SCIENCE IN AXIOMATIC PERSPECTIVE

MSc Thesis (*Afstudeerscriptie*)

written by

Antonio Florio
(born 10/08/1976 in Grumo Appula)

under the supervision of **dr. Sara Uckelman**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:
29/06/2012

Members of the Thesis Committee:
dr. Sara Uckelman
prof. Piet Rodenburg
prof. Martin Stokhof
prof. Benedikt Löwe

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

PREFACE

The axiomatic method counts two thousand and three hundred years circa. Suppes [61] has proposed the category of *Euclidean-Archimedean tradition* to refer to the axiomatic theories that have been developed before the invention/discovery of the non-Euclidean geometries. Among these theories the first axiomatic system that we know is Euclid's *Elements* [16], a mathematical tractate consisting of thirteen books in which three centuries of Greek mathematical knowledge were given an order and were presented as a unified theory.[1] Euclid produced another axiomatic theory, the *Optics* [15]. This represents a theory of vision in Euclidean perspective rather than a tractate on physical optics. It is interesting that Archimedes's *Treatise* [12], probably the first book on mathematical physics, is an axiomatic theory.

The axiomatic method in the Euclidean-Aristotelian tradition was transmitted during the medieval age and scholarship in history of science has established the use of the axiomatic method in scientific tractates through all periods from antiquity up to the sixteenth–seventeenth-century Scientific Revolution [5]. In the context of the Scientific Revolution an important axiomatic theory is Newton's *Principia* [41, 61].

The axiomatic method covers a too big period of history and philosophy of science and we cannot deal with it in this thesis. So we skip the analysis of the axiomatic method in the Euclean-Archimedean tradition and begin our analysis in the nineteenth century when the axiomatic method entered in the modern phase. As Suppes puts it [61, p. 225]: "The historical source of the modern viewpoint toward the axiomatic method was the intense scrutiny of the foundations of geometry in the nineteenth century. Undoubtedly the most important driving force behind this effort was the discovery and development of non-Euclidean geometry at the beginning of the nineteenth century by Bolyai, Lobachevski, and Gauss.".

---

[1] In Books I, II, and IV is contained plane geometry, in Book II are contained the rudiments of geometrical algebra, Books V–X develop number theory (Note that number theory is developed without stating the axioms. An axiomatic treatment of number theory will appear only at the end of the nineteenth century, as we will see.) and Books X–XIII develop solid geometry.

Contents

Part I

The Modern Style of the Axiomatic Method

INTRODUCTION:

According to Suppes non-Euclidean geometries were "the most important driving force" towards the development of the "modern viewpoint" of the axiomatic method [61, p. 225]. The reason is not difficult to guess. Already from the antiquity the non-intuitive and non-evident character of the 5th postulate of Euclidean geometry, which is equivalent to the proposition asserting the uniqueness of the parallel: For any given line ℓ and a point a external to ℓ there is exactly one line drown through a that does not intersect ℓ, had been pointed out. The problem with this proposition is that it is not evident. In fact if the two lines are extended to infinity how can we be sure that they don't intersect? Being the fifth postulate not an evident truth, throughout history there were several attempts to prove it but the demonstrations were never satisfying [44]. Only in the nineteenth century was it realized that it is coherent with but independent from the other postulates.[2]
In the 19th century Bolyai and Lobachevsky discovered/invented independently non-Euclidean geometries by allowing geometries with the negation of the fifth postulate. In Euclidean geometry the fifth postulate is equivalent to the postulate of the parallel. In hyperbolic geometry infinite many lines through a not intersecting ℓ are allowed. In elliptic geometry any line through a intersects ℓ [45]. So non-Euclidean geometries pointed out the idealization nested in the idea of an objective and observable space, that space which was the ground of intuition and evidence. This caused a real problem for the axiomatic method which, in the past, considered axioms as evident truths. The point was that one cannot rely on intuition neither evidence to state the axioms because otherwise how can one account for non-Euclidean geometries? On the other hand allowing non-Euclidean geometries would provoke the suspicion that the axioms are not evident and objective or intuitive truths, but then what is their status? Therefore with non-Euclidean geometries the threat of uncertainty about the axiomatic method was first experienced.
The modern style of the axiomatic method[3] is an historical category adapted from [24]. In [24] the author analyzes the history of mathematics from the nineteenth to the thirties of the twentieth century proposing the category

[2]The geometry that can be developed from the Euclid's axioms without the 5th postulate is called *absolute geometry*.

[3]Suppes [61] uses "modern viewpoint", as already remarked, to refer to the same method we analyze but we use modern style for the reasons that follow.

of *modernism* as key concept driving the development of the mathematical research of this period toward standards of rigor and certainty.
According to [24] one of the main factors driving the mathematical research into modernism, i.e., search for certainty and rigor, was exactly the invention-discovery of non-Euclidean geometries. And this seems quite understandable because once the non-Euclidean geometries introduced elements of uncertainty, in the choice of the axioms as well as in our intuition of geometrical objects, then mathematicians started to search for certainty. Even if not explicitly theorized in [24] as such, the axiomatic method was one of the places where rigor and certainty were searched. In fact during the epoch of modernism the destiny of the axiomatic method was linked to the search of the foundations of mathematics. According to [23] the debate on the foundation of mathematics was essentially a search for certainty. Several mathematical enterprises with a sharp modernist style, such as the arithmetization of analysis, the foundations of geometries, the birth and development of mathematical logic, and the axiomatization of set theory, took place in the course of the nineteenth and the beginning of the twentieth century [24] and are relevant in our analysis of the axiomatic method. These enterprises constituted some of the factors of which the partecipants to the debate on the foundations of mathematics were aspected to give an unified perspective and an allembracing outlook [6]. So our strategy is to start from the analysis of the axiomatic method in the debate and to recover through it these relevant factors for the analysis of the method.
As we'll see the axiomatic method, during the epoch of modernism, passed trough a process of reconceptualization and developed in the way we know it today. Given the above considerations, our operation consists in proposing the category of modernism to the the framework of the analysis of the axiomatic method. In this operation of adaption of this category we also introduce a forcing because in our use modernism is replaced by modern and we use the term 'style' in the expression 'modern style' of the axiomatic method. In this we respect the use of the term modernism which in according to [24] is also a style. Our further operation is to articulate the 'modern style' with respect to the demand of our methodological, rather than historical, analysis and the expression 'modern style of the axiomatic method' comes from this. In fact the operation of proposing the category of modern as adjective of the word style with respect to the axiomatic method allows us, on the one hand, to approach the study of the axiomatic method in the framework of modernism and, on the other hand, to articulate further the modern style in

stages which have a methodological, rather than historical, value.
We articulate the modern axiomatic style in two stages by analyzing the axiomatic theories and their contextual framework and then by generalizing from these theories their essential features. These futures allow the reader to discern which of the two stages any axiomatic theory belongs to.
We call hypothetical-deductive theories and formal theories the theories at stage 1 and 2 of the modern style of the axiomatic method for reasons that will become evident in the next two chapters. Equivalently we use expressions of the kind "the axiomatic method at stage 1" as well as "the axiomatic method at stage 2" in order to refer to the processes of axiomatization of a theory as an hypothetical-deductive system and as a formal system.
Hypothetical-deductive theories and formal theories are the two subsequent outcomes of the process of axiomatization of the theories in modern style, they are the subject matter of the next two chapter.

A remark on notation: In our analysis we distinguish two stages of the axiomatic method and, in according to this, when we name a theory we distinguish the stage of the axiomatic method at which the theory is stated: The notation capital T indexed with the name of the theory, as for example T_{PA} for the hypothetical-deductive theory of Peano arithmetic, is used for theories at stage 1; the notation $\mathbf{T}$ indexed with the name of the theory, as for example $\mathbf{T}_{PA}$ for the formal theory of Peano arithmetic, is used for theories at stage 2.

Chapter 1

HYPOTHETICAL-DEDUCTIVE THEORIES

1.1 Two Ideas of Axiomatic Arithmetic in the Debate on the Foundations of Mathematics

The debate on the foundations of mathematics was undertaken from three mathematical schools, logicism, intuitionism, and formalism, from the end of the nineteenth century to the thirties of the twentieth century. It involved a wide range of philosophical problems. Probably the most known of these problems is the ontological-metaphysical problem of the nature of the mathematical objects about which the three schools held different positions.[1]

[1]Quine [49] has sustained the thesis that the three mathematical schools which undertook the debate were disputing essentially on an update-to-numbers version of the medieval debate on the nature of universals, i.e., the three schools represented three doctrines: Logicism represented the realist position holding that numbers exist in a sort of platonic world which is more real than the phenomenical flow of appearances of the empirical world; intuitionism held a conceptualist view for which numbers are considered as product of human creativity; formalists represented the nominalist version for the belief that numbers are just names, *flatus vocis*. As Snapper [54] notes, the metaphysical-ontological dispute concluded with three failures: Russell's paradox showed that the logicist entities were contradictory, the fact that many important mathematical objects were not intuitionistically plausible excluded the conceptualist view from the scenario, and Gödel's results reduced

Clearly ontological-metaphysical positions constrain to take particular stances in other domains, such as epistemology, semantics, methodology, of the philosophical discourse. We are interested in methodology. The link of the metaphysical problem with the analysis of the axiomatic method is immediate.[2] It was established already from the first of the mathematical schools involved in the debate, logicism, and took place on arithmetic, which in the next section we will see to be the fundamental theory among the axiomatics. In fact the modernist attention for a sharp ontological-metaphysical position on the nature of the mathematical entities produced as a consequence the rigorous definitions of natural numbers, which from millennia had been assumed and worked out without any trouble. In shifting the attention from the metaphysical/ontological problem on the nature of numbers to the definitions of the natural number concept we are already in the domain of the analysis of the axiomatic method because one of the three factors involved in an axiomatic system is exactly the nature and the role of the definitions, the others being the status of the axioms and the deduction of the theorems by them.

There were two ways by which the definition of naturals was achieved. Both ways find their roots in Cantor's work on cardinal and ordinal arithmetic, but restricted respectively to $\aleph_0$ and ω, that is to say, to the domain of the infinity of the finite numbers. The main protagonists of the enterprise of the definition of the natural numbers were Frege [19], Dedekind [10], and Peano [47]. Frege adopted a cardinal-oriented definition while Dedekind and Peano adopted an ordinal-oriented definition of natural numbers.

Frege's idea was that mathematics had to be founded on logic, thus he held the view that numbers are logical entities. To achieve the foundation of mathematics on logic, Frege had first to revolutionize logic [18] and then he had to show that numbers are logical entities [19] and that the principles of mathematics follow by logical chains of deductive reasoning from logical principles [20, 21]. He had a foundational project that we can call bottom-up in the sense that first one must derive natural numbers and axioms of

drastically the power of capturing an ontology in a formal system.

[2] The elements for the analysis of the axiomatic method emerge from the positions of two of the three mathematical schools involved in the debate, logicism and formalism. Intuitionism, with its emphasis on intuition, on the free activity or creation of the mathematician, and with its solipsism, held a position of criticism with respect to the axiomatic method [6]. Moreover intuitionism is considered by [24] as not representative of modernism but rather as representative of the old view and style of doing mathematics and as such it does not enter in our analysis of the modern style of the axiomatic method.

arithmetic by means of rules and logical principles and then one needs to found analysis on arithmetic, which was possible given the arithmetization of analysis [24], thus reducing mathematics to logic.[3]

Nevertheless, as it is well known, Russell's paradox provoked the failure of Frege's system and so the derivation of the axioms of arithmetic from logical principles which Frege achieved in [20, 21] was meaningless since from a contradiction it is possible to derive everything. Russell, more than any other, tried to save the logicist view, for which the axioms of arithmetics and in general the principle of mathematics are logical principles, from his own paradox. For that he wrote *Principia* [66] and invented the theory of types [50, 51, 66]. But in order to achieve from logic the derivation of the axioms of arithmetic, which we'll see next, the hard thing that Russell had to show was the derivation of the axiom of infinity, which asserts that there is an infinite set of naturals. But the existence of an infinite set of objects is a problematic thing to obtain by logic only. Here began for Russell a long way that did lead to link the axiom of infinity to the theory of types in his *Principia*, on which we do not linger in this thesis. Nowadays it can hardly be said that Russell did succeed in the logical derivation of the existence of the actual infinite.

Anyway we could rephrase Frege's 'cardinal-oriented definition' of natural numbers as follows:

$0 = \{X | x \neq x\}$;

$1 = \{X | \exists x \in X \wedge \forall y \in X (x = y)\}$;

$2 = \{X | \exists x, y \in X, x \neq y \wedge \forall z \in X (x = z \vee y = z)\}$; and so on;

But the problem is not with the definition as such, which we still use, but with the role of the definitions in the axiomatic system. The crucial point is that Frege and Russell's idea that numbers can be defined independently from the axioms and that their meaning is their logical meaning and that from these concepts the axioms of arithmetic can be derived by means of deductive reasoning was wrong. If that had been the case the axiomatic theories would have been found once and for all on a sure, certain, and completely deductive method. In conclusion the idea that Frege and Russell propounded of the axiomatic method was that axioms are true propositions derived from basic logical definitions. Given that the theorems are derived by

[3] Whereas, in the best of the Kantian tradition, geometry, for Frege, was founded on spatial intuition. We suspect that Frege was unaware of non-Euclidean geometries although we don't have any reference to support this.

the axioms and by deductive chains of reasoning also the theorems are true. In this way certainty, the target of modernism, would have been achieved in mathematics by the axiomatic method. But this idea of the method revealed paradoxical. Nevertheless the destiny of the axiomatic method did not end up with the logicist's failure and the search for certainty was still a work in progress.
Dedekind first [10] and Peano later [47] defined natural numbers in a different approach with respect to the cardinal approach of Frege. In fact, in Dedekind-Peano's axiomatic systems, the order of the elements in the sequence of natural numbers is more fundamental than the size of those elements. We do not present the original Dedekind-Peano axiomatizations but, following [27], we rephrase them in an efficient and elegant way as it is needed in a set theoretic construction of number systems. We call it $T_{PA_{1-8}}$:

T_{PA_1} There exists an infinite set $\mathbb{N}$ and a function $s : \mathbb{N} \to \mathbb{N}$;
T_{PA_2} The function s is not surjective: There exists $0 \in \mathbb{N}$ such that $s(n) \neq 0$ for every $n \in \mathbb{N}$;
T_{PA_3} The function s is injective: If $s(m) = s(n)$, then m=n;
T_{PA_4} If $F \subseteq \mathbb{N}$, and $0 \in F$, and if for every n, if $n \in F$ then $s(n) \in F$, then $F = \mathbb{N}$.

The first axiom asserts the existence of the infinite set of natural numbers. The second and the third axioms, while defining the successor function, respectively assert the existence of 0, which is not successor of any number or equivalently it has no predecessor or it is the least number, and assert that two different numbers cannot have the same successor. The third axiom implies that n and $s(n)$ are not the same number which would be the case if two different numbers have the same successor.[4] So the third axiom asserts the existence of an infinite series which is also guaranteed by the first axiom. The fourth axiom is known as the principle of mathematical induction, PMI, and it is the base for developing number theory. It can be proved that the well-ordering of the naturals implies the PMI. This fact makes transparent an ordinal approach to the definition of naturals because ordinals are a universal system for representing wellordering. The result is that natural numbers are just finite ordinals. Now, since we know that in the finite case ordinal and cardinal numbers depict the same elements or are different characterizations of the same elements, we can see that the ordinal-oriented definition of naturals does not exclude the cardinal-oriented one and so we can refer to naturals indistinguishably as ω or $\aleph_0$.

[4]In fact if $n = s(n)$ then n and $n - 1$ have the same successor.

Now we have to define the operations and add them as axioms.[5] It is customary today to define operations by recursion. It suffices to define the recursive functions for addition and multiplication, as the other operations can be obtained from them. Addition is defined as follows:

T_{PA_5} $x + 0 = 0$
T_{PA_6} $x + s(y) = (x + y)s$

and multiplication:

T_{PA_7} $x \times 0 = 0$
T_{PA_8} $x \times s(y) = (x \times y) + x$

At this point our axiomatic system is ready. It consists of three undefined ideas or primitive concepts, 0, number, and successor, and 8 axioms. Starting from them number theory can be effectively developed [27]. We use to say that this axiomatic system allows to develop arithmetic 'from with in', i.e., it allows to prove the theorems of arithmetic by the axioms.
But it should be noted that arithmetic does not need an axiomatization to be developed. In fact it had been developed from millennia without any axiomatization [43]. So to speak, the axiomatic system of arithmetic does not come first but it builds on an already well developed field of knowledge. In fact the axiomatic method, with respect to arithmetic, is a method which serves the purpose of easily teaching and transmitting the knowledge about the arithmetical domain, of conferring rigor, elegance, and hopefully certainty to the arithmetical knowledge, and therefore of founding the arithmetical knowledge on a number of principles or axioms or fundamental propositions from which the theory can be developed from with in. For these reasons we maintain that the axiomatic method is a method of foundation for a particular science and so we say that its foundational feature is 'particular'.[6]
Note that the axioms satisfy infinitely many interpretations. For example let's agree that 0 means 1, then starting the series of naturals with 1 all the axioms are satisfied. Next start with 2, the same holds, and so forth.

[5] Dedekind and Peano were able to effectively derive the operations by the axioms in their systems. But we'll follow the modern trend and add them as axioms without complication.

[6] In fact we will see in the next section and in the next chapter that the axioms of arithmetic configure not only as foundation of the arithmetical knowledge but as the foundation of mathematics and science in general, and than we will say that the foundational feature of the axiomatic method is 'general'.

This just tells us that every set of objects satisfying the axioms is a representative of the natural numbers. Russell [50] considered this as a weakness of Peano's system and that's why he wanted to define numbers as logical entities, because only as logical entities would numbers have had a precise meaning, their logical meaning, that evidently our axiomatic system cannot guarantee. So, while in Frege and Russell's idea of axiomatic arithmetic the meaning of the terms was their logical meaning and the axioms had to be derived from logical deduction from basic definitions, in T_{PA} the meaning of the terms is not explicitly defined but it is implicit in the axioms which in fact define the terms in an implicit way.

We could say that in the logicist's idea of axiomatics the definitions come first and then the axioms follow by deduction and for this fact they are true whereas in our axiomatic system of arithmetic the axioms and definitions come together. So our question is: If the axioms of arithmetic are not obtained by deduction what is their status? Once intuition and evidence have been ruled out the only available alternative is that axioms are just hypotheses.

Evidence and intuition cannot be methodological directives even for making sure of the axioms of a so simple theory as arithmetic, in fact among the axioms there is the axiom of infinity which asserts the existence of an actual infinite totality and the actual infinite was (and so still is) a counterintuitive notion. Moreover in the epoch of modernism any appeal to intuition and evidence would have been regarded with suspicion given Russell and other's paradoxes and the modern style axiomatic method was emerging to make sure that mathematical knowledge was certain knowledge and as such it did not need any appeal to intuition and evidence. Moreover for more complex theories than arithmetic, such as the geometrical and the physical theories, evidence and intuition, some decades later, were put out of context because theory laden [64].

To consider axioms as hypotheses was in the project of Hilbert's axiomatics. Cantini [6] names Hilbert's project 'the new axiomatics' and points out as Hilbert in [29] suggested that, in general, any system of axioms must contain the exact description of the basic relations which hold between the simplest concepts of the theory, that is, the axioms must be at the same time definitions of basic concepts, which in fact is what we have seen happening in T_{PA}. Relying on Cantini, to this, in Hilbert's meditation, follows the most important question about an axiomatic system which is built in such a way, the problem of consistency: to show that the set of axioms does not produce contradiction. The point is that when axioms are derived from nowhere and implicitly define the basic concepts of a theory we cannot assume their being true and so in order to 'believe the axioms' one needs the proof that the system of axioms in consistent.[7] Otherwise if one is sure that the axioms are true why would him need to show their consistency given that from true axioms would follow true theorems? Clearly the question of coherence makes sense only when a threat of uncertainty about the axioms is present. Uncertainty about the axioms means that they are for us just hypotheses. So it happened that in the epoch of modernism the inventor of the modern style axiomatic method struggled for certainty and his way out from uncertainty was the search for coherence. Moreover Hilbert, being nominalist, sustained that mathematical entities are fictional entities and their existence can be assumed only once the set of axiom is consistent, otherwise, if the system of axiom is not coherent, one must accept the non existence of the mathematical objects [6]. So the problem of consistency of an axiomatic system is linked, on the one hand, to the consideration of the axioms as hypotheses and, on the other hand, to the existence of the mathematical objects, and these two are the two faces of the same coin since axioms as hypotheses define implicitly the basic objects of the theory.

In conclusion let's remark the three futures which have emerged from our analysis of the axiomatic system of arithmetic. (1) The axiomatization of arithmetic builds on an already well developed arithmetical knowledge. We call this the 'building on a field of knowledge feature' of the axiomatic method. (2) The axiomatization of arithmetics serves the purpose of founding the arithmetical knowledge on a number of fundamental propositions or axioms, and as such it serves the purpose of giving elegance, certainty, and rigor, with respect to the modernist canons, to the arithmetical knowledge. We call this the 'particular foundational feature' of the axiomatic method. (3) We have seen that Frege and Russell's idea that the basic definitions of arithmetic are derived from logic and that the axioms of arithmetic are derived from the basic definitions plus logic and that therefore axiomatic arithmetic is a purely deductive theory is wrong. Otherwise we have seen that axioms are starting points, neither true or false, but just hypotheses which implicitly define the basic concepts of the theory and by which all the theorems can be deduced by means of mathematical proofs, that is the the-

[7]Call this 'the strong way' to believe the axioms. At the end of the next chapter we will come beck to this and we will distinguish it from the 'weak way' to believe the axioms.

ory can be developed from with in. We call this the 'hypothetical-deductive future' of the axiomatic method.[8] We have seen also that the axioms of arithmetic being just hypotheses and defining implicitly the basic concepts of the theory then the problem of consistency arises since to believe the axioms one needs the proof that they do not produce a contradiction.
In the next section we persist on point (3) focusing on the problem of coherence of arithmetic with respect to the coherence of the other theories. In doing so a strengthening and generalization of point (2) emerges.

1.2 Stage 1: The Place of Arithmetic in the Axiomatics

Interest in number theory was the result of a long path in the history of mathematics. In brief, mathematics was born with a dichotomy represented by arithmetic and geometry conceived respectively as being the study of quantity and measure, or equivalently of discrete and continuous, of number and figure, or of time and space. With the passing of time arithmetic became algebra which in nineteenth century was divided into abstract algebra (Boole, De Morgan) and number theory (Cantor, Dedekind, Peano). Geometry, already well developed by the Greeks in the synthetic approach, did not progress much further until Descartes introduced the coordinates and the analytic approach on which Newton and Leibniz based the infinitesimal calculus. In the nineteenth century, while mathematics appeared still divided in two parts, algebra and analysis, where analysis represented the evolution of geometry, once again the study of discrete and continuous magnitudes, the arithmetization of analysis took place. The enterprise consisted in the reduction of the continuous to the discrete, or equivalently of analysis to arithmetic, that is, of real numbers to rationals and, rationals being ordered pairs of integers, to integers and, by the bijection between integers and naturals, to naturals. The program of the arithmetization was quickly fulfilled: Descartes transformed synthetic geometry by the analytic approach, Newton and Leibniz with calculus reduced it to analysis, and analysis was reduced to arithmetic by arithmetization. But what are numbers in general? What

[8]From this feature the axiomatic theories at stage 1 get their name, i.e., hypothetical-deductive theories. Features (1) and (2) can also be recovered in feature (3), but for the purpose of our analysis we keep them distinguished.

was known about natural numbers?

These questions were fundamental because if all of mathematics can be reduced to arithmetic, then one has to be very precise in the definition of naturals. As we have seen in the previous section, there were two ways. One consisted in reducing numbers to logic and in deriving arithmetical axioms from logical definitions, thus founding mathematics on logic. If that had been successful, then mathematics would have been founded once and for all as a purely deductive system whose axioms are truths and their meaning is their logical meaning. This was the logicist program. But as we have seen the derivation of the axioms of arithmetic by purely logical means was unachievable.[9] The other way was to consider numbers as implicitly defined by the axioms of arithmetic which have to be considered as starting points, as hypotheses neither true or false, from which to develop the theory. This was the trend that was taking place in the mathematics of the nineteenth century after the discovery of non-Euclidean geometries. In fact the axioms of non-Euclidean geometries were in evident contrast with what were considered evident truths, such as, for example, the fact that the shortest distance between two points is a straight line, or with what were assumed indisputable truth from millennia, such as, for example, the Pythagorean theorem which is equivalent to the fifth postulate of Euclidean geometry.

The new idea that non-Euclidean geometries had raised first was that axioms are not true or false but just hypotheses which are used to derive the theorems and implicitly define the concepts of the theory, as for example line, triangle, etc. In the previous section we have seen that Hilbert was the first who theorized the axioms as hypotheses. Being axioms just hypotheses then the axiomatic method is not a purely deductive method because the axioms are deduced from nowhere and then we call the axiomatic method the hypothetical-deductive method, hypothetical for the status of the axioms and deductive for the deduction of the theorems by them. The axiomatic method as hypothetical-deductive method represents for us stage 1 of the development of the axiomatic method. At this stage the axioms are hypotheses neither true nor false from which to derive the theorems; they define implicitly the primitive concepts which do not have meaning until an interpretation is given for them.

[9]The neo-logicist program thinks that this can still be done. Nevertheless, for the time being, nobody has been able to show it without any doubt and our opinion is that it cannot be done.

At this point a question arise: If the axioms are not truths, then do the entities the axioms are about really exist?

The question arises because if the axioms are not evident truths then the existence of the entities the theory is about can be doubted, especially if different theories give different and sometimes, as in the case of non-Euclidean geometries, contradictory descriptions of them.

The problem of existence of the mathematical entities the axiomatic theory is about is solvable by showing the coherence of the theory so axiomatized. In fact, if the theory is consistent, then we can convince ourself that the entities the theory talks about could exist and that we are justified to believe their existence. Otherwise if the theory is inconsistent everything can be proved and thus the worry is that the entities the theory talks about do not really exist.

In principle consistency can be shown by giving a model for the theory. In fact the question of coherence that the non-Euclidean geometries had raised was solvable by showing a model for them. For example, hyperbolic geometry was accepted by the mathematical community as a (coherent) mathematical theory only when Beltrami's pseudosphere was shown to be its model [44]. After that other models for it, such as Klein's model and Poincare's model, were offered. Likewise the sphere and semisphere represent the models for spheric and elliptic geometry and thus they represent their consistency [44]. But this is quite strange since the sphere is an object of Euclidean space and thus in order to show the consistency of a non-Euclidean geometry we appeal to Euclidean geometry which contradicts the non-Euclidean [40].

Now consider the case of a model for the entire geometry as axiomatized by Hilbert [28]. To show a model for that would be an almost impossible task because its model would be space as a whole and apart from our intuitions about it, which do not represent anymore a trustable source for mathematics, we do not have such a model. In fact Hilbert begun his axiomatic production with the axiomatization of Geometry [28]. In the nineteenth century, in the context of the enterprise of the foundations of geometry [24], important works towards the axiomatization had been already done. The axiom of completeness and the Pasch axiom were stated and the axiomatizations of geometry of Pasch [46] and Pieri [48] were already available. Hilbert's novelty in this context consisted in addressing the problem of consistency. He first understood that in geometry the proof of the consistency can be achieved building a field of number such that the relations between the numbers in this field corresponds to the geometrical axioms [6]. So an eventual contradiction in the proofs from the axioms must hold also in the arithmetic of such a field. In this way the proof of consistency of the axioms of geometry must correspond to the proof of consistency of the axioms of analysis.[10] This is a relative notion of consistency. Nevertheless also a model for analysis is a problematic affair and such a model would guarantee only a relative proof of consistency since reals reduce to rationals and rationals to naturals. But then the problem reverts to the consistency of the arithmetic of the natural numbers. In fact the consistency of the arithmetic of the naturals does not collapse to the consistency of another theory and as such it constitutes an absolute proof of consistency for all of axiomatics. Axiomatics configures a sort of net in which each theory is connected to each other and all together cover the domain of mathematics. The consistency of axiomatics reverts to the consistency of arithmetic.

Let's remark that mathematics pictured as an axiomatic system [38] has its foundation on the consistency of arithmetic. So we would like to have an absolute proof of consistency of arithmetic. The first thought would be to give a model for it. But a model for arithmetic is a model with an infinite domain, the domain of the natural numbers. For example consider the axiom of arithmetic that say that every two different numbers do not have the same successor. We cannot establish the truth of this axiom by testing case by case. So this is really an impasse that needs to be settled. How? We already know the answer is to formalize the axiomatic theory of arithmetic at stage 1 in a logical language, as Gödel did, and answer the problem of coherence in this language. But this will be an affair for the next chapter. For the time being let's point out that we have reached a new face of the foundational feature of the axiomatic method. In fact in the previous section we have pointed out that the axiomatic method, with respect to axiomatic arithmetic, is a foundational method in the particular sense of point (2). In this section we have seen that arithmetic is the fundamental science among the axiomatics because all of the mathematical axiomatic theories reduce to the consistency of axiomatic arithmetic. From this it follows that the axiomatic method is a foundational method not only in a particular sense, in so much it serves as foundation of a particular science, but also in a general sense, in so much it presents a 'tension'[11] toward the foundation of all the

[10]In [29, 6] Hilbert addressed also the problem of the axiomatization of physics maintaining that also the consistency of the physical theories must be reduced to the consistency of analysis because to points in space can be assigned reals.

[11]We say "presents a tension" because we are not yet able, at stage 1 of the method, to

axiomatic mathematical theories.
In the next chapter we investigate the general foundational feature of the axiomatic method and we extend it to science in general. Before than that, in the next section, we occupy of the axiomatization of probability theory as a case study in which we point out all the features of the axiomatic method so far outlined. Let's remind that they are three: (1) the building on feature; (2) the particular/general foundational feature; (3) the hypothetical-deductive feature.

1.3 A case-study: The Axiomatization of Probability Theory

In this section we analyze Kolmogorov's axiomatization [31] of probability theory to show to the reader that each time he faces with an axiomatic theory he can recognize whether it is a system at stage 1 of the modern style of the axiomatic method by investigating if the theory presents the features we have so far introduced. So in the next three paragraphs we make clear that the axiomatization of probability theory does not come first but it builds on a preexisting field of knowledge, that axiomatic probability serves as a particular foundation for probability theory, that axiomatic probability theory is an hypothetical-deductive system.

Probability theory was born in the sixteenth century in the attempts made to analyze games of chance and it was developed in its theoretical part during the seventeenth century especially by Pierre de Fermat, Blaise Pascal, and Christiaan Huygens. In the eighteenth century and at the beginning of the nineteenth century, with the work of Thomas Bayes and Pierre-Simon Laplace, it was a well-developed theory [30, 33, 25, 55]. Especially with Laplace [33], who defined the probability of an event, $\mu(E)$, as the ratio of the number of favorable, f, and possible, p, i.e., $\mu(E)=f/p$, the classical or *apriori* or aleatoric interpretation of probability was established.
In the nineteenth century the major success of probability developed along two main channels and from those it reaches almost every field of scientific knowledge. These two channels are physics and logic.

---

settle the problem of the general foundation.

Probability entered physics by the statistical mechanics of Ludwig Boltzmann [3]. In statistical mechanics entropy is the link to obtain macroscopic information from microscopic configurations. Intuitively we imagine that to a certain macroscopic condition of equilibrium of a system there corresponds a multitude of microscopic configurations. If we look at a physical system from a microscopic point of view we can describe it with a probability distribution which gives, for any microscopic configuration, the possibility to observe it with a certain probability. It was Boltzmann [3, 8] who derived the relation between entropy and probability distribution: $S = k \log W$ where S is entropy, k is a constant (called by Plank the Boltzmann constant), and W is the number of possible ways (microstates) in which we can obtain the macrostate in consideration. For Boltzmann a system develops spontaneously towards the more probable configurations and, because the spontaneous evolution happens towards states in which entropy and disorder increase, it follows that a new face of entropy is linked to probability: *The real transformations are characterized by an increase of disorder, entropy, and probability.* With the combination of the discovery of Boltzmann with the principles of thermodynamics, we arrive at a very important concept: *In any real and spontaneous process the disorder of the universe increases.*
Together with physics, the other important channel that made probability ready and available in the framework of contemporary science was mathematical logic. The date of the birth of mathematical logic is usually referred to George Boole's *Investigation of the laws of thought, on which are founded the mathematical theories of logic and probabilities* [4], where Boole saw mathematical logic and probability theory intimately linked each other.[12] With respect to probability the insightful point to be recognized in Boole's work is that, once we consider propositional logic as a Boolean algebra[13] and we consider 0, 1, to represent true and false and $+, \cdot, -$, to represent respectively disjunction, conjunction, and negation, then we can consider also 0, 1, respectively as impossibility and certainty, and hence we can use the propositions of logic to represent reasoning about certainty-uncertainty.
Starting with Boole's and Boltzmann's works, which attracted the attention of many, probability became the subject of systematic mathematical studies

[12]Surprisingly enough, Boole published nothing more on logic after the *Investigation of the laws of thought* but he continued to publish on probability.

[13]That is as a structure $B = (B, 0, 1, +, \cdot, -)$, where B is a set; $0, 1, \in B$; $+$, $\cdot$, are binary operations and $-$ is a unary operation on B satisfying the commutative, associative, distributive, idempotence, and de Morgan laws.

and the need for rigor in the mathematics of the nineteenth and twentieth century evidently applied also to it [24, 23]. In this period the philosophical interpretations of probability were settled [64].

The classical aleatoric interpretation of Laplace [33] was objective but it suffered an inconvenience. In fact if probability is the ratio between favorable and possible cases it is not always possible to know them apriori. Beside this, are possible cases also equiprobable? An answer to this question can be given by the principle of sufficient reason which, roughly speaking, says that when there are no reasons to assign different probability measures to two events then the same probability can be attributed to them. But the problem with this answer is that it endorses a subjective interpretation of probability and so it is not an answer to the problem of the objective-aleatoric view.

In the first part of the twentieth century the systematic studies on probability endorsed two different perspectives, the frequentist and the subjectivist views. The frequentist or statistical or *aposteriori* view was introduced by von Mises [65]. It is an objective and aposteriori view since the probability is given by the observation of the events that already happened. We say that the *frequency* of an event, $f(E)$, is the ratio of the number of the cases in which the event happened, n, and the number of the trials, N, i.e., $f(E) = n/N$. The probability of an event, $\mu(E)$, is the limit of such a frequency, i.e., $\mu(E) = \lim_{N\to\infty} f(E)$. The subjectivist view was developed especially by De Finetti [9]. It consists in considering the probability of an aleatoric event as the belief, based on some evidence, that a subject has of that event. In this context probability discovers again its oldest inspiration as a theory about games of chance.

The systematization of probability theory culminated, two years after Gödel's limitative results, with the foundations of modern probability in Kolmogorov's axiomatization [31], which is a strange fate if you think that the epoch of the search for certainty [24, 23] concluded with the axiomatization of uncertainty.

Our historico-conceptual excursus should be enough to point out as feature (1) of the modern style axiomatic method at stage 1 is respected in Kolmogorov's axiomatization of probability theory. Indeed we would like to remark that the theory of probability reached an axiomatic arrangement only at a very advanced stage of its development, that is only when the framework was so mature to be susceptible of philosophical interest and interpretations.[14] The same discourse holds for all the axiomatic theories,

[14]Indeed we know only one philosophical interpretation which is subsequent to Kol-

think of axiomatic set theory, or of axiomatic arithmetic, or of axiomatic quantum mechanics which only recently, once it has become a well-trusted and differently interpretable theory [32], scientists are trying to axiomatize. It should be noted that the feature of the axiomatic method at stage 1 of building on a already well developed field of knowledge implies that the axiomatic method is not a method of discovery, i.e., an axiomatization serves the purpose of systematizing, transmitting, teaching, and further developing the theory but it does not serve the purpose of discovering it. And this introduce the next feature of the modern style axiomatic method at stage 1, that of being a method of foundation.

In general the calculus of probabilities studies situations in which the outcomes are indeterministic and although its axiomatization does not answer the philosophical question about what probability is, it gives an extremely simple setting in which to deal with the study of the mathematical models and concepts the calculus applies to. So let's see the axioms of Kolmogorov probability theory, T_K:

T_{K_1} F is a field of sets;
T_{K_2} F contains the set S;
T_{K_3} To each set in F is assigned a non negative real number $\mu(A)$. This is called the probability of event A;
T_{K_4} $\mu(S)$ equals 1;
T_{K_5} Given two incompatible events A, B, then $\mu(A \vee B) = \mu(A) + \mu(B)$

Having introduced the axioms it is an easy task to remark that the axiomatization of probability serves a foundational purpose, which is feature (2) of the axiomatic theories at stage 1 in our analysis. And of course it does that since you can see that it is at the stake a number of very basic and fundamental propositions by which all the theorems can be derived so that the theory can be developed, as we use to say, from with in. Indeed already Kolmogorov was aware of his foundational enterprise in his axiomatization, as you can see from the title of Kolmogorov's book [31]. Today, behind Kolmogorov's awareness, we can say that Kolmogorov's effectively did succeed in the foundational enterprise for two reasons: First because we can define almost all the probabilistic concepts starting from the basic definitions of

mogorov's axiomatization, Popper's interpretation, which is meant to recover the role of probability in quantum mechanics [64] and on which we do not linger in this thesis.

this theory, from basic measures, to independence, evidence and conditioning, updating, Bayesian networks, fuzzy control systems, and so forth. We can even implement quantum mechanics probabilistic experiments if we use linear algebra and complex numbers and matrices to represent probabilities [26]; and second because the practice of using Kolmogorov's axiomatization has established that all the three philosophical interpretation of probability, that we have introduced in the previous paragraph, are respected in his axiomatic system [26, 64].

Our next point is to show that T_K is a hypothetical-deductive system, which is feature (3) of the axiomatic theories at stage 1 in our analysis. With respect to (3) note that in the calculus of probabilities, as axiomatized by Kolmogorov, the definitions are not stated explicitly but the axioms implicitly define the basic concepts of the theory. So the definitions are only matter of fixing the terminology, which is a conventional matter. The basics terms are the following: The calculus of probability studies situations in which the outcome is uncertain and we call *aleatoric experiment* any operation which outcome is uncertain. An *aleatoric event*, A, is the result of an aleatoric experiment. The set of the outcomes is called a *sample space* S, and an *event* $A \subseteq S$. There is a function $\mu(x)$ in S, which assign probabilities to events, such that it associates to each event in S a real number, $\mu : E \in S \to \mu(E) \in \mathbb{R}$. An event is *certain* if its probability is 1 and *impossible* if its probability is 0. Two events A, B are *compatible* if they can be verified together, or *incompatible* if they cannot be verified together. Compatible events A, B can be *dependent*, if the presence of A modifies the probability of verification of B. This last basic term is implicitly defined in Bayes' theorem[15] which is a consequences of the axioms. All the other terms can be defined by means of mathematical definitions only.[16]
So at this point we have seen that T_K respects an important characteristic of the hypothetical-deductive feature of the modern style of the axiomatic method at stage 1, the characteristic that axioms implicitly define the basic concepts of the theory. There remain to make clear that the axioms of probability are just hypotheses.

[15]Given two compatible and dependent events, the probability of B once A is verified is $\mu(B/A) = \mu(A \wedge B)/\mu(A)$.

[16]Such definitions are composed of a *definiendum* and a *definiens*. The meaning of the definiendum is given by the terms already defined and which constitute the definiens. The relation between definiendum and definiens is that of equivalence.

At the first sight it could seem strange to consider the axioms of probability as hypotheses since they are so simple that they could appear evident and quite intuitive. But, once again, evidence and intuition are not trustable sources of mathematical knowledge and, although they may serve epistemological purposes, they certainly cannot be methodological guidelines. Moreover consider that events are sets and that the theory of set is highly exposed to paradoxes. Further consider that to events are associated reals numbers which are an uncountable set. There are plenty of reason for not considering the axioms of probability as evident truths. Now consider also that in an axiomatic system in which the axioms are at the same time definitions of the basic terms the problem of coherence arise since one can maintain that if the system is inconsistent that the entities it talks about do not really exist. But the question of coherence makes sense only if axioms are considered as hypotheses, neither true or false, by which to develop the theory. Otherwise, if axioms are true, why would we ask for coherence since from true propositions, if the mathematical proofs are correct, follow true propositions? In fact the question of coherence was made explicit by Kolmogorov soon after he stated the axioms [31, p. 2]: "Our system of Axioms I–V is *consistent*. This is provided by the following example. Let E (our S simple sample space) consist of the single element T and let's F consist of E and of the null set 0. $P(E)$ (our $\mu(S)$) is then set equal to 1 and $P(0)$ (our $\mu(0)$)is the set equal to 0."
As you can see Kolmogorov solves the problem of consistency by giving a model that satisfies the axioms and, in our analysis, this makes explicit that he himself considered the axioms as hypotheses. But this brings us to return to feature (2) of our analysis. In fact in §1.2 we distinguished a 'particular foundational feature' from a 'general foundational feature'. So let's note that axiomatic probability theory, as well as all the hypothetical-deductive systems, needs a proof of relative consistency, which is achieved showing a model that satisfies the axioms. Now note that we maintained that the axioms serve as foundation of probability theory and so the consistency of the axioms, besides making explicit the hypothetical-deductive feature of the axiomatic system which was feature (3), serves also the foundational feature because if the axioms are consistent then we can 'believe them' and so be sure of our foundation. So in this 'particular' sense of foundation the axiomatic method can be considered also as a method of justification in the sense that if the system is consistent we are justified to 'believe the axioms'. We call this the 'strong way' to believe the axioms in so much we believe the axioms for a methodological feature, coherence. So, in our jargon, the particular

foundational feature implies a strong way of justification. At the end of the next chapter we will distinguish the 'strong way' from the 'weak way' of believing the axioms which is at the stake in the analyses of the kind of [35, 36] and which, conversely, is implied from the general foundational feature of the axiomatic method. For the time being let's note that an axiomatic system such as probability theory, as well as all the hypothetical deductive systems, 'struggles' also for an absolute proof of consistency since we associate reals to events, and reals can be reduced to rationals, and rationals to integers, and integers to naturals. So it is needed the arithmetic of the natural to be non inconsistent, in this sense it is needed what we called in §1.2 the 'general foundational feature' of the axiomatic method, otherwise everything would crash. As we know, after Gödel, we cannot show the consistency of a so simple theory as Peano arithmetic but this does not mean that it is inconsistent. This will be one of the central problems of the next chapter.

Chapter 2

FORMAL THEORIES

2.1 Formalization and Elementary Arithmetic

An hypothetical-deductive theory, as for example T_{PA} or T_K, is not a formal theory. But it can be formalized. In fact what we formalize is an hypothetical-deductive theory. So we say that a formal theory "builds on" an hypothetical-deductive theory. Equivalently we say that the axiomatic method at stage 2 builds a formal theory out of an hypothetical-deductive theory.

Clearly the axiomatic method at stage 2, building on an hypothetical-deductive theory, inherits all the three features, which we have seen in the previous chapter, of the method at stage 1. But, differently from the axiomatic method at stage 1, the axiomatic method at stage 2 serves also the purpose of "investigating the theories from without", i.e., of investigating the mataproperties of the axiomatic systems.[1] We are manly interested in the metaproperty of coherence, and in a minor measure in that of completeness. In our analysis the coherence of arithmetic constitutes the general foundational feature of the axiomatic method. In fact we pointed out that the axiomatic method at stage 1, together with its particular foundational feature, has a tension toward a general foundational feature which, as we saw in the previous chapter, reduces to the problem of coherence of arithmetic. But we were not able to investigate the general foundational feature. At stage 2 we can in-

[1]We use the expression "investigating the theory from without" to mark the difference with the particular foundational feature of the axiomatic method at stage 1 which, as we said, had the characteristic of "developing the theory from with in", i.e., of deriving the theorems from a number of fundamental propositions or axioms.

vestigate the problem of the coherence of arithmetic in the formal theory of arithmetic[2] and therefore we can investigate the general foundational feature of the axiomatic method.

The formalization, or equivalently the axiomatic method at stage 2, consists in the embedding of the hypothetical-deductive theories in a logical language. In our analysis we choose classical logic because it is the strongest logic we know, in the sense that a stronger theory than classical logic is inconsistent. We choose a first-order language with identity with its items[3]: variables; the connectives; the two quantifiers; some primitive terms, for example belongs is the undefined or primitive term of set theory; zero, number, successor, of arithmetic; or betweenness and equidistance of Euclidean geometry in Tarski' style [62]. The simplicity but also the expressive limit of this language consists in the fact that it allows quantification on individual variables only.

The formalization consists in making explicit the extralogical constants corresponding to the primitive concepts of the hypothetical-deductive theory T (at stage 1), which constitute the alphabet $AL_{\mathbf{T}}$ of the theory $\mathbf{T}$ (at stage 2). The language of the formal theory $LA_{\mathbf{T}}$ is constituted by the well-formed formulas (*wff*) defined inductively as in first-order logic but containing as extralogical constants only those of $AL_{\mathbf{T}}$. A subset of the wff, $Ax_{\mathbf{T}}$, is taken to represent the axioms of the theory.

So in order to formalize an hypothetical-theory all that we need is to define $AL_{\mathbf{T}}$ and $Ax_{\mathbf{T}}$ in a first-order language. The deductive apparatus of the formal theory is the calculus of first-order logic and it serves the purpose of formalizing and making explicit the deductive rules which are there but they are implicit in the hypothetical-deductive theory we want to formalize. The particular choice of the calculus does not affect the development of the theory and we just assume one of them at our disposal. Thanks to the first-order deductive apparatus of the formalized theory, every time we produce a proof we can inspect it and be sure that it is correct. In this way proofs themselves become very precise objects, logical objects. Moreover first-order calculus is sound and complete. Thus by correctness and completeness we have that $Ax_{\mathbf{T}} \vdash A$ iff $Ax_{\mathbf{T}} \models A$.[4]

[2] This was effectively done by Gödel.

[3] This is not necessary in order to build a formal theory out of an hypothetical-deductive theory and some times it would be not sufficient. However first-order logic suffices for our purposes and it is the most used logic to formalize a theory.

[4] Naturally, with respect to the precision and rigor achieved by the formalization, formal theories constitute a great success for the modernist project.

A theory **T** is an *axiomatic formal theory* when we can decide if a formula is a wff or not; what the truth condition of any sentence is; which of the wff are the axioms of the theory; and which sets of wff count as proofs from the axioms.[5] The hypothetical deductive theories formalized in first-order logic become axiomatic formal theories and are called *elementary theories*.
In what follows we state the standard elementary theory of arithmetic, $\mathbf{T}_{PA}$. It is a first-order language on the structure $[N, 0, S, +, \times]$ where 0 is an individual constant and $S, +, \times$ are functional constant symbols. Its axioms are the following:

$\mathbf{T}_{PA_1}$ $\forall x(0 \neq Sx)$
$\mathbf{T}_{PA_2}$ $\forall x \forall y(Sx = sy \rightarrow x = y)$
$\mathbf{T}_{PA_3}$ $\forall x(x + 0 = x)$
$\mathbf{T}_{PA_4}$ $\forall x \forall y(x + Sy = S(x + y))$
$\mathbf{T}_{PA_5}$ $\forall x(x \times 0 = 0)$
$\mathbf{T}_{PA_6}$ $\forall x \forall y(x \times Sy = (x \times y) + x)$
$\mathbf{T}_{PA_7}$ $[(\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall x \varphi(x)]$

As it is evident $\mathbf{T}_{PA}$ is the formal version of the hypothetical-deductive system of arithmetic, T_{PA}, of §1.1. In general it is evident when a theory is a formal theory, one just notes that because of the logical language in which it is stated. It is clear also that behind a formal theory there is an hypothetical-deductive theory. One just takes out logic and what is left is an hypothetical-deductive theory.[6] On the other hand any hypothetical-deductive theory can be rephrased as a formal theory. It requires some representational work but surely it can always be done. Formalization is an ideal in the sense that it can be worked on and on and the care one puts in the formalization depends on what metaproperties of the theory one wants to address. In our analysis we are interested in the metapropseries of consistency and completeness. $\mathbf{T}_{PA}$ is equivalent to the arithmetic of the *Principia* [66] which Gödel proved to be incomplete and incompletable and of which Gödel proved that consistency cannot be proved [53]. So we rephrase Gödel's theorems as stating that a consistent and axiomatic formal theory which extends $\mathbf{T}_{PA}$ is incomplete and incompletable and its coherence cannot be shown.

[5] Note that the fact that we can decide if an array of strings of symbols is a proof or if a sentence is true or false does not imply that we can effectively decide the set of theorems or of truths of a formal axiomatic theory.

[6] One can always investigate if the theory is an hypothetical-deductive theory in the way we did it with Kolmogorov's probability theory, i.e., by investigating if the three feature of the axiomatic method at stage 1 are respected.

In §1.2 we explained that axiomatics is a sort of net which covers the domain of mathematics and that the consistency of axiomatics reverts to the consistency of arithmetic, and therefore the general foundational feature of the axiomatic method depends on the coherence of arithmetic. Now we have just pointed out that we have a proof that we cannot prove the consistency of axiomatic arithmetic. This is really a limit which needs to be settled. Before settling this problem we explain how the picture of axiomatics as a net which consistency reverts to arithmetic is extendable to science in general and after that we discuss the implications of Gödel's theorems for our general foundational feature of the axiomatic method.

2.2 To Axiomatize a Theory is to Define a Set-Theoretical Predicate

During the epoch of modernism already Hilbert raised the problem of the axiomatization of physics: It was the sixth of his open problems of mathematics [29].[7] In the first half of the twentieth century not only the axiomatization of physics but that of all scientific disciplines, in the framework of a formal philosophy of science of analytic inspiration, was pursued [64]. Protagonists of such an enterprise were the logical empiricists. Among them notoriously Rudolf Carnap, Ernst Nagel, and Hans Reichenbach were the promoters of the so called 'syntactic approach' to the axiomatization [39, 32]. In their project the axiomatization of the scientific theories had to be obtained in the same way in which from a mathematical hypothetical-deductive system is obtained a formal system, that is, by using a first-order language, by making explicit the extralogical constants of the theory, and by stating the axioms in the formal language. Clearly the additional thing to take care was the connection of the theory to the empirical world. The logical empiricists than thought to add some machinery to the formalization. The extralogical terms of the first-order language of the theory, $LA_{\mathbf{T}}$, had to be partitioned into two disjoint sets, one consisting of observational terms, OT, and the other of theoretical terms, TT. Correspondingly $LA_{\mathbf{T}}$ was divided in two sub-languages, $LA_{\mathbf{T}}OT$ and $LA_{\mathbf{T}}TT$, which together formed $LA_{\mathbf{T}}$, and which, of course, had

[7] Note that attempts to the axiomatization of physics were not new, as Newton's *Principia* [41] and Archimede's *Treatise* [12] testify. It is the style of the axiomatization, which we call modern style, that is new.

to contain mixed propositions. $LA_{\mathbf{T}}OT$ had to be interpreted assigning to each term an entity in the world. $LA_{\mathbf{T}}TT$ had to be interpreted in two ways: (a) by *theoretical postulates* or axioms which define the internal or implicit relations between the terms of $LA_{\mathbf{T}}TT$ without using the terms of $LA_{\mathbf{T}}OT$ and (b) by *bridge principles* which characterize mixed sentences and relate $LA_{\mathbf{T}}TT$ and $LA_{\mathbf{T}}OT$.
The syntactic approach ended up as a failure under the criticism of Quine, Hanson, and Kuhn [64] meanly because of the problem of observation which was considered theory laden [39, 32, 17]. We do not linger on this 'species' of the modern style at stage 2 of axiomatization of the scientific theories in this thesis.
In the fifty of the twenty century, once the epoch of modernism was already over [24][8], a different approach to the axiomatization of the theories was propounded. It is called the 'semantic approach' to mark the difference with the approach of the logical empiricists and because, roughly speaking, it considers a theory to be a set of models. This approach is still actual [39, 32]. Despite the fact that the modernist epoch in mathematics as an historical category does not fit well with the semantic approach to the axiomatization of the theories, from the conceptual point of view we consider the semantic approach as belonging to the modern style of the axiomatic method at stage 2 for two reasons: First because it makes use of the elementary theory of sets, which evidently is a theory at stage 2 of the method, and second because, as we will make clear, it embeds the general foundational feature of the modern style of the axiomatic method, which, as it should be clear by now, is the feature of the axiomatic method at stage 2 which our analysis in this chapter is all about.
The problem that we face in addressing the general foundational feature of the axiomatic method is to interconnect all theories to arithmetic. We solve the problem by considering the first-order formalized version of ZF set theory [11], call it $\mathbf{T}_{ST}$, and defining in it a predicate which embeds the axioms of the scientific theory we want to formalize.
The way in which the embedding of the scientific theories in $\mathbf{T}_{ST}$ is achieved is conveyed by the title of this section which is a slogan that has been formulated by Suppes [61, 60, 58, 57] and has been welcomed and developed with great

[8]The epoch of modernism concluded in the thirties of the twenty century with the advent of computation. In [24] it is sustained that the advent of computation opened a new phase which still endures today and which presents as its most important feature the concept of mechanization.

success by a community of scientists and philosophers of science [39].
The slogan can be explained as follows: Sets are the most abstract objects we can think of and thus, rather than going directly to formalize the scientific theories in a first-order language starting from some primitive notion from which it is known that the theory can be developed, we can take a scientific theory as a set of objects satisfying some axioms and define such a set as a one-place predicate in $\mathbf{T}_{ST}$. Such a predicate representing a theory is called a *Suppes predicate*.[9]
By transforming a theory into a set of the formal theory of sets, the theory takes a precise place in the hierarchy of $\mathbf{T}_{ST}$, i.e., it is one of the V_i of the hierarchy, and we can be sure that the primitive undefined notion from which it can be stated is $\in$ and that all the other notions can be defined by means of the theory definitions of first-order logic in the formal axiomatic system of set theory that now becomes $\mathbf{T}_{ST+S}$ where $_S$ represents the theory we have formalized by the definition of a set-theoretical predicate for it.

We consider first a variation of an example proposed by Suppes of the theory of groups [61]. AL_G is composed of a two place functional symbol $*$ and the constant a. $\mathbf{T}_G$ is composed of the following axioms:

$\mathbf{T}_{G_1}$ $\forall x \forall y \forall z (x * (y * z) = (x * y) * z)$
$\mathbf{T}_{G_1}$ $\forall x ((x * a) = x)$
$\mathbf{T}_{G_1}$ $\forall x \exists y (x * y) = a$

Next we define A to be a group with respect to the operation $*$, and the (identity) element a if and only if A is a set, $*$ is a function on A, $a \in A$, and for all x, y, z in A axioms $\mathbf{T}_{G_{(1-3)}}$ are satisfied. This definition is the definition of a three-place predicate which talks about a set and two other objects. To make it a one-place predicate we define the predicate being a group as an algebra $= [A, *, a]$. Having the notion of algebra at our disposal in set theory we define a group as one place predicate as follows: An algebra $= [A, *, a]$ is a group iff and only if for all x, y, z in A, $\mathbf{T}_{G_{(1-3)}}$ are satisfied.
In the same way the set-theoretical predicates of being a ring, field, or any of all the other algebraic theories can be stated.[10] Every theory of this sort finds its precise place in the hierarchy of set theory .

[9] It should be noted that the slogan works in one direction only, since obviously not every set-theoretical predicate represents a theory.

[10] Actual mathematical practice recognizes the integers $\mathbb{Z}$ to be a model of the algebraic elementary theory of rings, interpreting $+, \times, 0, 1$ as addition, multiplication, 0 and 1 in the set of integers. Likewise the rationals $\mathbb{Q}$ are a model of the elementary theory of

I have presented, differently from Suppes[11], the axioms of group theory in first-order logic, that is, I have presented the formal axiomatic theory of group and the way to embed it in formal axiomatic set theory. The important point to appreciate is that the same method applies when we want embed any axiomatic theory of stage 1 into $\mathbf{T}_{ST}$. The axiomatic systems at stage 1 are not formalized. In this case not only we do embed the theory but at the same time we formalize it. This point is important for us since it marks and at the same time destroys the boundaries between stage 1 and 2 of the axiomatic method. Moreover it is important because the real necessity in order to define a Suppes predicate and therefore in order to embed a theory in $\mathbf{T}_{ST}$ is the existence of an axiomatic system at stage 1 for the theory we want define as a Suppes predicate since once an axiomatization at stage 1 exists we can be sure that a Suppes predicate for that can be defined.[12] Probably it is possible to give an axiomatization at stage 1 for every mathematical theory and, relying on [38], that is sure. Which implies that it is possible to formalize the mathematical theories as Suppes predicates.

As an example consider Kolmogorov probability theory, T_K, which we discussed in §1.3. It is a theory at stage 1. We saw that T_K consists of five axioms:

T_{K_1} F is a field of sets;

T_{K_2} F contains the set S;

T_{K_3} To each set in F is assigned a non negative real number $\mu(A)$. This is called the probability of event A;

T_{K_4} $\mu(S)$ equals 1;

T_{K_5} Given two incompatible events A, B, then $\mu(A \vee B) = \mu(A) + \mu(B)$

Now we call the set of all outcomes a *sample space*, as we did in §1.3. Next, let S be a sample space, then we say that:

(1) If A and B are events, then so are $A \cup B$, $A \cap B$, $A \setminus B$.

---

fields. In particular the rationals and the reals $\mathbb{R}$ are ordered fields. For our purpose it is important to remark that these theories can be defined as Suppes predicates.

[11] Suppes presents the axioms of group in a non-formalized version.

[12] Clearly if there exists an axiomatic system at stage 2, in any nth-order language or in whatever logic, then it is always possible to obtain a system at stage 1. If you have an axiomatic theory at stage 2, i.e., a formal axiomatic theory, you take just the informal version of that, that is, you consider the theory without logic.

(2) The sample space S is also an event and $S = 1$, which means it is certain. Conversely $\emptyset = 0$, which means it is an impossible event.

(3) To each event is assigned a positive real number $\mu(E)$ that we call the probability or *measure* of the event (E).

(4) For a decreasing sequence of pairwise disjoint events $A_1 \supset A_2 \supset .... \supset A_n \ldots$, $\lim_{n\to\infty} \mu(A_n) = 0$.

Thus we characterize a probabilistic space, Π, as a triple (S, F, μ) where S is a sample space; an algebra on it is the set F of subsets closed under union and complementation and, given that to each event is assigned a real number and the system of reals is an ordered field, the algebra is a field; the function μ is a probability function which associates real numbers to events; we need (4) in a system with infinitely many events.
Now to the definitions (1–4) we add the axioms of T_K. Axioms plus (1–4) form the definition of the probabilistic space Π. In fact the axioms of T_K are the most important part of the definition of being a probabilistic space. So we define Π to be a probabilistic space with respect to a sample space S, an algebra F closed under the operations of union and complementation, and the function μ which assigns real numbers to the subsets of F, if and only if Π is a set, S is a sample space on Π, F is an algebra on Π, μ is a probability function on Π and for all A, B, C in Π the axioms $T_{K_{1-5}}$ are satisfied.
But this definition is the definition of a four-place predicate which talks about a set and three other objects. To transform the above definition into that of a one-place predicate Π = is a probabilistic space we, having the notion of algebra at our disposal in set theory, define the probabilistic space as follows: An algebra = $[S, F, \mu]$ is a probabilistic space Π iff for all A, B, C in Π, $T_{K_{1-5}}$ are satisfied. We see that T_K, defined as the predicate being a probabilistic space, has become a set. Such a set finds its place in the hierarchy of $\mathbf{T}_{ST}$ and evidently it gets formalized in the first-order language of that theory which becomes $\mathbf{T}_{ST+K}$.

The doubt is whether it is always possible to axiomatize at stage 1 a real science which is not mathematics. If a scientific theory cannot be axiomatized at stage 1 then we cannot define a Suppes predicate in order to formalize it. Indeed we believe that every scientific theory, when it reaches a mature stage of its development, can be axiomatized at stage 1, but we don't need to argue for it since our idea is to look at science by the axiomatic theories, i.e., by the axiomatic method, and if there is a field of knowledge that cannot be axiomatized at stage 1 then it means that that portion of knowledge does not belong to our perspective and we cannot look at it. Clearly we try to encompass as much as possible in our perspective but certainly we don't aim to encompass everything.

The method of defining a Suppes predicate works for physics. Its great successes are the formal axiomatization, as an extension of set theory, of theory of measurement [59, 61], of classical particle mechanics [61], of the theory of forces in classical physics [61], of general relativity theory [13], and of some parts of quantum mechanics [61]. But also some parts of genetics and evolutionary theory [37] and some parts of learning theory [61] have been formalized as Suppes predicates, and this list is not exhaustive. Indeed our view is that the obstacles and difficulties in formalizing scientific theories in this way could be overcome. Clearly it is an enterprise that falls beyond the limits of a single human mind and requires the work of specialized scientists. But theoretically the enterprise seems achievable and our main concern here is that it is theoretically possible. We can make also a little contribution to the enterprise in what follows.

Special relativity is a theory of motion and propagation of light which shows that space alone does not exist, i.e., what does exist is space-time.[13] It is a theory one would consider easy to axiomatize. In fact Einstein [14] already worked it out in an 'axiomatizable framework', as will become clear in what follows.

The theory was inspired by the failure of Michelson-Marley's experiment [8] meant to establish the existence of a non-observable entity called ether which was supposed to be the medium through which light flows. Given the failure of the experiment, Einstein did what an educated scientist had to do, that is, he applied abduction: He dropped one of the axioms of Newton's kinematics T_{Nk} (see below), asserting the existence of an absolute coordinate system, which implies the absoluteness of time and the existence of the ether, and replaced it with two principles asserting that all the coordinate systems are equivalent and that light travels at the same velocity in all directions. He had also to weaken another axiom of T_{Nk}, as we'll see.

The work of axiomatizing special relativity in first-order logic has been done

[13] General relativity connects space-time with gravity which results in the curvature of space-time. We do not consider general relativity here since a Suppes predicate for that has been already defined [13].

[34].[14] The authors have worked in the following way: First they have stated the axioms of special relativity theory in a non-formalized version, that is, at stage 1, and they have worked on the theory from within, that is, they have proved the theorems of the theory. Second they have formalized the theory in a first-order logical language and have worked on the theory from without, that is, they have proved the metaproperties of the theory. In doing so they have confirmed our thesis about the development of the axiomatic method, i.e., the axiomatic method develops from a pre-existing field of knowledge, it builds an hypothetical-deductive theory on that field, which we called stage 1, and it builds a formalization on that hypothetical-deductive theory, which we called stage 2.
We use the axiomatic system at stage 1 of special relativistic kinematics [34], T_{SRk} (see below), but we reach stage 2 by definition of a Suppes predicate which we call *being a relativistic kinematics*, Ⓡ, which leads us to have a formal axiomatic system for special relativity theory, $\mathbf{T}_{ST+SRk}$, which can prove the same theorems of T_{SRk}, but as we will remark it is not equivalent to it.
First, following [34], we state the axioms of Newtonian kinematics T_{Nk} (at stage 1):

T_{Nk_1} Each observer "lives" in a 4-coordinate system or *worldview*. The observer in his own coordinate system is motionless in the origin, i.e., his *worldline* is the time-axis.

T_{Nk_2} Motion is straight: Let o be an arbitrary observer and let b be a body. Then in o's 4-coordinate system the worldline of b is a straight line, i.e., in an observer's worldview all the worldlines of bodies appear as straight lines.

T_{Nk_3} Motion is permitted: In the worldview or 4-coordinate system of any observer it is possible to move through any point p in any direction with any finite speed.

T_{Nk_4} Any two observers observe the same events, i.e., if according to o_1 bodies b_1 and b_2 have met, then the same is true in the 4-coordinate system of any o_2.

T_{Nk_5} Absolute time: Any two observers agree about the amount of time elapsed between two events. (Hence temporal relationships are absolute.)

Next we state, following [34], the axioms of relativistic kinematics T_{SRk} (at stage 1):

$T_{SRk_1} = T_{Nk_1}$
$T_{SRk_2} = T_{Nk_2}$

[14] Clearly such an axiomatic theory is an axiomatic theory at stage 2. But we want an axiomatic theory at stage 2 which embeds relativity theory in $\mathbf{T}_{ST}$ and thus we are going to define the Suppes predicate of being a relativistic kinematics to achieve this.

$T_{SRk_3} = T_{Nk_{3^-}}$ Slower-than-light motion is possible: in the worldview of any observer, through any point in any direction it is possible to move with any speed slower than that of light (Here, light-speed is understood as measured at that place and in that direction where we want to move).

$T_{SRk_4} = T_{Nk_4}$

T_{SRk_5} Light Axiom: The speed of light is finite and direction independent in the worldview of any observer.

So, as in [34], we define $T_{Nk^-} = T_{Nk_{1,2,3^-,4}}$ and we see that $T_{SRk} = T_{Nk^-} + T_{SRk_5}$. In [34] it is shown that $T_{Nk^-} + T_{SRk_5} \vdash$ negation of T_{Nk_5}. Now we should remark that this confirms also the thesis, sustained in the previous chapter, about coherence and existence and incoherence and non-existence. In fact Newton's kinematics implied an absolute coordinate system and when it was shown, by Michelson-Marley's experiment, that light travels at the same velocity in any direction, the negation of the absolute system of coordinates was established and so the theory was disproved. At that point not only was the absolute system substituted with a multiplicity of equivalent systems but the non-existence of the ether was also declared since, as we have sustained, if a theory is incoherent then the entities it talks about do not exist.

Now our business: We define the Suppes predicate for being a relativistic kinematics, Ⓡ.

Special relativity is a theory of bodies in motion in the space, so we need a four-dimensional coordinate system with one time dimension t and three space dimensions x, y, z, $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. Given a body b in the four-dimension system or *worldview* there is a function f that for each instant of time t tells us the position of b in the three-dimensional space. So the function f : space$\rightarrow$time, tells us the positions of b in the worldview and its motion is called the *worldline* of b.

So we see that in the axiomatization of special relativity we need: A set B of bodies, a four coordinate system in which bodies move, a set of quantities Q assigned to bodies which move. Our four-dimensional coordinate system is $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ so we need real numbers and we know that a model for them is an ordered field, so we need F. Coordinate systems are observers which are special kinds of bodies, so we need a one place relation O on B. We need also a one place relation Ψ on B for representing photons which are another special kind of bodies representing light. Finally we need a worldview relation W which is a 6-place predicate W, i.e., o sees the body b at $txyz$.

So we define a relativistic kinematics as a structure Ⓡ $= [F, B, O, \Psi, W]$.

$[F, B, O, \Psi, W]$ is a relativistic kinematics iff for all bodies α, β, γ in ®, $T_{SRk_{1-5}}$ are satisfied. The set defined by this predicate finds its place in the hierarchy of $\mathbf{T}_{ST}$ and evidently it gets formalized in the first-order language of set theory and we call the theory so obtained $\mathbf{T}_{ST+SRk}$.
In [34] it is shown that T_{SRk} can prove all the theorems of special relativity theory so we infer that also $\mathbf{T}_{ST+SRk}$ can do that. But $\mathbf{T}_{ST+SRk}$ can prove also all the theorems of set theory so the two are not equivalent. But so much so good for the good news. We have to turn our attention to some limitative results. In fact in [34] is shown that T_{SRk} is consistent whereas we cannot prove the consistency of $\mathbf{T}_{ST+SRk}$, and we can prove that we cannot prove that, as we'll see in the next section.

Before to pass to the next section there is to address the problem of how theories do interconnect each other. This is one of the hardest problems of philosophy of science, as [32] testifies. We can do just a little here. The problem can be divided in three subquestions:
(a) How do the theories interconnect with respect to the mathematical? Consider for example the probabilistic space, $\Pi = [S, F, \mu,]$, and the relativistic kinematics, ® $= [F, B, O, \Psi, W]$, we have gained above. In both structures it is contained a substructure represented by F which is the structure of the ordered field. So we can see that the interconnection between different theories is that of intersection between the sets of numbers which represent the quantities assigned to the objects of the theories, i.e., in our case Π and ® intersect on F. So with respect to the mathematical the interconnections take place as intersections mostly on the algebraic structures of the ring, field, and group. Of course this is a very poor way to answer the problem of the interconnection and it is unsatisfying; it just points out that the interconnection of the theories must be searched somewhere else and so it is the prelude for the next question.
(b) How do the world must configure if we want the the intersections of the theories to be possible? The answer to this question does not belong to the methodological analysis. In the next chapter, when we will introduce some philosophical concepts, we will answer this problem.
(c) How can the theory change be accounted for? First there is to recognize that much depend on the historico-philosophical orientation one holds. In fact there are two opposite positions disputing over this point: One considers science to be a discontinuous enterprise and the other a cumulative one. With respect to the former the theory change is not an option, with

respect to the latter it should be the role, but indeed a very difficult aspect to account for.[15] Let's agree that we take the cumulative stance and let's consider for example the theory change between Newtonian mechanics and relativistic mechanics. The answer which usually is maintained [32] is that the two theories coincide with respect to bodies which travel at a very law velocity with respect to that of light and so they both 'save the phenomena' which hold when bodies interact at low velocities, while relativistic mechanics, with respect to Newtonian mechanics, 'save also the phenomena' which holds between bodies at very high velocity. In the last decades the theory change has been investigated especially in the framework of the partial structures [7, 32].[16] Partial structures allow to treat the theory change between theories, which evidently needs to be addressed in a case by case approach, with a general method, that is, by the employment of partial structures. We do not linger on this problem in this thesis since it would require a too big framework and so it would drive us away from what is the central subject of this thesis.

2.3 Stage 2: The Limits of the Method

In our view of science in the perspective of the axiomatic method at stage 2 science is configured as an extension of $\mathbf{T}_{ST}$ and we call the most comprehensive of these extensions *the theory of all axiomatic scientific theories*, $\mathbf{T}_{ST+MS_C}$, where MS_C stands for the class of scientific theories axiomatized with Suppes predicates. Note that MS_C is configured as a class rather than as a set since its members are scientific theories (or Suppes predicates) and there will always be new scientific theories which at some point will enter in it since science is not a finished enterprise.[17]

We are mainly interested in the consequences of Gödel's theorems for $\mathbf{T}_{ST+MS_C}$. Gödel showed that arithmetic, if consistent, is incomplete and incompletable

[15] Nowadays particular difficulties come from the inconsistency of relativistic mechanics and quantum mechanics [32].

[16] Let D be a nonempty set, an n-place *partial relation* R over D is a triple (R_1, R_2, R_3), where R_1, R_2, and R_3 are mutually disjoint sets, with $R_1 \cup R_2 \cup R_3 = D^n$, and such that: R_1 is the set of n-tuples that (we know that) belong to R; R_2 is the set of n-tuples that (we know that) do not belong to R, and R_3 is the set of n-tuples for which it is not known whether they belong or not to$R$. A *partial structure* A is an ordered pair $(D, R_i)_{i\in I}$, where D is a nonempty set, and $(R_i)_{i\in I}$ is a family of partial relations defined over D.

[17] Note that in our perspective and intuitively science equals $\mathbf{T}_{ST+MS_C}$.

and that, if consistent, cannot prove its consistency [40, 53]. In §2.2 we rephrased Gödel's results as: Every axiomatic formal extension of $\mathbf{T}_{PA}$, if consistent, is incomplete and its consistency cannot be shown. $\mathbf{T}_{PA}$ is representable in $\mathbf{T}_{ST}$. So we directly infer that $\mathbf{T}_{ST}$, if consistent, is incomplete and its consistency cannot be shown and we derive that also $\mathbf{T}_{ST+MS_C}$, being an extension of $\mathbf{T}_{ST}$, if consistent is undecidable and its consistency cannot be shown. And perhaps that will not come as a surprise. This is the motive for why in the previous section we said that we cannot prove that $\mathbf{T}_{ST+SRk}$ is consistent. In fact now we have pointed out that $\mathbf{T}_{ST+SRk}$ cannot prove $\mathrm{Cons}(\mathbf{T}_{ST+SRk})$.

Gödel's theorems are undecidability assertions which settle the limits of the axiomatic method and therefore of science in the axiomatic perspective. Nevertheless the fact that the consistency of $\mathbf{T}_{ST+MS_C}$ cannot be shown in $\mathbf{T}_{ST+MS_C}$ does not imply that the system is inconsistent and thus it cannot imply that the entities the theories talk about do not exist. This is so because if a system is inconsistent than the entities the system talks about do not exist but Gödel's second limitative result tell us that if arithmetic is consistent, its consistency cannot be shown and it does not tell us that the system is inconsistent. So we relax our constraint on consistency of the 'system science' and we assume that arithmetic is consistent, which is, in fact, just what Godel's says. If arithmetic were inconsistent we could prove everything, even its consistency, but if consistent we cannot prove that otherwise we contradict Gödel's first theorem.[18]

2.4 Foundational Projects

Before we pass to the next chapter we discuss Tarski's formal elementary theory of geometry [62] to mark the difference between the general foundational feature of of the axiomatic method as it emerged in our analysis and

[18] Informally the argument is the following: Assume $\mathbf{T}_{PA}$ derives its own consistency. By Gödel's first theorem we have G, the Gödel's sentence that says of itself 'I'm not provable', is true iff it is not provable. Assume we can derive such a sentence 'G is true iff G is not provable' in $\mathbf{T}_{PA}$, as effectively can be done. So we have, by modus ponens, that $\mathbf{T}_{PA}$ derives its consistency implies G, which contradicts the first theorem. This just tells us that the second theorem is just an instance of the first, i.e., it is another undecidability result in the sense that there is a sentence asserting the consistency of arithmetic which is true but that we cannot prove and since (we have the strong suspicion) that the sentence is true, then evidently we cannot prove its negation.

the particular foundational project of geometry of Tarski.
To give the foundations of geometry is the stated purpose of Tarski's elementary theory. Indeed Tarski discusses three elementary theories for Euclidean Geometry, $\mathbf{T}_{\mathbb{E}_2}, \mathbf{T}_{\mathbb{E}_{2'}}, \mathbf{T}_{\mathbb{E}_{2''}}$. In what follows we are interested in $\mathbf{T}_{\mathbb{E}_2}$ only.
$LA_{\mathbb{E}_2}$, is a first-order language where variables range over elements of a fixed set, i.e, over points of the space. It has only two primitive concepts, betweenness and equidistance, respectively represented in $AL_{\mathbb{E}_2}$ by the extralogical constant β, which is a three place relation $\beta(xyz)$, i.e., y lies between x and z, and by the extralogical constant δ, which is a four place relation $\delta(xyzu)$, i.e., x is as distant from y as z is from u. $\mathbf{T}_{\mathbb{E}_2}$ is constituted by 13 axioms. There is no need for us to state such axioms, you can look at them in [62]. Tarski's geometry is consistent, complete, and decidable, as Tarski shows, and that's why it is, for Tarski, the best system for the foundations of Euclidean geometry.[19]
The important point to appreciate is that it is possible to define a Suppes predicate for being a Tarski space, say $\bigstar$. It is possible to define such a predicate since we have Tarski's theory which is an axiomatic system at stage 2 and we need just an axiomatic system at stage 1 in order to define a Suppes predicate for that theory and we know that once we have an axiomatic at stage 2 we can always obtain an axiomatic at stage 1. But the situation with geometry is quite complex and it requires to take in account the axiomatizations of geometry of Pasch [46], Pieri [48], Hilbert [28], and Tarski [56]. So we prefer not to formulate a Suppes predicate for Euclidean geometry based only on Tarski [56, 62]. But since to define such a predicate requires to take in consideration so big a framework and it would require time, we just assume that the Suppes predicate of being *an Euclidean space*, $\bigstar$, can be defined and we discuss what that would imply, postponing the task to effectively define this predicate in future work.
Assuming we have such a predicate $\bigstar$, then it is possible to embed $\mathbf{T}_{\mathbb{E}_2}$ in $\mathbf{T}_{ST}$. The elementary theory so obtained, $\mathbf{T}_{ST+\mathbb{E}_2}$, is an extension of $\mathbf{T}_{ST}$ and belongs to $\mathbf{T}_{ST+MS_C}$. But note that $\mathbf{T}_{ST+\mathbb{E}_2}$ is undecidable and its consistency cannot be proved, as we have explained above.
$\mathbf{T}_{\mathbb{E}_2}$ represented for Tarski the most suitable system for the foundation of

[19]Tarski's theory has a model which is isomorphic with the Cartesian space over some real closed field and therefore it is coherent. Now if there is a model such that a sentence holds in that model iff it is valid in the given theory, then the theory is complete (and coherent). Tarski proves that. Now $\mathbf{T}_{\mathbb{E}_2}$ being complete and axiomatized, then it follows that it is decidable.

geometry. But in our framework Tarski's theory does not found anything. At most we can concede that it is a relative foundation of Euclidean geometry, that is, what we called a particular foundation. But our thesis is that the real foundation of Euclidean geometry would be our $\mathbf{T}_{ST+\mathbb{E}_2}$ which we have assumed to be definable. This is so because $\mathbf{T}_{ST+\mathbb{E}_2}$ belongs to $\mathbf{T}_{ST+MS_C}$, and so it embeds the general foundational feature of the axiomatic method.

At this point our analysis of the axiomatic method at stage 2 is over.
In the preceding chapters, among the features of the modern style of the axiomatic method, we distinguished a particular foundational feature from a general foundational feature. In chapter 1 we said that the particular foundational feature configures the axiomatic method also as a method of justification: We sustained that the relative proofs of consistency of the theories serves also as justification of them in the sense that if the theories are coherent then one is justified to believe the axioms (which are just hypotheses) and is justified to believe that the objects the theories talk about do exist (otherwise they do not). In this sense we said that the particular foundational feature of the axiomatic method constrains a strong way to believe the axioms, and it is a strong way since justification, in this sense, is methodologically constrained, i.e., it holds if the system is coherent.
In chapter 2 we have addressed the general foundational feature of the axiomatic method and we have seen that the impossibility of proving coherence already holds on a theory so simple as $\mathbf{T}_{PA}$ and it is a pathological limit of science in our axiomatic perspective.
At this point there is to reconsider the problem of justification. Now justification cannot be anymore achieved methodologically since we cannot believe the axioms of any theory belonging to $\mathbf{T}_{ST+MS_C}$ by appealing to coherence. So the question is: How do we are justified to believe the axioms? The only way left is a case by case study of the reasons we have to believe them, which is an indispensable effort since in the classical analytic philosophy knowledge is configured as true and justified belief. Evidently in our axiomatic context the most important analysis of the reasons why we believe the axioms is the analysis of the reasons why we believe the axioms of set theory since set theory configures as the 'door' through which the scientific theories get axiomatized as Suppes predicates. An important work in the justification of the axioms of set theory has been done by [35, 36]. We call the analysis of the kind of [35, 36] the 'weak way' to believe the axioms since we do not believe them for a methodological constraint. That's why we said that

the general foundational feature of the axiomatic method constrains a weak way to believe the axioms.[20] In the weak way the analysis of justification of the axioms, although indispensable, is pretty much an epistemological affair achieved by the discussion of the reasons why scientists believe the axioms, and as such it fulls behind our methodological analysis and we do not linger on it.

However it should be noted that the problem of the justification of the axioms is not the same as the problem of the 'discovery' of the axioms, i.e., how do the scientists come up with a set of axioms? The axiomatic method is not a method of discovery and never it will be so. That's because it is a method of foundation of a knowledge that we already hold. So if one for example comes up with a set of axioms for quantum mechanics, for which we still lack an axiomatization, one does not discover anything new but he makes a new foundation. In this sense we can maintain that the axiomatic method is a method of discovering foundations. Now the process of 'coming up' with the axioms being a foundational affair and our analysis being an analysis of a foundational method, i.e., the axiomatic method, we cannot discard the problem of discovering the axioms. Evidently it is not a problem that can be studied case by case since if we go to analyze the axioms of the theories one by one all that we can get is a justification of those, which is not what we wanted. On the other hand the problem of coming up with the axioms, when addressed in a general way, presents the difficulty of being psychologically committed, that is, of being very much dependent from the cognitive and subjective activities of the scientists who discover the axioms. So we are between two obstacles, on the one hand, we have to avoid the particularistic analysis of justification and, on the other hand, we have to be not dependent from the psychological discourse, if we want our analysis to be strictly methodological. Nevertheless there is a way in which we can address the problem of discovery of the axioms in a methodological setting and it consists in gaining the place where in the process of axiomatization the discovery or the choice of the axioms happens, leaving aside the process in the minds of the scientists through which it happens. But this is a task for the next chapter since in order to address this problem we need to introduce some concepts belonging to the philosophical image of the world.

[20] Note that weak and strong way are defined with respect to the absence or presence of methodological constraints and they are not perspective-free.

Part II

The Philosophical Image

INTRODUCTION:

On the basis of the analysis of the preceding chapters science in the axiomatic perspective configures 'pathologically undecidable, incomplete, and we cannot prove its consistency'. So our analysis of the axiomatic method, which we maintained was born in order to found scientific knowledge on certainty and rigor with respect to the standards of modernism, has turned out to produce an image-conception of science which returns to man a sense of profound uncertainty. Uncertainty about science itself, given the limits of the axiomatic method, about the world, given that scientific theories are theories about the world, and about man himself given that a theory, as for example a physical theory, or a biological theory, or whatever, is also a theory which produces an image of man-in-the-world, respectively man as a physical system, as a biological one, and so forth for all scientific dimensions. In conclusion our analysis of the axiomatic method returns a conception-of-man-in-the-world which takes uncertainty to be its main feature.
Sellars has sustained the thesis that [52, p. 37]: "For the philosopher is confronted not by one complex many dimensional picture, the unity of which, such as it is, he must come to appreciate; but by two pictures of essentially the same order of complexity, each of which purports to be a complete picture of man-in-the-world, and which, after separate scrutiny, he must fuse into one vision. Let me refer to these two perspectives, respectively, as the *manifest* and the *scientific* images of man-in-the-world."[21]
So in agreement with Sellars there should be, besides our 'scientific in axiomatic perspective image', also another image "of essentially the same order of complexity". This is the philosophical image or, as it were, the manifest image. Of course a "separate scrutiny" of this image, which would be a titanic enterprise, cannot be our affair. Nevertheless it seems natural to us to think that a philosophical image which can be fused "into one vision" with our 'scientific in axiomatic perspective image' is already available. This manifest image is that which considers the world or, if you wish, reality as made of phenomena, science to be explanations or prediction of those, and therefore man-in-the world as man-in-a-phenomenological-world.[22] So in what

[21] Sellars explains that with the term image he means essentially *conception*. We do the same.

[22] This is a very old philosophical image which dates beck to the naturalist Greek philosophy. By the end of the eighteenth century it received its highest fortune in the Kantian theorization and at the beginning of the twentieth century it was assumed, starting with

follows our operation consists in merging “into one vision” our ‘scientific in axiomatic perspective image’ with the ‘phenomenological image’. The merging of these two images has the purpose of answering the two problems which arose in the analysis of the axiomatic method in the preceding chapters but which remained not answered by that analysis, they are the problem of the discovery of the axioms and that of the interconnection of the theories. The merging “into one vision” our ‘scientific in axiomatic perspective image’ with the ‘phenomenological image’ is pursued in so much it accomplishes this task and not any further. However it may be needed to remark that the merging of the two seems natural to us but this does not mean that the latter fallows necessarily from the former. There could be somebodies who do not agree with the operation of linking the methodological analysis to the phenomenological image and who, after the methodological analysis, would not take any step further; or others who do not agree with our ‘link’ and would propose another different image, with respect to the phenomenological, to fuse “into one vision” with the methodological analysis. So the operation should be considered as the personal perspective of the writer which is open to rational discussion and criticism on the opportunity and explanatory force of the ‘link’.

Husserl, from some of the continental philosophies, especially from existentialism. In the field of philosophy of science the talk about phenomena was very usual in the age of the logical empiricism as [7, 64] testifies. Recently the view that science essentially search to explain and predict phenomena has been defended by James Bogen and James Woodward [2]. The work of Bogen and Woodward has attracted the attention of many and their ‘phenomenological image’, although with some reservations and the due divergences, has been accepted from the most part of the philosophers of science as [32] testifies. It is essentially from their work that our discourse is inspired.

Chapter 3

THEORIES and REALITY

3.1 Axiomatic Method and Phenomena

The crucial difference between our 'scientific in axiomatic perspective image' and the phenomenological image is about the aboutness, that is, 'the reality which scientific theories are about'.[1] As it is evident from our analysis in the preceding chapters, axiomatic theories, at each stage, are theories about sets of objects.[2] In our analysis of the axiomatic theories never the word phenomenon occurred.[3] Nevertheless according to the meditation of some philosophers science essentially deals with the discovery, description, and forecast of the phenomena happening in reality, call this the phenomenological image. So, according to this view, scientific theories are about phenomena.

Our question is: Is there a way to merge the axiomatic method with the conception that science is supposed to 'save'-cover the phenomena?

Bogen and Woodward [2], while defending the traditional phenomenological image, have introduced an element of novelty with respect to the tradition which consists in the distinction between data and phenomena and in the emphasis on both of them with respect to the articulation of the phe-

[1]Usually scientific theories are interested only in the knowledge of some part of reality, as for example the mathematical reality, the biological reality, and so forth.

[2]About the aboutness of the axiomatic theories there is not difference between an hypothetical-deductive theory, a formal theory, a Suppes predicate, since all of them are about sets of objects.

[3]It is a word that comes from the ancient Greek '$\phi\alpha\iota\nu o\mu\varepsilon\nu o\nu$' which in fact means to be manifest.

nomenological image. According to them [2, p. 305]: "Data, which play the role of evidence for the existence of phenomena, for the most part can be straightforwardly observed. However, data typically cannot be predicted or systematically explained by theory. By contrast, well-developed scientific theories do predict and explain facts about phenomena. Phenomena are detected through the use of data, but in most cases are not observable in any interesting sense of that term."
Bogen and Woodward's conception is relevant for our purpose since, it seems to us, it can be merged with our axiomatic perspective in two steps. First in according to the phenomenological image we can hold that scientists start with the observation of the data and 'detect' through them the phenomena.[4] The reader will find useful the following figure (Fig. 3.1) which pictures the two levels of the phenomenological image.

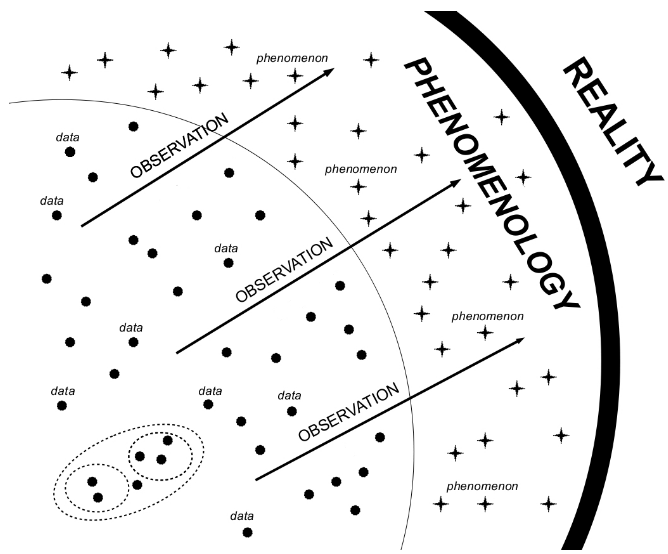

Figure 3.1: The phenomenological image of reality

Once some of the phenomena that compose reality have been detected it is needed a second step: The task of the scientist consists in choosing a small number of the detected phenomena as fundamental and in fixing these as

[4]Nothing is said in [2] about the process of 'detecting' phenomena by data. Plausibly the process has to be an inductive inference. We also do not discuss it and we take for granted that phenomena are 'detected' through the use of data.

principles, or axioms, or postulates, or fundamental laws. From the principles, by the deductive method of inference, not only the phenomena previously detected but also all the other phenomena constituting reality can now be derived and so reality can be known. The phenomena derived by the principles are called also theorems or laws depending on the context.
The reader will find useful the following figure (Fig. 3.2) which pictures the process of choosing the principles and the deductive inference from principles to phenomena. Principles are also called axioms.

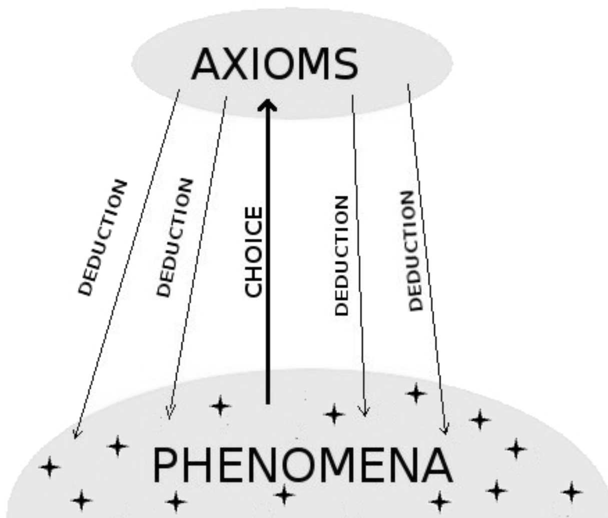

Figure 3.2: The choice of the axioms and the deductive inference of the phenomena

Note that principles (or axioms) and phenomena are at the same level since principles are phenomena too. So we have three kinds of phenomena, those reached through the use of data, the axioms, chosen among them, and those reached by deduction. These constitute the phenomenological level which our methodological investigation merges "into one vision" with.
Note that this setting answer the problem of discovery which emerged in the analysis of the axiomatic method. As already remarked the axiomatic method is not a method of discovery but it is a method of foundation of a field of knowledge. But, in order to apply to a pre-existing field of knowledge, this field of knowledge was previously discovered. The discovery is a discovery of phenomena through the data. Subsequently, when some of the phenomena

have been discovered, then the scientist chooses some of them from which all the others can be derived and in this sense he discovers a foundation or invents a theory. So the discovery in the axiomatic method is a choice of the fundamental phenomena. Of course in this choice many contextual factors are relevant and they need to be addressed in a case by case study as well as clearly the process of invention-discovery is a psychological one and as such it does not enter in our philosophico-methodological discourse. We want just point out that the choice happens among phenomena, which although is not a conclusive answer to the problem of discovery, it is a clarification of the context and place of discovery in the axiomatic method.

Finally note that the axiomatic method, once get fused 'into one vision' with the phenomenological image, is strong enough to capture also the experimental method which, as we have remarked in the previous chapter, was the main problem for the failure of the syntactic view of the theories. In this thesis we cannot enter in this problem but we want just report the words of Muller about this to signal that the problem can be effectively addressed in the perspective we propose and to point out that our operation of merging the phenomenological image with the methodological analysis does not come out from the blue. Muller, discussing the semantic view of the theories, says [39, §2]: "Let $Dt(T)$ be the set of data structures that are obtained until historical time t from the measurement-results of experiments or observations relevant for T, which are extracted from 'the phenomena' that T is supposed to save. Call T *observationally adequate* at t iff for every data structure $D \in Dt(T)$, there is some structure (model) $M \in T$ such that D is embeddable in M, where 'embeddability' is broadly construed as some morphism from D into (some part of) M.[5] [...] and further [39, §2]: "The embeddability-relation constitutes the connexion between (i) theory T and (ii) the phenomena that T is supposed to describe, to explain or to predict. To save the phenomena is to embed the data structures. The nexus between (i) and (ii) codifies the empirical essence of science: without it, there simply is no science-as-we-know-it."

[5] And he explains that with 'some morphism' he means [39, §2]: "isomorphism, partomorphism, monomorphism, epimorphism, partial isomorphism and perhaps more;"

3.2 Phenomenological Atomism

Having found in the previous section the place of discovery in the axiomatic method by merging together the methodological analysis with the phenomenological image there remains to answer the problem of the interconnection of the theories. We do it now building on the manifest image of science we have proposed. Roughly speaking we, building on [2], have maintained the thesis that axiomatic theories explain, or capture, or 'save' the phenomena. So now we can approach the problem of the interconnection of the theories by the interconnection of the phenomena happening in or constituting reality. With respect to this problem we want offer a perspective-image which does not answer the problem of the interconnection of that particular theory with that other particular theory but which offers a general framework in which a case by case study is not excluded.

In oder to accomplish our general explanatory task we consider two exquisitely philosophical notions which in the history of philosophy have attracted the attention of many: the whole and the part.

The conception that the whole is bigger than the part was the tenth of the axioms of Euclid's *Elements* and as such it was left unquestioned until Cantor showed that the opposite view, that wholes can have the same size as a proper part, has compelling reasons to be held true.[6] In the last century the author that more than any other, meditating on the work of Cantor, has discussed these two notions in an ontological-metaphysical perspective is Russell. He explained that both views are paradoxical [50]: The view that the whole is bigger than the part provokes Zeno's paradox of Achilles and the Tortoise,

[6]Historians have discovered fascinating discussions of the idea that some wholes can be of the same size as a proper part. Ivo Thomas [63] has found a discussion of equinumerosity between sets in the 12th century writings of Adam of Balsham. E. J. Ashworth [1] has found a discussion of the topic in the 15th century writings. Other better known discussions can be found in the 13th century writings of Duns Scotus and in the 16th century writings of Bruno, *De l'infinito universo et mondi* and *La cena delle ceneri*. A discussion of the views of Duns Scoto and Bruno can be found in [42]. Indeed during the course of history, the conception that the whole can be equinumerous with a proper part was raised many times but then it was dismissed even by those that believed in parts, rather than in the whole, as constituting reality. One was Leibniz who, even while proposing a metaphysics of monads, maintained that if the whole is equal to the part then the set of naturals and the set of multiples of two must be equal, but that, he concluded, is a contradiction. This was also Galileo's view, who first showed the equinumerosity of natural numbers and perfect squares and after denied that concluding that the ideas of less, equal, and greater apply to finite sets, but not to infinite sets [22, pp. 31–33].

while the view that the whole is equal to the part provokes what Russell calls Tristram Shandy's paradox.

Differently from Russell, being our discourse a phenomenological one and not an ontological or metaphysical one, we do not care about the ontological or metaphysical constitution of reality but we care about its phenomenological aspect, i.e., we do not care about the entities constituting reality and, assuming they there are, we care about the phenomena constituting reality. So we call phenomenological *holism* the belief that the phenomena happening in reality form a whole which is bigger than any of its parts and of the sum of them and we call phenomenological *atomism* the belief that the phenomena happening in reality does not form a whole but that reality it is the result of the interconnections of phenomenological parts which form different whole-parts and which give rise to different configurations of reality.

Having distinguished both views we opt for the phenomenological atomism in order to answer the problem of the interconnection of the theories. That is to say, theories, such as the biological theories or the physical theories and so on, in our perspective, capture the phenomena, that is, the biological phenomena, the physical phenomena and so on. Now we can think of these theories as parts in the sense they refer to parts of the phenomenological reality, the biological reality, the physical, the sociological, and so forth. Now, for example, one can look to reality through relativity theory plus non-Euclidean geometry plus some chemical theory plus Darwinian evolutionary theory plus some sociological theory, or one can look to reality through quantum mechanics and probability theory, or through Newtonian mechanics and so on. In general one looks to reality equipped with many theories and such a reality does not form a whole in the sense of the whole which is bigger than its parts but this reality is the result of the phenomenological configurations given by the theories one holds. So in this way we get an atomistic configuration of reality which is the result of the interconnections of different parts. One can think of this image of reality through the metaphor of the spectrum of colors. If you assign a color to each theory you know, for example you assign the blue color to probability theory and the yellow color to evolutionary theory, you will see a violet world. And so for all the theories that you know you will get 'different realities', which are different whole-parts. The interconnections of the theories is essentially interconnections of the phenomena in reality. And this answer also our general problem of the interconnection of the theories which, so to speak, happens in the reality, among the phenomena, and not in the abstract domain of the logico-mathematical ingredient of the theories which, as we think, is not suitable for such an explanation. Once one holds such a phenomenological- atomistic image of reality then the question of the interconnections of the theories makes sense and can be pursued in a case by case study.

Bibliography

[1] E.J. Ashworth. An Early Fifteenth Century Discussion of Infinite Sets. *Notre Dame Journal of Formal Logic*, XVIII, 2:232–234, 2006.

[2] James Bogen and James Woodward. Saving the Phenomena. *The Philosophical Review*, 97, 3:303–352, 1988.

[3] Ludwig Boltzman. Analytischer beweis des zweiten hauptsatzes der mechanischen warmetheorie aus den satzen uber das gleichgewicht der lebendingen kraft. *Sitzungsberichte Akad. Wiss.*, 63, 1871.

[4] George Boole. *Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*. London, 1854.

[5] M. Cagett. *The Science of Mechanics in the Middle Ages*. Madison, Wisc., The University of Wisconsin Press, 1959.

[6] Andrea Cantini. *I fondamenti della matematica*. Loesher Editore: Torino, 1979.

[7] Newton C. A. Da Costa and Steven French. *Science and Partial Truth: A Unitary Approach to Models and Scientific Reasoning*. Oxford University Press, 2003.

[8] J. T. Cushing. *Philosophical Concepts in Physics*. Cambridge University Press., 1998.

[9] B. de Finetti. Sul significato soggettivo della probabilitá. *Fundamenta Mathematicae*, 17:298–329, 1931.

[10] Richard Dedekind. *Was sind und was sollen nie Zahlen?* Vieweg, Braunschweig, 1888.

[11] K. Devlin. *The Joy of Sets: Fundamentals of Contemporary Set Theory.* Springer-Verlag, 2008.

[12] E. J. Dijksterhius. *Archimedes.* Humanities Press: New York, 1956.

[13] Newton C. A. Da Costa, F. A. Doria, and Roland Chuaqui. A Suppes Predicate for General Relativity and Set-Theoretically Generic Spacetimes. *International Journal of Theoretical Physics*, 29, 9:95–112, 1990.

[14] Albert Einstein. On the Electrodynamics of Moving Bodies. *Annalen der Physik*, 17:298–329, 1905.

[15] Euclid. Optics. *Journal of the Optical Society of America, H. E. Burton translation*, 35:357–372, 1945.

[16] Euclid. *Elements.* David E. Joyce, ed. 1997, c. 300 BC.

[17] Luciano Floridi. Semantic Conceptions of Information. *Stanford Encyclopedia of Philosophy*, 2005.

[18] Gottlob Frege. *Begriffsschrift, a Formula Language, Modeled upon that of Arithmetic, for Pure Thought.* In van Heijenoort, 1874.

[19] Gottlob Frege. *The Foundations of Arithmetics.* The Longman Library. 2007, 1884.

[20] Gottlob Frege. *Grundgesetze der Arithmetik 1.* Olms, Hildescheim, 1893.

[21] Gottlob Frege. *Grundgesetze der Arithmetik 2.* Olms, Hildescheim, 1903.

[22] Galileo Galilei. *Dialogues Concerning Two New Sciences.* Crew and de Salvio trans. Dover, New York, 1954.

[23] M. Giaquinto. *The Search for Certainty: A Philosophical Account of the Foundations of Mathematics.* Clarendon Press, Oxford, 2002.

[24] J. Gray. *Plato's Ghost: The Modernist Transformation of Mathematics.* Princeton University Press, 2008.

[25] Andres Hald. *A History of Probability and Statistics and Their Applications before 1750.* Wiley-Interscience, 2003.

[26] Joseph Halpern. *Reasoning about Uncertainty.* Mit Press, Cambridge, 2003.

[27] A.G. Hamilton. *Numbers, Sets and Axioms: The Apparatus of Mathematics.* Cambridge University Press, 1982.

[28] David Hilbert. *Foundations of Geometry.* 10th ed., L. Unger. Chicago: Open Court, 1971, 1899.

[29] David Hilbert. Mathematical Problems. *Bulletin of the American Society*, 8:437–479, 1902.

[30] Christiaan Huygens. *The Value of All the Chances in Games of Fortune.* Printed by S. KEIMER for T. WOODWARD, London, 1714.

[31] Andrei Kolmogorov. *Foundations of the Theory of Probability.* Translation edited by Nathan Morrison, University of Oregon, CHELSEA PUBLISHING COMPANY NEW YORK.

[32] J. Ladyman and D. Ross. *Every Thing Must Go: Metaphysics Naturalized.* Oxford University Press, 2007.

[33] Pierre Simon Laplace. *A Philosophical Essay on Probabilities.* English edition, New York: Dover Pubblications, 1814.

[34] Hajnal Andreka, Judit X. Madarasz, and Istvan Nemeti. Logic of Space-Time and Relativity Theory. *HandBook of Spatial Logics, edited by Marco Aiello, Ian E. Pratt-Hartmann, Johan F.A.K. van Benthem, Springer: The Netherlands*, pages 622–727, 2007.

[35] Penelope Maddy. Beleaving the Axioms. I. *The Journal of Symbolic Logic*, 53, 2:481–511, 1988.

[36] Penelope Maddy. Beleaving the Axioms. II. *The Journal of Symbolic Logic*, 53, 3:736–564, 1988.

[37] Joao Carlos Marques Magalhaes and Decio Krause. Suppes Predicate for Genetics and Natural Selection. *Journal of Theoretical Biology*, 29:141–153, 2000.

[38] S. Maracchia. *La matematica come sistema ipotetico-deduttivo.* Le Monnier, Firenze, 1975.

[39] F. A. Muller. Reflections on the Revolution at Stanford. *Synthese, DOI:10.1007/s11229–009–9669–7*, 2009.

[40] Ernest Nagel, J.R Newman. *Gödel's Proofs*. New York University Press, 2001.

[41] Isaac Newton. *Principia*. University of California Press, Cajori translation., 1946. Originally published in 1687.

[42] Piergiorgio Odifreddi. *Le menzogne di Ulisse: L'avventura della logica da Parmenide ad Amartya Sen*. Longanesi, Milano, 2004.

[43] Øystein Ore. *Number Theory and Its History*. Dover Publications, New York, 1976.

[44] Dario Palladino, Claudia Palladino. *Le geometrie non euclidee*. Le Bussole, Carocci: Roma, 2004.

[45] E. Agazzi, Dario Palladino. *Le geometrie non euclidee e i fondamenti della geometria*. EST Mondadori, Milano, 1978.

[46] M. Pasch. *Vorlesungen über neure Geometrie*. Verlag von Julious Springer: Leipzig, 1882.

[47] Giuseppe Peano. *Arithmetices principia, nova methodo exposita*. Bocca, Torino, 1889.

[48] M. Pieri. Della geometria elementare come sistema ipotetico-deduttivo. Monografia del punto e del metodo. *Memorie della R. Accademia delle scienze di Torino*, 48:173–222, 1897–8.

[49] W.V.O. Quine. On What There Is. *Review of Metaphysics*, 2:21–38, 1948.

[50] Bertrand Russell. *Principles of Mathematics*. Routledge classics, London and New York, 1903.

[51] Bertrand Russell. *Introduction to Mathematical Philosophy*. Spokesman, 1919.

[52] Wilfrid Sellars. Philosophy and the Scientific Image of Man. *Science, Perception and Reality*, 2:35–78, 1963.

[53] Peter Smith. *An Introduction to Gödel's Theorems.* Cambridge Introductions to Philosophy, 2007.

[54] E. Snapper. The Three Crises in Mathematics: Logicism, Intuitionism and Formalism. *Mathematics Magazine*, 52:207–16, 1979.

[55] Stephen M. Stigler. *The History of Statistics: The Measurement of Uncertainty before 1900.* Oxford Science Pubblications, 1986.

[56] Leon Henkin, Patrick Suppes, and Alfred Tarski. *The Axiomatic Method.* North Holland Publishing Company: Amsterdam, 1959.

[57] Patrick Suppes. A Comparison of the Meaning and Use of Models in Mathematics and the Empirical Sciences. *Synthese, 12*, pages 287–301, 1960.

[58] Patrick Suppes. What is a Scientific Theory. *In Sidney Morgenbesser (ed.), Philosophy of Science Today. Basic Books. New York*, pages 55–67, 1967.

[59] Patrick Suppes. Archimede's Anticipation of Conjoint Measurement. *In Role and importance of logic and methodology of science in the study of history of science. Moscow: Nauka*, 1971.

[60] Patrick Suppes. The Axiomatic Method in Empirical Sciences. *L. Henkin (ed.), proceedings of the Tarski Symposium, Proceedings of Symposia in Pure Mathematics, 25. Providence, RI: American Mathematical Society*, pages 465–479, 1974.

[61] Patrick Suppes. Axiomatic Methods in Science. *M.E. Carvallo (ed.), Nature, Cognition and System II, Kluwer Academic Publishers. Netherlands*, pages 205–232, 1992.

[62] Alfred Tarski. What is Elementary Geometry? *Proc. Intern. Symposium, Studies in Logic and Foundations of Mathematics, North Holland, Amsterdam*, pages 16–29, 1959.

[63] I. Thomas. A 12th Century Paradox of the Infinite. *The Journal of Symbolic Logic*, 23:133–134.

[64] Giovanni Boniolo, Paolo Vidali. *Filosofia della scienza.* Bruno Mondadori, Milano, 1999.

[65] Richard von Mises. *Wahscheinlichkeit, Statstik und Wahrheit*. Springer, Wien, 1928.

[66] Bertrand Russell, Alfred North Whitehead. *Principia Matematica*. Cambridge University Press, 1910–12.