;\&\; \text{for each max f.i.p. } \mathbb{X} \subseteq E_i(w) \restriction_{\bigcup \mathbb{C}_i} \\ & & \qquad\qquad \text{we have that } \bigcap \mathbb{X} \subseteq [\![\varphi]\!] \end{array}$$

For both the evidence modality and the belief operator we have that the left conjunct makes use of the set of sets of worlds representation of the context. As we argued above, in section 4.1, the context also defines the level of specificity of the alternatives that is available to the agent. Therefore, the agent only has evidence for, or believes in, the sets of worlds that are included in the context. Propositions that are more (or less) informative than the propositions in the context are not true under the evidence operator and the belief operator, since they are not relevant alternatives in the current context.

All the worlds w, that are absent in all the alternatives provided by the context, are irrelevant alternatives in this context. Therefore, just as with knowledge evaluated in a contextual epistemic model, in the right conjunct of the truth definition of the evidence operator we restrict all the pieces of evidence, $X \in E(w)$, to the set of worlds that are not irrelevant, $X \cap \mathbb{C}$. Then we check whether such a restricted evidence set which supports the formula φ exists. In the non contextual case we just checked whether there is an evidence set which supports the formula φ, without first restricting the evidence sets relative to the context.

In the right conjunct of the truth definition of the belief operator, we also make sure that the irrelevant alternatives are not taken into account.

Thereto, we restrict all the pieces of evidence to the sets of worlds that are not irrelevant $E_i(w) \upharpoonright_{\bigcup \mathbb{C}_i}$ (see definition 5 for the details). Then, just as with the original belief operator, we have that each set satisfying the maximal finite intersection property must make φ true.

For the evidence modality it does not matter whether we first restrict the evidence set of the agent to the union of her context and afterwards see if a piece of evidence in this new set exists that is nonempty and included in the truth set of φ, or that we first check if a piece of evidence in her original evidence set exists, that intersected with the union of the context is nonempty and included in the truth set of φ.

Fact 1 (Restricting evidence for $\Box_i$ is order independent)**.** *Let $M = \langle W, E, \mathbb{C}, V\rangle$ be a contextual evidence model. For some formula $\varphi \in \mathcal{L}^{CE}$ we have that the truth for the evidence modality is independent of the order in which we restrict to the relevant alternatives in the context:*

$$\mathcal{M}, w \models \Box_i\varphi \quad \textit{iff} \quad [\![\varphi]\!] \in \mathbb{C}_i \ \& \ \exists X \in E_i(w) \upharpoonright_{\bigcup \mathbb{C}_i} \ s.t. \ \emptyset \neq X \subseteq [\![\varphi]\!]$$

Suppose that $E_i = \{X_1, ..., X_n\}$. Then $E_i \upharpoonright_{\bigcup \mathbb{C}_i} = \{X_1 \cap \bigcup \mathbb{C}_i, ..., X_n \cap \bigcup \mathbb{C}_i\}$. Thus, $\exists X \in E_i(w)\ s.t.\ \emptyset \neq (X \cap \bigcup \mathbb{C}_i) \subseteq [\![\varphi]\!]$ if and only if $\exists X \in E_i(w) \upharpoonright_{\bigcup \mathbb{C}_i}\ s.t.\ \emptyset \neq X \subseteq [\![\varphi]\!]$.

However, this order independence of restricting the pieces of evidence, which holds for the evidence modality, does not hold for the belief operator.

Fact 2 (Restricting evidence for B_i is order dependent)**.** *Let $M = \langle W, E, \mathbb{C}, V\rangle$ be a contextual evidence model. For some formula $\varphi \in \mathcal{L}^{CE}$ we have that the truth for the belief operator is dependent on the order in which we restrict to the relevant alternatives in the context:*

$$\mathcal{M}, w \models B_i\varphi \not\Leftrightarrow$$

$$[\![\varphi]\!] \in \mathbb{C}_i \ \& \text{ for each max f.i.p. } \mathbb{X} \subseteq E_i(w) \text{ we have that } (\bigcap \mathbb{X} \cap \bigcup \mathbb{C}_i) \subseteq [\![\varphi]\!] \tag{4.1}$$

This can be illustrated by the following counter example. Assume there is only one agent $\{i\} = Ag$. Take $W = \{w_1, w_2\}$, $E_i = \{(w_1, \{w_1\}), (w_1\{w_1, w_2\})\}$, $\mathbb{C}_i = \{\{w_2\}\}$ and $V(p) = \{w_1\}$, see picture 4.1. Then the only max f.i.p. set in $E_i(w_1)$ is $\{\{w_1\}, \{w_1, w_2\}\}$. It follows that, its intersection intersected with the union of all the sets in the context, $\bigcap\{\{w_1\}, \{w_1, w_2\}\} \cap \bigcup\{\{w_2\}\} = \{w_1\} \cap \{w_2\} = \emptyset$, is empty. Thus, with this truth definition we have

$M, w \models B_i p$ (and $M, w \models B_i \neg p$). However, if we first restrict every piece of evidence to the union of the context, $E_i(w) \upharpoonright_{\bigcup \mathbb{C}_i} = \{\{w_1\}, \{w_1, w_1\}\} \upharpoonright_{\{w_2\}} = \{\{w_1\} \cap \{w_2\}, \{w_1, w_2\} \cap \{w_2\}\} = \{\emptyset, \{w_2\}\}$. Then the only max f.i.p. set is $\{w_2\}$. Therefore, with the real truth definition, we have $M, w \models \neg B_i p$ (and $M, w \models B_i \neg p$).

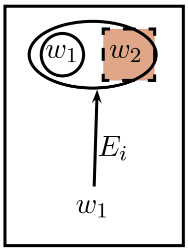


Figure 4.1: counter example

Thus, the truth definition (24) for the belief operator and equation 4.1 are not the same. We have that the belief operator is indeed dependent on the order in which we restrict to the relevant alternatives in the context. Since we do not want the pieces of evidence, that completely fallout of the context, to influence an agent's belief, the above counter example shows why we choose for 24 as the truth definition for the belief operator and not the one suggested by equation 4.1.

4.3.2 Characterization Results

We already saw above, in section 2, that in the original evidence logic by van Benthem and Pacuit ([21]) the evidence modality satisfies the minimal classical modal logic, and the belief operator satisfies the modal logic KD. Since, in contextual evidence logic, the evidence modality and the belief operator are totally dependent on the context, even upwards monotonicity, the requirement for even the minimal modal logic is not satisfied in general.

However, as we will show here, classes of contextual filters satisfying a certain set theoretical operation exists, such that corresponding rules, that are not valid on contextual evidence frames in general, are valid on frames based on this class. We call a rule $\frac{\varphi}{\psi}$ valid on a frame, $\mathcal{F}$, if for an arbitrary valuation function V and for all worlds $w \in W$ we have that $\langle \mathcal{F}, V \rangle, w \models \varphi$, then we have for all worlds $v \in W$ that $\langle \mathcal{F}, V \rangle, v \models \psi$.

Furthermore, there exist classes of contextual filters satisfying a certain set theoretical operation, such that corresponding formulas, that are not valid on contextual evidence frames in general, are valid on frames based on this class. We call a formula φ valid on a frame, $\mathcal{F}$, if for every valuation V and every world $w \in W$, we have that $\langle \mathcal{F}, V \rangle, w \models \varphi$.

We first discuss the frame correspondence with respect to the evidence modality and thereafter with respect to the belief operator.

Evidence modality.

We have that the evidence logic by van Benthem and Pacuit satisfies the principles of the minimal modal logic, that is the upward monotonicity rule (see equation 4.2) is valid on all evidence frames.

$$\frac{\varphi \to \psi}{\Box_i \varphi \to \Box_i \psi} \tag{4.2}$$

The first observation is very straightforward. Namely, if the contextual filter is equal to the set of all possibilities $\mathbb{C}_i = \wp(W)$ for all agents i, we have that the contextual evidence logic is equal to the evidence logic. Thus, it follows that if the contextual filter equals the set of all possibilities, we have that $\Box_i$ satisfies the principles of the minimal classical modal logic. However, as we will see later on, there are more contexts that satisfy upward monotonicity.

We continue with the class of contexts, which contains all contexts that are join-semilattice. For completeness we give the definition of a join-semilattice.

Definition 25 (Join-semilattice)**.** A join-semilattice is a partially ordered set $\mathbb{X}$ such that for every element $X, Y \in \mathbb{X}$ we have:

$$X \cup Y \in \mathbb{X}$$

We can derive the following frame validity for frames with a join-semilattice as context.

Fact 3. *For all contextual evidence frames, $\mathcal{F} = \langle W, E, \{\mathbb{C}_i \mid i \in Ag\} \rangle$ we have that: if for an arbitrary agent $i \in Ag$ her contextual filter $\mathbb{C}_i$ is a join-semilattice, then the formula $(\Box_i \varphi \land \Box_i \psi) \to \Box_i(\varphi \lor \psi)$ is valid on $\mathcal{F}$.*

Take an arbitrary contextual evidence frame $\mathcal{F}$, with $\mathbb{C}_i$ a join-semilattice for an arbitrary $i \in Ag$. Furthermore, take an arbitrary valuation function V, an arbitrary world $w \in W$ and two arbitrary formulas $\varphi, \psi \in \mathcal{L}^{CE}$. Assume that $\langle \mathcal{F}, V \rangle, w \models \Box_i \varphi$ and $\langle \mathcal{F}, V \rangle, w \models \Box_i \psi$. Then we have, by definition 24, that (1) $[\![\varphi]\!] \in \mathbb{C}_i$, (2) $[\![\psi]\!] \in \mathbb{C}_i$ and (3) $\exists X \in E_i(w)$ *s.t.* $\emptyset \neq (X \cap \bigcup \mathbb{C}_i) \subseteq [\![\varphi]\!]$. By (1), (2) and the assumption that $\mathbb{C}_i$ a join-semilattice is we have that

$[\![\varphi \vee \psi]\!] \in \mathbb{C}_i$. By (3) we have that $\exists X \in E_i(w)$ *s.t.* $\emptyset \neq (X \cap \bigcup \mathbb{C}_i) \subseteq [\![\varphi \vee \psi]\!]$. Thus, $\mathcal{M}, w \models \Box_i(\varphi \vee \psi)$.

Note that if we have an arbitrary contextual evidence frame, $\mathcal{F}$, such that the formula $(\Box_i\varphi \wedge \Box_i\psi) \to \Box_i(\varphi \vee \psi)$ for an arbitrary $i \in Ag$ is valid on $\mathcal{F}$, then we do not necessarily have that the contextual filter $\mathbb{C}_i$ in $\mathcal{F}$ is a join-semilattice. Consider for example the contextual evidence frame, $\mathcal{F}$, such that for all worlds $w \in W$ the evidence relation $E_i(w) = \emptyset$. Then for all valuations V, all worlds $w \in W$ and all formulas $\varphi, \psi \in \mathcal{L}^{CE}$, independently of the contextual filter $\mathcal{C}_i$, we have that $\langle \mathcal{F}, V \rangle, w \not\models \Box_i\varphi \wedge \Box_i\psi$. Thus, $\langle \mathcal{F}, V \rangle, w \models (\Box_i\varphi \wedge \Box_i\psi) \to \Box_i(\varphi \vee \psi)$. Therefore, the formula is valid on $\mathcal{F}$, independent of its contextual filter. In particular the formula is valid on $\mathcal{F}$, where the contextual filter of agent i in $\mathcal{F}$ is no join-semilattice.

However, if the contextual filter $\mathbb{C}_i$ for an arbitrary agent i is no join-semilattice, then we can construct a contextual evidence frame $\mathcal{F}$, such that the formula $(\Box_i\varphi \wedge \Box_i\psi) \to \Box_i(\varphi \vee \psi)$ for an arbitrary $i \in Ag$ is not valid on $\mathcal{F}$. Namely, if $\mathbb{C}_i$ is not a join-semilattice, we have that there exists $X, Y \in \mathbb{C}_i$ such that $X \cup Y \notin \mathbb{C}_i$. Note that $X, Y \neq \emptyset$, since their union is not in the context while they themselves are. Now take $E_i(w_1) = \{X, Y\}$ for an arbitrary $w_1 \in W$. Then we have that $\mathcal{M}, w_1 \models \Box_i p$ and $\mathcal{M}, w_1 \models \Box_i q$. However, since $[\![p \vee q]\!] = (X \cup Y) \notin \mathbb{C}_i$, we have that $\mathcal{M}, w_1 \not\models \Box_i(p \vee q)$.

Fact 3 showed that if the contextual filter is a join-semilattice, every frame built on this contextual filter is known to validate a very weak axiom. As we will see now, it is not enough for a contextual filter to be a join-semilattice in order for a frame built on this contextual filter to validate the monotonicity rule. This can be clarified by noticing that a frame based on a join-semilattice context does not validate the rule in equation 4.3 and moreover, also does not validate the rule in equation 4.4. These are not valid, since both formulas enlarge the evidence set where it is not demanded that the enlarged part is in the context, and therefore also not demanded that their join is in the context.

Fact 4. *There exists a contextual evidence frame, $\mathcal{F} = \langle W, E, \{\mathbb{C}_i \mid i \in Ag\}\rangle$, such that: if for an arbitrary agent $i \in Ag$ her contextual filter $\mathbb{C}_i$ is a join-semilattice, then the following two formulas are not valid on $\mathcal{F}$.*

$$\Box_i(\varphi \wedge \psi) \to \Box_i\varphi \tag{4.3}$$

$$\Box_i\varphi \to \Box_i(\varphi \vee \psi) \tag{4.4}$$

We can use the same counterexample for both formulas. Take $W = \{w_1, w_2\}$, $E_i(w_1) = \{\{w_1,\}\}$, $\mathbb{C}_i = \{\{w_1\}\}$, $V(p) = \{w_1, w_2\}$ and $V(q) = \{w_1\}$, for illustration see picture 4.2. For sure we have that $\mathbb{C}_i$ is a join-semilattice. Against 4.3 we have that $\mathcal{M}, w_1 \models \Box_i(p \wedge q)$, but not $\mathcal{M}, w_1 \models \Box_i p$. Against 4.4 we have that $\mathcal{M}, w_1 \models \Box_i q$, but not $\mathcal{M}, w_1 \models \Box_i(p \vee q)$.

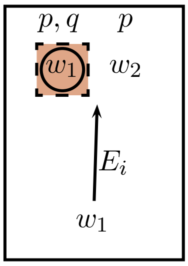


Figure 4.2: counter example

Thus, the above negative result shows us we need a stronger restriction on the context in order for the frames based on this context to satisfy the principles of the minimal classical modal logic. We will see that this stronger restriction on the context we need is upward closure.

Definition 26 (Upward closed)**.** A partially ordered set $\mathbb{X}$ is upward closed if for every element $\emptyset \neq X \in \mathbb{C}$ and X' such that $X \subseteq X'$ we have that:

$$X' \in \mathbb{C}$$

It turns out that all we need in order for the contextual evidence frame to satisfy the upward monotonicity rule (equation 4.2), is that the contextual filters are upward closed.

Fact 5. *For all contextual evidence frames, $\mathcal{F} = \langle W, E, \{\mathbb{C}_i \mid i \in Ag\}\rangle$, we have that: if for an arbitrary agent i her contextual filter $\mathbb{C}_i$ is upward closed, then the upward monotonicity rule in equation 4.2 above is valid on $\mathcal{F}$.*

Take an arbitrary contextual evidence frame $\mathcal{F}$, with $\mathbb{C}_i$ for an arbitrary $i \in Ag$ upward closed. Furthermore, take an arbitrary valuation function V such that for every world $w \in W$ and every two formulas $\varphi, \psi \in \mathcal{L}^{CE}$, we have that $\mathcal{M}, w \models \varphi \to \psi$. Then we have that (1) $[\![\varphi]\!] \subseteq [\![\psi]\!]$. Assume for an arbitrary world $v \in W$ we have $\mathcal{M}, v \models \Box_i \varphi$. Then we have, by definition 24, that (2) $[\![\varphi]\!] \in \mathbb{C}_i$ and (3) $\exists X \in E_i(v)$ *s.t.* $\emptyset \neq (X \cap \bigcup \mathbb{C}_i) \subseteq [\![\varphi]\!]$. By (1), (2) and the assumption that $\mathbb{C}_i$ is upward closed (and by the fact that the truth set of φ is non empty, since by (3) there exists a non empty set which is included in the truth set of φ), we have that $[\![\psi]\!]\in\mathbb{C}_i$. By (1) and (3) we have that $\exists X\in E_i(v)$ *s.t.* $\emptyset \neq (X \cap \bigcup \mathbb{C}_i) \subseteq [\![\psi]\!]$. Thus, $\mathcal{M}, v \models \Box_i\psi$.

Note that, again, we do not have that if the upward monotonicity rule is valid on an arbitrary contextual evidence frame $\mathcal{F}$, the contextual filter of $\mathcal{F}$ is upward closed. Again the same frame as before, where the evidence relation is empty, will independent of the contextual filter validate upward monotonicity of the evidence modality.

However, just as before, we can construct a contextual evidence frame $\mathcal{F}$ that does not validate the upward monotonicity rule, whenever the contextual filter is not upward closed. If $\mathbb{C}_i$ is not upward closed, we have that there exists $X \in \mathbb{C}_i$ and $X' \subseteq W$, with $X \subseteq X'$ and $X' \notin \mathbb{C}_i$. Take a valuation function V such that $V(p) = X$ and $V(q) = X'$. Then we have that $[\![q]\!] = X' \notin \mathbb{C}_i$. That means that $M, w \not\models \Box_i q$ for an arbitrary E. For all worlds $w \in W$ we have that $\mathcal{M}, w \models p \to q$ since $V(p) \subseteq V(q)$. Now take $E_i(w_1) = \{X\}$ for an arbitrary $w_1 \in W$. By the definition of upward closure, we have that $X \neq \emptyset$. Thus, then we have that $\mathcal{M}, w_1 \models (p \to q)$ and $\mathcal{M}, w_1 \models \Box_i p$, but $\mathcal{M}, w_1 \not\models \Box_i q$.

Belief operator.

We saw earlier in section 2.3 that the belief modality in the original evidence logic satisfies KD, that is

1. $B_i(\varphi \to \psi) \to (B_i\varphi \to B_i\psi)$
2. $B_i\varphi \to \neg B_i\neg\varphi$
3. Rule of necessitation (see equation 4.5 below).

$$\frac{\varphi}{B_i\varphi} \tag{4.5}$$

Before, when we discussed the characteristics for the evidence modality, we already noted that if the contextual filter is equal to the set of all possibilities $\mathbb{C}_i = \wp(W)$ for all agents i, we have that the contextual evidence logic equals the evidence logic. Therefore, we have that the axioms and rule stated above are valid on frames based on the contextual filter $\mathbb{C}_i = \wp(W)$. However, just as with the evidence modality, there is more that can be said.

If we look at axiom 1, we need the dual of upward closure, downward closure, in order for a contextual evidence frame to validate 1.

Definition 27 (Downward closed)**.** A partially ordered set $\mathbb{X}$ is downward closed if for every element $\emptyset \neq X \in \mathbb{C}$ and X' such that $X \subseteq X'$ we have that:

$$X' \in \mathbb{C}$$

Then we have the following frame validity, for frames based on a downward closed contextual filter.

Fact 6. *For all contextual evidence frames, $\mathcal{F} = \langle W, E, \{\mathbb{C}_i \mid i \in Ag\}\rangle$, we have that: if for an arbitrary agent i her contextual filter $\mathbb{C}_i$ is downward closed, then the formula $(\varphi \to \psi) \to (B_i\varphi \to B_i\psi)$ is valid on $\mathcal{F}$.*

Take an arbitrary contextual evidence frame $\mathcal{F}$, with $\mathbb{C}_i$ downward closed for an arbitrary agent $i \in Ag$. Furthermore, take an arbitrary valuation function V, an arbitrary world $w \in W$ and two arbitrary formulas $\varphi, \psi \in \mathcal{L}^{CE}$. Assume that $\langle \mathcal{F}, V\rangle, w \models B_i(\varphi \to \psi)$ and $\langle \mathcal{F}, V\rangle, w \models B_i\varphi$. Then we have, by definition 24, that (1) $[\![\varphi \to \psi]\!] \in \mathbb{C}_i$, (2) for each max f.i.p. $\mathbb{X} \subseteq E_i(w) \restriction_{\bigcup \mathbb{C}_i}$ we have that $\bigcap \mathbb{X} \subseteq [\![\varphi \to \psi]\!]$ and (3) for each max f.i.p. $\mathbb{X} \subseteq E_i(w) \restriction_{\bigcup \mathbb{C}_i}$ we have that $\bigcap \mathbb{X} \subseteq [\![\varphi]\!]$. By (1), and the assumption that $\mathbb{C}_i$ is downward closed, we have that $[\![\psi]\!] \in \mathbb{C}_i$ (and by (3), that the truth set of φ is non empty and overlaps with the truth set of $\varphi \to \psi$). By (2) and (3) we have that for each max f.i.p. $\mathbb{X} \subseteq E_i(w) \restriction_{\bigcup \mathbb{C}_i}$ we have that $\bigcap \mathbb{X} \subseteq [\![\psi]\!]$. Thus, $\mathcal{M}, w \models \Box_i(\psi)$.

Note that, again, if we have for an arbitrary contextual evidence frame, $\mathcal{F}$, that the formula $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$ for an arbitrary $i \in Ag$ is valid on $\mathcal{F}$, then we do not necessarily have that the contextual filter $\mathbb{C}_i$ in $\mathcal{F}$ is downward closed. When the evidence relation of the frame is empty, then independent of the contextual filter, the above formula is valid this frame.[2]

For validity 2 above, we will need the same assumption as was made originally by van Benthem and Pacuit ([21]), to get the validity. Namely, every agent knows its space[3], $W \in E_i(w)$ for all agents i.

[2] In this case we cannot even construct the evidence set of the frame such that we are sure that when the contextual filter is not downward closed, we have that the formula $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$ is not valid on $\mathbb{F}$. This is true, since we cannot control that the frame satisfies $\Box_i\varphi$ (without making further assumptions on the contextual filter), except for when $[\![\varphi \to \psi]\!] = [\![\varphi]\!]$. Namely, then $[\![\psi]\!] = [\![\varphi]\!]$ as well and therefore the formula holds also in this case.

[3] In footnote 4 of chapter 2 we noted that it is enough to suppose that every agent has at least one evidence set. This is no longer the case in contextual evidence logic.

Fact 7. *With the assumption that* $\{(w, W) \mid w \in W\} \in E_i$*, we have for all contextual evidence frames,* $\mathcal{F} = \langle W, E, \{\mathbb{C}_i \mid i \in Ag\}\rangle$ *that:* $B_i\varphi \to \neg B_i\neg\varphi$ *is valid on* $\mathcal{F}$*.*

Take an arbitrary contextual evidence frame $\mathcal{F}$. Furthermore, take an arbitrary valuation function V, an arbitrary world $w \in W$ and two arbitrary formulas $\varphi, \psi \in \mathcal{L}^{CE}$ such that $\mathcal{M}, w \models B_i\varphi$. Then we have that (1) $[\![\varphi]\!] \in \mathbb{C}_i$ and (2) for each max f.i.p. $\mathbb{X} \subseteq E_i(w) \restriction_{\bigcup \mathbb{C}_i}$ that $\bigcap \mathbb{X} \subseteq [\![\varphi]\!]$. Then, there are two cases. Either $\bigcup \mathbb{C}_i = \emptyset$, but then, by (1), we have that $[\![\varphi]\!] = \emptyset$. That means that $[\![\neg\varphi]\!] = W$ (and W is by definition a non empty set). Therefore, $[\![\neg\varphi]\!] \notin \mathbb{C}_i$. Thus, $\mathcal{M}, w \not\models B_i\neg\varphi$, which means that $\mathcal{M}, w \models \neg B_i\neg\varphi$. The second case is that $\bigcup \mathbb{C}_i \neq \emptyset$. By the assumption that $W \in E_i(w)$ and that $\bigcup \mathbb{C}_i \neq \emptyset$ we have that there exists a max f.i.p. $\mathbb{X} \subseteq E_i(w) \restriction_{\bigcup \mathbb{C}_i}$. By definition we have that $\bigcap \mathbb{X} \neq \emptyset$. That means that by (2) we cannot have for every max f.i.p. $\mathbb{X}' \subseteq E_i(w) \restriction_{\bigcup \mathbb{C}_i}$ that $\bigcap \mathbb{X}' \subseteq [\![\neg\varphi]\!]$. Thus, $M, w \models \neg B_i\neg\varphi$.

Finally, we need to explore the class of contexts, such that the contextual evidence model satisfies the rule of necessitation (rule 3 above).

Fact 8. *For all contextual evidence frames,* $\mathcal{F} = \langle W, E, \{\mathbb{C}_i \mid i \in Ag\}\rangle$*, we have that: if for an arbitrary agent* $W \in \mathbb{C}_i$*, then the necessitation rule in equation 4.5 above is valid on* $\mathcal{F}$*.*

Take an arbitrary contextual evidence frame $\mathcal{F}$, with $W \in \bigcup \mathbb{C}_i$ for an arbitrary $i \in Ag$. Furthermore, take an arbitrary valuation function V such that for every world $w \in W$ and for every two formulas $\varphi, \psi \in \mathcal{L}^{CE}$, we have that $\mathcal{M}, w \models \varphi$. Then we have that $[\![\varphi]\!] = W$. By assumption we therefore have that $[\![\varphi]\!] \in \mathbb{C}_i$. Furthermore, we surely have that the intersection of each $\mathbb{X} \subseteq E_i(w) \restriction_{\bigcup \mathbb{C}_i} = E_i(w) \restriction_W = E_i(w)$ that satisfies that each maximal finite intersection property is included in $[\![\varphi]\!] = W$. Thus, $\mathcal{M}, w \models B_i\varphi$.

For right to left, we use contra position. We have that the truth set of a validity $\varphi \in \mathcal{L}^{CE}$ equals the set of all worlds W. If $W \notin \bigcup \mathbb{C}_i$, we have that $[\![\varphi]\!] \notin \mathbb{C}_i$. Thus, for an arbitrary $\mathcal{M}$, with contextual filter $\mathbb{C}_i$ for an arbitrary agent i and an arbitrary world $w \in W$ we have that $\mathcal{M}, w \not\models B_i\varphi$.

Thus, with the above facts, we have that the belief operator satisfies the logic KD if the contextual filter $\mathbb{C}_i$ is downward monotone for all agents $i \in Ag$ and $W \in \mathbb{C}$. That means that the top element has to be in the context together with the downwards closure (with exception of the empty set). Thus $\mathbb{C}_i = \wp(W) - \{\emptyset\}$ in order to satisfy the principles of the logic KD.

Chapter 5

Dynamic Contextual Evidence Logic.

Fortunately, the context is not a static filter in an agent's belief state. Different situations trigger different contexts. Therefore, we add dynamic operations in the language for actions that change the context.

First, we will give two examples of dynamic actions that operate on an agent's contextual filter. As we will see, under certain conditions, these two operators together are enough to manipulate the contextual filter in any possible way. Moreover, we will point out a connection to public announcement logic and some characterization results will be given. Both will provide a further insight on these actions.

5.1 Updating the Context

Two actions one can think of, that influence the context, are widening the agent's view and narrowing the agent's view. These are natural actions, when one thinks of the background of the agent as co-determining the current context. If then, certain changes take place which influence the current context, these changes, as well as the background of the agent, determine the relevant alternatives for the agent after these changes took place. We have that these context changes can make more alternatives available, widening the view of the agent, or make less alternatives available, narrowing the view of the agent.

From this moment on, we will make the language dynamic. As a consequence, it becomes non trivial on which model the truth set of a formula is evaluated. As we will see later, the only part of the model that is changed by our new dynamic actions is the contextual filter in the model. Therefore, the model on which the truth set is evaluated after one of these actions stays the same, except for the contextual filters in the model. In order to deal with the non triviality of which contextual filter is used, we will, for some model $\mathcal{M} = \langle W, E, \mathbb{C}, V \rangle$ and some set of formulas $\Phi \subseteq \mathcal{L}^{CE}$, write $[\![\Phi]\!]_{\mathbb{C}}$ for $[\![\Phi]\!]_{\mathcal{M}}$ if W, E and V are clear from the context.

Although we take the context to be a set of sets of worlds, we will, on some occasions, refer to the context as being a set of formulas $\Phi \subseteq \mathcal{L}$. Hereby, we mean that the set of formulas Φ determines for some model $\mathcal{M}$ the unique context $[\![\Phi]\!]_{\mathcal{M}}$.

In the following, we will define the model after an update with the narrowing view action, followed by the definition of the model after an update with the widening view action.

Definition 28 (Narrowing the view)**.** Let $\mathcal{M} = \langle W, E, \mathbb{C}, V \rangle$ be a contextual evidence model and $\Phi \subseteq \mathcal{L}^{CE}$ a context. The model narrowed to the context Φ equals $\mathcal{M}^{\cap\Phi} = \langle W, E, \{\mathbb{C}_i \cap [\![\Phi]\!]_{\mathbb{C}} \mid i \in Ag\}, V \rangle$.

Thus, narrowing the view is publicly defined by removing all the alternatives from the contextual filter of each agent, that are not present in the context we update with[1]. Hereby, the view of the agent gets narrowed, since less alternatives are relevant after the update than before. One can also see this update action as focusing; when the agents view gets narrowed, she focuses on the context she updates with, respecting her background context.

Widening the view goes similar, only then all the alternatives from the updated context are now added (instead of removed) to each agents contextual filter. Thus, more alternatives become relevant after the update, which causes that the view of the agent gets widened.

Definition 29 (Widening the view)**.** Let $\mathcal{M} = \langle W, E, \mathbb{C}, V \rangle$ be a contextual evidence model and $\Phi \subseteq \mathcal{L}^{CE}$ a context. The model widened with the context Φ equals $\mathcal{M}^{\cup\Phi} = \langle W, E, \{\mathbb{C}_i \cup [\![\Phi]\!]_{\mathbb{C}} \mid i \in Ag\}, V \rangle$.

In the language, we use the dynamic modalities $[\cap\Phi]$ and $[\cup\Phi]$ to refer respectively to narrowing the context and widening the context. The truth

[1]Note that this may lead to an empty contextual filter in the updated model. Although it may seem unwanted, there could be situations in which such a clash exists between the background of an agent and the contextual changes she is confronted with, that her ability to judge based on her evidence is blanc.

conditions for these actions are straightforward:

$$\begin{array}{lll} \mathcal{M}, w \models [\cap\Phi]\varphi & \textit{iff} & \mathcal{M}^{\cap\Phi}, w \models \varphi \\ \mathcal{M}, w \models [\cup\Phi]\varphi & \textit{iff} & \mathcal{M}^{\cup\Phi}, w \models \varphi \end{array} \tag{5.1}$$

Logically speaking it would be interesting to also add a dynamic modality that reverses the context. However, as we will see towards the end of this section, the normal set theoretical properties of $\cap$, $\cup$ and complement will be respected only under a uniformity constriction of the set of formulas that occurs in the scope of the operator. Furthermore, it seems that complementing the context, which means that an agent operating in some context gets feedback from the world that completely reverses her context, does not seem a natural action. Therefore, we will not define this operator here (although it would be very straightforward to do so).

In order to be able to state a set of reduction axioms for these dynamic operators, we need to add a family of contextual evidence modalities, $\Box_i^{\Xi}$, and a family of contextual belief operators, B_i^{Ξ}, to the language.

Definition 30 (Contextual modalities)**.** We extend the grammar in definition 23 with $\Box_i^{\Xi}\varphi$ and B_i^{Ξ} where, given that $\Phi \subseteq \mathcal{L}^{CE}$ is a set of formulas, we have that Ξ can be recursively defined as follows,

$$\begin{array}{lll} \Xi & \to & () \\ \Xi & \to & (\cap\Phi)\Xi \\ \Xi & \to & (\cup\Phi)\Xi \end{array}$$

Next, we give the truth definition for these contextual modalities.

Definition 31 (Truth)**.** Let $\mathcal{M} = \langle W, E, \mathbb{C}, V\rangle$ be a contextual evidence model. Truth of a formula $\varphi \in \mathcal{L}^{CE}$ is inductively defined by definition 24 and, with Ξ as defined above, extended with the following truth formulas for the family of contextual evidence and belief:

$$\begin{array}{lll} \mathcal{M}, w \models \Box_i^{\Xi}\varphi & \textit{iff} & [\![\varphi]\!]_{[\![\Xi]\!]_{\mathbb{C}}} \in [\![[\Xi]_i]\!]_{\mathbb{C}} \ \& \\ & & \exists X \in E_i(w) \ \textit{s.t.}\ \emptyset \neq (X \cap \bigcup [\![[\Xi]_i]\!]_{\mathbb{C}}) \subseteq [\![\varphi]\!]_{[\![\Xi]\!]_{\mathbb{C}}} \end{array}$$

$$\begin{array}{lll} \mathcal{M}, w \models B_i\varphi & \textit{iff} & [\![\varphi]\!]_{[\![\Xi]\!]_{\mathbb{C}}} \in [\![[\Xi]_i]\!]_{\mathbb{C}} \ \& \text{ for each max f.i.p. } \mathbb{X} \subseteq E_i(w)\restriction_{\bigcup [\![[\Xi]_i]\!]_{\mathbb{C}}} \\ & & \text{we have that } \bigcap \mathbb{X} \subseteq [\![\varphi]\!]_{[\![\Xi]\!]_{\mathbb{C}}} \end{array}$$

where $[\![\Xi]\!]_{\mathbb{C}} = \{[\![[\Xi]_i]\!]_{\mathbb{C}} \mid i \in Ag\}$, with

$$\begin{array}{lll} [\![[]_i]\!]_{\mathbb{C}} & = & \mathbb{C}_i \\ [\![[(\cap\Phi)\Xi]_i]\!]_{\mathbb{C}} & = & [\![[\Xi]_i]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}} \\ [\![[(\cup\Phi)\Xi]_i]\!]_{\mathbb{C}} & = & [\![[\Xi]_i]\!]_{\mathbb{C}\cup[\![\Phi]\!]_{\mathbb{C}}} \end{array} \tag{5.2}$$

and

$$\begin{array}{rcl} \mathbb{C} \cap [\![\Phi]\!]_{\mathbb{C}} & = & \{\mathbb{C}_i \cap [\![\Phi]\!]_{\mathbb{C}} \mid i \in Ag\} \\ \mathbb{C} \cup [\![\Phi]\!]_{\mathbb{C}} & = & \{\mathbb{C}_i \cup [\![\Phi]\!]_{\mathbb{C}} \mid i \in Ag\} \end{array} \tag{5.3}$$

Similar to the original evidence modality, we have that the left conjunct of the definition rules out every possibility that is not available in the context, only here the contextual filter is updated with Ξ. The right conjunct restricts the pieces of evidence to the relevant alternatives. Actually, all the work is done by the definition of the truth set of Ξ, which is taken to act as the total replacement of the contextual change, in which the contextual evidence modality keeps track of these contextual changes, combined with the original contextual filter in the original model.

For the contextual belief operator, we see that the definition resembles the definition of the original belief operator in the same way as the contextual evidence modality resemblance the original evidence modality. Namely, we have replaced the contextual filters $\mathbb{C}$ everywhere with $[\![\Xi]\!]_{\mathbb{C}}$ and the contextual filter $\mathbb{C}_i$ for agent i with $[\![[\Xi]_i]\!]_{\mathbb{C}}$. We will further explain what is happening here.

We have that the contextual information Ξ of the contextual modalities is an arbitrary combination of $(\cap\Phi)$ and $(\cup\Phi)$, where Φ can stand for every possible set of formulas, and for every occurrence of $(\cap\Phi)$ or $(\cup\Phi)$ we have that Φ can be a different set of formulas. For example, $\Xi = (\cup\Phi_1)(\cap\Phi_2)(\cap\Phi_3)$. Hereby we have that Ξ changes the context of the model $\mathbb{C}_i$ in such a way that the contextual filter is now equal to something like: $(((\mathbb{C}_i \cup [\![\Phi_1]\!]) \cap [\![\Phi_2]\!]) \cap [\![\Phi_3]\!])$.

Now that we roughly sketched the idea we will show for the example $\Xi = (\cup\Phi_1)(\cap\Phi_2)(\cap\Phi_3)$ what the definition of $[\![[\Xi]_i]\!]_{\mathbb{C}}$ does. We have that:

$[\![[\Xi]_i]\!]_{\mathbb{C}} =$

$[\![[(\cup\Phi_1)(\cap\Phi_2)(\cap\Phi_3)]_i]\!]_{\mathbb{C}} =$

$[\![[(\cap\Phi_2)(\cap\Phi_3)]_i]\!]_{\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}}} =$

$[\![[(\cap\Phi_3)]_i]\!]_{(\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}})\cap[\![\Phi_2]\!]_{\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}}}} =$

$[\![[]_i]\!]_{((\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}})\cap[\![\Phi_2]\!]_{\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}}})\cap[\![\Phi_3]\!]_{(\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}})\cap[\![\Phi_2]\!]_{\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}}}}} =$

$[((\mathbb{C} \cup [\![\Phi_1]\!]_{\mathbb{C}}) \cap [\![\Phi_2]\!]_{\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}}}) \cap [\![\Phi_3]\!]_{(\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}})\cap[\![\Phi_2]\!]_{\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}}}}]_i =$

$((\mathbb{C}_i \cup [\![\Phi_1]\!]_{\mathbb{C}}) \cap [\![\Phi_2]\!]_{\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}}}) \cap [\![\Phi_3]\!]_{(\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}})\cap[\![\Phi_2]\!]_{\mathbb{C}\cup[\![\Phi_1]\!]_{\mathbb{C}}}}$

As the definition shows us, we have been sloppy with notation when we described the idea of the contextual information given by Ξ. Namely, we did not specify the model in which the truth sets of the set of formulas are being evaluated. As we can see by following the definition, we have that the sets of formulas Φ_n are not all evaluated in the same current model, as might be a bit unexpected. This feature will have consequences for the reduction axioms in this logic, with respect to the reduction axioms in familiar logics like public announcement logic [3]. We will come back to this in section 5.2.

Furthermore, when we will come to the reduction axioms later, we will see tha,t except for the interesting truth definition of the contextual information Ξ, another important feature in definition 31 is that the truth set of the formula, that occurs inside the scope of the contextual modality, is evaluated in the model with the updated context instead of in the old model. See section 5.2 for further details.

The following substitution laws are valid for the family of contextual evidence and belief.

Fact 9 (Uniform substitution)**.** *For some set of formulas $\Phi \subseteq \mathcal{L}$, two formulas $\varphi, \psi \in \mathcal{L}$ and an agent $i \in Ag$, we have that:*

If for all contextual filters $\mathbb{C}$ we have
$[\![\varphi]\!]_{\mathbb{C}} = [\![\psi]\!]_{\mathbb{C}}$ & $\varphi \in \Phi$ *then*

1. $\Box_i^{\Xi_1(\cap\Phi)\Xi_2}\chi \leftrightarrow \Box_i^{\Xi_1(\cap(\Phi/_{\{\varphi\}}\cup\{\psi\}))\Xi_2}\chi$
2. $\Box_i^{\Xi_1(\cup\Phi)\Xi_2}\chi \leftrightarrow \Box_i^{\Xi_1(\cup(\Phi/_{\{\varphi\}}\cup\{\psi\}))\Xi_2}\chi$
3. $B_i^{\Xi_1(\cap\Phi)\Xi_2}\chi \leftrightarrow B_i^{\Xi_1(\cap(\Phi/_{\{\varphi\}}\cup\{\psi\}))\Xi_2}\chi$
4. $B_i^{\Xi_1(\cup\Phi)\Xi_2}\chi \leftrightarrow B_i^{\Xi_1(\cup(\Phi/_{\{\varphi\}}\cup\{\psi\}))\Xi_2}\chi$

First observe that since we have for all contextual filters $\mathbb{C}$ that $[\![\varphi]\!]_{\mathbb{C}} = \bigcup_{\varphi' \in \Phi}\{[\![\varphi']\!]_{\mathbb{C}}\}$, $[\![\varphi]\!]_{\mathbb{C}} = [\![\psi]\!]_{\mathbb{C}}$ and $\varphi \in \Phi$ imply that $[\![\Phi]\!]_{\mathbb{C}} = (\bigcup_{\varphi' \in \Phi/_{\{\varphi\}}}\{[\![\varphi]\!]_{\mathbb{C}}\}) \cup \{[\![\psi]\!]_{\mathbb{C}}\} = [\![\Phi/_{\{\varphi\}} \cup \{\psi\}]\!]_{\mathbb{C}}$.

Verifying the first substitution law with the above observation. Suppose $M, w \models \Box_i^{\Xi_1(\cap\Phi)\Xi_2}\chi$. By definition 31 that means $[\![\chi]\!]_{[\![\Xi_1(\cap\Phi)\Xi_2]\!]_{\mathbb{C}}} \in [\![[\Xi_1(\cap\Phi)\Xi_2]_i]\!]_{\mathbb{C}}$ & $\exists X \in E_i(w)$ *s.t.* $\emptyset \neq X \cap \bigcup([\![[\Xi_1(\cap\Phi)\Xi_2]_i]\!]_{\mathbb{C}}) \subseteq [\![\chi]\!]_{[\![\Xi_1(\cap\Phi)\Xi_2]\!]_{\mathbb{C}}}$. We have that by abbreviation 5.2 that $[\![\Xi_1(\cap\Phi)\Xi_2]\!]_{\mathbb{C}} = [\![\Xi_2]\!]_{[\![\Xi_1]\!]_{\mathbb{C}} \cap [\![\Phi]\!]_{[\![\Xi_1]\!]_{\mathbb{C}}}}$. By the above observation that equals $[\![\Xi_2]\!]_{[\![\Xi_1]\!]_{\mathbb{C}} \cap [\![\Phi/_{\{\varphi\}} \cup \{\psi\}]\!]_{[\![\Xi_1]\!]_{\mathbb{C}}}}$. Again by abbreviation 5.2 that means $[\![\Xi_1(\cap(\Phi/_{\{\varphi\}} \cup \{\psi\}))\Xi_2]\!]_{\mathbb{C}}$. Therefore we have $[\![\chi]\!]_{[\![\Xi_1(\cap(\Phi/_{\{\varphi\}}\cup\{\psi\}))\Xi_2]\!]_{\mathbb{C}}} \in [\![[\Xi_1(\cap(\Phi/_{\{\varphi\}} \cup \{\psi\}))\Xi_2]_i]\!]_{\mathbb{C}}$ & $\exists X \in E_i(w)$ *s.t.* $\emptyset \neq X \cap \bigcup([\![[\Xi_1(\cap(\Phi/_{\{\varphi\}} \cup \{\psi\}))\Xi_2]_i]\!]_{\mathbb{C}}) \subseteq [\![\chi]\!]_{[\![\Xi_1(\cap(\Phi/_{\{\varphi\}}\cup\{\psi\}))\Xi_2]\!]_{\mathbb{C}}}$. By definition 31, this is the same as $M, w \models \Box_i^{\Xi_1(\cap(\Phi/_{\{\varphi\}}\cup\{\psi\}))\Xi_2}\chi$.

The proof for the second substitution law follows straight from the same proof, replacing the appropriate intersections with unions. Furthermore, this proof for the first substitution law also provides us the proof for the third substitution law, since it gives a proof for $[\![\Xi_1(\cap\Phi)\Xi_2]\!]_{\mathbb{C}} = [\![\Xi_1(\cap(\Phi/_{\{\varphi\}} \cup \{\psi\}))\Xi_2]\!]_{\mathbb{C}}$. The equivalence of these two statements makes that we can replace them in the definition for contextual belief without changing the truth value. This will give us the third substitution law. Again, the proof for the fourth substitution law follows from the proof for the third substitution laws, replacing the appropriate intersections with unions.

The following reduction axioms are very simple equivalences, since, as their proofs shows, the hard work is already done by definition 31.

1.N	$[\cap\Phi]p$	$\leftrightarrow$	p
1.W	$[\cup\Phi]p$	$\leftrightarrow$	p
2.N	$[\cap\Phi](\neg\varphi)$	$\leftrightarrow$	$\neg[\cap\Phi]\varphi$
2.W	$[\cup\Phi](\neg\varphi)$	$\leftrightarrow$	$\neg[\cup\Phi]\varphi$
3.N	$[\cap\Phi](\varphi\wedge\psi)$	$\leftrightarrow$	$[\cap\Phi]\varphi\wedge\ [\cap\Phi]\psi$
3.W	$[\cup\Phi](\varphi\wedge\psi)$	$\leftrightarrow$	$[\cup\Phi]\varphi\wedge\ [\cup\Phi]\psi$
4.N	$[\cap\Phi]\Box_i\varphi$	$\leftrightarrow$	$\Box^{\cap\Phi}\varphi$
4.W	$[\cup\Phi]\Box_i\varphi$	$\leftrightarrow$	$\Box^{\cup\Phi}\varphi$
5.N	$[\cap\Phi]\Box_i^{\Xi}\varphi$	$\leftrightarrow$	$\Box^{(\cap\Phi)\Xi}\varphi$
5.W	$[\cup\Phi]\Box_i^{\Xi}\varphi$	$\leftrightarrow$	$\Box^{(\cup\Phi)\Xi}\varphi$
6.N	$[\cap\Phi]B_i\varphi$	$\leftrightarrow$	$B_i^{(\cap\Phi)}\varphi$
6.W	$[\cup\Phi]B_i\varphi$	$\leftrightarrow$	$B_i^{(\cup\Phi)}\varphi$
6.N	$[\cap\Phi]B_i^{\Xi}\varphi$	$\leftrightarrow$	$B_i^{(\cap\Phi)\Xi}\varphi$
6.W	$[\cup\Phi]B_i^{\Xi}\varphi$	$\leftrightarrow$	$B_i^{(\cup\Phi)\Xi}\varphi$

The first 3 reduction axioms of both narrowing the view (N) and widening the view (W) state that these dynamic actions interact the same with the boolean connectives as the usual dynamic-epistemic actions.

Axioms 4.N and 4.W are actually special cases of axioms 5.N and 5.W respectively (with Ξ empty). Furthermore, axioms 6.N and 6.W are special cases of 7.N and 7.W. Finally, 5.W and 7.W follow from the proofs 5.N and 7.N respectively, replacing the appropriate intersections with unions. Therefore, we will only give the proofs of axiom 5.N and 7.N.

5.N $\langle W,E,\mathbb{C},V\rangle, w \models [\cap\Phi]\Box_i^{\Xi}\varphi \leftrightarrow$
by the truth definition of $[\cap\Phi]$ (equation 5.1)
$\langle W,E,\mathbb{C},V\rangle^{\cap\Phi}, w \models \Box_i^{\Xi}\varphi \leftrightarrow$
by definition 28
$\langle W,E,\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}},V\rangle, w \models \Box_i^{\Xi}\varphi \leftrightarrow$
by definition 31
$[\![\varphi]\!]_{[\![\Xi]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}}} \in [\![\Xi_i]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}}$ &
$\exists X \in E_i(w)$ *s.t.* $\emptyset \neq (X \cap \bigcup[\![\Xi_i]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}}) \subseteq [\![\varphi]\!]_{[\![\Xi]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}}} \leftrightarrow$

by definition of $[\![\Xi]\!]$ (equation 5.2)
$[\![\varphi]\!]_{[\![(\cap\Phi)\Xi]\!]_{\mathbb{C}}} \in [\![[(\cap\Phi)\Xi]_i]\!]_{\mathbb{C}}$ &
$\exists X \in E_i(w)\ s.t.\ \emptyset \neq (X \cap \bigcup [\![[(\cap\Phi)\Xi]_i]\!]_{\mathbb{C}}) \subseteq [\![\varphi]\!]_{[\![(\cap\Phi)\Xi]\!]_{\mathbb{C}}} \leftrightarrow$
by definition 31
$\langle W, E, \mathbb{C}, V\rangle, w \models \Box_i^{(\cap\Phi)\Xi}\varphi$

7.N $\langle W, E, \mathbb{C}, V\rangle, w \models [\cap\Phi]B_i^{\Xi}\varphi \leftrightarrow$
by the truth definition of $[\cap\Phi]$ (equation 5.1)
$\langle W, E, \mathbb{C}, V\rangle^{\cap\Phi}, w \models B_i^{\Xi}\varphi \leftrightarrow$
by definition 28
$\langle W, E, \mathbb{C} \cap [\![\Phi]\!]_{\mathbb{C}}, V\rangle, w \models B_i^{\Xi}\varphi \leftrightarrow$
by definition 31
$[\![\varphi]\!]_{[\![\Xi]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}}} \in [\![[\Xi]_i]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}}$ & for each f.i.p. $\mathbb{X} \subseteq E_i(w)\restriction_{\bigcup [\![[\Xi]_i]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}}}$ we have that $\bigcap \mathbb{X} \subseteq [\![\varphi]\!]_{[\![\Xi]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}}} \leftrightarrow$
by definition of $[\![\Xi]\!]$ (equation 5.2)
$[\![\varphi]\!]_{[\![(\cap\Phi)\Xi]\!]_{\mathbb{C}}} \in [\![[(\cap\Phi)\Xi]_i]\!]_{\mathbb{C}}$ & for each f.i.p. $\mathbb{X} \subseteq E_i(w)\restriction_{\bigcup [\![[(\cap\Phi)\Xi]_i]\!]_{\mathbb{C}}}$ we have that $\bigcap \mathbb{X} \subseteq [\![\varphi]\!]_{[\![(\cap\Phi)\Xi]\!]_{\mathbb{C}}} \leftrightarrow$
by definition 31
$\langle W, E, \mathbb{C}, V\rangle, w \models B_i^{(\cap\Phi)\Xi}\varphi$

It might be surprising that the formulas are so simple. However, the only thing we have to take care of is that the formulas in Ξ and the formula inside the scope of the modality, φ, are evaluated in the right model. Since definition 31 already evaluates the truth sets of φ and the formulas in Ξ in the right updated context, there is no need to force this in the combination actions.

In the next subsection we will explore the reduction axioms when we evaluate the formulas in Ξ and the formula inside the contextual modalities as is conventional in the literature.

5.1.1 Characterization Results

Above we explored the behavior of contextual evidence frames for some structures on the context. However, it would still be interesting to explore how evidence will change under contextual change.

We start out with a global observation that will give immediate results in the logic.

Fact 10 (Subset preservation)**.** *We have for all W, E, V and two contextual filters $\mathbb{C}$ and $\mathbb{C}'$ that*

$$\langle W, E, \mathbb{C}, V\rangle, w \models \Box_i\varphi \Rightarrow \langle W, E, \mathbb{C}', V\rangle, w \models \Box_i\varphi$$

$$if\ \mathbb{C}'_i \subseteq \mathbb{C}_i\ and\ [\![\varphi]\!]_{\mathbb{C}'} \in \mathbb{C}'_i.$$

This may not seem a surprising fact, since if the truth set of φ is still a possibility and there are less possibilities left than in the first place, it most certainly must be the case that the evidence for φ remains in the smaller context.

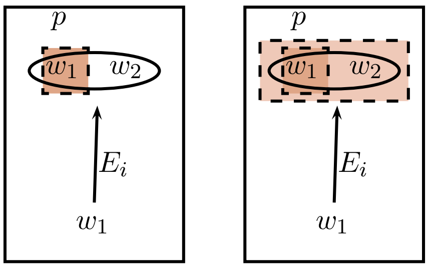
However, the following fact may be more surprising (at first sight).

Fact 11 (No superset preservation)**.** *We have for all W, E, V and two contextual filters $\mathbb{C}$ and $\mathbb{C}'$ that*

$$\langle W, E, \mathbb{C}', V \rangle, w \models \Box_i\varphi \not\Rightarrow \langle W, E, \mathbb{C}, V \rangle, w \models \Box_i\varphi$$

$$if\ \mathbb{C}'_i \subseteq \mathbb{C}_i\ and\ [\![\varphi]\!]_{\mathbb{C}} \in \mathbb{C}_i.$$

We can give the following counter example. Assume there is only one agent $\{i\} = Ag$. Let $W = \{w_1, w_2\}$, $E_i = \{(w_1, \{w_1, w_2\})\}$ and $V(p) = \{w_1\}$, for illustration see picture 6.1. Furthermore, take context $\mathbb{C}' = \{\{w_1\}\}$ and $\mathbb{C} = \{\{w_1\}, \{w_1, w_2\}\}$. Then we have that $\mathbb{C}' \subseteq \mathbb{C}$ and $[\![p]\!]_{\mathbb{C}} \in \mathbb{C}$. Thus, $\langle W, E, \mathbb{C}', V \rangle, w_1 \models \Box_i p$, but $\langle W, E, \mathbb{C}, V \rangle \not\models \Box_i p$.



(a) With context $\mathbb{C}'$. (b) With context $\mathbb{C}$.

Figure 5.1: counter example for superset preservation.

Although this result, showing that the superset construction is not preserved might look surprising, it is what we aimed for from the beginning in this logic. Namely, when 'Janet Jackson is on television' and 'Bush is on television' are the only two alternatives in the context, then the evidence for 'there is a man on television', is evidence for 'Bush is on television'. However, when the context also allows for the alternative 'Will Ferrell is on television', it may be clear that the evidence for 'there is a man on television' in this superset context is not evidence for 'Bush is on television'. Although 'Bush is on television' is still a relevant alternative, the appearance of another relevant alternative, namely 'Will Ferrell is on television', makes that the agent does not have evidence anymore for 'Bush is on television'.

It follows from the fact that we do have subset preservation that the following formulas are validities in the contextual evidence logic.

1.	$\Box_i\varphi$	$\rightarrow$	$[\cap\Phi]\Box_i\varphi$	*if* $[\![\varphi]\!]_{\mathbb{C}\cap\Phi} \in [\![\Phi]\!]_{\mathbb{C}}$
2.	$[\cup\Phi]\Box_i\varphi$	$\rightarrow$	$\Box_i\varphi$	*if* $[\![\varphi]\!]_{\mathbb{C}\cap\Phi} \notin [\![\Phi]\!]_{\mathbb{C}}$

Clearly, since we do not have that the preservation of evidence holds when the context gets extended, these are only one way validities.

We also have the following validities, due to the preservation of the set theoretical properties of narrowing and widening the view, under the restriction that all the formulas in Φ are true in the same worlds independent of the context.

Fact 12 (Context change)**.** *The following three statements are equivalent under the condition that* $[\![\Phi]\!]_{\mathbb{C}} \equiv [\![\Phi]\!]_{\mathbb{C}\cap\Phi} \equiv [\![\Phi]\!]_{\mathbb{C}\cup\Phi}$*:*

1. $\langle W, E, \mathbb{C}, V\rangle, w \models [\cap\Phi][\cup\Phi]\Box_i\varphi$
2. $\langle W, E, \mathbb{C}, V\rangle, w \models [\cup\Phi][\cap\Phi]\Box_i\varphi$
3. $\langle W, E, [\![\Phi]\!]_{\mathbb{C}}, V\rangle, w \models \Box\varphi$

We have that (1) $\langle W, E, \mathbb{C}, V\rangle, w \models [\cap\Phi][\cup\Phi]\Box_i\varphi = \langle W, E, \mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}, V\rangle, w \models [\cup\Phi]\Box_i\varphi = \langle W, E, (\mathbb{C} \cap [\![\Phi]\!]_{\mathbb{C}}) \cup [\![\Phi]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}}, V\rangle, w \models \Box_i\varphi$. Due to set the above condition that equals $\langle W, E, (\mathbb{C} \cap [\![\Phi]\!]_{\mathbb{C}}) \cup [\![\Phi]\!]_{\mathbb{C}}, V\rangle, w \models \Box_i\varphi$. Due to set theory that equals (2) $\langle W, E, [\![\Phi]\!]_{\mathbb{C}}, V\rangle, w \models \Box_i\varphi$. Again due to set theory that equals $\langle W, E, (\mathbb{C} \cup [\![\Phi]\!]_{\mathbb{C}}) \cap [\![\Phi]\!]_{\mathbb{C}}, V\rangle, w \models \Box_i\varphi$. Again by the above condition that equals $\langle W, E, (\mathbb{C} \cup [\![\Phi]\!]_{\mathbb{C}}) \cap [\![\Phi]\!]_{\mathbb{C}\cup[\![\Phi]\!]_{\mathbb{C}}}, V\rangle, w \models \Box_i\varphi = \langle W, E, \mathbb{C} \cup [\![\Phi]\!]_{\mathbb{C}}, V\rangle, w \models [\cap\Phi]\Box_i\varphi = \langle W, E, \mathbb{C}, V\rangle, w \models [\cup\Phi][\cap\Phi]\Box_i\varphi$ (3).

5.2 Global versus local conditionals

In the following we will only make the observations with respect to the evidence operator. Although practically the same observations can be made for belief, after spelling them out for the evidence modality it would be redundant to do the same for the belief operator. Therefore, the focus in this section will be on the evidence modality.

In the previous section we saw that the contextual evidence ($\Box_i^{\Xi}\varphi$) was defined in a way that the formulas occurring in the contextual information (Ξ) and the formula inside the the contextual evidence modality (φ) were updated with respect to the context specified by the contextual information (Ξ). We thereby noted that this definition is different from the definition of the conventional conditional operators ([3]), where both the conditional and the formula occurring inside the operator are evaluated in the current model. We will first make this observation precise.

For comparison, we will state the definition of conditional evidence and contextual evidence respectively.

$$\mathcal{M}, w \models \Box_i^{\varphi}\psi \quad \textit{iff} \quad \exists X \in E_i(w) \ \textit{s.t.} \ \emptyset \neq X \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}} \tag{5.4}$$

$$\begin{aligned} \mathcal{M}, w \models \Box_i^{\Xi}\varphi \quad \textit{iff} \quad & [\![\varphi]\!]_{[\![\Xi]\!]_{\mathbb{C}}} \in [\![[\Xi]_i]\!]_{\mathbb{C}} \ \& \\ & \exists X \in E_i(w) \ \textit{s.t.} \ \emptyset \neq (X \cap \bigcup [\![[\Xi]_i]\!]_{\mathbb{C}}) \subseteq [\![\varphi]\!]_{[\![\Xi]\!]_{\mathbb{C}}} \end{aligned} \tag{5.5}$$

We see that the second conjunct of the definition of contextual evidence (equation 5.5) looks very similar to the definition of conditional evidence (equation 5.4). In both definitions the pieces of evidence are restricted to some set of worlds, indicated by the uppercase of the box operator.

However, this definitions show us that, with conditional evidence, the formulas φ and ψ are evaluated in the current model, while, with contextual evidence, the formulas in Ξ and φ are evaluated in properly updated models.

Furthermore, this difference influences the reduction axioms. See 5.6 for the reduction axiom of conditional evidence for public announcements ([21]) and 5.7 for the reduction axiom of contextual evidence for narrowing the view.

$$[!\varphi]\Box^{\alpha}\psi \quad \leftrightarrow \quad (\varphi \rightarrow \Box^{\varphi \wedge [!\varphi]\alpha}[!\varphi]\psi) \tag{5.6}$$

$$[\cap\Phi]\Box^{\Xi}\psi \quad \leftrightarrow \quad \Box^{(\cap\Phi)\Xi}\psi \tag{5.7}$$

The reduction axioms show us that after a public announcement is made, the announcement has to be brought recursively inside the conditional and the formula in the scope of the evidence modality. After an update that narrows the view of the agent there is no need to bring this update recursively inside the evidence modality, since this is already taken care of by the definition.

We detected a fundamental difference in the way the contextual evidence and the conditional evidence are defined. Every formula occurring inside the contextual evidence modality (either in the superscript or just in the scope of the modality) is evaluated in the updated contextual filter. We will call this a global conditional, since the condition (on the context) globally determines the model in which the formulas are to be evaluated. The conditional of conditional evidence, on the other hand, is a local conditional, since the conditional only influences the evidence locally. All formulas occurring in the conditional evidence modality are evaluated outside the scope of the conditional. Thus, the conditional has only local power.

This difference between local and global conditions raises a couple of questions, such as whether contextual evidence can be defined by evaluating the contextual information locally and, if so, why choose one or the other definition? Furthermore, can conditional evidence be defined with a global conditional and, if so, why choose one or the other definition? Finally, what price is has to be pay for the simple reduction axioms when the conditional is interpreted locally?

We will first define contextual evidence when the contextual information is defined locally. Then, we will give a definition of conditional evidence when the conditional is interpreted globally. Afterwards, we will see that the different definitions will not influence the complexity of the reduction axioms, However, as we will finally argue, it does have other consequences. These consequences will lead to the conclusion that conditional evidence can indeed best be interpreted locally, while contextual evidence makes more sense when interpreted globally.

5.2.1 Local Contextual evidence

In the following, we will see what happens if the definition of contextual evidence, just as the definition of conditional evidence, evaluates all its formulas in the current model.

Definition 32 (Truth)**.** Let $\mathcal{M} = \langle W, E, \mathbb{C}, V \rangle$ be a contextual evidence model. Truth of a formula $\varphi \in \mathcal{L}^{CE'}$ is inductively defined by definition 24 and, with Ξ as defined by definition 30, extended with the following truth formula for the contextual evidence prime:

$$\begin{aligned} \mathcal{M}, w \models \Box_i'^{\Xi}\varphi \quad \textit{iff} \quad & [\![\varphi]\!]_{\mathbb{C}} \in \mathbb{C}_i[\![\Xi]\!]_{\mathbb{C}} \ \& \\ & \exists X \in E_i(w) \ s.t. \ \emptyset \neq (X \cap \bigcup \mathbb{C}_i[\![\Xi]\!]_{\mathbb{C}}) \subseteq [\![\varphi]\!]_{\mathbb{C}} \end{aligned}$$

where $\mathbb{C}[\![\Xi]\!]_{\mathbb{C}} = \{\mathbb{C}_i[\![\Xi]\!]_{\mathbb{C}} \mid i \in Ag\}$, with

$$\begin{array}{lcl} \mathbb{C}_i[\![]\!]_{\mathbb{C}} & = & \mathbb{C}_i \\ \mathbb{C}_i[\![(\cap\Phi)\Xi]\!]_{\mathbb{C}} & = & \mathbb{C}_i \cap [\![\Phi]\!]_{\mathbb{C}}[\![\Xi]\!]_{\mathbb{C}} \\ \mathbb{C}_i[\![(\cup\Phi)\Xi]\!]_{\mathbb{C}} & = & \mathbb{C}_i \cup [\![\Phi]\!]_{\mathbb{C}}[\![\Xi]\!]_{\mathbb{C}} \end{array} \tag{5.8}$$

The recursion axioms for the contextual evidence prime modality will now look as follows.

$$\begin{array}{llcl} 5.N' & [\cap\Phi]\Box_i'^{\Xi}\varphi & \leftrightarrow & \Box'^{(\cap\Phi)[\cap\Phi]\Xi}[\cap\Phi]\varphi \\ 5.W' & [\cup\Phi]\Box_i'^{\Xi}\varphi & \leftrightarrow & \Box'^{(\cup\Phi)[\cup\Phi]\Xi}[\cup\Phi]\varphi \end{array}$$

where $[\cap\Phi]\Xi$ is inductively defined as,

$$\begin{array}{lcl} [\cap\Phi]() & = & () \\ [\cup\Phi]() & = & () \\ [\cap\Phi](\cap\Psi)\Xi & = & (\cap[\cap\Phi]\Psi)[\cap\Phi]\Xi \\ [\cup\Phi](\cap\Psi)\Xi & = & (\cap[\cup\Phi]\Psi)[\cup\Phi]\Xi \\ [\cap\Phi](\cup\Psi)\Xi & = & (\cup[\cap\Phi]\Psi)[\cap\Phi]\Xi \\ [\cup\Phi](\cup\Psi)\Xi & = & (\cup[\cup\Phi]\Psi)[\cup\Phi]\Xi \end{array} \tag{5.9}$$

and

$$\begin{array}{lcl} [\cap\Phi]\{\Psi_1, ..., \Psi_n\} & = & \{[\cap\Phi]\Psi_1, ..., [\cap\Phi]\Psi_n\} \\ [\cup\Phi]\{\Psi_1, ..., \Psi_n\} & = & \{[\cup\Phi]\Psi_1, ..., [\cup\Phi]\Psi_n\} \end{array} \tag{5.10}$$

As $5.W'$ follows from the proof for $5.N'$, replacing the appropriate intersections with unions, we will only give the proof of axiom $5.N'$.

$5.N'$ $\langle W, E, \mathbb{C}, V\rangle, w \models [\cap\Phi]\Box_i'^{\Xi}\varphi \leftrightarrow$
by the truth definition of $[\cap\Phi]$ (equation 5.1)
$\langle W, E, \mathbb{C}, V\rangle^{\cap\Phi}, w \models \Box_i'^{\Xi}\varphi \leftrightarrow$
by definition 28
$\langle W, E, \mathbb{C} \cap [\![\Phi]\!]_{\mathbb{C}}, V\rangle, w \models \Box_i'^{\Xi}\varphi \leftrightarrow$
by definition 32
$[\![\varphi]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}} \in \mathbb{C}_i[\![\Xi]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}}$ &
$\exists X \in E_i(w)$ *s.t.* $\emptyset \neq (X \cap \bigcup \mathbb{C}_i[\![\Xi]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}}) \subseteq [\![\varphi]\!]_{\mathbb{C}\cap[\![\Phi]\!]_{\mathbb{C}}} \leftrightarrow$
by rewrite rules 5.8, 5.9 and 5.10
$[\![[\cap\Phi]\varphi]\!]_{\mathbb{C}} \in \mathbb{C}_i[\![(\cap\Phi)[\cap\Phi]\Xi]\!]_{\mathbb{C}}$ &
$\exists X \in E_i(w)$ *s.t.* $\emptyset \neq (X \cap \bigcup \mathbb{C}_i[\![(\cap\Phi)[\cap\Phi]\Xi]\!]_{\mathbb{C}}) \subseteq [\![[\cap\Phi]\varphi]\!]_{\mathbb{C}} \leftrightarrow$
by definition 31
$\langle W, E, \mathbb{C}, V\rangle, w \models \Box_i'^{(\cap\Phi)[\cap\Phi]\Xi}[\cap\Phi]\varphi$

As the reduction axioms 5.N' and 5.W' show us, defining the contextual evidence modality locally, indeed gives us reduction axioms of the form we are used to within public announcement logic (see 5.6).

There is still a prominent difference, however, between 5.N and 5.W in comparison with 5.6. Namely, 5.6 has an extra requirement which state the φ must be actually true. That implies that the public announcement is presupposed to be truthful. Contexts, on the other hand, do not have the requirement of being truthful.

5.2.2 Global conditional evidence

In the previous section we saw that contextual evidence could also be defined locally, instead of globally. Here, we will show that conditional evidence could also be defined globally, instead of locally.

In order to define the conditional evidence modality as a global conditional, we need to change the notation of the conditional evidence a bit.

Definition 33 (Conditional evidence)**.** Given a fixed set of propositional variables At and a group of agents Ag, with $i \in Ag$. Let $\mathcal{L}^{PE'}$ be the smallest set of formulas generated by the grammar from definition 17, where $\Box_i^{\varphi}\psi$ is replaced with $\Box_i^{'\xi}\psi$, with ξ recursively defined as follows:

$$\begin{array}{lll} \xi & \to & \top \\ \xi & \to & (\varphi \cap \xi) \end{array}$$

The idea of this definition is that we can write some formula φ always as $(\varphi \cap \top)$. Thus, changing the notation of $\Box_i^{\alpha}\varphi$ into $\Box_i^{\xi}\varphi$ does not influence expressive power of the conditional evidence at all.

Furthermore, we will introduce the notation of a restricted evidence set.

$$E_i(w) \restriction_X = \{Y \cap X \mid \exists Y \in E_i(w) \ s.t. \ Y \cap X \neq \emptyset\}$$

$$E \restriction_X = \{w, E_i(w) \restriction_X \mid w \in W \cap X\}$$

Now we can give the global truth definition for conditional evidence.

Definition 34 (Truth)**.** Let $\mathcal{M} = \langle W, E, V\rangle$ be an evidence model. Truth of a formula $\varphi \in \mathcal{L}^{PE'}$ is inductively defined by definition 18, replacing the truth definition for conditional evidence by the following definition with ξ defined as above:

$$\begin{array}{lll} \mathcal{M}, w \models \Box_i^{'\xi}\varphi & \textit{iff} & \exists X \in E_i(w) \ s.t. \\ & & \emptyset \neq (X \cap [\![\xi]\!]_{\mathcal{M}}) \subseteq [\![\varphi]\!]_{\langle [\![\xi]\!]_{\mathcal{M}}, E\restriction_{[\![\xi]\!]_{\mathcal{M}}}, \{V(p) \cap [\![\xi]\!]_{\mathcal{M}} \mid p \in At\}\rangle} \end{array}$$

where $[\![\xi]\!]_{\mathcal{M}}$ is inductively defined as:

$$\begin{array}{lcl} [\![\top]\!]_{\mathcal{M}} & = & [\![\top]\!]_{\mathcal{M}} \\ [\![(\varphi \cap \xi)]\!]_{\mathcal{M}} & = & [\![\xi]\!]_{\mathcal{M}^{!\varphi}} \end{array} \quad (5.11)$$

The recursion axiom for the conditional evidence prime modality will now look as follows.

$$\text{(PA6')} \quad [!\varphi]\Box_i'^{\xi}\psi \quad \leftrightarrow \quad (\varphi \to \Box_i'^{(\varphi\cap\xi)}\psi)$$

We will prove this equivalence.

(PA6') $\mathcal{M}, w \models [!\varphi]\Box_i'^{\xi}\psi \Leftrightarrow$
by the truth definition of $[!\varphi]$ (equation 2.1)
$\mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}^{!\varphi}, w \models \Box_i'^{\xi}\psi \Leftrightarrow$
by definition 34
$\mathcal{M}, w \models \varphi \Rightarrow$
$\exists X \in E_i^{!\varphi}(w)\ s.t.\ \emptyset \neq (X \cap [\![\xi]\!]_{\mathcal{M}^{!\varphi}}) \subseteq$
$[\![\psi]\!]_{\langle [\![\xi]\!]_{\mathcal{M}^{!\varphi}}, E^{!\varphi}\restriction_{[\![\xi]\!]_{\mathcal{M}^{!\varphi}}}, \{V^{!\varphi}(p)\cap[\![\xi]\!]_{\mathcal{M}^{!\varphi}} \mid p\in At\}\rangle} \Leftrightarrow$
Since for all formulas $\varphi, \psi \in \mathcal{L}^{PE'}$ we have $[\![\psi]\!]_{\mathcal{M}^{!\varphi}} \subseteq [\![\varphi]\!]_{\mathcal{M}}$
$\mathcal{M}, w \models \varphi \Rightarrow$
$\exists X \in E_i(w)\ s.t.\ \emptyset \neq (X \cap [\![\xi]\!]_{\mathcal{M}^{!\varphi}}) \subseteq$
$[\![\psi]\!]_{\langle [\![\xi]\!]_{\mathcal{M}^{!\varphi}}, E\restriction_{[\![\xi]\!]_{\mathcal{M}^{!\varphi}}}, \{V(p)\cap[\![\xi]\!]_{\mathcal{M}^{!\varphi}} \mid p\in At\}\rangle} \Leftrightarrow$
by rewrite rule 5.11
$\mathcal{M}, w \models \varphi \Rightarrow$
$\exists X \in E_i(w)\ s.t.\ \emptyset \neq (X \cap [\![(\varphi\cap\xi)]\!]_{\mathcal{M}}) \subseteq$
$[\![\psi]\!]_{\langle [\![(\varphi\cap\xi)]\!]_{\mathcal{M}}, E\restriction_{[\![(\varphi\cap\xi)]\!]_{\mathcal{M}}}, \{V(p)\cap[\![(\varphi\cap\xi)]\!]_{\mathcal{M}} \mid p\in At\}\rangle} \Leftrightarrow$
by definition 34
$\mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}, w \models \Box_i'^{(\varphi\cap\xi)}\psi \Leftrightarrow$
by defintion of $\to$
$\mathcal{M}, w \models (\varphi \to \Box_i'^{(\varphi\cap\xi)}\psi)$

Just as in the previous section, we can see the similarity between the reduction axiom PA6' and the reduction axiom 5.7, in which both conditional evidence and contextual evidence are defined globally. Due to the fact that public announcements are defined to be true, the same difference as before shows in this global view.

5.2.3 Complexity

The fact that we can create very simple reduction formulas by changing the definition of the conditional operators, giving the conditionals global instead of local power, raises the question what this means for the complexity of the formulas. The definition of global conditional evidence, as introduced in the previous section, shows that the global versus local conditionals does not influence the complexity. In fact we can notice that the global conditional operators are dynamic operators. Namely, for every intersection ($\cap$) in the conditional ξ, we need to update the model once with a public announcement (see rewrite rule 5.11). So the amount of intersections in the conditional determines the amount of $[!\varphi]$ reductions in the local setting. These reductions both happen in the conditional (when we intersect the evidence set with the truth set of the conditional) and in the formula in the scope of the operator (when we evaluate this formula in the new model, updated with the conditional). Thus, the complexity for both local and global definitions is the same. Only, the local conditional operator really is a static operator, whereas the global conditional operator is actually a dynamic operator[2].

Moreover, this observation that the complexity of the model update stays the same for the global and local interpretation, holds also for the contextual evidence modality. In this definition model change is less prominent, since we only change the contextual filter of the model. However, the actions also only change the contextual filter of the model. Thus, the same recursion as we saw with the conditional evidence modality is applied in the contextual evidence modality. Here as well, we have that the amount of intersections ($\cap$) and unions ($\cup$) in the contextual information (Ξ) determines the amount of model updates needed. We can see in the recursion axiom, every time an action takes place, an intersection or union is added to the contextual information. As a consequence, the complexity of the needed amount of model updates is equal.

5.2.4 Consequences

Frank Ramsey argued in a footnote that

> if two people are arguing 'if p will q?' and are both in doubt as to p, they are adding p hypothetically to their stock of knowledge and arguing on that basis about q. [14, p. 247]

"Putting p hypothetically to their stock" can be interpreted local or global, that is, one can assume p and then argue on that basis.

We have that the difference between local and global conditionals is only observable in iterated modalities.

[2]Strictly speaking it is the combination of the dynamic action and the evidence modality.

It makes sense to have something like: $K^p(\neg Kp)$, I know that under the assumption that p is true, I do not know that p is true. The same holds for conditional evidence: $\Box^p(\neg\Box p)$, I have evidence that under the assumption that p is true, I do not have evidence that p is true. However, for contextual evidence, such an argumentation would go as follows: $\Box'^{\cap\{p\}}(p)$, I have evidence that under the assumption that my view is narrowed to $\{p\}$, I do not have evidence for p. This argumentation makes no sense, since if your context was not narrowed to $\{p\}$, you would know it. Namely, in that case there is some alternative you are considering in which p is not true.

In general, we have that the conditional hypothetically assumes a certain property of the actual world, and then reasons from there, keeping the agents information the same. However, when the conditional assumes some property of the agents information state, then this influences the agents knowledge for sure. For example, $K^{(Kp)}Kp$, I know that under the assumption that I know that p, I will know that p. This example better resembles what is going on in the contextual case. Namely, in the contextual case, the contextual information is always some property of the agent's information state. However, in the conditional evidence case, the conditional can be independent of the information state of the agent.

For example, $\Box^{\cap\{p\}}\Box p$, I have evidence that under the hypothetical assumption that I narrow my view to $\{p\}$, I have evidence that p. This should always be the case, since it is true that narrowing ones view to $\{p\}$ leads to having evidence for p, if one has an evidence set that contain a p world. The original, local reading is meant to find out what the actual world looks like. One hypothetically adds that actually the conditional is true to her stock of knowledge, and sees what can be derived from this hypothesis. However, seeing the contextual information as such a hypothetical property of the world seems redundant. This would mean that one adds the hypothesis to the world that she is actually having a wider or a more narrowed view of the world than she has in the model. Although an agent might not be able to access the actual contextual filter see is in at the moment, the moment that she hypothetically proposes to widen or narrow her view, this is actually here new context. For example, consider the shop example from chapter 4.2. An agent might say: 'the shop is open, since it is 10 o'clock'. Although there might just have been a robbery in which case I don't know whether the shop is open or closed'. In the first sentence she is operating within a default context, which filters out all the alternatives specifying that a robbery has taken place in the shop. In the second sentence, she is viewing her evidence in a wider view. Namely, she gives the information, that when it is actually considered as an option outside her default context, that the shop has just been robbed, she would not know whether the shop is open or closed. This is the major difference between hypothetically assuming formulas to be true and hypothetically assuming contextual filters to be narrower or wider.

Furthermore, a more technical support for the concept that conditional evidence could be better interpreted locally, does not hold for contextual evidence. Within public announcements and conditional evidence, we can only express the removal of possibilities overtime, thereby narrowing the uncertainty of the agent. Once the agent is no longer in uncertainty about some proposition, there is no turning back. When conditional evidence is interpreted locally, we can always express the global interpretation by conditioning the iterated operators as well. However, if we were to interpret the conditional evidence globally, we would have no way to express the local interpretation. With respect to contextual evidence we do not have this problem. Since no such one way route exists to update the context here, we can both widen and narrow the view. Therefore, by interpreting the contextual evidence operator globally, we can still express the local setting by hypothetically changing the context back again. As we belief this is just what happens in conversations.

Chapter 6

Applications.

In this sections we will show that the framework we discussed in this thesis can handle the examples we used for the motivation of the context dependence of knowledge and beliefs in section 3. We will discuss two examples that cover most discussed phenomena.

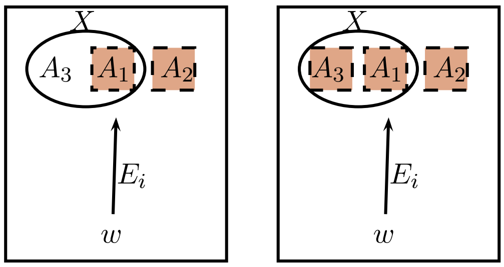
6.1 The red wall

Suppose are passing a wall and observe that the wall is red. You are inclined to believe that the wall is red. Namely, in a default context, only the alternatives 'the wall is red and looks red' (A_1) and 'the wall is not red and does not look red' (A_2) are relevant alternatives. The evidence (X) you receive by perception, provides evidence for all alternatives where you would perceive the wall as red. Since there is only one relevant alternative in the context in which you perceive the wall as red, you have evidence in this context for A_1. The evidence of perception is the only evidence we are considering in this example, so it follows immediately that you believe that the wall is red and looks red.

However, the moment that someone asks you whether the wall is not actually white and cleverly illuminated to look red, the alternative that 'the wall is white and cleverly illuminated to look red' (A_3) becomes a relevant alternative as well. Now both A_1 and A_3 are included in the evidence piece X, so X cannot distinguish between A_1 and A_3. So based on your evidence and the contextual filter in this situation, you are doubting whether A_1 or A_3 is the case. In a default context the alternative $A_1 \vee A_3$ is not available, so you immediately collect more evidence that enables you to rule out one of the alternatives. So for example you search for the light source and check whether it emits red light. If so, you have got evidence for A_3 and otherwise for A_1. Here the definition of believe leads you to believe respectively A_3 or A_1.

Note that, in the same way as before, a new alternative A_4 can be introduced in the context, which competes with the previous believed alternative. Then, again more evidence is needed to distinguish A_4 from the previous believed alternative. Note that, due to the representation of the context as a set of sets of worlds (instead of sets of worlds) we are able to explain the need to find more evidence to distinguish between the relevant alternatives.

Furthermore, after you observed that the wall is red and you are still in the default context, providing only A_1 and A_2 as relevant alternatives, it could also be that the alternative A_3 becomes relevant through evidence instead of through a non informative question. Namely, if you would, by accident discover a red light shining on the wall, that observation actually triggers two actions in your information state. It creates a context change, which makes the alternative A_3 a relevant alternative in the current context, and it adds the evidence (Y), that 'the light source emits red light', to your information state. Taken together these actions, now results in your believe that A_3 is the case.



(a) In the first context. (b) In the second context.

Figure 6.1: the red wall.

6.2 Who's on television?

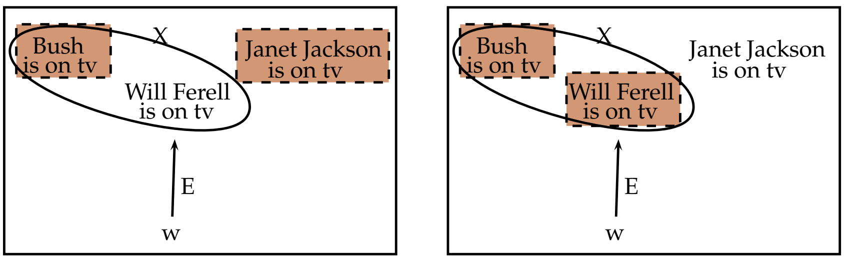
In the previous chapters, we saw how the framework discussed in this thesis handles examples in which more alternatives become available to the agent. In the first scenario a non informative question causes the context change, and in the second scenario the context change was caused by an informative discovery. However, both scenarios explain the widening the view action of the context. So, what can we say about narrowing the view?

As Lewis mentions in his article, we have that once an alternative is brought up to our attention, we cannot simply ignore it again. However, ''(W)e might quickly strike a tacit agreement to speak just as if we were ignoring it; and after just a little of that, doubtless it really would be ignored.' ([12, p.560]). We can see from this that we are not ignoring possibilities by bringing our attention to the possibility that must be ignored, but by bringing our attention to the relevant alternatives that are left. This is exactly the reason why we chose the narrowing the view action as an action focusing only on the left relevant alternatives.

However, since narrowing the view means narrowing the view of the agent by attending to other possibilities, it is often the case that by an action that narrows the view of the agent, it also adds alternatives if they were not there before (although most of the time these alternatives were available before the context change anyway).

For example, by a question such as 'Is Bush or or Janet Jackson on television?' we are narrowing the view of the agent to the alternatives that Bush is on television and Janet Jackson is on television. However, if one of these alternatives was not available in the context of the agent before the question was asked, the question widens the view of the agent with these alternatives as well.

See picture 6.2 as an illustration of how the contextual evidence model explains that the evidence (X) 'there is a man on television' can discriminate between the alternatives 'Bush is on television' and 'Janet Jackson is on television', but fails to discriminate between the alternatives 'Bush is on television' and 'Will Ferell' is on television'.



(a) In the first context. (b) In the second context.

Figure 6.2: who's on television?

Chapter 7

Overall Reflections and Future Work.

In this chapter we will reflect on what has been done in this thesis, focusing on what has been achieved. Furthermore, we will highlight some of the questions that are left over for further research in this field.

7.1 Achievements

In this section we will give a short overview of the questions we have addressed in this thesis and what has been achieved doing this. First we will give a summary, which describes the global achievements, and later on we will highlight some topics which we think deserve extra attention.

7.1.1 Summary

In this thesis we have argued for the context dependence of epistemic operators, like knowledge and beliefs, in the lines of the relevant alternatives theory. According to the relevant alternatives theory, knowledge of p can be granted to every agent who can rule out the relevant alternatives in which p is false. So, in order to have knowledge, she need to be able to rule out only the relevant alternatives, not every alternative. For belief the same holds true; an agent beliefs p if she has evidence against all relevant alternatives in which p is false. Subsequently, we argued that the relative alternatives are determined by the context, or by the circumstance of evaluation. Since knowledge and beliefs are dependent on the relevant alternatives, and the relevant alternatives are dependent on the context, we have that knowledge and beliefs are dependent on the context. This context dependency, explains how we can have the knowledge and the beliefs we have, even though we do not have evidence against all other competing alternatives.

Furthermore, we brought the context dependence of beliefs into practice. In order to do so, we had to represent the context as a set of relevant alternatives. We thereby defined the relevant alternatives as sets of worlds, instead of worlds. By this transformation we were able to distinguish between the levels of information we were dealing with in a certain context. To incorporate this rich definition of the context into the logic, we were in need of the rich evidence logic for beliefs. We extended this evidence logic to a contextual evidence logic, such that where you first believed on the basis of maximal consistent evidence sets, you now belief on the basis of maximal consistent evidence sets within the relevant alternatives.

Moreover, we proposed dynamic actions on the logic which are able to model the context switches. These actions can widen or narrow the view of the agent. Some characteristics and logical consequences of belief with respect to these two actions have been specified.

Finally, we discussed two examples that were used to motivate the relevant alternatives theory and showed that our contextual evidence logic can explain what happens with the context during a conversation, and how this influences the information state of an agent. These examples show that contextual evidence logic can model how non informative acts can lead the agent to revise her believe state, and also how informative and non informative acts operate on the relevant alternatives.

7.1.2 Logical omniscience - context dependence?

In the overview, we started this thesis out with a fundamental problem of epistemic logics; that they assume logical omniscience. Essentially, this means that if q is logically implied by p and agent i knows that p, then agent i knows q. However, as is explained by the zebra example (adapted from Dretske), it is possible that an agent knows that she is looking at zebras and does not know that she is not looking at mules cleverly disguised as zebras. Due to Lewis's insight, stating that probably the knowing that we are looking at zebras and the doubting whether we are looking at mules cleverly disguised as zebras does not happen at the same time, we were able to overcome the difficulty of logical omniscience in contextual evidence logic of this sorts. Namely, according to Lewis, the knowing that we are looking at zebras happens first in a certain context, and then a context switch takes place after which we are not sure anymore. Since we have proposed a logic in this thesis which can model these dynamic context switches, we have created an epistemic logic that does not suffer from 'the logical omniscience problem' as described by the zebra example.

Note that the way we overcome this part of the problem of logical omniscience is not solved by creating a logic which is totally non logical omniscience, since within one and the same context the agents are still logical omniscience. However, due to a dynamic context change it can happen that the agent knows p in some context and q is implied by p, but the agent may not know q in another context. As a consequence of logical omniscience within contexts, the agent does not know p either in this new context.

7.1.3 Evidence versus context

It is the case that in a lot of the examples that support the context dependence of epistemic operators, the evidence relation delivers the same results as the context. For instance, in the zebra example we have that the alternative that we are looking at mules cleverly disguised as zebras is so implausible according to our experience (which is evidence), that we indeed do not want to consider it. Furthermore, looking at a red wall leads to the conclusion that the wall is red, since as a child we have all learned that a wall with that color is a red wall. Only at the moment we receive the evidence that this wall is in fact white and cleverly illuminated to look red, the evidence leads us to believe that the wall is in fact not red.

However, as we tried to convince the reader in this thesis, there is indeed a difference between evidence and context. Evidence always comes in the form of information, whereas context can come in any form, forcing the agent to switch her attention. For sure the agent's attention is switched when new evidence is provided. However, as we saw before, a (non-informative) question can change the attention of the agent as well.

Furthermore, except for the difference that context can come as informative and non informative information, there certainly is a difference between ruling out alternatives by evidence, or by ignoring the alternatives. As discussed in the thesis, this is exactly the frame problem researchers in artificial intelligence have to deal with.

7.1.4 Relation to other approaches

Several approaches have been made to dynamically model the consequences of contextual changes due to questions (for a recent approach see [20] and further references in there). These approaches solve issues, related to the context changes discussed here, only they also take more question specific information in consideration, for example that a question usually is considered to propose a partition on the set of worlds.

Another approach by Holliday ([11]) brings the relevant alternatives theory and epistemic logic together. Holliday uses the basic epistemic logic and adds an extra relevant alternatives relation on top of the epistemic relation in the model. In the approach we described above, we added an extra contextual filter. However, it can easily be seen that when this contextual filter is defined per world, it is just a neighborhood function of relative alternatives on top of the neighborhood function we had for evidence. In that sense we extended the approach of Holliday, which is based on epistemic logic, to evidence logic. The motivation for this extension to evidence logic is explained by one of the maxims of Grice: be as informative as is asked for within your abilities, as is discussed in section 4.1 above. Overall, the discussion in this thesis is more language driven. However, the proposed approaches for modeling the relative alternatives theory seem compatible.

7.2 Future work

While we can now model some more natural epistemic behavior of human beings by making use of contextual evidence logic, we still have a long way to go. In this section we discuss (only) two ideas of further research in line we with this thesis.

7.2.1 On the basis of evidence

As we have emphasized throughout this thesis, we only tried to model knowledge and beliefs that are based on evidence. For example, you can know something because you proved it is the case. Then you know it based on your evidence. This does not state that knowledge and beliefs can only exist on the basis of evidence. As Lewis ([12]) explains, we can indeed have beliefs even though we forgot how these beliefs arose in the first place. In further research it would be interesting to investigate how time make the agents forget their reasons, but keep their beliefs and knowledge.

7.2.2 Reliability of the evidence

In this thesis we did not say anything about how certain information must or can be interpreted as evidence. However, much can be said about this. Namely, if some expert tells you something about his field of expertise, you might consider this reliable evidence. If, on the other hand, you look something up on the Internet and some weird looking site gives you information, you might want to consider this as unreliable evidence. The reliability of evidence therefore comes in degrees. A consequence of evidence coming in degrees is that beliefs will come in degrees as well[1].

The following two questions arise when evidence will come with a degree of reliability. The first question is how reliable a certain alternative is, given all the evidence that is available to the agent. Thus, in other words, how will different pieces of evidence be combined? The second question is whether knowledge is a superlative of degree of beliefs. That is, is knowledge just based on extremely reliable evidence?

Combined evidence

In this thesis we worked with two valued evidence. Which means we only discriminated between evidence supporting an alternative exist and no evidence supporting that alternative exist. However, if we have evidence that p is the case and evidence that $p \wedge q$ is the case, we are inclined to say that we have stronger evidence for p than for q. Namely, we have two evidence pieces telling us p and only one telling us q. Now, if we would also have evidence for $\neg q$ the situation becomes even more complicated. Nonetheless if we can say something about the reliability of the evidence we received, we might be able to say something extra. If the reliability of the evidence that $\neg q$ is low, then again we still believe p and q. There is a lot more to say about combining of evidence pieces, and theories like the Dempster-Shaffer method ([17]) and the equal weight view ([6]) might be able to help us to resolve these issues.

Another difficulty is due to the connectedness of pieces of evidence. Namely, we can judge the reliability of the evidence on several bases. First, we can consider the source unreliable. Second, we might consider our perception in general unreliable (for example because we are tired or on drugs). Third, we might consider our perception unreliable for this particular evidence (for example because there was a lot of noise so we did not hear the information so well, or it happened far away so we did not see the information so well). Fourth, the source himself can tell you he is not sure about the information he is giving you. If you might consider a source unreliable, this has impact on all the evidence coming from this source. If you consider

[1] Of course one can set a threshold as well for beliefs, which keeps beliefs discretely valued. However, with this approach it also becomes natural to speak about weaker and stronger beliefs.

your perception unreliable in general, this has impact on all the evidence you receive at this moment. For combining pieces of evidence, it of course matters whether the reliability of the pieces are independent or dependent on each other (for example if they are coming from the same the source or thy are perceived at the same moment makes them dependent).

Furthermore, when we think about dynamically updating our information state, we do not only consider our beliefs based on the pieces of evidence anymore, but also the reliability we assign to evidence pieces, sources and our perception. It would be very interesting to investigate in the future what this would mean for belief revision.

Is knowledge a superlative of degree?

If the degree of reliability of pieces of evidence is taken into account, we have that another topic one can think about is what distinguishes knowledge from beliefs. In chapter 2 we made a proposal of evidence logic for knowledge, where knowledge is dependent on very secure evidence. However, if we have more and less reliable beliefs in general, does knowledge then just happen to be a very strong belief? Of course this is not plainly acceptable; there is more to knowledge. For example, someone can never ascribe another person knowledge of some proposition p, if she herself knows that p is false. Nonetheless, one can ascribe some other person a very strong belief of p.

On the other hand there are enough examples imaginable of the sort: 'I have reasons to belief that this statement is true, although I do not know it before I have proved it'. Here, the proof can be seen as very reliable evidence, whereas normally one would assume that the reasons to believe are weaker.

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