Finding the phase transition for Friedman's long finite sequences

MSc Thesis (*Afstudeerscriptie*)

written by

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Contents

Chapter 1

Introduction

We say that a phase transition occurs if there is a big change in the behaviour of a system due to a small change in some parameter. The most well-known phase transitions are probably the melting of ice if the temperature gets above $0°$C and the boiling of water if the temperature gets above $100°$C under normal pressure. Phase transitions have also been discovered in many mathematical and computational disciplines such as statistical physics, evolutionary graph theory, percolation theory, Markov chains, computational complexity and artificial intelligence.

In the last decades there have been a large number of independence results. Mathematically interesting theorems have been found that require strong systems to prove them. They are for example not provable in Peano arithmetic. Usually these theorems have a parameter which is set to a specific natural value. However, if we let this parameter decrease then at some point the theorem becomes provable and a phase transition occurs.

1.1 The Paris Harrington theorem

Peano arithmetic (PA) tries to axiomatise the properties of the natural numbers. Gödel showed that there are assertions about the natural numbers that are neither proved nor refuted by the Peano axioms. He also showed that these assertions exist for any axiomatisation. These assertions were especially constructed for the purpose of being independent of PA, so the question then became if there would exist natural mathematically interesting assertions that are indendent of PA. The first of them was discovered by Paris and Harrington around 1977.

The theorem they discovered which is not provable in PA says that, given natural numbers p, k and n there exists a natural number r that is so large that for any mapping P from the k-element subsets of $\{1, \ldots, r\}$ to $\{1, \ldots, p\}$ there exists $H \subseteq \{1, \ldots, r\}$ such that H has at least n elements, every k-element subset of H has the same value under P and if h is the least element of H then

H has at least $I(h)$ elements. Here, I is the identity function.

If this last demand that H contains at least $I(h)$ elements is omitted then the theorem is provable in PA. So if I is replaced by a constant function then the theorem is provable in PA. It can be shown that for every k the theorem is unprovable if the kth iterate of the logarithm is used and that the theorem is provable if the inverse of the superexponential function is used. The phase transition can be described even more precisely by letting k depend on the argument. See [9].

1.2 Tree sequences

Kruskal proved that for every infinite sequence $T_1, T_2, \ldots$ of finite trees there exist $i < j$ such that T_i is embeddable in T_j (i.e. there exists an inf preserving one to one mapping from T_i into T_j). Friedman showed that for every k there exists N so large that for every sequence $T_1, \ldots, T_N$ of finite trees such that T_i has at most $k + I(i)$ nodes (here I is the identity function again) there exist $i < j \leq N$ such that T_i is embeddable in T_j but that PA does not prove this. Again, if the identity function is replaced with a constant function it is clear that PA will prove the statement. So somewhere between the constant function and the identity function there will be a phase transition from provability to unprovability.

Matousek and Loebl showed that we have provability for $\frac{1}{2}\log$ and unprovability for $4\log$ where log is the binary logarithm. Weiermann showed in [6] that the phase transition threshold is extremely sharp and that for the function $r\log$ there is provability for $r \leq \rho$ and unprovability for $r > \rho$ for a certain real number $\rho \approx 0.63957769\ldots$.

1.3 Summary

The phase transition that we will focus on is about fast growing sequences, which were discovered by Friedman [4]. PA can prove that the length of these sequences remains finite, but $I\Sigma_2$ cannot. $I\Sigma_2$ is the subsystem of PA where the induction scheme is limited to induction of Σ_2 formulas. This system can prove the totality of a function if and only if it is multiple recursive. The multiple recursive functions are the functions that can be defined using the elementary functions and nested recursion schemes with some finite number of variables. So it can prove the totality of the primitive recursive functions. It can also prove the totality of the Ackermann function which is double recursive. The multiple recursive functions are the same as the $< \omega^{\omega^\omega}$ recursive functions. Hence, $I\Sigma_2$ proves the totality of the Hardy functions H_α for $\alpha < \omega^{\omega^\omega}$ but does not prove the totality of $H_{\omega^{\omega^\omega}}$. In section 4, which follows section 5 from [4], we will first show that the growth of the length of these sequences is ω^{ω^ω} recursive (this is a direct consequence of a theorem from [14]), which implies that $I\Sigma_3$ (and thus PA) can prove that the length of these sequences remains finite. Then, using lemma 3.2.5 it will be shown that this growth is so fast that it eventually dominates every $< \omega^{\omega^\omega}$ recursive function. From theorem 3.3.3 it then follows that $I\Sigma_2$ cannot prove that these sequences remain of finite length.

Section 5 is based on an article with Weiermann [10]. In section 5.1 we show that if the parameter function grows very slowly, it is easy to find an upper bound on the length of the sequences. In section 5.2 we make a construction which shows that for a slowly growing function f the length of the sequences doesn't grow significantly slower than for the identity function. To nicely characterize the phase transition we use that $I\Sigma_2$ proves the totality of H_α iff $\alpha < \omega^{\omega^\omega}$. This known fact is proved in section 3.3 using a theorem from [1] and the notion of ordinal recursion.

In the last section we study a similar phase transition which seems easier to characterize. It is again about sequences but now an extra condition is introduced which makes the sequences grow so fast that PA is no longer able to prove that they remain finite. The phase transition here is very similar to the one in section 5.

Chapter 2

Explosive sequences

In [4] Friedman defines a property $\mathcal{F}$ of sequences over $\{1, \cdots, k\}$, $k \in \mathbb{N}$. He shows that sequences with property $\mathcal{F}$ are finite, that the maximum length of a sequence over $\{1\}$ with property $\mathcal{F}$ is 3, that the maximum length of a sequence over $\{1, 2\}$ with property $\mathcal{F}$ is 11 and that the maximum length of a sequence over $\{1, 2, 3\}$ with property $\mathcal{F}$ is bigger than $A_{7198}(158386)$. Here, A_{7198} is the 7198th branch of the Ackermann function.

2.1 Introduction of a function parameter

We generalize property $\mathcal{F}$ to property $\mathcal{F}_f$ which depends on a function f.

Definition 2.1.1. Suppose we have a sequence $s = a_1, a_2, a_3, \cdots$ and a function f. We select a sequence of subsequences from s:

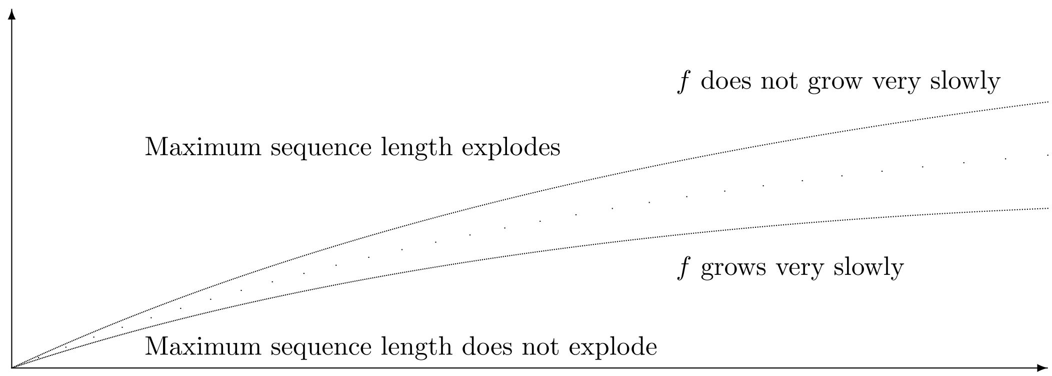
$$(a_1, \ldots, a_{1+f(1)}), \quad (a_2, \ldots, a_{2+f(2)}), \quad (a_3, \ldots, a_{3+f(3)}), \cdots .$$

We call the elements in this sequence the *parts* of s. If there do not exist two parts such that the first is a subsequence of the second (i.e. there are no $i < j$ such that $(a_i, \ldots, a_{i+f(i)})$ is a subsequence of $(a_j, \ldots, a_{j+f(j)})$) then we say that s has property $\mathcal{F}_f$.

The property $\mathcal{F}$ is $\mathcal{F}_I$ where I is the identity function. The ordinary proof of the finiteness of the sequences remains the same with the introduction of a function parameter. We will give the proof in the next subsection. So we can investigate how the growth of the maximum length of these sequences depends on the function f.

Definition 2.1.2. For a function f, let L_f be the function which maps a positive integer k to the maximum length that a sequence over $\{1, \ldots, k\}$ with property $\mathcal{F}_f$ can have.

It turns out that there is some sort of critical function f such that L_f grows fast, but if g grows only a little bit less fast than f then L_g grows relatively slowly. We call this a phase transition. The intuitive picture of the phase transition is sketched in the figure below.



In [4] Friedman proves that L_I grows so fast that $I\Sigma_2$ cannot prove its totality, but $I\Sigma_3$ can. The system $I\Sigma_n$ is the fragment from PA where the induction scheme is limited to Σ_n formulas. So if we want to look at the phase transition from the provability perspective the question becomes for which f we have

$$I\Sigma_2 \vdash \forall k \exists n L_f(k) = n$$

and for which f we have

$$I\Sigma_2 \not\vdash \forall k \exists n L_f(k) = n.$$

2.2 Existence of the maximum length of sequences with property $\mathcal{F}_f$

It is not immediately clear that the functions L_f are well-defined. We will prove the well-definedness of L_f in the same way as Friedman does in [4] for L_I. Actually the function f does not affect the proof. The proof method is from Nash-Williams [18]. So we are going to prove that the length of sequences over $\{1, \ldots, k\}$ with property $\mathcal{F}_f$ is bounded in k. We start by proving the following lemma:

Lemma 2.2.1. *For every infinite sequence $s_1, s_2, \ldots$ of finite sequences over $\{1, \ldots, k\}$ there are $i < j$ such that s_i is a subsequence of s_j.*

Proof. Suppose for a contradiction that the lemma is false. Then there is a sequence $s_1, s_2, \ldots$ which is a counterexample to the lemma. Call such a sequence bad. We now construct what Nash-Williams calls a minimal bad sequence.

Let s_1 be a sequence of minimal length over $\{1, \ldots, k\}$ which starts some bad sequence. Let s_2 be a sequence of minimal length over $\{1, \ldots, k\}$ such that s_1, s_2 starts some bad sequence. Let s_3 be a sequence of minimal length over $\{1, \ldots, k\}$ such that s_1, s_2, s_3 starts some bad sequence. Continue in this way to obtain a minimal bad sequence. Since no s_i can be empty we can pick an infinite subsequence $s_{i_1}, s_{i_2}, \ldots$ whose first terms are all the same. This is also a bad sequence. If we let $s^{'}_{i_1}, s^{'}_{i_2}, \ldots$ be the sequence which results by chopping off the first terms this sequence is still bad. The sequence $s_1, \ldots, s_{i_1-1}, s^{'}_{i_1}, s^{'}_{i_2}, \ldots$ is also bad, contradicting the choice of s_{i_1} because the length of $s^{'}_{i_1}$ is shorter than the length of s_{i_1} and $s_1, \ldots, s_{i_1-1}, s^{'}_{i_1}$ also starts some bad sequence. □

If there would be an infinite sequence $a_1, a_2, \ldots$ over $\{1, \ldots, k\}$ with property $\mathcal{F}_f$ then $(a_1, \ldots, a_{1+f(1)}), (a_2, \ldots, a_{2+f(2)}), \ldots$ is an infinite sequence of finite sequences over $\{1, \ldots, k\}$ so by the lemma there exist $i < j$ such that $(a_i, \ldots, a_{i+f(i)})$ is a subsequence of $(a_j, \ldots, a_{j+f(j)})$ contradicting the assumption that $a_1, a_2, \ldots$ has property $\mathcal{F}_f$. Hence, sequences over $\{1, \ldots, k\}$ with property $\mathcal{F}_f$ are finite. The last thing we have to show to complete the proof of the well-definedness of L_f is that this implies that the lengths of sequences over $\{1, \ldots, k\}$ with property $\mathcal{F}_f$ are bounded in k. This we do with the help of Königs Tree Lemma. So we construct a tree of sequences over $\{1, \ldots, k\}$. This is actually fairly natural, we can take the empty sequence at the bottom of the tree and say that a sequence s is below a sequence t in the tree if s is an initial segment of t. If s is an initial segment of t and t has property $\mathcal{F}_f$ then clearly s also has property $\mathcal{F}_f$. So we can look at the subtree consisting of the sequences over $\{1, \ldots, k\}$ with property $\mathcal{F}_f$. An infinite path in this tree would give us an infinite sequence with property $\mathcal{F}_f$, so infinite paths do not exist. The branching in this tree is clearly finite and thus we can apply König's Tree Lemma and conclude that the tree is finite. Hence, the maximum length a sequence over $\{1, \ldots, k\}$ with property $\mathcal{F}_f$ can have does indeed exist.

Chapter 3

Function hierarchies

To classify the growth rate of the functions L_f we need some theory on function hierarchies. The main goal of this section is lemma 3.2.5. This lemma is used in lemma 4.2.21, which is essential in the proof of the main theorem of section 4 (4.2.1). In section 3.3 it is shown that in a theorem from [1] a version of the Hardy hierarchy which uses the Ackermann function can be replaced by the standard Hardy hierarchy (theorem 3.3.3). This will enable us to draw conclusions about provable totality of a function if we know where it is in the ordinal recursive hierarchy. It also allows us to give a nice description of the phase transition in section 5. The Hardy functions were first introduced by Hardy in [17].

The ordinal recursive functions use an elementary notation system for ordinals $< \epsilon_0$. They are defined with a recursion scheme which uses previously defined functions, so we have to start with some set of basic funcions. It seems natural to choose a small set of functions here so we'll let the elementary functions be the set of basic functions, although it wouldn't make a difference for our purposes if we would use the primitive recursive functions as the basic ones instead.

3.1 Elementary functions

The elementary functions can be thought of as the "normal" functions such as plus, times, minus, division, remainder, exponentiation, prime decomposition etc. Formally they are defined as the smallest class of functions that contains

- the successor function
- the zero function
- the projection functions
- addition

- multiplication
- modified subtraction

and is closed under the following operations

- composition
  If f is an n-ary elementary function and $g_1, \ldots, g_n$ are m-ary elementary functions then the function $h(x_1, \ldots, x_m) = f(g_1(x_1, \ldots x_m), \ldots, g_n(x_1, \ldots, x_m))$ is an elementary function.
- bounded sum
  If f is an elementary function that has $n+1$ arguments then the function $g(x, x_1, \ldots, x_n) = f(0, x_1, \ldots, x_n) + \ldots + f(x, x_1, \ldots, x_n)$ is also elementary.
- bounded product
  If f is an elementary function that has $n+1$ arguments then the function $g(x, x_1, \ldots, x_n) = f(0, x_1, \ldots, x_n) \cdot \ldots \cdot f(x, x_1, \ldots, x_n)$ is also elementary.

We say that a relation P is represented by a function p if

$$p(x_1, \ldots, x_n) = 0 \Leftrightarrow P(x_1, \ldots, x_n).$$

A relation is elementary if it is represented by an elementary function. The relation $x \leq y$ for example is represented by $x \dot{-} y$. If P is represented by p and Q is represented by q then

$$\begin{array}{rll} P \& Q & \text{is represented by} & p + q, \\ P \vee Q & \text{is represented by} & p \cdot q, \\ \neg P & \text{is represented by} & 1 \dot{-} p, \\ P \Rightarrow Q & \text{is represented by} & (1 \dot{-} p) \cdot q. \end{array}$$

Hence, the elementary relations are closed under boolean combinations.

We will now define the least number operator. This will be useful in the definition of a pairing function that will be used to introduce a notation system for ordinals $< \epsilon_0$ which we need in the next subsection. Let

$$(\mu t \leq x)[f(t) = 0]$$

stand for the least $t \leq x$ such that $f(t) = 0$ and zero if there is no such t. We proof that if f is elementary, then $(\mu t \leq x)[f(t) = 0]$ is elementary.

Let $a(x) = 1 \dot{-} (1 \dot{-} x)$, so $a(x)$ is zero if $x = 0$ and $a(x)$ is one if $x \neq 0$. If we start with zero and add one for every t such that for all $u \leq t$, $f(u) > 0$ then we end up with $(\mu t \leq x)[f(t) = 0]$ if $f(y) = 0$ for some $y \leq x$. So at each stage we add

$$\prod_{u \leq t} a(f(u)).$$

If there exists $y \leq x$ such that $f(y) = 0$ we have that

$$(\mu t \leq x)[f(t) = 0] = \sum_{t \leq x} \left(\prod_{u \leq t} a(f(u)) \right).$$

We have

$$1 \dot{-} \prod_{t \leq x} a(f(t)) = \begin{cases} 1 & \exists y \leq x \; f(y) = 0 \\ 0 & \neg \exists y \leq x \; f(y) = 0 \end{cases}$$

and thus we can define $(\mu t \leq x)[f(t) = 0]$ as

$$(\mu t \leq x)[f(t) = 0] = \sum_{t \leq x} \left(\prod_{u \leq t} a(f(u)) \right) \cdot \left(1 \dot{-} \prod_{t \leq x} a(f(t)) \right).$$

We will now use this to define a pairing function w and projection functions m_1 and m_2. Let

$$w(x, y) = \left(\sum_{t \leq x+y} t \right) + y.$$

We now want to be able to extract x and y from $w(x, y)$. First we will extract $x + y$ from $w(x, y)$. It is easy to see that $x + y$ is the least z such that

$$\sum_{t \leq z} t \leq w(x, y).$$

Here the least number operator is useful. Let

$$v(x) = (\mu t \leq x) \left[\sum_{u \leq t} u \leq x \ \& \sum_{u \leq t+1} u > x \right].$$

We see that $v(w(x, y)) = x + y$, so if we define

$$\begin{aligned} m_2(x) &= x \dot{-} \sum_{t \leq v(x)} t, \\ m_1(x) &= v(x) - m_2(x) \end{aligned}$$

then we have

$$\begin{aligned} m_1(w(x, y)) &= x, \\ m_2(w(x, y)) &= y \end{aligned}$$

which is what we were after.

This pairing function will help us define a notation system for ordinals $< \epsilon_0$. An ordinal $\alpha < \epsilon_0$ can be written in the Cantor normal form

$$\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k} + n$$

with n and k natural numbers, $\alpha \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k > 0$ and each α_i (i = 1, 2, ..., k) of the same form. We want to use a code for the sequence $\alpha_1, \alpha_2, \ldots, \alpha_k, n$ as a notation for α. We use our pairing function to construct codes for sequences. Let w_n, r_i and s_i be given by

$$w_n(x_0, \ldots, x_n) = w(n, w(x_0, \ldots, w(x_{n-1}, x_n) \ldots)),$$
$$r_i(x) = m_1(m_2^{i+1}(x)) \text{ and } s_i(x) = m_2^{i+1}(x)$$

where $m_2^0(x) = x$ and $m_2^{i+1}(x) = m_2(m_2^i(x))$. We see that if $x = w_n(x_0, \ldots, x_n)$ then $r_i(x) = x_i$ (i = 0, ..., n-1) and $s_n(x) = x_n$. The definition of the codes is by recursion. For an ordinal α, let $\bar{\alpha}$ denote the code of α. Since the α_i $(i = 1, 2, \ldots, k)$ in the Cantor normal form of α are less than α we may assume that their codes have already been defined. We define

$$\bar{\alpha} = w_k(\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_k, n).$$

Let $\langle x \rangle$ be the ordinal denoted by x. We define the denotation relation D and the ordering on notations $\prec$ by simultaneous recursion. The relation $D(x)$ will hold iff x is the notation of an ordinal and $x \prec y$ will hold iff $D(x), D(y)$ and $\langle x \rangle < \langle y \rangle$.

Definition 3.1.1.

$$D(x) \equiv (\forall i \leq m_1(x))[r_i(x) > 0 \& D(r_i(x))] \& (\forall i < m_1(x) \dot{-} 1)[r_i(x) \succeq r_{i+1}(x)]$$

where $x \succeq y$ is short for $y \prec x \vee x = y$.

$$\begin{aligned}
c_1 &\equiv m_1(x) = 0 \;\& m_1(y) > 0 \\
c_2 &\equiv m_1(x) = m_1(y) \& (\forall i < m_1(x))[r_i(x) = r_i(y)] \& s_{m_1(x)}(x) < s_{m_1(y)}(y) \\
c_3 &\equiv m_1(x) \geq m_1(y) > 0 \& (\exists i < m_1(y))((\forall j < i)[r_j(x) = r_j(y)] \;\& r_i(x) \prec r_i(y)) \\
c_4 &\equiv m_1(y) > m_1(x) > 0 \& ((\exists i < m_1(x))((\forall j < i)[r_j(x) = r_j(y)] \& r_i(x) \prec r_i(y)) \vee \\
&\quad\; (\forall i < m_1(x))[r_i(x) = r_i(y)]) \\
x \prec y &\equiv D(x) \;\& D(y) \;\& (c_1 \vee c_2 \vee c_3 \vee c_4)
\end{aligned}$$

Since for $x > 0$ it is the case that $m_1(x) < x$, $r_i(x) < x$ and $s_i(x) < x$ for all i this definition is valid.

3.2 Ordinal recursion

We define the class $\mathcal{R}^\alpha$ of α recursive functions for limit ordinals $\alpha < \epsilon_0$ as in [13]. We will only use $\mathcal{R}^\alpha$ for $\alpha \leq \omega^{\omega^\omega}$ but it is natural to go up to ϵ_0 and it doesn't make things more complicated.

Definition 3.2.1. For limit ordinals $\alpha < \epsilon_0$ we define $\mathcal{R}^\alpha$ to be the smallest class of functions that contains the elementary functions and is closed under elementary operations and the following recursion scheme:

$$
\begin{aligned}
\phi(x,0) &= f(x) \\
y>0 \;\&\; D(y) \;\&\; y \prec \bar{\alpha} \Rightarrow \phi(x,y) &= g(x,y,\phi(x,\theta(x,y))) \\
y>0 \;\&\; (\neg D(y) \vee \neg y \prec \bar{\alpha}) \Rightarrow \phi(x,y) &= 0
\end{aligned}
$$

where f, g and θ have been defined previously and for $y > 0$

$$D(y) \Rightarrow [D(\theta(x,y)) \;\&\; \theta(x,y) \prec y].$$

If $\omega \leq \alpha \leq \omega^{\omega}$ we have that $\mathcal{R}^{\alpha}$ is exactly the class of primitive recursive funtions. If $\alpha = \omega^{\omega}$ then $\mathcal{R}^{\alpha}$ contains the Ackermann function and as α increases, $\mathcal{R}^{\alpha}$ will contain ever faster growing functions. In [13] the following is proved about the class $\mathcal{R}^{\alpha}$.

Lemma 3.2.1. *The ordinal recursion in the definition above can be replaced by primitive recursions and ordinal counting functions give by the scheme*

$$
\begin{aligned}
s(x,0) &= 0 \\
y>0 \;\&\; D(y) \;\&\; y \prec \bar{\alpha} \Rightarrow s(x,y) &= s(x,\theta(x,y))+1 \\
y>0 \;\&\; (\neg D(y) \vee \neg y \prec \bar{\alpha}) \Rightarrow s(x,y) &= 0
\end{aligned}
$$

where θ has been defined previously.

Proof. First we prove that this replacement doesn't create functions that are not in $\mathcal{R}^{\alpha}$. Clearly this is the case for the ordinal counting functions, since the recursion scheme is just a special case of the ordinal recursion scheme. For the case of primitive recursion, let

$$
\begin{aligned}
h(x,0) &= f(x) \\
h(x,y+1) &= g(x,y,p(x,y)).
\end{aligned}
$$

Define $\theta(x,y) = \overline{\langle y\rangle - 1}$ if y is a successor ordinal and $\theta(x,y) = 0$ otherwise. Let ϕ be defined by α-recursion. Now $h(x,y) = \phi(x,\bar{y})$.

For the other direction we show that the α-recursion scheme can be obtained by primitive recursion and the α-recursion scheme for counting functions. Let ϕ be defined by α-recursion and let s be the corresponding counting function (i.e. the θ in the recursion scheme for s is the same as the θ in the recursion scheme for ϕ). Let θ' be defined by primitive recursion as

$$
\begin{aligned}
\theta'(x,y,0) &= y \\
\theta'(x,y,n+1) &= \theta(x,\theta'(x,y,n))
\end{aligned}
$$

and let T' be defined by primitive recursion as

$$
\begin{aligned}
T'(x,y,q,0) &= f(x) \\
T'(x,y,q,t+1) &= g(x,\theta'(x,y,t+1\dot{-}q),T'(x,y,q,t)).
\end{aligned}
$$

If we now set $T(x,y,t) = T'(x,y,t,t)$ we see that T is defined by recursion as

$$\begin{aligned} T(x,y,0) &= f(x) \\ T(x,y,t+1) &= g(x,y,T(x,\theta(x,y),t)) \end{aligned}$$

and we have $\phi(x,y) = T(x,y,s(x,y))$ □

Lemma 3.2.2. *If $\alpha \geq \omega^2$ then primitive recursion can be replaced by an ordinal counting function.*

Proof. Suppose f is defined by primitive recursion as follows

$$f(x,0) = g(x)$$
$$f(x,y+1) = h(x,y+2,f(x,y)).$$

Define

$$\theta(w(x,z),y) = \begin{cases} \overline{\omega \cdot n + w(a,b+1)} & \text{if } \langle y \rangle = \omega \cdot n + w(a+1,b) \\ \overline{\omega \cdot n + w(h(x,z-n,b),0)} & \text{if } \langle y \rangle = \omega \cdot (n+1) + w(0,b) \\ 0 & \text{if } \langle y \rangle = w(0,b) \end{cases}$$

and

$$q(x,y) = \begin{cases} \overline{w(0,g(x))} & \text{if } y = 0 \\ \overline{\omega \cdot (y-1) + w(g(x),0)} & \text{if } y > 0 \end{cases}.$$

Now

$$f(x,y) = s(w(x,y+1),q(x,y+1)) \dot{-} s(w(x,y),q(x,y)).$$

□

Lemma 3.2.3. *Given an α-recursive ordinal counting function s defined with elementary θ there is an α-recursive ordinal counting function s' defined with elementary θ' such that s is elementary in any function that dominates s'.*

Proof. We will construct s' in such a way that the function f defined by

$$\begin{aligned} f(0,x,y) &= y \\ f(z+1,x,y) &= \theta(x,f(z,x,y)) \end{aligned}$$

satisfies $f(z,x,y) \leq s'(w(x,z),g(y))$ for some elementary g (so f is elementary in s') and $s'(x,y) \geq s(x,y)$. Then we have that

$$s(x,y) = \mu z \leq s'(x,y)[f(z,x,y) = 0].$$

We define θ' as follows.

$$\theta'(w(x,z),\overline{\lambda + w(n,m)}) = \begin{cases} \overline{\lambda + w(n',m)} & \text{if } \theta(x,\overline{\lambda+n}) = \overline{\lambda+n'} \text{ and } \lambda > 0 \\ \overline{\lambda' + w(n',m+\overline{\lambda'+n'})} & \text{if } \theta(x,\overline{\lambda+n}) = \overline{\lambda'+n'} \text{ and } \lambda' < \lambda \\ \overline{w(n,m) \dot{-} 1} & \text{if } \lambda = 0 \end{cases}$$

Here λ is 0 or a limit ordinal. □

Lemma 3.2.4. *Given α-recursive ordinal counting functions s and s' defined with elementary θ and θ' and elementary f and f'. If α is closed under addition then there is an α-recursive ordinal counting function s'' defined with elementary θ'' and an elementary function f'' such that for all x, y, $s''(x, f''(y)) \geq \max(s'(x, f'(y)), s(x, f(y)))$*

Proof. We define $f''(y) = \overline{\lambda + w(n, f(y))}$ if $\lambda + n = \langle f(y)\rangle + \langle f'(y)\rangle$ and

$$\theta''(x, \overline{\lambda + w(n, m)}) = \overline{\lambda' + w(n', m)}$$

where $\overline{\lambda' + n'} = \overline{\langle m\rangle + \langle\theta'(x, \overline{\alpha})\rangle}$ if $\lambda + n = \langle m\rangle + \alpha$ for some $\alpha > 0$ and $\overline{\lambda' + n'} = \theta(x, \overline{\lambda + n})$ otherwise. Except if the result would be of the form $\overline{0 + w(0, m)}$, in that case we change the result to 0. □

Definition 3.2.2. For ordinals α and β, let $\alpha - \beta$ be the least γ such that $\beta + \gamma = \alpha$ if $\alpha \geq \beta$ and $\alpha - \beta = 0$ otherwise.

The following lemma is from [5].

Lemma 3.2.5. *For ordinals $\omega^2 < \alpha < \epsilon_0$ that are closed under multiplication the use of the scheme for ordinal counting functions can be restricted to only allow the use of elementary θ without affecting the class $\cup_{\beta<\alpha}\mathcal{R}^\beta$.*

Proof. Suppose s' is an ordinal counting function defined with an ordinal $\beta < \alpha$ and a function $\theta'(x, y) = g(x, y, s(w(x, y), h(x, y)))$ where g and h are elementary and s an ordinal counting function defined with an ordinal $\gamma < \alpha$ and elementary θ. By lemmas (3.2.3), (3.2.4) and (3.2.2) we can take this as general form. We define an ordinal counting function s'' using the ordinal $\gamma \cdot \beta$ with an elementary θ'' such that for some elementary function f we have that $s''(x, f(x, y)) \geq s'(x, y)$. The result then follows with lemma (3.2.3). We define θ'' as

$$\theta''(x, \overline{\gamma \cdot \delta + \lambda + w(n, m)}) = \begin{cases} \overline{\gamma \cdot \delta + \lambda' + w(n', m+1)} & \text{if } \lambda + n > 0 \text{ and } \theta(w(x, \overline{\delta}), \overline{\lambda + n}) = \overline{\lambda' + n'} \\ \overline{\gamma \cdot \delta' + \lambda' + w(n, 0)} & \text{if } \lambda + n = 0 \text{ and } \overline{\delta'} = g(x, \overline{\delta}, m) \text{ and } \overline{\lambda' + n'} = h(x, \overline{\delta'}) \end{cases}$$

and f as $f(x, \overline{\delta}) = \overline{\gamma \cdot \delta + \langle h(x, \overline{\delta})\rangle}$. □

3.3 The Hardy functions

This hierarchy of functions is useful in determining if $I\Sigma_n$ or PA proves the totality of some function. The Hardy functions that are defined in [1] use the Ackermann function and the following norm on ordinals $< \epsilon_0$.

Definition 3.3.1. If we have $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ with $\alpha > \alpha_1 \geq \cdots \geq \alpha_n$ then we define the norm of α, $|\alpha| = n + |\alpha_1| + \cdots + |\alpha_n|$ and we define $|0| = 0$.

We will denote these functions by H'_α. They are defined by

$$\begin{aligned} H'_0(x) &= x \\ H'_{\alpha+1}(x) &= H'_\alpha(x+1) \\ H'_\lambda(x) &= H'_{\lambda[x]}(x) \end{aligned}$$

where λ is a limit ordinal and $\lambda[x]$ is the largest $\kappa < \lambda$ such that $|\kappa| \leq A(|\lambda|+x)$ where A is the Ackermann function. Since for every n there are only finitely many ordinals α with $|\alpha| < n$ such a κ does indeed exist.

Burr shows in [1] that

Theorem 3.3.1.

$$I\Sigma_2 \vdash \forall k \exists n \; f(k) = n$$

implies that for some $\alpha < \omega^{\omega^\omega}$ f *is dominated by* H'_α.

Because the proof is quite long, we do not give it here. We use a different version of the Hardy functions and show that this result still holds.

Definition 3.3.2.

$$\begin{aligned} H_0(x) &= x \\ H_{\alpha+1}(x) &= H_\alpha(x+1) \\ H_\lambda(x) &= H_{\lambda[x]}(x) \end{aligned}$$

where λ is a limit ordinal. In this case, if $\lambda = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n+1}$ with $\lambda > \alpha_1 \geq \cdots \geq \alpha_n + 1$ then $\lambda[x] = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \cdot (x+1)$ and if $\lambda = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ with $\lambda > \alpha_1 \geq \cdots \geq \alpha_n$ and α_n a limit ordinal, then $\lambda[x] = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n[x]}$.

From the definition it is clear that H_α is an α-recursive function. Since the Ackermann function is ω^ω-recursive we have that for $\alpha \geq \omega^\omega$ the function H'_α is an α-recursive function. After two lemmas we will show (theorem 3.3.2) that if f is an α-recursive function for some $\alpha < \omega^{\omega^\omega}$ then there exists $\beta < \omega^{\omega^\omega}$ such that f is dominated by H_β. It then follows directly that $I\Sigma_2$ proves the totality of H_α iff $\alpha < \omega^{\omega^\omega}$ (theorem 3.3.3).

Lemma 3.3.1.

$$\begin{aligned} &(a) \quad H_\alpha(n) < H_\alpha(n+1) \\ &(b) \quad \beta[m] < \alpha < \beta \Rightarrow H_{\beta[m]}(n) < H_\alpha(n) \\ &(c) \quad (\beta < \alpha \,\&|\beta| \leq n) \Rightarrow H_\beta(n) \leq H_\alpha(n) \end{aligned}$$

Proof. (a) and (b) are proved by simultaneous induction on α. The statements are clearly true for $\alpha = 0$. If α is a limit ordinal then

$$H_\alpha(n) = H_{\alpha[n]}(n) < H_{\alpha[n]}(n+1) < H_{\alpha[n+1]}(n+1) = H_\alpha(n+1).$$

From definition 3.3.2 it follows that $\beta[m] < \alpha[n]$ and thus

$$H_{\beta[m]}(n) < H_{\alpha[n]}(n) = H_\alpha(n).$$

If α is of the form $\lambda + k + 1$ with λ zero or a limit ordinal and $k < \omega$ then

$$H_\alpha(n) = H_{\lambda+k+1}(n) = H_{\lambda+k}(n{+}1) < H_{\lambda+k}(n{+}2) = H_{\lambda+k+1}(n{+}1) = H_\alpha(n{+}1).$$

If $\beta[m] \geq \lambda$ then there is some $l < \omega$ such that $\alpha = \beta[m] + l$ and we have

$$H_{\beta[m]}(n) < H_{\beta[m]}(n + l) = H_{\beta[m]+l}(n) = H_\alpha(n).$$

If $\beta[m] < \lambda$ then

$$H_{\beta[m]}(n) < H_\lambda(n) < H_\lambda(n + k + 1) = H_{\lambda+k+1}(n) = H_\alpha(n).$$

This completes the induction.

(c) is proved by induction on α. It is clearly true for $\alpha = 0$. If α is a successor ordinal and $\alpha = \gamma + 1$ we have

$$H_\beta(n) \leq H_\gamma(n) < H_\gamma(n + 1) = H_{\gamma+1}(n) = H_\alpha(n).$$

If α is a limit ordinal then $\beta \leq \alpha[|\beta|] \leq \alpha[n]$. If $\beta = \alpha[n]$ then

$$H_\beta(n) = H_{\alpha[n]}(n) = H_\alpha(n).$$

If $\beta < \alpha[n]$ then we have by the induction hypothesis

$$H_\beta(n) \leq H_{\alpha[n]}(n) = H_\alpha(n).$$

This ends the proof. $\square$

Definition 3.3.3. We say that $\mathrm{NF}(\alpha, \beta)$ holds if

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$$

with $\alpha > \alpha_1 \geq \cdots \geq \alpha_n$,

$$\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$$

with $\beta > \beta_1 \geq \cdots \geq \beta_m$ and $\alpha_n \geq \beta_1$.

Lemma 3.3.2.

(*a*) *NF*$(\alpha, \beta) \Rightarrow H_{\alpha+\beta}(n) = H_\alpha(H_\beta(n))$
(*b*) $H_{\omega^{m+1}}(n) = H_{\omega^m}^{n+1}(n)$
(*c*) *For each primitive recursive function f there exists m such that* $(\forall \vec{x})[f(\vec{x}) < H_{\omega^m}(\max\{\vec{x}\})$

Proof. (a) is proved by induction on β. Note that if $\mathrm{NF}(\alpha, \beta)$ then for all $\gamma < \beta$ $\mathrm{NF}(\alpha, \gamma)$. We have

$$H_{\alpha+0}(n) = H_\alpha(n) = H_\alpha(H_0(n)).$$

If $\beta = \gamma + 1$ then

$$H_{\alpha+\beta}(n) = H_{\alpha+\gamma+1}(n) = H_{\alpha+\gamma}(n{+}1) = H_\alpha(H_\gamma(n{+}1)) = H_\alpha(H_{\gamma+1}(n)) = H_\alpha(H_\beta(n)).$$

If β is a limit ordinal then we first note that $(\alpha+\beta)[n] = \alpha+\beta[n]$ and it follows that

$$H_{\alpha+\beta}(n) = H_{(\alpha+\beta)[n]}(n) = H_{\alpha+\beta[n]}(n) = H_\alpha(H_{\beta[n]}(n)) = H_\alpha(H_\beta(n)).$$

(b) follows from (a) and (c) follows from (b). □

The following theorem is extracted from [16] in [15].

Theorem 3.3.2. *For every $< \omega^{\omega^\omega}$-recursive function f there is $\beta < \omega^{\omega^\omega}$ such that f is dominated by H_β.*

Proof. By lemma 3.2.5 we have that f is of the form

$$f(x) = g(x, s(x, h(x)))$$

where g and h are elementary functions, there exists $\alpha < \omega^{\omega^\omega}$ such that for all x, $\langle h(x)\rangle$ is an ordinal $< \alpha$ and s is an ordinal counting function defined with an elementary θ. Let p be the primitive recursive function defined by

$$\begin{aligned} p(x,0) &= h(x) \\ p(x,y+1) &= \theta(x, p(x,y)). \end{aligned}$$

Since p, the norm function and the function $x, y \mapsto |\omega^x \cdot (\langle y\rangle + 1)|$ are all primitive recursive, there exists a strictly increasing primitive recursive function q such that

$$(\forall x, y)(p(x,y) \le q(x,y)) \text{ and } (\forall \xi, l)(|\omega^l \cdot (\xi+1)| \le q(|\xi|, l).$$

By lemma 3.3.2 we have that there is an m such that

$$q(q(n, k+2), l) < H_{\omega^m}(\max\{l, n, k\}), \quad \text{for all } l, k, n.$$

Let $\gamma(n,k) = \omega^m \cdot \langle p(n,k)\rangle$ and $b(n,k) = q(q(n,k+1), m)$. Then we have

$$\begin{aligned} |\gamma(n,k+1) + \omega^m)| &= |\omega^m \cdot (\langle p(n,k+1)\rangle + 1)| \\ &\le q(|\langle p(n,k+1)\rangle|, m) \\ &\le q(q(n,k+1), m) = b(n,k). \end{aligned} \tag{3.1}$$

and

$$\gamma(n,0) \le \omega^m \cdot \alpha \text{ and } |\gamma(n,0)| \le b(n,0), \tag{3.2}$$

$$b(n,k+1) < H_{\omega^m}(\max\{m,n,k\}). \tag{3.3}$$

Equation (3.3) implies

$$b(n,k+1) < H_{\omega^m}(b(n,k)). \tag{3.4}$$

We will now prove

$$p(n,k+1) \prec p(n,k) \Rightarrow H_{\gamma(n,k+1)}(b(n,k+1)) < H_{\gamma(n,k)}(b(n,k)). \tag{3.5}$$

The premise yields $\gamma(n,k+1)+\omega^m \leq \gamma(n,k)$ and thus, by (3.1) and lemma 3.3.1(a), (c),

$$H_{\gamma(n,k+1)+\omega^m}(b(n,k) \leq H_{\gamma(n,k)}(b(n,k)).$$

By (3.4), lemma 3.3.1(a) and lemma 3.3.2(a) we get

$$H_{\gamma(n,k+1)}(b(n,k+1)) < H_{\gamma(n,k+1)}(H_{\omega^m}(b(n,k))) \leq H_{\gamma(n,k+1)+\omega^m}(b(n,k)).$$

This proves (3.5).

From (3.5) it follows that

$$\min\{k : p(n,k+1) \not< p(n,k)\} \leq H_{\gamma(n,0)}(b(n,0)).$$

By (3.2), (3.3) and lemma 3.3.2(a) we obtain

$$\begin{aligned} H_{\gamma(n,0)}(b(n,0)) &\leq H_{\omega^m\cdot\alpha}(b(n,0)) \\ &\leq H_{\omega^m\cdot(\alpha+1)}(\max\{m,n\}) \\ &\leq H_{\omega^m\cdot(\alpha+1)+m}(n). \end{aligned}$$

□

The above theorem and theorem 3.3.1 now imply

Theorem 3.3.3.

$$I\Sigma_2 \vdash \forall k \exists n\ H_\alpha(k) = n \quad \Leftrightarrow \quad \alpha < \omega^{\omega^\omega}$$

Chapter 4

The function L_I

4.1 Place in the ordinal recursive hierarchy

We will define a primitive recursive function g^* that assigns ordinals less than ω^{ω^ω} to elements of $\{1,\ldots,k\}^*$ in such a way that if $a, b \in \{1,\ldots,k\}^*$, a is an initial segment of b and both a and b have property $\mathcal{F}_I$ then $f(a) > f(b)$. Then we can calculate $L_I(k)$ by counting down through ordinals less than ω^{ω^ω} where the nth ordinal is the largest ordinal of a sequence with property $\mathcal{F}_I$ in $\{1,\ldots,k\}^n$. The definition of this function g^* will be uniform in k, so this shows that L_I is ω^{ω^ω} recursive.

Definition 4.1.1. For a countable partial ordering A, let $\mathrm{Bad}(A)$ be the set of finite sequences $\langle a_1,\ldots,a_n\rangle$ of elements of A such that there are no $i < j$ such that $a_i \leq a_j$. If s is a sequence of elements of A and $a \in A$, let $s^\frown\langle a\rangle$ denote the concatenation of s with a. For $s \in \mathrm{Bad}(A)$ let $A_s = \{a \in A | s^\frown\langle a\rangle \in \mathrm{Bad}(A)\}$. Let $A_s(a) = A_{s^\frown\langle a\rangle}$.

Definition 4.1.2. A *reification* of a countable partial ordering A by an ordinal α is a function $f : \mathrm{Bad}(A) \to \alpha + 1$ such that if $s \in \mathrm{Bad}(A)$ and $a \in A_s$ then $f(s) > f(s^\frown\langle a\rangle)$

Let $A = \{1,\ldots,k\}$ with the ordering $a \leq b$ iff $a = b$. Let $g : \mathrm{Bad}(A) \to \{0,\ldots,k\}$ and for $s \in \mathrm{Bad}(A)$ let $g(s)$ be k minus the length of s. Since it is clear that the length of s kan be at most k this is well-defined. It is easy to verify that g is a reification of A by k. Let the ordering on A^* be the subsequence relation. We now define the function g^* from the start of this section and show that it has the desired property.

Lemma 4.1.1. *There exists a primitive recursive reification g^* of A^* by the ordinal $\omega^{\omega^{k+1}}$.*

Proof. For each $t \in \mathrm{Bad}(A^*)$ we will define a set C_t from which we will define $g^*(t) \leq \omega^{\omega^{k+1}}$. We will also define a mapping

$$h_t : (A^*)_t \to C_t$$

which will enable us to define $g^*(t^\frown\langle u\rangle)$ for $u \in (A^*)_t$. The form of C_t will be

$$C_t = \bigcup_{i\in I}\prod_{j\in J_i} B_{ij}$$

where I and J_i, $i \in I$ are finite index sets, $\bigcup$ denotes disjoint union and $\prod$ denotes cartesian product. Each B_{ij} will be of the form $B_{ij} = A_s$ or $B_{ij} = (A_s)^*$ for some $s \in \mathrm{Bad}(A)$.

We define the mapping h_t by primitive recursion. Let $C_{\langle\rangle} = A^*$ and set $h_{\langle\rangle}$ to be the identity map. If $t = t'^\frown\langle u\rangle \in \mathrm{Bad}(A^*)$ we can assume that $C_{t'}$ and $h_{t'} : (A^*)_{t'} \to C_{t'}$ have already been defined. Since $u \in (A^*)_{t'}$ it follows that $h_{t'}(u) \in C_{t'}$ and thus $h_{t'}(u) = \prod_{j\in J_i} B_{ij}$ for a unique $i \in I$. Hence $h_{t'}(u) = \langle c_j : j \in J_i\rangle$ where $c_j \in B_{ij}$ for all $j \in J_i$.

Let D be the set of elements $\langle d_j : j \in J_i\rangle$ such that for each $j \in J_i$, $d_j \in B_{ij}$ and for at least one $j \in J_i$ it is not the case that $c_j \leq d_j$. We can now define a natural mapping of D into $\bigcup_{l\in J_i}\prod_{j\in J_i} B'_{lj}$ with $B'_{lj} = B_{ij}$ if $j \neq l$ and $B'_{jj} = B_{ij}(c_j)$ if $l = j$. We define a mapping of B'_{lj} into B''_{lj}. We distinguish three cases.

Case 1. $l \neq j$. $B''_{lj} = B_{ij}$ and the mapping from B'_{lj} into B''_{lj} is the identity mapping.

Case 2. $l = j$ and $B_{ij} = A_s$ with $s \in \mathrm{Bad}(A)$. In this case $c_j = a \in A_s$. We define $B''_{lj} = A_s(a)$ and map B'_{lj} into B''_{lj} via the identity mapping.

Case 3. $l = j$ and $B_{ij} = (A_s)^*$ with $s \in \mathrm{Bad}(A)$. In this case $c_j = \langle a_m : m < n\rangle \in (A_s)^*$. For each $w \in B'_{lj}$ there is a smallest $p < n$ such that $\langle a_m : m \leq p\rangle \not\leq w$. So w has the form

$$w = w_0{}^\frown\langle b_0\rangle^\frown\ldots^\frown w_{p-1}{}^\frown\langle b_{p-1}\rangle^\frown w_p$$

where $w_m \in A_s(a_m)^*$, $b_m \in A_s$ and $a_m \leq b_m$. We define

$$B''_{lj} = \bigcup_{p<n}(A_s(a_0)^* \times A_s \times \cdots \times A_s(a_{p-1})^* \times A_s \times A_s(a_p)^*)$$

And $w \in B'_{lj}$ which is of the form above is mapped to $(w_0, b_0, \ldots, w_{p-1}, b_{p-1}, w_p)$.

The set C_t is now the set $C_{t'}$ where the term $\prod_{j\in J_i} B_{ij}$ is replaced by $\prod_{j\in J_i} B''_{ij}$ and cartesian products are distributed over the disjoint unions. We set $h_t(v) = h_{t'}(v)$ if $h_{t'}(v) \notin \prod_{j\in J_i} B_{ij}$. If $h_{t'}(v) \in \prod_{j\in J_i} B_{ij}$ then $h_t(v) \in D$ and we let $h_t(v)$ be the composition of $h_{t'}(v)$ and the mappings defined above.

It remains to define ordinals from the sets C_t. We define the ordinal value of A_s, $|A_s| = \omega^{\omega^{f(s)}}$ and the ordinal value of $(A_s)^*$, $|(A_s)^*| = \omega^{\omega^{f(s)+1}}$. The ordinal value of C_t is now defined as $|C_t| = \sum_{i\in I}\prod j \in J|B_{ij}|$ where $\sum$ means natural sum and $\prod$ means natural product. We set $g^*(t) = |C_t|$

The last step is to show that for $u \in (A_t)^*$ we have $|C_t| > |C_{t^\frown\langle u\rangle}|$. By additive and multiplicative indecomposability of $\omega^{\omega^{f(s)+1}}$ we have that in case 3

$$|B''_{jj} < \omega^{\omega^{f(s)+1}} = |(A_s)^*| = |B_{ij}|.$$

In case 2 we have

$$|B''_{jj} = |A_s(a)| = \omega^{\omega^{f(s^\frown\langle a\rangle)}}| < \omega^{\omega^{f(s)}} = |A_s| = |B_{ij}|$$

and in case 1 it is clear that $|B''_{ij}| = |B_{ij}|$. Hence, for each $l \in J_i$, $\prod_{j\in J_i} |B''_{lj}| < \prod_{j\in J_i} |B_{ij}|$. By additive indecomposability of $\prod_{j\in J_i} |B_{ij}|$ we now see that

$$\sum_{l\in J_i} \prod_{j\in J_i} |B''_{lj}| < \prod_{j\in J_i} |B_{ij}|$$

and this completes the proof. □

4.2 The fast growth of L_I

In this section we will show that L_I grows so fast that it eventually dominates every $< \omega^{\omega^\omega}$ recursive function and thus, by theorem 3.3.1, $I\Sigma_2$ does not prove the totality of L_I. We start by introducing the functions E and G_k which are similar to L_I, but simpler. After that we define a bijection between the set of finite sequences of positive integers and the ordinal ω^{ω^ω} in such a way that a descending sequence of ordinals less than ω^{ω^ω} of length n will correspond to a sequence of finite sequences $s_1, s_2, \dots, s_n$ in which there are no $i < j \leq n$ such that s_i is a subsequence of s_j. By lemma 3.2.5 function values of a $< \omega^{\omega^\omega}$ recursive function g correspond to elementary descending sequences of ordinals. We show that these sequences can be extended in a way which leads (for large enough arguments) to the domination of g by L_I.

Definition 4.2.1. The function E maps a number $k \geq 1$ to the maximum length that a sequence $x_1, \dots, x_n$ with the following two properties can have

1. Every x_i is a sequence over $\{1, \dots, k\}$ with length at most $i + 1$.

2. there are no $i < j \leq n$ such that x_i is a subsequence of x_j.

Lemma 4.2.1. $L_I(k) \leq 2E(k) + 1$

Proof. Let $x_1, \dots, x_n$ be a sequence over $\{1, \dots, k\}$ with property $\mathcal{F}$ of maximum length (thus $n = L_I(k)$). The sequence $(x_1, x_2), \dots, (x_{\lfloor n/2\rfloor}, x_{2\lfloor n/2\rfloor})$ now witnesses that $E(k) \geq \lfloor n/2\rfloor$, hence the lemma. □

This lemma gives us an upper bound for L_I in terms of E. We will also find a lower bound of L_I in terms of E. This is done by adding some new numbers to the x_i from the definition of E such that each x_i gets the right length and then concatenating them with separation marks in between. The positions of the separation marks will be given by the elements of the following sequence.

Definition 4.2.2. $a_1 = 6$, $a_2 = 9$, $a_{i+2} = 2a_i + 1$.

Lemma 4.2.2. $a_{i+1} - a_i \geq i + 2$

Proof. This is true for $i = 1, 2$. Suppose that the lemma holds for some i, then $a_{i+3} - a_{i+2} = 2a_{i+1} + 1 - (2a_i + 1) = 2(a_{i+1} - a_i) \geq 2(i + 2) \geq i + 4$ so it also holds for $i + 2$. □

Lemma 4.2.3. *For all $m \geq 6$ there is a unique i such that $a_i, a_{i+1} \in \{m, \ldots, 2m\}$.*

Proof. First we consider the sequence $a_1, a_3, a_5, \ldots$. If no element of this sequence is smaller than m then $m = 6$ and a_1 is the only element from this sequence in the interval $\{m, \ldots, 2m\}$. If some element from $a_1, a_3, a_5, \ldots$ is smaller than m then we can take j such that a_j is the largest element from this sequence which is smaller than m. We get $m \leq a_{j+2} = 2a_j + 1 < 2m + 1$ so $a_{j+2} \in \{m, \ldots, 2m\}$ and $a_{j+4} = 2a_{j+2} + 1 \geq 2m + 1$ and thus a_{j+2} is the only element from $a_1, a_3, a_5, \ldots$ in the interval $\{m, \ldots, 2m\}$. With analogous reasoning it follows that there is a unique element from the sequence $a_2, a_4, a_6, \ldots$ in the interval $\{m, \ldots, 2m\}$. Hence there are exactly two elements from $a_1, a_2, a_3, \ldots$ in $\{m, \ldots, 2m\}$ and since we have $a_1 < a_2 < a_3, \ldots$ there has to be a unique i such that a_i, a_{i+1} are in $\{m, \ldots, 2m\}$. □

We will use the above lemma in the proof of the lemma below which gives us a lower bound of L_I in terms of E.

Lemma 4.2.4. *For all $k \geq 8$, $E(k - 7) \leq L_I(k)$.*

Proof. Let $x_1, \ldots, x_n$ satisfy 1. and 2. in the definition of E and let $n = E(k-7)$. Append some $k - 6$'s to each x_i to obtain a sequence $x_1^{'}, \ldots, x_n^{'}$ such that the length of x_i' is $a_{i+1} - a_i - 1$. By lemma 4.2.2 this is possible. Since there are no $i < j \leq n$ such that x_i is a subsequence of x_j, there are no $i < j \leq n$ such that $x_i^{'}$ is a subsequence of $x_j^{'}$. We now define a sequence $y_1, \ldots, y_{a_{n+1}}$ over $\{1, \ldots, k\}$ with property $\mathcal{F}$.

- For $1 \leq i < 6$, set $y_i = k + 1 - i$.
- If there is a $j \leq n + 1$ such that $i = a_j$, set $y_i = k - 5$
- For $1 \leq i \leq n$, set $y_{a_i+1}, \ldots, y_{a_{i+1}-1} = x_i^{'}$

We now check that $y_1, \ldots, y_{a_{n+1}}$ has property $\mathcal{F}$. Let $i < j \leq a_{n+1}/2$. We show that $y_i, \ldots, y_{2i}$ is not a subsequence of $y_j, \ldots, y_{2j}$. We split the argument into two cases.

1. If $i \geq 6$ then by lemma 4.2.3 there are unique p, q such that $a_p, a_{p+1} \in \{i, \ldots, 2i\}$ and $a_q, a_{q+1} \in \{j, \ldots, 2j\}$. By the construction of the sequence y it follows that both $y_i, \ldots, y_{2i}$ and $y_j, \ldots, y_{2j}$ contain exactly two $k-5$'s. This means that if $y_i, \ldots, y_{2i}$ is a subsequence of $y_j, \ldots, y_{2j}$ then the piece between the two $k-5$'s in $y_i, \ldots, y_{2i}$ is a subsequence of the piece between the two $k - 5$'s in $y_j, \ldots, y_{2j}$. Hence $x_p^{'}$ is a subsequence of $x_q^{'}$. Since this cannot be the case for $p < q$ we get $p = q$. It must be the case that the piece of $y_i, \ldots, y_{2i}$ that comes before the first $k - 5$ is a subsequence the

piece of $y_j, \ldots, y_{2j}$ which comes before the first $k-5$. But from $p = q$ it follows that the first piece is longer than the second piece, so it cannot be a subsequence and thus we have a contradiction.

2. If $i < 6$ then y_i does not appear in $y_j, \ldots, y_{2_j}$, so $y_i, \ldots, y_{2i}$ cannot be a subsequence of $y_j, \ldots, y_{2j}$.

□

Lemma 4.2.5. *E is strictly increasing*

Proof. Let $k \geq 1$ and let $x_1, \ldots, x_n$ be a sequence of maximum length according to the definition of E. The sequence $x_1, \ldots, x_n, (k+1)$ shows that $n = E(k) < E(k+1)$. □

We now define the functions G_k which we will compare to E.

Definition 4.2.3. For each $k \geq 1$ the function G_k maps a number $m \geq 1$ to the maximum length a sequence $x_1, \ldots, x_n$ with the following two properties can have.

1. Every x_i is a sequence over $\{1, \ldots, k\}$ with length at most $i + m$.
2. There are no $i < j \leq n$ such that x_i is a subsequence of x_j.

By comparing definitions we see that $E(k) = G_k(1)$.

Definition 4.2.4. If f_1, f_2 are two functions from the positive integers to the positive integers then we say that f_1 dominates f_2 if for all n, $f_1(n) > f_2(n)$. We say that f_1 eventually dominates f_2 if there is an N such that for all $n > N$, $f_1(n) > f_2(n)$.

Lemma 4.2.6. *$G_k(n)$ is strictly increasing in each argument.*

Proof. Let $k, n \geq 1$ and let $x_1, \ldots, x_p$ be a sequence of maximum length according to the definition of G. The sequence $x_1, \ldots, x_p, (k+1)$ shows that $p = G_k(n) < G_{k+1}(n)$. The sequence $x_1k, \ldots, x_pk, (k)$ shows that $p = G_k(n) < G_k(n+1)$. □

Lemma 4.2.7. *E eventually dominates each G_k.*

Proof. It suffices to proof that for all $n > k \geq 1$, $G_k(n) < E(n)$. Let $x_1, \ldots, x_p$ be a sequence of maximum length according to the definition of G. The sequence $(n, 1), (n, 2), \ldots, (n, n), x_1, \ldots, x_p$ shows that $p = G_k(n) < E(n)$. □

We will now define a well-order on the finite sequences of positive integers.

Definition 4.2.5. Let x and y be finite sequences of positive integers. Let $\max(x)$ be the largest number in the sequence x. If x is empty, we set $\max(x) = 0$. If $\max(x) < \max(y)$ then $x < y$. For the case $\max(x) = \max(y)$ we define the order by recursion. Let $k = \max(x) = \max(y)$. If $k = 0$ then $x = y$ so there is nothing to define. If $k \geq 1$ then there are unique sequences $x_1, \ldots, x_n$

and $y_1, \ldots, y_m$ such that for all $1 \leq i \leq n$, $\max(x_i) < k$ and for all $1 \leq i \leq m$, $\max(y_i) < k$ and $x = x_1kx_2k \ldots kx_n$ and $y = y_1ky_2k \ldots ky_m$ (some of the x_i and y_i may be empty). If $n < m$ then $x < y$. If $n = m$ then $x < y$ if $(x_1, \ldots, x_n)$ is lexicographically less than $(y_1, \ldots, y_n)$ (here we use that the order for sequences z with $\max(z) < k$ is already defined).

Let α be the unique ordinal to which this order is isomorphic and let h be the order-preserving bijection from the set of finite sequences of positive integers to α.

Lemma 4.2.8. *For $k \geq 1$, the order type of the sequences x with* $\max(x) = k$ *is $\omega^{\omega^{k-1}}$.*

Proof. We use induction. For $k = 1$ it is easily verified that the lemma holds. Now suppose that the lemma holds for $k - 1 \geq 1$, then the sequences x with $\max(x) = k$ and with one k in x have order type $\omega^{\omega^{k-2}} \cdot \omega^{\omega^{k-2}}$ and the sequences x with $\max(x) = k$ and with two k's in x have order type $\omega^{\omega^{k-2}} \cdot \omega^{\omega^{k-2}} \cdot \omega^{\omega^{k-2}}$. In general, the sequences x with $\max(x) = k$ and with n k's in x have order type $(\omega^{\omega^{k-2}})^{n+1}$. So the order type of the sequences x with $\max(x) = k$ is

$$\sum_{n=2}^{\omega} (\omega^{\omega^{k-2}})^n = \omega^{\omega^{k-1}}.$$

□

Lemma 4.2.9. *The sequences x with* $\max(x) = k \geq 2$ *and $n \geq 1$ k's in x are mapped by h to the interval*

$$[\omega^{\omega^{k-2}\cdot(n)}, \omega^{\omega^{k-2}\cdot(n+1)}).$$

Proof. From lemma 4.2.8 we see that the sequences x with $\max(x) = k \geq 2$ is mapped to the interval

$$[\sum_{i=1}^{k-1} \omega^{\omega^{i-1}}, \sum_{i=1}^{k} \omega^{\omega^{i-1}}) = [\omega^{\omega^{k-2}}, \omega^{\omega^{k-1}}).$$

In the proof of that lemma we also saw that the order type of sequences x with $\max(x) = k \geq 2$ and with $m \geq 1$ k's in x is $\omega^{\omega^{k-2}\cdot(m+1)}$. Hence the sequences x with $\max(x) = k \geq 2$ and 1 k in x are mapped to the interval

$$[\omega^{\omega^{k-2}}, \omega^{\omega^{k-2}} + \omega^{\omega^{k-2}\cdot 2}) = [\omega^{\omega^{k-2}}, \omega^{\omega^{k-2}\cdot 2})$$

and the sequences x with $\max(x) = k \geq 2$ and 2 k's in x are mapped to the interval

$$[\omega^{\omega^{k-2}\cdot 2}, \omega^{\omega^{k-2}\cdot 2} + \omega^{\omega^{k-2}\cdot 3}) = [\omega^{\omega^{k-2}\cdot 2}, \omega^{\omega^{k-2}\cdot 3}).$$

In general, the sequences x with $\max(x) = k \geq 2$ and n k's in x are mapped to the interval

$$[\omega^{\omega^{k-2}} + \sum_{i=2}^{n} \omega^{\omega^{k-2}\cdot i}, \omega^{\omega^{k-2}} + \sum_{i=2}^{n+1} \omega^{\omega^{k-2}\cdot i}) = [\omega^{\omega^{k-2}\cdot n}, \omega^{\omega^{k-2}\cdot(n+1)}).$$

□

Lemma 4.2.10. *If $x_0, \ldots, x_n$, $n \geq 1$ are sequences (possibly empty) with for $0 \leq i \leq n$, $\max(x_i) < k \geq 2$ then*

$$h(x_0 k x_1 k \ldots k x_n) = \omega^{\omega^{k-2}\cdot n} + \omega^{\omega^{k-2}\cdot n} \cdot h(x_0) + \omega^{\omega^{k-2}\cdot(n-1)} \cdot h(x_1) + \ldots + h(x_n)$$

Proof. By lemma 4.2.9 the sequences y with $\max(y) < k$ are mapped by h onto $\omega^{\omega^{k-2}}$ and the sequences y with $\max(y) = k$ and n k's in y are mapped by h onto the interval

$$[\omega^{\omega^{k-2}\cdot(n)}, \omega^{\omega^{k-2}\cdot(n+1)}).$$

So if we would set

$$g(x_0 k x_1 k \ldots k x_n) = \omega^{\omega^{k-2}\cdot n} + \omega^{\omega^{k-2}\cdot n} \cdot h(x_0) + \omega^{\omega^{k-2}\cdot(n-1)} \cdot h(x_1) + \ldots + h(x_n)$$

then g is an order-preserving bijection from the sequences y with $\max(y) = k$ and with n k's in y to the interval mentioned above. Since such an order-preserving bijection is unique, it must be the case that h restricted to sequences y with $\max(y) = k$ and n k's in y equals g. □

Lemma 4.2.11. *If x, y are sequences over $\{1, \ldots, k\}$ and x is a subsequence of y then $h(x) \leq h(y)$.*

Proof. By induction on k. The case $k = 1$ is clear. Suppose it holds for $k - 1 \geq 1$. Write x as $x_1 k \ldots k x_n$ and y as $y_1 k \ldots, k y_m$ with $x_1, \ldots, x_n, y_1, \ldots, y_m$ sequences over $\{1, \ldots, k-1\}$. If x is a subsequence of y then $n \leq m$. If $n < m$ then clearly (by definition) $x < y$ and thus $h(x) \leq h(y)$. In case $n = m$ we get that x_1 is a subsequence of y_1, x_2 is a subsequence of y_2, etc. and x_n is a subsequence of y_m. Hence by the induction hypothesis $x_1 \leq y_1, x_2 \leq y_2, \ldots, x_n \leq y_m$ and thus $(x_1, x_2, \ldots, x_n)$ is lexicographically less than or equal to $(y_1, \ldots y_m)$ so by definition $x \leq y$ and thus $h(x) \leq h(y)$. □

We now use the norm on ordinals $< \epsilon_0$ from definition 3.3.1. In terms of the norm of an ordinal less than $\omega^{\omega^{\omega}}$ a bound on the length of the corresponding sequence will be derived.

Lemma 4.2.12. *For $k, n \geq 2$, $|\omega^{\omega^{k-1}\cdot(n-1)}| = kn - k + 1$.*

Proof. We have $|\omega^{\omega^{k-1}\cdot(n-1)}| = 1 + |\omega^{k-1} \cdot (n-1)| = 1 + |\omega^{k-1}| \cdot (n-1) = 1 + k \cdot (n-1) = kn - k + 1$. □

Lemma 4.2.13. *For* $0 < \alpha < \omega^{\omega^{k-1}}$ *and* $\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \ldots + \omega^{\beta_p}$ *with* $\alpha > \beta_1 \geq \beta_2 \geq \ldots \geq \beta_p$, $|\omega^{\omega^{k-1}\cdot(n-1)} \cdot \alpha| = p \cdot (n-1) \cdot k + |\alpha|$.

Proof. We have

$$\begin{aligned} & |\omega^{\omega^{k-1}\cdot(n-1)} \cdot \alpha| \\ = \;& |\omega^{\omega^{k-1}\cdot(n-1)+\beta_1} + \ldots + \omega^{\omega^{k-1}\cdot(n-1)+\beta_p}| \\ = \;& p + |\omega^{k-1} \cdot (n-1) + \beta_1| + \ldots + |\omega^{k-1} \cdot (n-1) + \beta_p|. \end{aligned}$$

Since for $1 \leq i \leq p$, $\beta_i < \omega^{k-1}$ it follows that

$$\begin{aligned} & p + |\omega^{k-1} \cdot (n-1) + \beta_1| + \ldots + |\omega^{k-1} \cdot (n-1) + \beta_p| \\ = \;& p + p \cdot |\omega^{k-1} \cdot (n-1)| + |\beta_1| + \ldots + |\beta_p| \\ = \;& p + p \cdot (n-1) \cdot |\omega^{k-1}| + |\beta_1| + \ldots + |\beta_p| \\ = \;& p + p \cdot (n-1) \cdot k + |\beta_1| + \ldots + |\beta_p| \\ = \;& p \cdot (n-1) \cdot k + |\beta_1| + \ldots + |\beta_p| + p \\ = \;& p \cdot (n-1) \cdot k + |\alpha| \end{aligned}$$

□

Lemma 4.2.14. *For all* $n \geq 2, k \geq 3$ *and sequences* $x_1, \ldots, x_n$ *over* $\{1, \ldots, k-1\}$, $|h(x_1 k \ldots k x_n)| \geq n + |h(x_1)| + \ldots + |h(x_n)|$.

Proof. By lemma 4.2.10 we have

$$\begin{aligned} & |h(x_1 k \ldots k x_n)| \\ = \;& |\omega^{\omega^{k-2}\cdot(n-1)} + \omega^{\omega^{k-2}\cdot(n-1)} \cdot h(x_1) + \omega^{\omega^{k-2}\cdot(n-2)} \cdot h(x_2) + \ldots + h(x_n)| \\ = \;& |\omega^{\omega^{k-2}\cdot(n-1)} \cdot (1 + h(x_1)) + \omega^{\omega^{k-2}\cdot(n-2)} \cdot h(x_2) + \ldots + h(x_n)| \\ = \;& |\omega^{\omega^{k-2}\cdot(n-1)} \cdot (1 + h(x_1))| + |\omega^{\omega^{k-2}\cdot(n-2)} \cdot h(x_2)| + \ldots + |h(x_n)| \end{aligned}$$

We now apply lemma 4.2.13 to see that

$$\begin{aligned} & |\omega^{\omega^{k-2}\cdot(n-1)} \cdot (1 + h(x_1))| + |\omega^{\omega^{k-2}\cdot(n-2)} \cdot h(x_2)| + \ldots + |h(x_n)| \\ & \quad \geq (n-1) \cdot (k-1) + |1 + h(x_1)| + |h(x_2)| + \ldots + |h(x_n)| \end{aligned}$$

and $(n-1) \cdot (k-1) \geq 2(n-1) \geq 2n - 2 \geq n + 2 - 2 = n$. □

Lemma 4.2.15. *If* x *is a sequence of length* n *over* $\{1, \ldots, k\}$ *then* $|h(x)| \geq n$

Proof. By induction on k. From the definitions one easily sees that for $k = 1$, $|h(x)|$ is the length of x. Suppose that the lemma holds for all sequences over $\{1, \ldots, k-1\}$. Let $x = x_1 k x_2 k \ldots k x_n$ with for $1 \leq i \leq n$, x_i a sequence over $\{1, \ldots, k-1\}$. Then $|h(x)| \geq n + |h(x_1)| + \ldots |h(x_n)|$ and by the induction hypothesis this is bigger than the length of x. □

Definition 4.2.6. The function H_k maps a positive integer n to the largest number m such that there exists a sequence of ordinals $\omega^{\omega^{k-1}} > \alpha_1 > \alpha_2 > \ldots > \alpha_m$. with for $1 \leq i \leq m$, $|\alpha_i| \leq n + i$.

Lemma 4.2.16. $G_k(n) \geq H_k(n)$

Proof. Let $\alpha_1, \ldots, \alpha_m$ be of maximum length according to the definition of $H_k(n)$. Consider $h^{-1}(\alpha_1), \ldots, h^{-1}(\alpha_m)$. By lemma 4.2.11 there are no $i < j \leq m$ such that $h^{-1}(\alpha_i)$ is a subsequence of $h^{-1}(\alpha_j)$ and by lemma 4.2.15 the lengths of $h^{-1}(\alpha_i)$ are not too big. Hence $G_k(n) \geq m = H_k(n)$. □

We now show that if we start a descending sequence of ordinals a little lower, but allow the growth of the norms to be as fast as an arbitrary branch of the Ackermann function, the maximum sequence lengths do not increase. This fast growth of the norms will then be used to dominate an arbitrary $< \omega^{\omega^{\omega}}$ recursive function.

Definition 4.2.7. The function I_k maps a positive integer n to the largest m such that there is a sequence of ordinals $\omega^{2k} > \alpha_1 > \alpha_2 > \ldots > \alpha_m$ with $|\alpha_i| \leq n + i$.

Lemma 4.2.17. $I_1(n) \geq 2^{n/2}$

Proof. We define ordinal sequences $a_{n,m}, n \geq 2m$ by recursion. We start the recursion by defining

$$a_{n,0} = n, n-1, \ldots, 0.$$

For $m > 0$ we define

$$a_{n,m} = \omega \cdot m + n - 2m, \ldots, \omega \cdot m, a_{2(n-m)+1,m-1}.$$

These sequences $a_{n,m}$ satisfy the requirements in the definition of $I_1(n-1)$. We define $R(n,m)$ to be the sum of the length of $a_{n,m}$ and n. From the recursive definition of the $a_{n,m}$ a recursive relation for R follows: For $m > 0$,

$$\begin{aligned} R(n,0) &= 2n+1 \\ R(n,m) &= R(2(n-m)+1, m-1) \end{aligned}$$

This recursive relation implies

$$R(n,m) = 1 + (n-m)2^{m+1} - \sum_{i=1}^{m}(i-2)2^i.$$

Since we have

$$\begin{aligned} \sum_{i=1}^{m}(i-2)2^i &= \sum_{i=2}^{m+1}(i-3)2^i - \sum_{i=1}^{m}(i-2)2^i \\ &= (m-2)2^{m+1} + 2 - \sum_{i=2}^{m} 2^i \\ &= (m-3)2^{m+1} + 6 \end{aligned}$$

we get

$$R(n,m) = (n-2m+3)2^{m+1} - 5.$$

Hence

$$I_1(2n-1) \geq a_{2n,n} = R(2n,n) - 2n = 3 \cdot 2^{n+1} - 2n - 5 \geq 2^n$$

and the lemma follows. □

Definition 4.2.8. We define the following version of the Ackermann function

$$\begin{aligned} A_1(n) &= 2n \\ A_{p+1}(n) &= A_p^n(1) \end{aligned}$$

Lemma 4.2.18. $I_1(n) \geq 2n = A_1(n)$

Proof. The sequence $\omega > 2 > 1 > 0$ shows that $I_1(1), I_1(2) \geq 4$ so for $n \leq 2$, $I_1(n) \geq 4$ and the lemma holds. Suppose this is the case for n, we prove that it holds for $n+2$. Let $\alpha_1 > \ldots > \alpha_p$ be according to the definition of $I_1(n)$. The sequence $\omega + \alpha_1, \ldots, \omega + \alpha_p, p-1, \ldots, 0$ shows that $I_1(n+2) \geq 2I_1(n) \geq I_1(n) + 4 \geq 2n + 4 = 2(n+2)$. The lemma follows by induction. □

Lemma 4.2.19. *$I_p(n) \geq A_p(n)$ for all $p \geq 1$ and $n \geq 4p + 50$.*

Proof. Lemma 4.2.18 proves the basis case. For the induction step we will construct a sequence according to the definition of $I_{p+1}(n)$ which is at least as long as $A_{p+1}(n) = A_p^n(1)$. Let $\omega^{2p} > \alpha_1 > \cdots > \alpha_{I_p(1)}$ and $|\alpha_i| \leq n + i$. Let $\omega^{2p} > \beta_1 > \cdots > \beta_{I_p^2(1)}$ and $|\beta_i| \leq I_p(n) + i$. Let $\omega^{2p} > \gamma_1 > \cdots > \gamma_{I_p^3(1)}$ and $|\gamma_i| \leq I_p^2(n) + i$ etc. We can put these sequences together into one sequence $\omega^{2p} \cdot n + \alpha_1 > \cdots > \omega^{2p} \cdot n + \alpha_{I_p(n)} > \omega^{2p} \cdot (n-1) + \beta_1 > \cdots > \omega^{2p} \cdot (n-1) + \beta_{I_p^2(n)} > \omega^{2p} \cdot (n-2) + \gamma_1 > \cdots > \omega^{2p} \cdot (n-2) + \gamma_{I_p^3(n)} > \ldots$ which is long enough, but the norm of the first term is too big. To fix this we put the following sequence in front. $\omega^{2p+1} + \nu_1 > \cdots > \omega^{2p+1} + \nu_{I_1(n-2p-2)}$. The sequence $\nu_1 > \cdots > \nu_{I_1(n-2p-2)}$ is the longest one possible according to the definition of $I_1(n-2p-2)$. By lemma 4.2.17 and the assumption $n \geq 4p+50$ it follows that $I_1(n-2p-2) \geq (2p+1) \cdot n + n$. So it follows that the norms of the terms in this sequence are now low enough and thus $I_{p+1}(n) \geq I_p^n(n)$. By the induction hypothesis $I_p^n(n) \geq A_p^n(n) \geq A_p^n(1) = A_{p+1}(n)$. □

Definition 4.2.9. Let $J_{k,m,p}$ be a function from the positive integers to the positive integers such that $J_{k,m,p}(n)$ is the length of the longest sequence $\alpha_1 > \cdots > \alpha_q$ such that $\alpha_1 < \omega^{\omega^{k-1} \cdot m}$ and $|\alpha_i| \leq A_p(i+n)$.

Lemma 4.2.20. *For all $k, m, p > 1$, $J_{k,m,p}$ is eventually dominated by H_{k+1}.*

Proof. Without loss of generality we can assume $p \geq 3$, $m \geq 4p + 54$ and $n \geq 2km + 1$. Let $\alpha_1 > \cdots > \alpha_q$ be a sequence of maximal length according to the definition of $J_{k,m,p}(n)$. We will construct a sequence according to the definition of $H_{k+1}(n)$ which is at least as long. By lemma 4.2.19 there exists a sequence $\beta_1^t > \cdots > \beta_{r_t}^t$ according to the definition of $I_{p+1}(km+t)$ with

$$r_t \geq A_{p+1}(km+t) \geq (2p+3) \cdot A_p(2km+1+t) + km + t.$$

The sequence $\omega^{\omega^{k-1} \cdot m} + \beta_1^0 > \cdots > \omega^{\omega^{k-1} \cdot m} + \beta_{r_0}^0 > \omega^{2p+2} \cdot \alpha_1 + \beta_1^1 > \cdots > \omega^{2p+2} \cdot \alpha_1 + \beta_{r_1}^1 > \omega^{2p+2} \cdot \alpha_2 + \beta_1^2 > \cdots > \omega^{2p+2} \cdot \alpha_2 + \beta_{r_2}^2 > \ldots$ now meets the requirements. □

We now use the theory about ordinal recursion to prove that L_I dominates every function that is α-recursive for some $\alpha < \omega^{\omega^\omega}$.

Lemma 4.2.21. *For $k \geq 1$, every $< \omega^{\omega^k}$ recursive function is dominated by some $J_{k,m,p}$.*

Proof. By lemma 3.2.5 it suffices to prove the lemma for a function of the form $f(x) = g(x, s(x, h(x)))$ where g and h are elementary and s is an ordinal counting function defined with an ordinal $\alpha < \omega^{\omega^k}$ and elementary θ. Choose m such that $\alpha < \omega^{\omega^{k-1} \cdot m}$ and p such that

$$\begin{array}{l} i) 2 + |\langle \theta^i(x, h(x)) \rangle| < A_p(x+i) \text{ for all } x \geq 1 \\ ii) g(x,q) < A_p(x+q) \end{array}.$$

The sequence $\omega + h(x) \succ \omega + \theta(\underline{x, h(x)}) \succ \underline{\omega + \theta(x, \theta}(x, h(x))) \succ \omega + \theta^3(x, h(x)) \succ \cdots \succ \omega + \theta^q(x, h(x)) = \omega \succ \underline{g(x,q)} \succ \underline{g(x,q) - 1} \succ \cdots \succ 0$ now proves the lemma. □

Theorem 4.2.1. *The function L_I eventually dominates every $< \omega^{\omega^\omega}$ recursive function*

Proof. Let g be a $< \omega^{\omega^\omega}$ recursive function. Then for some k, g is $< \omega^{\omega^k}$ recursive. By lemma 4.2.21 there are m and p such that g is dominated by $J_{k,m,p}$. By lemma lemma 4.2.20 it follows that g is eventually dominated by H_{k+1}. Lemma 4.2.16 now implies that g is eventually dominated by G_{k+1}. By lemma 4.2.7 g is eventually dominated by E. So E eventually dominates all $< \omega^{\omega^\omega}$ recursive function and thus we have that E eventually dominates the function $x \mapsto g(x+7)$. Hence, for sufficiently large x, $E(x) > g(x+7)$ and thus $E(x-7) > g(x)$. By lemma 4.2.4 we now have that for sufficiently large x, $L_I(x) > g(x)$. □

Chapter 5

The phase transition

Experience has shown that it is usually easy to prove the provability part of a phase transition. So wel start with that. We will try to make f as fast growing as possible under the condition that L_f is provably total in $I\Sigma_2$.

5.1 Provability

To show provability of the totality of L_f (in $I\Sigma_2$) we have to find a bound on the length that a sequence over $\{1, \ldots, k\}$ with property $\mathcal{F}_f$ can have. An easy way to do this would be to find for each k an l such that $|\{n|f(n) = l-1\}| > k^l$. By the pigeon hole principle it would follow that for every sequence $x_1 \ldots x_m$ over $\{1, \ldots, k\}$ with $m \geq \max\{n|f(n) = l-1\} + l - 1$ there are $i < j$ such that $x_i \ldots x_{i+f(i)} = x_j \ldots x_{j+f(j)}$. Hence $L_f(k) < \max\{n|f(n) = l-1\} + l - 1$. The sort of functions that accomplish this are the logarithmic ones. However, if we take f to be a logarithm with some fixed base then the argument above will not work when k gets too large. Therefore we use a logarithmic function with a slowly decreasing factor in front. This factor will be of the form $1/(g^{-1}(n))$. We define inverse functions as follows.

Definition 5.1.1. If $h : \mathbb{N} \to \mathbb{N}$ is any unbounded function then we define h^{-1} to be the function $m \mapsto \min\{n|h(n) \geq m\}$.

So we set

$$f(n) = \lfloor \frac{1}{g^{-1}(n)} \log_2 n \rfloor$$

where $g : \mathbb{N} \to \mathbb{N}$ is an increasing function which tends to infinity. The base of 2 in the logarithm is not important. It could as well be any other number > 1. We will now formalize the argument above and see that it works if $I\Sigma_2$ proves the totality of g.

Theorem 5.1.1. *If g is provably total in $I\Sigma_2$ then the function L_f is provably total in $I\Sigma_2$.*

Proof. We set $j = \max\{g(2\lceil \log_2 k \rceil), k^4\}$ and show that any sequence with length at least $4j + f(4j)$ cannot have property $\mathcal{F}_f$ because it will contain two windows that are the same. So suppose that we have a sequence w over $\{1, \ldots, k\}$ of length at least $4j + f(4j)$. For $i \geq j$ we have

$$f(i) \leq \frac{\log_2 i}{2 \log_2 k} = \frac{1}{2} \log_k i$$

and in particular $f(4j) \leq 1 + \frac{1}{2} \log_k j$. We now look at the windows of w which start at a position i for $j \leq i \leq 4j$. There are more than $3j$ such windows. These windows are sequences over $\{1, \cdots k\}$ with length at most $2 + \frac{1}{2} \log_k j$. The number of possible sequences over $\{1, \cdots k\}$ with length at most $2 + \frac{1}{2} \log_k j$ is limited by

$$\sum_{m=1}^{2+\lfloor \frac{1}{2} \log_k j \rfloor} k^m \leq 2k^{2+\lfloor \frac{1}{2} \log_k j \rfloor} \leq 2k^2 \sqrt{j}.$$

Using $k^4 \leq j$ we get $k^2 \sqrt{j} \leq j$, thus $2k^2 \sqrt{j} < 3j$ and we can conclude that indeed two windows must be the same which implies that $L_f(k) < 4j + f(4j)$. Since we assume that g is provably total in $I\Sigma_2$ the function $k \mapsto 4j + f(4j)$ is provably total in $I\Sigma_2$ and this proves the theorem. □

5.2 unprovability

In this section we will use the same functions f (which depends on g) and show that if every function that is provably total in $I\Sigma_2$ is eventually dominated by g, then L_f is not provably total in $I\Sigma_2$. So we have to show that the function L_f grows very fast. We do this by using sequences over $\{1, \ldots, k\}$ with property $\mathcal{F}$ and constructing out of them sequences over $\{1, \ldots, h(k)\}$ (where h is some elementary function) with property $\mathcal{F}_f$ that have about the same length.

The whole construction consists of the combination of three constructions. The first construction will give a sequence with property $\mathcal{F}_\phi$ where ϕ is a log like function. The second construction will transform this sequence a little so that it has property $\mathcal{F}_\psi$ where ψ is a small modification of ϕ such that ψ is non decreasing and has small enough values at small arguments. The third construction then finally gives us a sequence with property $\mathcal{F}_f$. In these constructions we will view numbers as finite sequences. We will use the following coding functions that depend on the number k which will stand for the cardinality of the set of elements of the input sequence. The function N maps a sequence $(a_1, \ldots a_q)$ to $1 + \sum_{i=1}^{q} (a_i - 1) k^{i-1}$ (we will only use inputs in which for every i, $1 \leq a_i \leq k$). The function p_j maps the number $N((a_1, \ldots a_q))$ to a_j, so p_j maps n to a number in $\{1, \ldots k\}$ that is equivalent to $\lfloor (n-1)/k^{j-1} \rfloor + 1$ modulo k.

5.2.1 Construction I

Input

The input of this construction is a sequence $x_1 x_2 \ldots x_n$ (with $n > 1$) over

$\{1,\ldots,k\}$ with property $\mathcal{F}$

Output
The output of this construction is a sequence $y_1 y_2 \ldots y_m$ with $m = \beta(\lfloor n/2 \rfloor)$ (β is defined below) over $\{1,\ldots,k^2+2\}$ with property $\mathcal{F}_\phi$. We will now define this function ϕ (the idea behind this definition is given in the informal description below). In this definition we need the following function. Let $\beta : \mathbb{N} \to \mathbb{N}$ be defined recursively by

$$\begin{array}{rcl} \beta(0) & = & 0 \\ \beta(b+1) & = & \beta(b) + (b+3)\cdot k^{b+2} - b. \end{array}$$

We now define ϕ as follows.

$$\phi(i) = \begin{cases} \beta^{-1}(i)+1 & \text{if } \beta^{-1}(i+\beta^{-1}(i)+1) = \beta^{-1}(i) \\ \beta^{-1}(i)+3 & \text{otherwise} \end{cases}$$

Example
If the input is 112222 ($k=2$) then the output will be

$$\substack{1\\0}\ \substack{1\\0}\ S\ \substack{1\\0}\ \substack{1\\1}\ S\ \substack{1\\1}\ \substack{1\\0}\ S\ \substack{1\\1}\ \substack{1\\1}\ T\ \substack{2\\0}\ \substack{2\\0}\ S\ \substack{1\\0}\ \substack{2\\0}\ \substack{2\\1}\ S\ \substack{1\\0}\ \substack{2\\1}\ \substack{2\\0}\ S\ \substack{1\\0}\ \substack{2\\1}\ \substack{2\\1}\ S\ \substack{1\\1}\ \substack{2\\0}\ \substack{2\\0}\ S$$

$$\substack{1\\1}\ \substack{2\\0}\ \substack{2\\1}\ S\ \substack{1\\1}\ \substack{2\\1}\ \substack{2\\0}\ S\ \substack{1\\1}\ \substack{2\\1}\ \substack{2\\1}\ T\ \substack{2\\0}\ \substack{2\\0}\ S\ \substack{2\\0}\ \substack{2\\0}\ \substack{2\\0}\ \substack{2\\1}\ S\ \substack{2\\0}\ \substack{2\\0}\ \substack{2\\1}\ \substack{2\\0}\ S\ \substack{2\\0}\ \substack{2\\0}\ \substack{2\\1}\ \substack{2\\1}\ S$$

$$\substack{2\\0}\ \substack{2\\1}\ \substack{2\\0}\ \substack{2\\0}\ S\ \substack{2\\0}\ \substack{2\\1}\ \substack{2\\0}\ \substack{2\\1}\ S\ \substack{2\\0}\ \substack{2\\1}\ \substack{2\\1}\ \substack{2\\0}\ S\ \substack{2\\0}\ \substack{2\\1}\ \substack{2\\1}\ \substack{2\\1}\ S\ \substack{2\\1}\ \substack{2\\0}\ \substack{2\\0}\ \substack{2\\0}\ S\ \substack{2\\1}\ \substack{2\\0}\ \substack{2\\0}\ \substack{2\\1}\ S$$

$$\substack{2\\1}\ \substack{2\\0}\ \substack{2\\1}\ \substack{2\\0}\ S\ \substack{2\\1}\ \substack{2\\0}\ \substack{2\\1}\ \substack{2\\1}\ S\ \substack{2\\1}\ \substack{2\\1}\ \substack{2\\0}\ \substack{2\\0}\ S\ \substack{2\\1}\ \substack{2\\1}\ \substack{2\\0}\ \substack{2\\1}\ S\ \substack{2\\1}\ \substack{2\\1}\ \substack{2\\1}\ \substack{2\\0}\ S\ \substack{2\\1}\ \substack{2\\1}\ \substack{2\\1}\ \substack{2\\1}\ T$$

where $\substack{a\\c}$ stand for $N((a,c+1))$, S stands for k^2+1 and T stands for k^2+2.

Informal description
The output in the example consists of three blocks because the input 112222 has three windows, namely 11, 122 and 2222. The end of each block is marked by a T (k^2+2). The ith block consists of repetitions of window i of the input ($x_i \ldots x_{2i}$) on the first line and these repetitions are counted in base k on the second line. So the ith block contains k^{i+1} repetitions since the length of window i of the input is $i+1$. These repetitions are separated by an S (k^2+1). The first repetition in each block is different. These repetitions only consist of the last two elements of the corresponding window of the input. This is because these two elements are new in the sense that the other elements in this window are also contained in the previous window. The function β maps a number i to the position of ith T. The function ϕ is defined in such a way that if a window of the output is contained in block i then this window contains exactly one S and a cyclic permutation of the corresponding window from the input. If a window

of the output starts in block i and ends in block $i+1$ then this window contains exactly one S and one T and a cyclic permutation of the window $x_{i+1} \dots x_{2i+2}$ from the input.

Formal description
We will still use $S = k^2 + 1$ and $T = k^2 + 2$ here. Given the input $x_1 \dots x_n$ over $\{1, \dots k\}$ with property $\mathcal{F}$ we will define the output $y_1 \dots y_m$ over $\{1, \dots, k^2+2\}$ and prove that this sequence has property $\mathcal{F}_\phi$. Let $m = \beta(\lfloor n/2 \rfloor)$. To improve readability we will denote $\beta^{-1}(i)$ by b in the next definition.

$$y_i = \begin{cases} k^2+2 & \text{if } \beta(b) = i \\ k^2+1 & \text{if } \beta(b) \neq i \text{ and} \\ & i - \beta(b-1) + b - 1 \text{ is a multiple of } b+2 \\ N((x_{b+j-1}, c+1)) \cdot k & \text{if } \exists q\,(i - \beta(b-1) + b - 1 = q \cdot (b+2) + j \text{ and} \\ & 1 \leq j \leq b+1 \text{ and } 0 \leq c < k \text{ and } \lfloor q/(k^{b+1-j}) \rfloor \equiv c \pmod{k}) \end{cases}$$

One can check that for every i exactly one of the three conditions is the case and in case of the third condition the values of j, q and c are uniquely determined.

Lemma 5.2.1. *Let $0 \leq i' \leq i$, $0 \leq j' \leq j$ and let $u_1 \dots u_i$ and $v_1 \dots v_j$ be sequences over $\{1, \dots, l\}$ and let $w > l$. If $u_{i'+1} \dots u_i w u_1 \dots u_{i'}$ is a subsequence of $v_{j'+1} \dots v_j w v_1 \dots v_{j'}$ then $u_1 \dots u_i$ is a subsequence of $v_1 \dots v_j$.*

Proof. Since $w \notin \{1, \dots, l\}$ it follows that it must be the case that $u_{i'+1} \dots u_i$ is a subsequence of $v_{j'+1} \dots v_j$ and $u_1 \dots u_{i'}$ is a subsequence of $v_1 \dots v_{j'}$. Hence $u_1 \dots u_i$ is a subsequence of $v_1 \dots v_j$. □

Lemma 5.2.2. *If $u_1 \dots u_i$ is a subsequence of $v_1 \dots v_j$ then for any function h the sequence $h(u_i) \dots h(u_{i'})$ is a subsequence of $h(v_1) \dots h(v_j)$ with the same embedding.*

Proof. Clear. □

Lemma 5.2.3. *If $i < j$, $j + \phi(j) \leq m$, $\phi(i) = \beta^{-1}(i) + 1$, $\phi(j) = \beta^{-1}(j) + 1$, $\beta^{-1}(i) < \beta^{-1}(j)$ and $y_i \dots y_{i+\phi(i)}$ is a subsequence of $y_j \dots y_{j+\phi(j)}$ then $x_{\beta^{-1}(i)} \dots x_{2\beta^{-1}(i)}$ is a subsequence of $x_{\beta^{-1}(j)} \dots x_{2\beta^{-1}(j)}$.*

Proof. From the definition of y_i it follows that both $y_i \dots y_{i+\phi(i)}$ and $y_j \dots y_{j+\phi(j)}$ contain exactly one element that is bigger than k^2 and since the one is a subsequence of the other these elements have to be equal. Let h equal the identity on numbers $> k^2$ and p_1 on numbers $\leq k^2$. There exist i', j' such that

$$h(y_i) \dots h(y_{i+\phi(i)}) = x_{i'+1} \dots x_{2\beta^{-1}(i)} w x_{\beta^{-1}(i)} \dots x_{i'}$$

and

$$h(y_j) \dots h(y_{j+\phi(j)}) = x_{j'+1} \dots x_{2\beta^{-1}(j)} w x_{\beta^{-1}(j)} \dots x_{j'}$$

where $w > k^2$. By lemmas 5.2.2 and 5.2.1 we conclude that $x_{\beta^{-1}(i)} \dots x_{2\beta^{-1}(i)}$ is a subsequence of $x_{\beta^{-1}(j)} \dots x_{2\beta^{-1}(j)}$. □

Lemma 5.2.4. *If $i < j$, $j + \phi(j) \leq m$, $\phi(i) = \beta^{-1}(i) + 1$, $\phi(j) = \beta^{-1}(j) + 3$ and $y_i \dots y_{i+\phi(i)}$ is a subsequence of $y_j \dots y_{j+\phi(j)}$ then $x_{\beta^{-1}(i)} \dots x_{2\beta^{-1}(i)}$ is a subsequence of $x_{\beta^{-1}(j)+1} \dots x_{2\beta^{-1}(j)+2}$.*

Proof. Let h be defined as in the proof of the previous lemma. There exist $i^{'}, j^{'}$ such that $\beta^{-1}(i) - 1 \leq i^{'} \leq 2\beta^{-1}(i)$, $\beta^{-1}(j) \leq j^{'} \leq 2\beta^{-1}(j)$ and

$$h(y_i) \dots h(y_{i+\phi(i)}) = x_{i'+1} \dots x_{2\beta^{-1}(i)} w x_{\beta^{-1}(i)} \dots x_{i'}$$

where $w = S$ or $w = T$ and

$$h(y_j) \dots h(y_{j+\phi(j)}) = x_{j'+1} \dots x_{2\beta^{-1}(j)} T x_{2\beta^{-1}(j)+1} x_{2\beta^{-1}(j)+2} S x_{\beta^{-1}(j)+1} \dots x_{j'}.$$

By lemma 5.2.2 the first sequence is a subsequence of the second. In case $w = S$ this will still be the case if we delete the T from the second sequence and then by lemma 5.2.1 we conclude that $x_{\beta^{-1}(i)} \dots x_{2\beta^{-1}(i)}$ is a subsequence of $x_{\beta^{-1}(j)+1} \dots x_{2\beta^{-1}(j)+2}$. In case $w = T$ the first sequence will still be a subsequence of the second if we delete the S from the second sequence and then by lemma 5.2.1 we conclude that $x_{\beta^{-1}(i)} \dots x_{2\beta^{-1}(i)}$ is a subsequence of $x_{\beta^{-1}(j)+1} \dots x_{2\beta^{-1}(j)+2}$. □

Lemma 5.2.5. *If $i < j$, $j + \phi(j) \leq m$, $\phi(i) = \beta^{-1}(i) + 3$, $\phi(j) = \beta^{-1}(j) + 3$, $\beta^{-1}(i) + 1 < \beta^{-1}(j) + 1$ and $y_i \dots y_{i+\phi(i)}$ is a subsequence of $y_j \dots y_{j+\phi(j)}$ then $x_{\beta^{-1}(i)+1} \dots x_{2\beta^{-1}(i)+2}$ is a subsequence of $x_{\beta^{-1}(j)+1} \dots x_{2\beta^{-1}(j)+2}$.*

Proof. Let h be as defined in the proof of the previous lemma. There exist $i^{'}, j^{'}$ such that $\beta^{-1}(i) \leq i^{'} \leq 2\beta^{-1}(i)$, $\beta^{-1}(j) \leq j^{'} \leq 2\beta^{-1}(j)$ and

$$h(y_i) \dots h(y_{i+\phi(i)}) = x_{i'+1} \dots x_{2\beta^{-1}(i)} T x_{2\beta^{-1}(i)+1} x_{2\beta^{-1}(i)+2} S x_{\beta^{-1}(i)+1} \dots x_{i'}.$$

and

$$h(y_j) \dots h(y_{j+\phi(j)}) = x_{j'+1} \dots x_{2\beta^{-1}(j)} T x_{2\beta^{-1}(j)+1} x_{2\beta^{-1}(j)+2} S x_{\beta^{-1}(j)+1} \dots x_{j'}.$$

By lemma 5.2.2 the first sequence is a subsequence of the second. Two applications of lemma 5.2.1 now yield the result. □

Lemma 5.2.6. *If $i < j$, $j + \phi(j) \leq m$, $\phi(i) = \beta^{-1}(i) + 3$ and $\phi(j) = \beta^{-1}(j) + 1$ then it cannot be the case that $y_i \dots y_{i+\phi(i)}$ is a subsequence of $y_j \dots y_{j+\phi(j)}$.*

Proof. The sequence $y_i \dots y_{i+\phi(i)}$ contains an S and a T while the sequence $y_j \dots y_{j+\phi(j)}$ does not contain an S or does not contain a T. □

Lemma 5.2.7. *If $i < j$, $j + \phi(j) \leq m$, $\phi(i) = \beta^{-1}(i) + 1$, $\phi(j) = \beta^{-1}(j) + 1$ and $\beta^{-1}(i) = \beta^{-1}(j)$ then it cannot be the case that $y_i \dots y_{i+\phi(i)}$ is a subsequence of $y_j \dots y_{j+\phi(j)}$.*

Proof. Since the sequences $y_i \dots y_{i+\phi(i)}$ and $y_j \dots y_{j+\phi(j)}$ have the same length, all we have to do is show that they are not equal. Let h equal the identity on numbers $> k^2$ and p_2 on numbers $\leq k^2$. If these sequences are equal then there exists a number c such that $0 \leq c \leq \beta^{-1}(i) + 1$ and

$$h(y_i) \dots h(y_{i+\phi(i)}) = h(y_j) \dots h(y_{j+\phi(j)}) = a_{c+1} \dots a_{\beta^{-1}(i)+1} w a_1 \dots a_c$$

where $w > k^2$. Let d be the number q that is used in the definition of y_i if $y_i \leq k^2$ and let d be one less than the q used in the definition of y_{i+1} otherwise. So $q = d$ is used in the definition of $y_i \dots y_{i+\beta^{-1}(i)-c}$ and $q = d+1$ is used in the definition of $y_{i+\beta^{-1}(i)-c+1} \dots y_{i+\phi(i)}$. The number d is uniquely determined by $a_1 \dots a_{\beta^{-1}(i)+1}$. This implies that $i = j$ which contradicts the assumption $i < j$. □

Lemma 5.2.8. *If $i < j$, $j + \phi(j) \leq m$, $\phi(i) = \beta^{-1}(i) + 3$, $\phi(j) = \beta^{-1}(j) + 3$, $\beta^{-1}(i) + 1 = \beta^{-1}(j) + 1$ then it cannot be the case that $y_i \dots y_{i+\phi(i)}$ is a subsequence of $y_j \dots y_{j+\phi(j)}$.*

Proof. Since the length of $y_i \dots y_{i+\phi(i)}$ is the same as the length of $y_j \dots y_{j+\phi(j)}$ it suffices to show that they are not equal. This is clear since both sequences must contain exactly one T. This is the T at position $\beta(\beta^{-1}(i))$ $(= \beta(\beta^{-1}(j)))$ and since $i < j$ there cannot exist a c such that $y_{i+c} = y_{j+c} = T$. □

Lemma 5.2.9. *There are no $i < j$ such that $y_i \dots y_{i+\phi(i)}$ is a subsequence of $y_j \dots y_{j+\phi(j)}$.*

Proof. For every $i < j$ the conditions of one of the lemmas 5.2.3 - 5.2.8 are satisfied and it either follows directly that $y_i \dots y_{i+\phi(i)}$ is not a subsequence of $y_j \dots y_{j+\phi(j)}$ or the assumption that $y_i \dots y_{i+\phi(i)}$ is a subsequence of $y_j \dots y_{j+\phi(j)}$ implies that there are $i^{'} < j^{'} \leq \lfloor n/2 \rfloor$ such that $x_{i^{'}} \dots x_{2i^{'}}$ is a subsequence of $x_{j^{'}} \dots x_{2j^{'}}$ which contradicts the assumption that the sequence $x_1 \dots x_n$ has property $\mathcal{F}$. □

5.2.2 Construction II

Input
The input of this construction are functions h, h' and a sequence $x_1 \dots x_n$ over $\{1, \dots, k\}$ with property $\mathcal{F}_h$. The functions $h, h^{'}$ must satisfy the conditions $\forall i(i + 1 + h(i+1) \geq i + h(i))$, $\forall i(i + 1 + h^{'}(i+1) \geq i + h^{'}(i))$ and $\forall i(0 \leq h(i) - h^{'}(i) \leq 1)$.

Output
The output of this construction is a sequence $y_1 \dots y_m$ with $m = \max\{i + h'(i) | \exists j \quad (i + h^{'}(i) < j + h(j) \leq n)\}$ over $\{1, \dots, 2k^2\}$ with property $\mathcal{F}_{h'}$.

example
If

$$h(i) = \begin{cases} 3 & \text{if } i \text{ is a multiple of 3} \\ 2 & \text{otherwise} \end{cases},$$

$h^{'}(i) = 2$ and $x_1 \dots x_n = 1122333$ then the output is

$$\begin{matrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 3 & 3 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{matrix}$$

where $\begin{matrix} a \\ b \\ c \end{matrix}$ stands for $N((a, b, c+1))$.

informal description
The function $h^{'}$ is almost the same as h. The only difference is that for some arguments the value of $h^{'}$ is one less. To ensure that for $i < j$ with $h^{'}(i) = h(i) - 1$ it is not the case that $y_i \dots y_{i+h'(i)}$ is a subsequence of $y_j \dots y_{j+h'(j)}$ we write the next element on the second line and indicate the difference between $h(i)$ and $h^{'}(i)$ at position i on the third line.

formal description
The sequence $y_1 \dots y_m$ over $\{1, \dots, 2k^2\}$ is defined as follows.

$$y_i = N((x_i, x_{i+1}, h(i) - h^{'}(i) + 1)).$$

We show that this sequence has property $h^{'}$. Assume that we have $i < j$ such that $j + h^{'}(j) \le m$.

Lemma 5.2.10. *If $h^{'}(i) = h(i)$ and $y_i \dots y_{i+h'(i)}$ is a subsequence of $y_j \dots y_{j+h'(j)}$ then $x_i \dots x_{i+h(i)}$ is a subsequence of $x_j \dots x_{j+h(j)}$.*

Proof. Applying lemma 5.2.2 to the function p_1 and the sequences $y_i \dots y_{i+h'(i)}$ and $y_j \dots y_{j+h'(j)}$ yields the result. □

Lemma 5.2.11. *If $h^{'}(i) = h(i) - 1$ and $y_i \dots y_{i+h'(i)}$ is a subsequence of $y_j \dots y_{j+h'(j)}$ then there exists $j^{'} \ge j$ such that $x_i \dots x_{i+h(i)}$ is a subsequence of $x_{j'} \dots x_{j'+h(j')}$.*

Proof. Let E be an embedding from $y_i \dots y_{i+h'(i)}$ into $y_j \dots y_{j+h'(j)}$. Since $p_3(y_i) = 2$ it must also be the case that $p_3(y_{E(i)}) = 2$. From the definition of y we see that this implies that $h^{'}(E(i)) = h(E(i)) - 1$ and from the conditions on h and $h^{'}$ and the definition of m it follows that there exists $j^{'}$ such that $j \le j^{'} \le E(i)$ and $j + h^{'}(j) < j^{'} + h(j^{'}) \le n$. Let $E^{'} : \{i, \dots, i + h(i)\} \to \{j^{'}, \dots, j^{'} + h(j^{'})\}$ be an extension of E with $E^{'}(i + h(i)) = E(i + h(i) - 1) + 1$. We claim that $E^{'}$ is an embedding from $x_i \dots x_{i+h(i)}$ into $x_{j'} \dots x_{j'+h(j')}$. For $i \le i^{'} < i + h(i)$ we have $y_{i'} = y_{E'(i')}$ and thus $p_1(y_{i'}) = p_1(y_{E'(i')})$ which can be rewritten as $x_{i'} = x_{E'(i')}$ and we also have $y_{i+h(i)-1} = y_{E'(i+h(i)-1)}$ and thus $p_2(y_{i+h(i)-1}) = p_2(y_{E'(i+h(i)-1)})$ which can be rewritten as $x_{i+h(i)} = x_{E'(i+h(i)-1)+1} = x_{E'(i+h(i))}$ □

The above two lemmas show that if the conditions on the input are satisfied then the output sequence does indeed have the property $\mathcal{F}_{h'}$.

5.2.3 Construction III

Input The input of this construction is a non decreasing function h, a function f and a sequence $x_1 \dots x_n$ over $\{1, \dots, k\}$ with property $\mathcal{F}_h$. The function f must satisfy the following conditions.

$$\text{For all } j, \quad |\{i \in \{j, \dots, j + f(j) - 1\} | f(i) \neq f(i + 1)\}| \leq 2 \tag{5.1}$$

and

$$\begin{gathered}\text{If } f(j) > f(j+1) + 1 \text{ then for every } i \text{ in the interval} \\ \{j + 1 + f(j + 1) + 1 \dots j + f(j)\} \text{ it is the case that } i + f(i) \geq j + f(j)\end{gathered} \tag{5.2}$$

Output The output of this construction is a sequence $y_1 \dots y_m$ over $\{1, \dots 3k^3 + s\}$ with property $\mathcal{F}_f$. The numbers m and s depend on n and the functions h and f

Example
If $h(i) = \lfloor \sqrt{i} \rfloor$, $x_1 \dots x_{15} = 113223333111122$ and f is given by the table

i	1	2	3	4	5	6	7	8	9	10	11	12	13
$f(i)$	2	2	1	1	1	3	3	3	3	2	2	3	3

then the output sequence is

$$\begin{array}{ccccccccccccc} 2 & 2 & 3 & 3 & - & - & - & - & - & 3 & 3 & 3 & 3 \\ - & - & 1 & 1 & 3 & 2 & - & - & - & - & - & - & - \\ - & - & - & - & - & 3 & 1 & 1 & 1 & 1 & 2 & 2 & - \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 1 & 1 & 2 & 2 \end{array}$$

where $\begin{array}{c} a \\ b \\ c \\ d \end{array}$ stands for $N((a, b, c, d))$. For clarity, the positions that are not important (i.e. are not used in the proof that the output has property $\mathcal{F}_f$) are marked with $-$.

Informal description
The part of $x_1 \dots x_{15}$ that consists of windows of length 2 is $x_1 \dots x_{3+h(3)} = 1132$.
The part of $x_1 \dots x_{15}$ that consists of windows of length 3 is $x_4 \dots x_{8+h(4)} = 2233331$.
The part of $x_1 \dots x_{15}$ that consists of windows of length 4 is $x_9 \dots x_{15} = 3111122$.
We have $f(1) = f(2) = 2$ so we start by using a beginning of $x_4 \dots x_{4+h(4)}$ and putting that on the first line. On the fourth line we write a 1 so we know that we are using the first line at these positions. We then have $f(3) = f(4) = f(5) = 1$ so we now use a beginning of $x_1 \dots x_{3+h(3)}$ and put that on the second line. On the fourth line we write a 2 so we know that we are using the second line at these

positions. Now $f(6) = f(7) = f(8) = f(9) = 3$ and we do the same thing again. Then $f(10) = f(11) = 2$ so we use a piece from $x_4 \dots x_{8+h(8)}$. Since we already used the beginning at two positions earlier we will now start at x_6 and jump to the first line again. The fact that we use the same piece again on the first line is a coincidence. In case there are arguments i such that $f(i+1) < f(i) - 1$ we will use a new number each time that this happens and put it on the positions $i + 1 + f(i+1) + 1, \dots, i + f(i)$. This will enable us to show that if the output does not have property $\mathcal{F}_f$ then the input cannot have property $\mathcal{F}_h$.
Formal description
We need the following three functions in the definition of y_i.
$\ell : \mathbb{N} \to \{1, 2, 3\}$ with $\ell(i) \equiv |\{j < i | f(j) \neq f(j+1)\}| + 1 \pmod 3$. This function tells us which coördinate is currently used.
$c(i) = |\{j < i | f(j) = f(i)\}|$. This function gives the number of sequences of length $f(i)$ that are already used.
$e(a, i) = \max(\{1\} \cup \{j \leq i | \ell(j) = a\})$. This function gives the latest position at which coördinate a was active.
Let m be maximal such that for all $i \leq m$ we have that

$$h(h^{-1}(f(i)) + c(i)) = f(i) \tag{5.3}$$

and

$$n \geq h^{-1}(f(i)) + c(i) + f(i) \tag{5.4}$$

(for each length we must have enough windows to use). We will first define a sequence $z_1 \dots z_m$ which we will then modify into a sequence $y_1 \dots y_m$.

$$\begin{aligned} z_i = N((\quad & x_{h^{-1}(f(e(1,i)))+c(e(1,i))+i-e(1,i)}, \\ & x_{h^{-1}(f(e(2,i)))+c(e(2,i))+i-e(2,i)}, \\ & x_{h^{-1}(f(e(3,i)))+c(e(3,i))+i-e(3,i)}, \\ & \ell(i))) \end{aligned} .$$

Let $z^0 = z$ and construct z^q (of the same length) out of z^{q-1} by letting j be the qth element in the set $\{r | r + f(r) > r + 1 + f(r+1)\}$ and setting $z_i^q = z_i^{q-1}$ if $i \notin \{1 + j + f(j+1) + 1 \dots j + f(j)\}$ and $z_i^q = 3k^3 + q$ otherwise. Let s be the least number such that $z^s = z^{s+1}$ and set $y = z^s$.

Lemma 5.2.12. *If $i < j$ and $y_i \dots y_{i+f(i)}$ is a subsequence of $y_j \dots y_{j+f(j)}$ then $y_i \dots y_{i+f(i)} = z_i \dots z_{i+f(i)}$.*

Proof. We will show that $z_i^s \dots z_{i+f(i)}^s$ does not contain a $3k^3 + s$. This implies that $z_i^s \dots z_{i+f(i)}^s = z_i^{s-1} \dots z_{i+f(i)}^{s-1}$ and thus $z_i^{s-1} \dots z_{i+f(i)}^{s-1}$ is a subsequence of $z_j^{s-1} \dots z_{j+f(j)}^{s-1}$. By repetition of the argument we conclude that $z_i^s \dots z_{i+f(i)}^s = z_i^0 \dots z_{i+f(i)}^0$ which proves the lemma. We now show that $z_i^s \dots z_{i+f(i)}^s$ does not contain a $3k^3 + s$. Suppose for a contradiction that it did. Let j' be the number such that $\{j' + 1 + f(j' + 1) + 1 \dots j' + f(j')\}$ is exactly the set of indices of $3k^3 + s$ elements in z^s. If $z_i^s = 3k^3 + s$ then by (5.2) we have that $i + f(i) \geq j' + f(j')$ and $j + f(j) \geq j' + f(j')$. Since $i < j$ this means that $z_i^s \dots z_{i+f(i)}^s$ contains more $3k^3+s$ elements than $z^s_j \dots z^s_{j+f(j)}$ contradicting the assumption that $z^s_i \dots z^s_{i+f(i)}$ is a subsequence of $z^s_j \dots z^s_{j+f(j)}$. If $z^s_i \neq 3k^3+s$ then let a be the least number such that $z^s_{i+a} = 3k^3+s$ and let b be the least number such that $z^s_{j+b} = 3k^3+s$. The assumption that $z^s_i \dots z^s_{i+f(i)}$ is a subsequence of $z^s_j \dots z^s_{j+f(j)}$ implies that $a \leq b$, but $i < j$ yields $b < a$ and we have the desired contradiction. □

Lemma 5.2.13. *If $i < j$, $i+f(i), j+f(j) \leq m$ and $y_i \dots y_{i+f(i)}$ is a subsequence of $y_j \dots y_{j+f(j)}$ then there exist i', j' such that $i' < j'$ and $x_{i'} \dots x_{i'+h(i')}$ is a subsequence of $x_{j'} \dots x_{j'+h(j')}$.*

Proof. Let E be an embedding from $y_i \dots y_{i+f(i)}$ into $y_j \dots y_{j+f(j)}$. Let $a = \min\{b+f(b) | j \leq b \leq j+f(j)\}$. From the construction of y we see that every element in $y_{a+1} \dots y_{j+f(j)}$ is $> 3k^3$. By lemma 5.2.12 it follows that E is an embedding from $y_i \dots y_{i+f(i)}$ into $y_j \dots y_a$ and it also follows that E is an embedding from $z_i \dots z_{i+f(i)}$ into $z_j \dots z_a$. By definition of a we have that $E(i) + f(E(i)) \geq a$ and thus E is also an embedding from $p_{p_4(z_i)}(z_i) \dots p_{p_4(z_i)}(z_{i+f(i)})$ into $p_{p_4(z_i)}(z_{E(i)}) \dots p_{p_4(z_i)}(z_{E(i)+f(E(i))})$. By the definition of z_i and the conditions (5.1), (5.3) and (5.4) we have for q, r with $q+f(q) \leq m$ and $0 \leq r \leq f(q)$

$$p_{p_4(z_q)}(z_{q+r}) = x_{h^{-1}(f(q))+c(q)+r}$$

and thus

$$\begin{aligned} p_{p_4(z_i)}(z_i) \dots p_{p_4(z_i)}(z_{i+f(i)}) &= \\ x_{h^{-1}(f(i))+c(i)} \dots x_{h^{-1}(f(i))+c(i)+h(h^{-1}(f(i))+c(i))} & \qquad \text{and} \\ p_{p_4(z_i)}(z_{E(i)}) \dots p_{p_4(z_i)}(z_{E(i)+f(E(i))}) &= \\ x_{h^{-1}(f(E(i)))+c(E(i))} \dots x_{h^{-1}(f(E(i)))+c(E(i))+h(h^{-1}(f(E(i)))+c(E(i)))}. & \end{aligned}$$

Set $i' = h^{-1}(f(i)) + c(i)$ and $j' = h^{-1}(f(E(i))) + c(E(i))$. Since (5.3) and h is non decreasing we see that $i' < j'$ in case $f(i) < f(E(i))$. In case $f(i) = f(E(i))$ we have $c(i) < c(E(i))$ since $i < E(i)$ and thus in this case we also have $i' < j'$. □

5.2.4 Putting the constructions together

We will now be able to show that L_f grows about as fast as L or as g depending on which grows slower.

Theorem 5.2.1. *Let $f(i) = \lfloor \frac{1}{g^{-1}(i)} \log_2 i \rfloor$ with g a strictly increasing function satisfying the condition $g(1) \geq 2$, $g(i+1) \geq g(i)^4$. Then the following holds.*

$$L_f(3 \cdot 2^{45}(k^2+2)^{48} + \lfloor \log_2 k \rfloor) \geq \min\{g(\lfloor \log_2 k \rfloor), \lfloor L(k)/2 \rfloor - 1\}.$$

Proof. Let $w = x_1 \dots x_n$ be a sequence over $\{1, \dots, k\}$ with property $\mathcal{F}$ and $n = L(k)$. Applying the first construction to it we obtain a sequence $w' = x'_1 \dots x'_{n'}$ over $\{1, \dots k^2+2\}$ with property $\mathcal{F}_p$ and $n' = \beta(\lfloor n/2 \rfloor)$. Then, using the second construction four times, we can get a sequence $w'' = x''_1 \dots x''_{n''}$

over $\{1, \ldots 2^{15}(k^2+2)^{16}\}$ with property $\mathcal{F}_{\beta^{-1}-1}$ and $n^{''} = n^{'} - 4$. Finally we want to apply the third construction to $w^{''}$ to produce a sequence $w^{'''} = x_1^{'''} \ldots x_{n^{'''}}^{'''}$ over $\{1, \ldots, 3 \cdot 2^{45}(k^2+2)^{48} + \lfloor \log_2 k \rfloor\}$ with property $\mathcal{F}_f$ and $n^{'''} \geq \min\{g(\lfloor \log_2 k \rfloor), \lfloor L(k)/2 \rfloor - 1\}$.

We verify that conditions (5.1) and (5.2) are met. If $f(i) \neq f(i+1)$ then $\lfloor \log_2 i \rfloor \neq \lfloor \log_2 (i+1) \rfloor$ or $g^{-1}(i) \neq g^{-1}(i+1)$. The interval $\{j, \ldots, j + f(j) - 1\}$ is contained in $\{j, \ldots, j + \lfloor \log_2 j \rfloor - 1\}$ and by the condition on g it is clear that both possibilities can occur at most once in this interval. Hence (5.1) is satisfied.

It is clear that on an interval $\{g(i)+1, \ldots, g(i+1)\}$ the function f is non decreasing. By the condition on g it follows that for all i, $f(g(i+1)+1) > f(g(i))$. Hence, if $i > j$ then $f(i)+1 \geq f(j)/2$. So if $i \in \{j+1+f(j+1)+1, \ldots, j+f(j)\}$ then $i + f(i) \geq j + f(j+1) + 1 + f(i) + 1 \geq j + f(j)/2 + f(j)/2 = j + f(j)$ and thus (5.2) is satisfied.

We will now show that $n^{'''} \geq \min\{g(\lfloor \log_2 k \rfloor), \lfloor L(k)/2 \rfloor - 1\}$. So we have to show that (5.3) and (5.4) hold for $i \leq \min\{g(\lfloor \log_2 k \rfloor), \lfloor L(k)/2 \rfloor - 1\}$. Since $f(i) < i$ we have for $i \leq \lfloor L(k)/2 - 1 \rfloor$

$$|\{j | j + h(j) \leq n^{''} \text{ and } h(j) = f(i)\}| = (f(i)+3)k^{f(i)+2} - f(i).$$

Since for all i, $f(g(i+1)+1) > f(g(i))$ it follows that for a fixed l there can be at most two values of $g^{-1}(j)$ such that $f(j) = l$. If we let $d = \max\{g^{-1}(j) | f(j) = l\}$ then $|\{j | f(j) = l\}| \leq 2 \cdot 2^{l \cdot d}$. Hence

$$\begin{aligned}
|\{j | j \leq g(\lfloor \log_2 k \rfloor) \text{ and } f(j) = f(i)\}| &\leq 2 \cdot 2^{f(i) \cdot \log_2 k} \\
&= 2 \cdot k^{f(i)} \\
&< (f(i)+3)k^{f(i)+2} - f(i) \\
&= |\{j | j + h(j) \leq n^{''} \text{ and } h(j) = f(i)\}|
\end{aligned}$$

and (5.3) follows.

If $i \leq \lfloor L(k)/2 - 1 \rfloor$ then by (5.3) $h^{-1}(f(i)) + c(i) < h^{-1}(f(i)+1) \leq h^{-1}(\lfloor L(k)/2 - 1 \rfloor) = \beta(\lfloor L(k)/2 - 1 \rfloor) + 1$. Since $f(i) \leq \lfloor L(k)/2 \rfloor \leq \beta(\lfloor L(k)/2 \rfloor) - 4 - \beta(\lfloor L(k)/2 - 1) - 1$, (5.4) follows. □

5.3 The phase transition

Using theorem 3.3.3 and combining theorems 5.1.1 and 5.2.1 we get the following description of the phase transition. Let $f_\alpha(i) = \frac{1}{H_\alpha^{-1}(i)} \log_2 i$.

Theorem 5.3.1. *$I\Sigma_2$ proves the totality of L_{f_α} if and only if $\alpha < \omega^{\omega^\omega}$.*

Chapter 6

Modification with gap-condition

In this chapter we find the phase transition if the normal subsequence relation is replaced by the subsequence relation with gap condition as defined in the section below. The gap condition makes it more difficult to embed sequences into each other. So the sequence length can grow much faster with this gap condition. It turns out that the functions f that will be interesting here are the same as in the previous chapter. As the sequence length can grow much faster with this gap condition, the lower bound will now be limited by an ϵ_0 recursive function instead of an ω^{ω^ω} recursive function. The result of this is that $I\Sigma_2$ is replaced by PA and ω^{ω^ω} by ϵ_0.

6.1 The gap condition

The gap condition is defined as follows [14].

Definition 6.1.1. We say that a sequence $x_1 \dots x_n \in \{1, \dots, k\}^*$ is embeddable in a sequence $y_1 \dots y_m \in \{1, \dots, k\}^*$ with gap condition if there is a strictly increasing function $h : \{1, \dots, n\} \to \{1, \dots, m\}$ such that for all $i \in \{1, \dots, n\}$, $x_i = y_{h(i)}$ and if $i < n$ then for all $j \in \{h(i)+1, \dots, h(i+1)\}$ we have $y_j \geq y_{h(i+1)}$. We denote this as $x_1 \dots x_n \preceq y_1 \dots y_m$.

Definition 6.1.2. A sequence $x_1x_2 \dots x_p$ over $\{1, \dots, k\}$ is *f-bad* if there are no $i < j$ such that $x_i \dots x_{i+f(i)} \preceq x_j \dots x_{j+f(j)}$. Let L'_f be the function which sends a number k to the maximum length an f-bad sequence over $\{1, \dots, k\}$ can have.

We use the following theorem from Schütte and Simpson to show that these sequences can get very long.

Theorem 6.1.1. *Let $\tilde{L}$ be the function which sends a positive integer k to the maximum n such that there exists a sequence $y_1, \dots, y_n$ where each $y_i \in$*

$\{1,\ldots,k\}^i$ *and there are no* $i < j$ *such that* $y_i \preceq y_j$*. The function* $\tilde{L}$ *eventually dominates every* $< \epsilon_0$ *recursive function.*

Proof. see [14]. □

Just as in the previous section we set

$$f(i) = \lfloor \frac{1}{g^{-1}(i)} \log_2 i \rfloor.$$

In section 5.1 we found an upper bound for $L_f(k)$ by showing that if a sequence is too long it will have two identical parts. Hence, this upper bound is still valid if we replace the subsequence relation by $\preceq$. Since it is clear that $L_f(i) \leq L'_f(i)$ the lower bound from the previous section also remains valid, but in the next subsection we will derive a better lower bound for L'_f.

6.2 a Lower bound for L'_f

Theorem 6.2.1. *If there exists a sequence* $y_1,\ldots,y_n$ *such that* $y_i \in \{1,\ldots,k\}^i$, $1 \leq i \leq n$ *and there are no* $i \neq j \leq n$ *such that* $y_i \preceq y_j$ *then*

$$L'_f(2k^2) \geq \min(n, g(\lfloor \frac{1}{4} \log_2 k \rfloor - 1)).$$

Proof. Suppose such a sequence $y_1,\ldots,y_n$ exists. The idea is to concatenate a number of y_i's in order to build an f-bad sequence $x_1 \ldots x_p$ with $p = \min(n, g(\lfloor \frac{1}{4} \log_2 k \rfloor - 1))$. This will happen in such a way that every $x_j \ldots x_{j+f(j)}$ contains at least one complete y_i. But we will need to redesign the y_i in such a way that the first element of a y_i is always smaller than any element that is not the first element of some $y_{i'}$ and every reoccurence of the same y_i in the concatenation will be recognizably different from the previous occurences. This will imply that $x_j \ldots x_{j+f(j)} \preceq x_{j'} \ldots x_{j'+f(j')}$ is impossible. Since otherwise take the y_i that is present in $x_j \ldots x_{j+f(j)}$. If the first element of this y_i is mapped into some $y_{i'}$ then all of its elements have to be mapped to that $y_{i'}$ since otherwise the gap condition cannot apply (the first element of the next $y_{i''}$ will be too small). This will mean that $y_i \preceq y_{i'}$ and thus $i = i'$. Since each occurence of y_i in the concatenation will be different from its previous occurences the contradiction will follow.

We will now describe how the y_i are redesigned. We will make sure that we have at most k^i occurences of y_i in the concatenation. We will make $x_1 \ldots x_p$ a sequence over $\{1,\ldots,2k^2\}$. We can view $\{1,\ldots,2k^2\}$ as $\{1,2\} \times \{1,\ldots,k\} \times \{0,\ldots,k-1\}$ with the lexicographic order. So each $x_j \in \{1,\ldots,2k^2\}$ has three coördinates. If an x_j is starting some y_i we set the first coördinate to 1, else we set it to 2. This will ensure that the start of some y_i is always smaller than any x_j that is not starting some $y_{i'}$. We set the projection of $x_1 \ldots x_p$ on the second coördinate to be an actual concatenation of a sequence of y_i's with possible repetitions. The third coördinate is used as a counter: if the projection on the second coördinates of $x_j \dots x_{j+i-1}$ is y_i then the projection on the third coördinates of $x_j \dots x_{j+i-1}$ is the number of times that y_i is used before in base k. This gives us the limit of k^i for the number of times that we can use y_i.

It remains to define positions $a_1 < a_2 < \dots$ at which some y_i begins in a way that respects the limit on the number of times that y_i can be used and that ensures that each $\{j, \dots, j + f(j) + 1\}$ contains at least two of these positions. At each position a_m we start the y_i that fits (so $i = a_{m+1} - a_m$). The function f decreases at some points, but let's first assume that f is increasing for the sake of simplicity. When we define the position a_m we want to keep enough space in reserve so that the only thing that can force us to use small y_i next is a small value of $f(j)$. We can achieve this by setting $a_1 = 1$, $a_{m+1} = 1 + \lfloor f(1)/2 \rfloor + 1$ (the division by 2 is to make sure that we reserve enough space for the next y_i so that we aren't unnecessarily forced to use a small y_i next), $a_{m+2} = \min(a_m + 1 + f(a_m + 1) + 1, a_{m+1} + \lfloor f(a_{m+1})/2 \rfloor + 1)$ (the first part of the minimum takes care of the condition that there are at least two starting positions in every interval $\{j, \dots, j + f(j) + 1\}$ and the second part reserves enough space for the next y_i). We can also make this work for non-increasing functions f by taking the minimum. We use the following inductive definition

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= \min(\{j + \lfloor f(j)/2 \rfloor + 1 : j \geq 1\}) \\
a_{m+2} &= \min(\{j + f(j) + 1 : a_m < j < a_{m+1}\} \cup \{j + \lfloor f(j)/2 \rfloor + 1 : a_{m+1} \leq j\})
\end{aligned}$$

From this definition it is clear that each $\{j, \dots, j + f(j) + 1\}$ contains at least two of these positions. We note that for $a_m \leq j < a_{m+1}$ we have that

$$a_{m+1} \leq j + \lfloor f(j)/2 \rfloor + 1.$$

Using this we see that

$$a_{m+2} - a_{m+1} \geq \min(\{\lfloor f(j)/2) \rfloor : a_m < j < a_{m+2}\}).$$

So for every time that y_i is used there exists m such that $a_{m+2} - a_{m+1} = i$ and by the above there must be a j such that $a_m < j < a_{m+2}$ and $f(j) \leq 2i + 1$. Therefore, every j such that $f(j) \leq 2i + 1$ can only produce two occurences of y_i. Hence, the limit of k^i for the number of times we can use y_i will be satisfied if

$$|\{j : f(j) \leq 2i + 1\}| \leq k^i/2.$$

The number of j such that

$$\lfloor \frac{1}{g^{-1}(j)} \log_2 j \rfloor \leq 2i + 1$$

is at most the number of j such that

$$\frac{1}{g^{-1}(j)} \log_2 j < 4i$$

which is equivalent to

$$j < \left(2^{4g^{-1}(j)}\right)^i$$

and

$$\left(2^{4g^{-1}(j)}\right)^i \leq \left(2^{\log_2 k - 4}\right)^i = \left(\frac{1}{16}k\right)^i < k^i/2.$$

Hence, the same y_i is not used too often and this ends the proof. □

The lower bound for $L_f^{'}$, theorem 6.1.1 and 6.2.1 now give the following.

Theorem 6.2.2. *Let g be a strictly increasing function from the positive integers to the positive integers and let $f(i) = \lfloor \frac{1}{g^{-1}(i)} \log_2 i \rfloor$. Let $h(k) = \lfloor \sqrt{k/2} \rfloor$. If $k \geq 2^{18}$ then*

$$\min\{g(\lfloor \frac{1}{4} \log_2 h(k) \rfloor - 1), \tilde{L}(h(k))\} \leq L_f(k) \leq 8g(2\lceil \log_2 k \rceil)$$

Corollary 6.2.1. *Let $f_\alpha(i) = \frac{1}{H_\alpha^{-1}(i)} \log i$. We have*

$$PA \vdash \forall k \exists n \; L'_{f_\alpha}(k) = n$$

if $\alpha < \epsilon_0$ and

$$PA \nvdash \forall k \exists n \; L_{f_{\epsilon_0}}(k) = n.$$

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