Categorical Structuralism and the Foundations of Mathematics

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ABSTRACT

Structuralism is the view that mathematics is the science of structure. It has been noted that category theory expresses mathematical objects exactly along their structural properties. This has led to a programme of categorical structuralism, integrating structuralist philosophy with insights from category theory for new views on the foundations of mathematics.

In this thesis, we begin by by investigating structuralism to note important properties of mathematical structures. An overview of categorical structuralism is given, as well as the associated views on the foundations of mathematics. We analyse the different purposes of mathematical foundations, separating different kinds of foundations, be they ontological, epistemological, or pragmatic in nature. This allows us to respond to both the categorical structuralists and their critics from a neutral perspective. We find that common criticisms with regards to categorical foundations are based on an unnecessary interpretation of mathematical statements. With all this in hand, we can describe "schematic mathematics", or mathematics from a structuralist perspective informed by the categorical structuralists, employing only certain kinds of foundations.

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1

STRUCTURALISM

In this chapter, structuralism as a philosophy of mathematics is introduced. We shall go through the concepts central to this philosophy, such as structure, system, and abstraction. Certain problems in the philosophy of mathematics will be closed using them, while others left as open as before; we shall see wherein the difference lies. Finally, this chapter aims to provide an overview of the ontology and epistemology of mathematics from a structuralist perspective.

1.1 WHAT IS STRUCTURALISM?

Structuralism in the philosophy of mathematics is perhaps best summed up with its slogan: "Mathematics is the science of structure". A structuralist would describe mathematics as not being concerned with numbers, calculation, triangles, geometric figures, or any such objects. These may all occur in mathematics of course, but they are not its subject. The subject of mathematics is, on the structuralist account, something akin to pattern, relational structure or form.

Structuralism is perhaps best introduced by contrasting it with previous philosophies of mathematics. Many classic philosophies of mathematics take mathematics to be about mathematical objects, such as numbers or geometric figures. It is these objects, abstract as they may be, existing independently of the human mind or not, that form the basic "building blocks" of mathematics. Platonism, one of the most well-established philosophies of mathematics, has been characterised by Michael Resnik as revolving around an analogy between mathematical objects and physical ones.[1] To the platonist, mathematical objects are, in a way, like physical things, and like physical things, they may possess certain qualities (such as abstractness) and not possess others (such as extension or colour).[2] On the platonist account in particular, these objects have a certain independence: they exist regardless of anything external, be it the human mind thinking of these

1 [Resnik 1981], pp. 529

2 It is customary in the philosophy of mathematics to refer to theories positing the (mind-)independent existence of mathematical objects as platonism, after being so dubbed by Bernays in the 1930s (see [Bernays 1935]). There are many ways in which these theories are nothing like the philosophy of Plato, even on the subject of mathematics. Following contemporary custom, I shall nevertheless refer to such theories straightforwardly as "platonism".

objects, symbols referring to them, or physical objects exemplifying them in some way. To the structuralist, by contrast, it makes no sense to speak of mathematical objects *per se*. To be a mathematical object at all is to be part of a larger mathematical structure: no number 2 without a structure of natural numbers, no triangle without a geometry. The structuralist holds that mathematical objects are not truly independent, but at the very least dependent on the structure they are part of, and moreover, that they don't have any intrinsic properties. Whatever properties an object may have are merely relational ones, describing the object as it relates to other objects within the structure. It is through these two means that structuralism is usually characterised: through this dependence or through the lack of intrinsic properties.

1.1.1 *Structuralism as a matter of abstraction*

Turning to the "intrinsic properties account" first, what is typical of structuralism is that mathematical objects are nothing more than positions within a structure. We consider objects as mere "empty spaces" within a structure; that is to say, objects are nothing more than their relational properties within the structure, and in particular, they have no further internal structure or intrinsic properties. Michael Resnik most prominently developed this account of structuralism and described it as follows:

> In mathematics, I claim, we do not have objects with an "internal" composition arranged in structures, we have only structures. The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are structureless points or positions in structures. As positions in structures, they have no identity or features outside of a structure.[3]

What is put to the forefront here is a degree of abstraction characteristic of structures. When dealing with structures, objects may be involved, but everything about them is disregarded except for the relation they have within a structure. The structure, in turn, is nothing but the whole of these relations. Typically, a structure can be characterised through a rule or an array of rules. Examples of structures are typically geometric or algebraic. An easy one to grasp in particular is the structure of a group: a group consists of a domain D of objects with an associative operator $*$ on them, an inverse for every element of the domain, and an identity element e s.t. $a * e = a = e * a$ for all a in the domain. One may find that certain objects in other areas of mathematics form a group. The objects in the domain may be

3 [Resnik 1981], pp. 530

complicated mathematical objects themselves. For the group theorist, though, this is irrelevant. What is studied are the relations between objects in the group and through this, the group itself. The structuralist claim is then: as in group theory, so in all of mathematics. One may find something in the physical world that can be regarded as the group $\mathbb{Z}/60\mathbb{Z}$, such as the behaviour of the long hand on a grandfather clock, but one only engages in mathematics when one takes such an abstract view of it as to study merely the relations that hold on it. In such a case, one considers a minute as merely an empty point in the structure. Resnik further elaborates on the status of such points:

> A position is like a geometrical point in that it has no distinguishing features other than those it has in virtue of being that position in the pattern to which it belongs. Thus relative to the equilateral triangle ABC the three points A, B, C can be differentiated, but considered in isolation they are indistinguishable from each other and the vertices of any triangle congruent to ABC. Indeed, considered as an isolated triangle, ABC cannot be differentiated from any other equilateral triangle.
> ([Resnik 1981], pp. 532)

Thus, the differentiation between objects relies on a prior notion of structure. It should be noted that this is still not the strongest formulation of structuralism. The consideration of a geometrical point *as* a point in the mathematical sense, that is, not as a physical dot on paper but as an entity with a length of 0 in every dimension requires the consideration of a mathematical structure.

We find another expression of this account of structuralism in the works of Nicholas Bourbaki, characterising elements as having an unspecified nature prior to their connection by relations. Relations are in turn made intelligible by stating the axioms true of them, thus characterising the structure as an object of mathematical study:

> [...] to define a structure, one takes as given one or several relations, into which [elements of a set whose nature has not been specified] enter [...] then one postulates that the given relation, or relations, satisfy certain conditions (which are explicitly stated and which are the axioms of the structure under consideration)[4]

As part of their larger programme emphasising the role of the axiomatic method in mathematics, Bourbaki thus puts the axiomatic nature of the relations in the forefront.

Another way to characterise structure is in terms of roles and objects filling that role. A relational structure can take many shapes; it

4 [Bourbaki 1950] pp. 225-226, quoted in [Shapiro 1997]

can be the structure of natural numbers, of a programming language, or of a game of Tic-tac-toe. The objects within a structure then are roles that must be played in the structure. The structure of mathematical numbers calls for something to fill the role of the second number; the structure of Tic-tac-toe needs symbols for both players. These roles can be filled in many ways; traditionally, crosses and circles are the symbols used in Tic-tac-toe, but this is obviously not fundamental to the game as a structure. The properties of the game don't change if we use squares and triangles instead - in particular, the game will still be always a draw if both players play perfectly. Mathematics, then, is the study of structures and the roles therein *qua* roles. The mathematician completely disregards whatever fills any particular role in a structure, and then proceeds to see what he can still show about the structure. As such, it is a *mathematical* result that Tic-tac-toe always results in a draw if both players play perfectly.[5]

Stewart Shapiro introduces the term *System* for any collection of objects with interrelations among them. A structure is then the abstract form of such a system taken only as interrelations between abstract objects, disregarding any feature of the objects, physical or otherwise, that is not of this nature.[6] The Arabic numbering system or sequences of strokes may then both be considered systems expressing the natural number structure. It is important to note that the system/structure dichotomy is a relative one. A particular mathematical structure may be found in other mathematical structures, and thus serve as a system as well. For example, the set theorist might recognise the ordinals $\emptyset, \{\emptyset\}, \{\{\emptyset\}, \emptyset\}, \ldots$ as a system expressing the natural numbers structure. Likewise, he might find the same structure in the series $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots$. It is a particular claim of the structuralist that neither of these sets *are* the natural numbers. They are merely different systems expressing this structure. $\{\{\emptyset\}, \emptyset\}$ and $\{\{\emptyset\}\}$ may both fill the role of 2 in the natural number structure, but that does not make them the number 2.[7]

The notion of "object" itself in a structural framework does leave some room for explication. In particular, the link between a structure and that what it is abstracted from, and Stewart Shapiro's system/structure dichotomy in particular, leave room for two different interpretations of the notion of object. Based on the structuralist slogan "Mathematical objects are places in structures", Shapiro calls these the *places-are-offices* and the *places-are-objects* perspectives.

5 There is an argument to be made that further properties are necessary for a structure to be mathematical in nature; for one, deductive proofs need to be applicable as a tool to investigate the structure. Since we want to concern ourselves with philosophy of mathematics rather than with general epistemology, we leave this issue open for now and refer to mathematical structures simply as "structures".

6 [Shapiro 1989] pp. 146

7 A rather famous argument based on this inequality was made by Benacerraf in [Benacerraf 1965]. We will come back to this in section 1.2.2.

One can regard a place as a role to be filled, as an "open office", so to speak. Borrowing an analogy from Shapiro, we can consider the structure of the American federal government. This structure features political positions such as "Senior Senator for New York" as its objects, and relations such as "x elects y" as its relations. A relation within this structure might be "The president nominates judges for the Supreme Court". Nevertheless, we often use structural terms in the context of a particular system. For example, we may truthfully utter the sentence: "The president has a Kenyan father". This does not express a structural truth about the system, about the office of the president as it relates to other positions in the government, as a place in the structure. It instead talks about a particular system instantiating this structure by way of referring to objects within the structure; "The President" is used to refer to Barack Obama. This is the *places-are-offices* perspective; we refer to positions in the structures as offices to be filled, always with a specific interpretation or exemplification in mind. Our example of a relation in this structure, however, did not refer to the object "President" in this way. When we express that the president nominates judges for the supreme court, we aren't talking about Barack Obama, or about any holder of the office of president in particular; rather, we are expressing a property of the position itself. This perspective regards a position as an object in itself, to be considered independently from any particular way to fill the position. This view is called the *places-are-objects* perspective.[8] Unlike the *places-are-offices* perspective, it has no need of a system, or of any background ontology of objects that may fill the offices.[9]

1.1.2 *The identity of structures*

The system/structure dichotomy suggests a relation between the structure on one hand and the structured, the system, on the other. In fact, there should be a way for two systems to exemplify the same structure. To make this precise, Resnik took a relation between different structures as a starting point. The principal relation between different structures is one of *congruence* or *structural isomorphism*. A congruence relation exists when there is an isomorphism between two structures. An isomorphism is traditionally taken as the method of saying that two structures are the same: and two structures A and B are isomorphic if there is a bijective relation $f : A \longrightarrow B$ on the objects and relations on A s.t. for every relation $R_1, R_2, ...R_n$ on A, if aR_xb, then $f(a)f(R_x)f(b)$.[10] It is not a rare occurence that two structures are

8 Some philosophers, such as Resnik, deny that there is such an object, and *a fortiori*, that there is such a perspective. Statements like these can be interpreted alternatively as generalisations over all the occupants of the office. See section 1.3.2 for a discussion of this view.

9 [Shapiro 1997], pp. 82

10 [Shapiro 1997], pp. 91

not isomorphic because they do not feature the exact same relations, even though they very well *could* be through a matter of definition. Resnik cites the example of the natural numbers with the "less than" operator $<$ and the natural numbers with a successor function S.[11] In order to be able to say in such cases that we are still talking about one structure rather than two distinct ones, a weaker notion than isomorphism has been introduced. One can call two structures *structurally equivalent* if there exists a third structure, object-isomorphic to both structures, and with relations that can be defined in terms of the relations of both structures.[12] For example, let $\mathbb{N}_S$ be the natural numbers with a successor relation S but no "less than" relation, and let $\mathbb{N}_<$ be the natural numbers with no successor relation but with a "less than" relation $<$. We can then formulate a third structure $\mathbb{N}_3$ with the relation $<$ as in $\mathbb{N}_<$ and with a relation S defined as follows: aSb iff $b < a \wedge \neg\exists c : b < c < a$. Now $\mathbb{N}_3$ is object-isomorphic to $\mathbb{N}_S$ and $\mathbb{N}_<$, and all of its relations can be defined in terms of the relations of $\mathbb{N}_S$ and $\mathbb{N}_<$. Hence, we can conclude that $\mathbb{N}_S$ and $\mathbb{N}_<$ are structurally equivalent. This construction serves to free us from needing to claim that these two entities are not the same, because they are not strictly isomorphic, even though they are intuitively different depictions of the exact same kind of mathematical structure.

The process distinguishing a certain structure within another is called Dedekind abstraction: certain relations among the objects are emphasised, and features irrelevant to these interrelations are left out completely. The result is a new structure which then again stands in an isomorphic relation with the old.[13]

The relation connecting the structured with the structure may also connect arrangements of concrete objects, such as physical objects or symbols on paper, with an abstract structure. In such a case the arrangement or system is said to *instantiate* the structure. This notion of a relation between arrangements or systems of concrete objects with abstract concrete objects is not epistemologically simple. In particular, it presupposes that the concrete objects can be regarded as having structure of their own in some way. There are many theories on how such a connection can take place: the Platonist holds that a concrete object may participate into an eternal, abstract Form, the Aristotelian that we gain the structure through a mental process of abstraction, and the Kantian that it is a feature of human consciousness to add such structure to the world in order to understand it. The structuralist view is not limited to one of these theories and may be combined with a number of views on the matter, but the viewpoint does suggest a movement away from theories connecting mental or ideal objects with concrete objects (such as traditional Platonism) and

11 [Resnik 1981], pp. 536
12 [Resnik 1981], pp. 535-536
13 We will come back to Dedekind abstraction in section 1.2.1.

towards theories able to handle connections between entire abstract structures and systems.

1.1.3 *Structuralism as a matter of dependence*

The other account of structuralism, and the one that prompted a comparison with platonism, is the dependence account. On this view, the main thesis of structuralism is that mathematical objects are dependent on one another or on the structure they are a part of. This contrasts most sharply with platonism, as that latter theory relies first and foremost on a notion of independence. It is typically stated explicitly that mathematical objects are independent of the human mind. The independence of mathematical objects goes further than that, though: a number may be said to be independent of any concrete physical objects and of other mathematical objects, such as triangles. The strength of this argument relies on a notion of truth: it is a particularly strong intuition that mathematical truths are "static", and that changing the properties of certain objects should leave mathematics unaffected.[14]

When establishing such a thing as a dependence relation among concrete objects, it seems obvious that objects may depend on other objects at the very least if we take the notion of dependence to be an existential one: for example, the existence of a particular table is not dependent on the existence of a chair, but it seems to be dependent on the existence of atoms and molecules. An existential dependence relation tends to hold between concrete objects and relations holding among them as well; two objects cannot be of the same size if they do not exist first, while two objects may very well be said to exist without there existing a same-size relation between them. Another way to characterize dependence is through identity; in such a case, X can be said to depend on Y if Y is a constituent of some essential property of X.[15] Considering that the existence of the relata is essential to the relation, the conclusion may be made that, for concrete objects, the object is prior to any relations that may hold on it.

The structuralist holds that in the case of mathematics, this priority is inverted. The mathematical object depends on the existence of a certain relational structure.[16] At the very least, the structuralist claims

14 Of course, this is a crude picture of mathematical platonism, meant merely as contrast with the structuralist account. Some of the more obvious problems with regards to the independence of mathematical objects are readily answered by platonists. Traditionally, the necessity of all mathematical objects has been posed, thereby positing the whole of mathematics, as it were, "at once", and avoiding situations in which a mathematical object depends on an entity that doesn't exist. Hale and Wright ([Hale & Wright 2001]) characterise the independence of mathematical objects as merely independence from objects of another sort, not from each other.

15 [Linnebo 2008], pp. 78

16 Whether the converse holds, i.e. whether we can think of mathematical structures while completely ignoring the possibility of objects in them, is a different and per-

that there is no such thing as priority between mathematical objects and their relations. Mathematical objects exist simply *as* part of the structure they are part of. It is perhaps easiest to illustrate this using the example of numbers, due to Shapiro.[17] Whereas the traditional, "object-based" Platonist would hold that all numbers simply exist, independently of us and of each other, the structuralist holds that the relation between numbers is what *makes* them numbers. It constitutes an essential feature without which they would not be numbers. The structure of natural numbers is such that there is a first number and a successor relation; numbers, as objects, depend wholly on this structure, and all their properties derive from it. Numbers can in this sense be seen as simply *being* positions within this structure: "3" is the third position in it, "4" the fourth, and so forth.

In "Structuralism and the notion of dependence", Øystein Linnebo has argued that the "intrinsic properties account" of mathematical structuralism reduces to dependence claims. He distinguishes this view further into two accounts: one claiming that mathematical objects have no non-structural properties, and one claiming that they have no internal composition or intrinsic properties. Dealing with the latter first, for an object to have any internal composition, or more generally, any intrinsic properties, is for it to have properties that it would have regardless of the rest of the universe. Thus, for an object not to have any intrinsic properties is for it only to have properties that it has on account of the rest of the structure. On the structuralist claim that a mathematical object is no more than its position in a relational structure, this equates to the claim that a mathematical object is dependent in all its properties on the structure.

The claim that mathematical objects have no non-structural properties is more directly challenged. The obvious candidate for a definition of a structural property comes from the "abstraction account" of structuralism: a property is structural if and only if it is preserved through the process of Dedekind abstraction. This account runs into straightforward counterexamples. Numbers seem to have more properties than merely structural ones: they can be expressed using Arabic numerals, they are abstract, et cetera. Some of these properties, such as abstractness, are even necessary properties, making a weakened claim that mathematical objects have no non-structural necessary properties false as well. Linnebo suggests that a yet weaker claim may suffice, though: mathematical objects have no non-structural properties that matter for their identity. This statement can then be equated with a dependence claim again: it is equivalent with the statement that mathematical objects depend for their essential properties on the structure they are in. Thus on the structuralist account

haps more subtle point. It seems that there are at the very least certain mathematical structures that operate without objects. Category theory, for example, can be formulated using only the relational notion of a morphism.

17 [Shapiro 2000] pp. 258

of mathematics, there is an "upward dependence": objects depend on the structure to which they belong, as opposed to the "downwards dependence" typical of physical objects, where larger, more complex entities cannot exist without their parts.[18]

Linnebo further argues that set theory, on the usual iterative conception of set, cannot fit into a structuralist framework. This is because the dependence relation in set theory is fundamentally "downwards". On the iterative conception of set, a set is anything that exists in some place in the iterative hierarchy of sets. On the first stage, we have the empty set and any possible set of *urelemente* we may wish to have in our universe. Each subsequent stage contains all sets consisting of some combination of previous sets.[19] The totality of such stages then encompasses the totality of all sets. Linnebo claims that set theory is thus a counterexample to the dependence claim of structuralism: sets depend "downwards" on their constituents, out of which they were formed, and not "upwards" on any sets containing them.[20] A particularly strong example is that of the singleton: it is clear that the singleton set of some object depends on that object, but it is hard to imagine the object as being dependent on the singleton it is contained in.[21]

Linnebo himself has characterised the dependence relation in two ways: the claim that any mathematical object is dependent on all other objects in the structure ("ODO"), and the claim that mathematical objects are dependent on the structure they are part of ("ODS").[22] The example of the singleton seems, at first sight, a counterexample to the first claim. It is less clear why it would be a counterexample to the second, though. A more thorough analysis of the situation is wanted. Let A be some singleton set: let $A = \{B\}$. It is clear that a singleton set depends for its identity on the element it contains. But it seems hard to argue that it does not also rely on the entire set-theoretic structure.

Consider that, in order for the argument to work, the singleton set here must be a pure mathematical object. It is not a collection of one object in any metaphysical sense involving more properties than the mathematically given ones. The singleton set A is *entirely* given by the fact that it is part of the set-theoretic hierarchy, and the totality of $\in$-relations defining it: in this case, $B \in A$. It is a set, and what it means to be a set is for it to occur at some stage in the set-theoretic

18 [Linnebo 2008], pp. 66-68
19 [Boolos 1971], pp. 220-222
20 [Linnebo 2008], pp. 72
21 [Fine 1994] pp. 5
22 [Linnebo 2008], pp. 67-68.

hierarchy. This is exactly equivalent with the following conjunctive statement:

> *A* can be formed out of the objects it contains through a single application of the ∈-relation, and all the objects it contains are present at some earlier stage in the hierarchy. (1)

Thus, there is a clear dependence, vital for the identity of the singleton set *A* as a mathematical entity, on the set-theoretic hierarchy as a whole.[23] The case can be made even stronger when we consider that, since any set is fully given by the sets it stands in the ∈-relation with, it depends fundamentally on this relation.[24] After all, we may imagine a situation wherein there is such a thing as the ∈-relation, but not this particular set (due to e.gȧ change in the axioms governing the existence of certain sets), but we cannot conceive of a set without conceiving of it as containing elements. Dependence on this relation can hardly be considered downward: if we take ∈ as a primitive notion, it is simply captured by the axioms prescribing its use. It seems difficult to get closer to the structuralist claim that this equates to dependence on the very structure of set theory, and hence ODS. If we take ∈ not to be a primitive, but to be a relation given by the pairs of relata it connects, then any set is dependent on all those other sets related somehow by ∈, and we come back to the other of the structuralists' two dependence claims, ODO.

The platonist might balk here, claiming that the equation of a mathematical entity with its version in a limited mathematical system is an incorrect one. For example, a set *S* may turn out to have properties and relations in the full set-theoretic universe *V* with the usual Zermelo-Fraenkel axioms that it does not have in, say, a finitist limitation of it. Likewise, *S* may have more properties when we add more axioms, such as ones stating the existence of inaccessible cardinals. If we consider *S* "in its full splendor" then, not limited by any specific

23 An alternative formulation is to simply ask that the objects it contains are sets themselves. (In our example, this is simply *B*.) This, in turn, is the case if they can be formed through a single application of the ∈-relation out of the objects it contains, and that all the objects it contains are sets themselves. The downward dependence continues. This manoeuvre does little more than buy time, though. On the iterative conception of set this process must end somewhere: at a set containing either only *urelemente* or at the empty set. It seems impossible to formulate why these are in turn sets without referring to the definition of the hierarchy, on account of which they are. Non-well-founded set theories may be trickier on this regard, but in those cases one may ask whether there is a downward dependence at all. In either case, the dependence on the ∈-relation is clearly present still.

24 The very idea of depending on a relation is not uncontroversial. One might consider that a relation always presupposes its relata, and hence putting a relation on top of the hierarchy of dependence makes little sense. For now, I will leave this with a suggestion that there may be no need to presuppose relata in mathematics. What is *prima facie* prior here are those mathematical terms taken as primitives. A more thorough way to avoid this problem is given by Awodey, who substitutes the relation for the morphism. We shall turn to this in detail in section 2.2.3.

axiomatisation, it may be considered independent of them. The structuralist answer to this is to simply grant this. On the structuralist account, mathematics is about objects only in as far as they are characterised mathematically, i.e. in a structure. If there is such a thing as a "true" S with all the properties it "should" have, in such a way that it cannot be captured by any mathematical system, then it is neither a mathematical object in the structuralist sense, nor the kind of object the mathematician actually studies in practice.

Of course, a dependence between a singleton set and the whole set-theoretic framework does not exist if we consider a set on the naive conception. On this conception, a set is any extension of a predicate. To take another singleton example, consider the set containing only Queen Elizabeth II. On this account, we can disregard the second conjunct of (1) and thus the dependence on the set-theoretic structure. But to the structuralist, this is simply to say that naive set theory is *not* a proper mathematical structure. The set theorist has known all along.

As Linnebo points out, many other areas of mathematics seem to behave straightforwardly in a way in line with structuralism. Chief amongst these are algebraic structures such as groups. More generally, this holds for any structure gained through Dedekind abstraction: for consider such a structure, consisting only of objects as determined by their relations, with all other features left out. The dependency here is clearly "upwards" in a non-roundabout way. the grander structure of a group, for example, determines the behaviour of its objects. In particular, consider again the relations $R_1, \ldots, R_n$ of a particular structure. Let $a(x)$ denote the arity of R_x. We can then consider relation $R = R_1 \times R_2 \times \ldots \times R_n$, which holds between $x_1, \ldots, x_z$ and $y_1, \ldots, y_z$ if and only if $x_1, ..., x_{a(1)} R_1 y_1, \ldots, y_{a(1)}$, and $x_{a(1)+1}, \ldots, x_{a(2)} R_2 y_{a(1)+1}, \ldots, y_{a_2}$ and so forth up to R_n. This relation R then effectively functions as simple combination of all the relations $R_1, ..., R_n$. In particular, this single relation can now be said to fully determine the group. The behaviour of any particular object in a group is defined by its relations with other objects, which is in turn given entirely by R. We thus have complete dependence of the objects of the group upon the structure as characterised by R. Thus, we can characterise any mathematical structure gained through Dedekind abstraction by its structure, considered as the whole of its relations.

1.1.4 *Taking stock*

We can conclude that characterising structuralism in terms of its objects, in particular through their supposed lack of internal structure, does not suffice. Rather, we can establish structures as determining their objects, or as gained through Dedekind abstraction. These accounts may be considered equivalent, as they straightforwardly imply one another. There seems to be no particular reason to take the dependency view of structuralism *over* views based on the abstract nature of structures, although Linnebo's demand for more attention to the notion of dependency in structuralist philosophy seems well-placed.

Summarising the structuralist account of mathematics, we can state the following:

1. One engages in mathematics when one treats any arrangement of objects, concrete or abstract, merely in terms of the relations that hold amongst the objects therein.

2. The whole of such relations is a structure, and is typically characterised by rules establishing the behaviour of the relations.

3. Structures are obtained through a process of *Dedekind abstraction* from other structures.

4. As a consequence, structures are only determined up to isomorphism.[25]

5. Mathematical objects are dependent on the structure they are part of. In particular, they are thus also only determined up to isomorphism.

The philosophy of mathematics has always been concerned with ontological questions, regarding the existence or status of mathematics, and epistemological questions of how we can gain mathematical knowledge. The structuralist view has shifted the focus of these questions: rather than philosophise about mathematical objects, we now ponder the structures they are part of. This shift in attention has allowed certain questions regarding mathematical objects to be solved. Other, more dire problems, such as the platonist thesis of a mind-independent existence of mathematical objects, have simply shifted along: they are now questions about structures as a whole. In the following paragraphs, we shall go through these systematically. First we shall deal with questions regarding the ontology and identity of mathematical objects, second we shall pay attention to the ontology of structures, and finally, attention shall be paid to questions of epistemology on which the structuralist account can shed new light.

1.2 THE IDENTITY OF MATHEMATICAL OBJECTS

The question "What are mathematical objects?" is not merely a question of dependence or independence. It is a general demand in philosophy that we be able to identify an object and be able to differentiate

25 More precisely, they are determined up to structural equivalence, but this difference is mathematically nigh-trivial.

it from different objects. This echoes a dictum by Quine: "No entity without identity". There are a few philosophical problems with regards to the identity relation and mathematical objects: in particular, Frege's "Caesar Problem", relating to the identity between mathematical objects and non-mathematical ones, and Paul Benacerraf's challenge in "What Numbers Could Not Be", relating to the identity between different kinds of mathematical objects. The structuralist view of mathematics does not purport to answer all problems in the philosophy of mathematics, but these identity problems seem to have a tendency to fold to a structuralist analysis. To aid in this endeavour, a more precise look at Dedekind abstraction is due first.

1.2.1 *Dedekind abstraction*

The term "Dedekind abstraction" was introduced by William Tait to describe the process of obtaining new types of objects in mathematics. The canonical example is the acquisition of the natural numbers from a different, more complicated mathematical system, such as as the collection of all ordinal numbers $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$.[26] This process goes back to Dedekind's 1888 article "Was sind und und was sollen die zahlen?". In this article, Dedekind took up the challenge of giving a mathematically precise notion of the natural numbers. He was responding to a mathematical challenge at the time; Frege, amongst others, took up this same challenge and identified numbers with extensions, which are then captured by sets. In [Resnik 1981], Michael Resnik remarks that the importance of this work today does not lie in its mathematical value, but rather in the philosophical interpretation it gives of notions such as the natural number.

A key difference between Frege and Dedekind in their interpretation of the natural numbers is that while the former identified them with a singular kind of mathematical object, Dedekind emphasised their generality. To Frege, numbers *were* a kind of set. Dedekind's approach, on the other hand, was to identify a specific kind of system within other mathematical objects: the simply infinite system. If one can specify a successor function such that there is a unique successor $S(n)$ for each n in the domain N and an initial object 0, such that induction holds (in second-order logic, $\forall X(0 \in X \land \forall x((x \in X) \longrightarrow (S(x) \in X)) \longrightarrow N \subseteq X)$), then one is dealing with a simply infinite system $(N, 0, S)$. The direction where Dedekind is going seems clear: the initial element 0 has to do with the natural number 0, $S(0)$ with 1, et cetera. However, the natural numbers are not simply identified with the objects in any such system. Rather, an extra step is taken:

> If, in considering a simply infinite system N, ordered by a mapping ϕ, one abstracts from the specific nature

26 [Tait 1986], footnote 12

> of the elements, maintains only their distinguishability, and takes note only of the relations into which they are placed by the ordering mapping ϕ, then these elements are called *natural numbers* or *ordinal numbers* or simply *numbers*, and the initial element 1 is called the initial number (*Grundzahl*) of the number series N.[27]

Here, the notion of abstraction is made explicit. When we talk of "Dedekind abstraction", this an be seen as a process in two steps. The first step is decidedly mathematical: one has to show that specific relations hold in some mathematical structure, connecting a collection of objects within the system. The second step is to consider the objects thus connected as no longer within the original system, but as a *new* mathematical structure, featuring only the relations shown in the first step and the objects involved by these relations. This structure can then be seen as a new object of study for the mathematician.[28] Thus, our earlier quick characterisation of Dedekind abstraction as a matter of simply emphasising certain relations and leaving out others should be seen as, while true, oversimplified. It makes it seem as a simple matter of picking and choosing from a system that is already clear, whereas in reality, it will often be mathematically nontrivial to locate a particular mathematical structure within some system.

Dedekind's account gains strength through a categoricity proof: any two simply infinite systems $(N, 0, S), (N', 0', S')$ are isomorphic. Thus, whenever we find that the relation S holds on some domain N with some initial object 0, we may consider ourselves to be talking about the same structure: the natural numbers. The source of our structure, the mathematical system it was once abstracted from, does not matter at all. Once we have characterised a certain structure, we have identified it: there are, after all, no mathematical properties of the structure to be found outside the scope of our structure. The appeal to the structuralist should be clear: the natural numbers are, after having been acquired through a process of abstraction, of such a nature that the only mathematical properties that hold of them are the relational properties essential to the natural numbers structure, and the natural numbers are unique and identifiable only up to isomorphism.

27 [Dedekind 1888] par. 73, quoted in [Parsons 1990], pp. 307

28 There is nothing in principle preventing this from being a vacuous exercise; one could emphasise relations so fundamental to the original system that after the process of abstraction, we are left with a new structure that is structurally equivalent to the old. For example, if we start out with a natural number structure with an ordinary addition operator $+$, and leave out the successor function S, we do not change anything: in the new system, one could define S again through $S(x) = x + 1$. Whether one would still consider such processes a matter of Dedekind abstraction is a semantic choice of little philosophical interest; but for the remainder of this thesis, it may be assumed that when we mention Dedekind abstraction, a non-trivial abstraction is intended.

1.2.2 *Benacerraf's Problem and the Caesar Problem*

With this new tool in hand, we can turn to a problem raised by Benacerraf in his famous 1965 article "What Numbers Could Not Be" ([Benacerraf 1965]). In this, he sketches a picture of two children raised and taught mathematics in slightly different ways. While both are taught that numbers are to be identified with particular kinds of sets, the devil is in the details. The first child is taught that numbers are the Von Neumann ordinals: 0 is the empty set $\emptyset$, 1 is $\{\emptyset\}$, 2 is $\{\emptyset, \{\emptyset\}\}$, and so forth. In particular, the ordinals are transitive: $a \in b$ if and only if $a \subsetneq b$. The "less than" relation can be easily defined on these ordinals as follows: $a < b$ iff $a \subsetneq b$ iff $a \in b$. Since the natural numbers are identified with these ordinals, it then follows that since $3 < 7$, $3 \in 7$.

The second of the two children is taught a similar thing. However, he learns that the natural numbers are a different set of ordinals, the Zermelo numerals: 0 is $\emptyset$, 1 is $\{\emptyset\}$, 2 is $\{\{\emptyset\}\}$, and generally $n+1$ is $\{n\}$. On this account, $a \in b$ if b is the direct successor $S(a)$ of a, not if $a < b$ in general. For purposes of ordinary arithmetic, the two children will agree. For each child, $3 + 7 = 10$ and 11 is a prime number. However, there are set-theoretic matters that drive a wedge between them. Whereas the first child will insist that $3 \in 7$, the second finds that only the direct successor of a number contains that number, and hence while $6 \in 7$, $3 \notin 7$.

It is notable that the mathematician has no way to settle the matter.[29] Both accounts of the natural numbers lead to a consistent arithmetic. In fact, both accounts lead to the same arithmetic: any notion that can be expressed in the language of arithmetic can be thus expressed regardless of the set-theoretic identity of the numbers, and any question formulated in that language will have the same answer regardless of that identity. The questions that the children will disagree on are matters of set theory. What makes the issue particularly thorny is that there is no set-theoretic answer to the question, either. The difference between the two accounts is a result of a different definitional choice. Neither derives from a previous mathematical result; such a thing would be impossible, the natural numbers not being part of the language of set theory prior to such a definition. Nevertheless, it is clear that these different identities of arithmetic cannot both be true. $3 \in 7$ cannot be both true and false; and more directly, $\{\{\emptyset\}\} = 2 = \{\emptyset, \{\emptyset\}\} \neq \{\{\emptyset\}\}$ is a straightforward inconsistency.

Our understanding of Dedekind abstraction can then be used to shed light upon this problem. Both formulations of the natural num-

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29 The mathematician may have more subjective reasons to prefer one series of ordinals over the other. Considerations of e.g. mathematical beauty may play a role in such a choice. This goes beyond the scope of this essay; for our present point, it is sufficient that there is no *strictly mathematical* way to establish which set of ordinals has the best claim to "being" the natural numbers. See [Paseau 2009].

bers can be seen as applications of this technique. We can identify the natural numbers, *as* a simply infinite structure, in either of the ordinals. More accurately perhaps, we can acquire two different systems of natural numbers through abstraction from the set-theoretic universe V. By Dedekind's result, these two systems are then isomorphic to one another. And since structures are only determined up to isomorphism, this means that both systems exemplify the very same structure. The structuralist would simply hold that both series of ordinals instantiate the natural numbers structure, if provided with the correct successor function.

The question of the identity of mathematical objects is then tackled by restricting domain on which questions with identity statements are considered meaningful. Benacerraf sought such a solution to the problem in his original formulation:

> "For such questions to make sense, there must be a well-entrenched predicate C, in terms of which one then asks about the identity of a particular C, and the conditions associated with identifying C's as the same C will be the deciding ones. Therefore, if for two predicates F and G there is no third predicate C which subsumes both and which has associated with it some uniform conditions for identifying two putative elements as the same (or different) C's, the identity statements crossing the F and G boundary will not make sense."[30]

Within a contemporary structuralist framework, we can make this notion more precise. Whereas Benacerraf could not yet formulate the conditions for identifying two elements under the single predicate C, we may now avoid finding such a predicate and such conditions altogether. Rather than finding a specific predicate, we can be certain that identity is unproblematic within a single structure. We can simply take the mathematical rules already governing identity within such a structure be decisive in the matter. Moreover, and perhaps more importantly, this is all there is to say on the identity of mathematical objects. Their identity is always relative to the structure they are in, and it is simply nonsensical to ask for an identification of a position within a structure with an object outside it.

We may, of course, choose to do so as a matter of convention - as it is convention to associate the natural numbers with the Von Neumann ordinals in set theory - but even if such identities are regarded as truth, they are truths of convention in a way that identities within a structure will never be. A group of mathematicians cannot simply decide that 12 is a prime number within the ordinary structure of natural numbers; such mathematical properties of 12 are set in stone by the axioms of the number structure. Any identity such as

30 [Benacerraf 1965] pp. 65

$2 = \{\emptyset, \{\emptyset\}\}$, however, can be overturned at any moment. Such an identity can be seen either as acknowledgement that the natural numbers structure can be obtained through Dedekind abstraction from a particular collection within set theory, or even simply as a notational shorthand for the Von Neumann ordinals. We should likewise interpret certain other common associations between different structures, for example the association between the numbers $1, 2, 3...$ in the natural number structure, and the "same" numbers in the rationals, reals, or complex number structure. These too are not strictly identifications. The natural number 3 has a direct successor in 4, but the real number 3 does not; the real number $-e$ has no square root, but the complex number does. Generally, the ontology of mathematical objects is always relative to their structure. Cross-structure identifications are at best not strictly speaking identities, and at worst nonsensical.

The realist with regards to mathematical objects, in particular the Fregean, faces an even more general version of this particular problem. If mathematical objects are granted existence on par with physical entities or other objects, the identity relation needs to be defined over all these objects, lest we be unable to individuate between certain independently existing objects. To Frege, the notions of identity and object are always unambiguous.[31] This leads to a particularly strange version of the above identity problem, which does not concern identity across different areas of mathematics, but rather across mathematical and non-mathematical realms. This problem came to be known as the *Caesar Problem*, after an example by Frege: what is the truth value of the identity $4 =$ Julius Caesar? Without a criterion to decide the matter, this is a rather awkward open problem. The relativity of identity to a particular structure allows us to conclude that there simply is no answer to this question, that it is nonsensical. There is no identity criterion that crosses the boundary between a structure and what lies outside it; identity within a structure is part and parcel of that structure. As a welcome consequence, the mathematician is then always free to create a new structure, taking care to have a mathematical criterion for identity on its objects, without having to worry about its relation to other mathematical structures or non-mathematical objects.[32]

We should take care to note that ordinary usage of "is" is open to multiple interpretations. We are familiar with using "is" to signify identity, and the "is" of role-filling ("$\{\emptyset, \{\emptyset\}\}$ is 2") should be rather unproblematic as well. We have seen that one may also associate objects within one structure with objects within another. Another case worthy of special attention is our tendency to associate numbers with real objects; for example, we may want to consider it true that the

31 [Shapiro 1997], pp. 80-81
32 *Ibidem*, pp. 81

number of planets in the solar system is eight. This is to be interpreted as finding a particular structure in these objects: the cardinal structure eight. The cardinal number structures are perhaps the simplest of structures. They consist simply of a fixed number of objects with no relations amongst them, or exclusively with an identity relation on each single one. Thus, we can see the cardinal number eight, as a structure, exemplified in the planets of the solar system.

If we want to do calculations with this, it is fine to again associate this cardinal structure with the natural number 8 within the natural numbers, but again, this is not strictly an identity. This means that certain complex sentences, such as "The number of planets in the solar system is one more than the number of cardinal virtues", require a bit of interpretation. First we have to acquire two cardinal structures (the 8-structure "||||||||" and the 7-structure "|||||||"), then we have to associate them with places 8 and 7 in a natural number structure, and finally we can determine the truth of this sentence by checking whether $8 = 7 + 1$ is true within that structure. Perhaps such an interpretation offends our sense of mathematical beauty or simplicity, but that does not mean it is not accurate; it may even explain why the sentence itself strikes us as rather awkward and artificial.

Cross-structure identifications over mathematical objects like these do allow us to illustrate a certain relativity in the perspective we take on mathematical objects. What is a mathematical object within one structure is merely a system exemplifying it within another. The number 2 is an object in and of itself in ordinary arithmetic, and we refer to it as an object (using Shapiro's terms, from the *places-are-offices* perspective) when we say that 2 is prime. When working in another structure, however, we may make statements about 2 as an office, to refer to an object within the structure taking the role of 2. In the context of Zermelo-Fraenkel set theory, for example, "2 has two elements" uses 2 not to refer to the natural number directly, but to the set $\{\emptyset, \{\emptyset\}\}$ filling the role of that number.[33] There is a certain ambiguity in how we interpret such a statement; it does not seem intuitively wrong to see "2" here as mere shorthand for $\{\emptyset, \{\emptyset\}\}$. This does not, however, do justice to the mathematical import of calling the set "2" and not "X" or "*Johnny*". The symbol "2" is not neutral but carries a certain weight: it is primarily associated with a certain number - the second natural number, or the cardinal 2 ("||") - and thus with certain well-known structures, such as the natural numbers, the reals, or simply the cardinal number structure 2. While we cannot say that it is *wrong* to use 2 as a mere shorthand for a certain set, we capture more of what is happening mathematically when we see it as a *places-are-offices* reference to that set. Using this interpretation, we make explicit what is implicit in calling the set "2": that a certain structure, in this case the natural numbers, can be obtained

33 [Shapiro 1997] pp. 83

within the set-theoretic universe V we are working in. The tool with which we do this is simply the Dedekind abstraction we encountered earlier - though once we are familiar with the process of abstracting a specific structure from another, terms such as "2" and $\{\emptyset, \{\emptyset\}\}$ may grow to seem increasingly synonymous.

1.3 THE ONTOLOGY OF STRUCTURES: THREE SCHOOLS

We have seen that certain problems in the philosophy of mathematics get a clear answer when reformulated in structuralist terms. The concept of structure is a powerful tool that can allow us to explain certain phenomena that we struggled with previously. Like any concept, though, it has a limited capacity to explain and clarify, and as a result, certain problems remain stubborn in the face of a structuralist account. And though we may gain a new perspective on these issues, a reformulation in structuralist terms may truly feel like little more than just a reformulation. A significant number of ontological questions with regards to mathematics seem to fall in this category. Certain ontological questions, after introducing the concept of structure, suddenly clearly concern objects or relations within a structure, or multiple such objects in multiple structures. The Caesar problem was one such concern. These are suddenly placed in the middle of a rich framework of concepts, and may fold to a bit of analysis. Others, though, when viewed from a structuralist framework, simply shift along, turning into questions about a structure *as a whole*. And although new insights may still be gathered by regarding these questions as questions regarding structures rather than simply regarding objects, it is less clear that this helps us forward in any large way.

Issues that fall in this second, stubborn category include traditional ontological questions regarding the way in which mathematical objects - now structures - exist. Are they to be found "out there" in the world of phenomena, like physical objects? Are they constructions of our minds, bound to our psychology? Do they exist independently of both our minds and physical worlds? Do they exist at all? These questions - matters of realism and antirealism, psychologism and nominalism - remain as open questions about structures. A few of these have gotten thorough reformulations in structuralist terms, and it is these we will focus on in the following section.

1.3.1 Ante rem *and* in re *Structuralism*

Shapiro's system/structure dichotomy equips us particularly well to discuss the main ontological division amongst structuralists: the distinction between *ante rem*, *in re*, and *eliminative* structuralism. Leaving aside the latter for now, the difference between the first two is one of ontological priority between structures and the systems expressing a structure. According to the *in re* structuralist, structures exist, but only in as far as they are expressed in systems. The game of Tic-tac-toe, as a structure (i.e. as studied by the mathematician), exists because there are games of Tic-tac-toe that are actually played, embodying this structure in the world. Thus, mathematical structures do exist, and they exist within the objects expressing the structure, or as a result of certain features of human cognition on certain objects. Hence, *in re* structuralism: structures exist *in the things* of our ordinary perception.

In re structuralism is the structuralist equivalent of a much older position within the philosophy of mathematics, Aristotelean realism. In both philosophies there is a one-way dependence relation between mathematical or abstract entities and physical ones, and the physical come first. In both cases, this has advantages and disadvantages compared to other viewpoints such as platonism. Advantages include an arguably simpler epistemology, since we need not conceive of abstract objects independently of the world around us, and a clear explanation of the applicability of mathematics. If we gain mathematical structures from physical objects and phenomena, it does not stretch the imagination to say that perhaps these mathematical structures can be used to explain them.

The main problem haunting Aristotelean realism and *in re* structuralism alike is a matter of ontological poverty. There is an infinite wealth of mathematical objects and structures that the mathematician investigates in practice, but it is not clear that each and every one of these can be said to exist "in things" of the physical world. Perhaps the most straightforward example is literal infinity, which is not exemplified in the world if the world is finite.[34] The consequences can be dire. For the *in re* structuralist, there is no such thing as a structure of chess games if there is no game of chess; likewise, there cannot be a natural number structure if there is no simply infinite object or constellation of objects in the physical world. Likewise, given that the world around us is not structured spatially according to the laws of Euclidian geometry, there can be no such thing as an Euclidian square or triangle. Given how familiar we are with just such objects,

34 A historically common method to avoid problems involving infinities such as these is to distinguish actual from potential infinity. A potential infinity is not a finished whole; it exists merely as a limit of finite entities (such as numbers), as a practical way to speak of such a limit. The position denying the existence of actual infinities held popularity for a long time, but has dwindled since Cantor's mathematical use of, and philosophical defense of, actual infinities as full-fledged entities in and of themselves. Even if one would seek such a way out in the modern day, it seems to be outright prevented by the structuralist view of mathematics. After all, if each natural number depends on the structure of the natural numbers as a whole, the finite numbers depend on an actually infinite entity. Perhaps one may feel called to develop a finitist-structuralist account in which we consider structures as potential entities, but here we are not concerned with structures as anything other than a finished whole.

this leaves a burden of explanation with the *in re* structuralist. What is required to make this philosophy work is an account of how there are all these objects "out there" nonetheless, in the physical world, in our minds, or in some interaction between our minds that world. The poverty of actual systems in relation to the wealth of structures dependent on just these systems is a question that requires answer.

If *in re* structuralism can be considered an equivalent to Aristoteleanism, then *ante rem* structuralism is the younger brother of Platonism. On this view, there are such things as structures regardless of any instantiation, in our minds, in the world or otherwise. Structures exist as the "one over many" unifying all the different systems instantiating a particular structure, and they exist independently of these systems. The *ante rem* structuralist reverses the existential priority between structures and physical objects; or more generally, between structures and systems. There can be a structure that is not instantiated in any particular system, and the mathematician may indeed study such structures. On the other hand, there can't be any system, physical or otherwise - and thus, something that is *structured* in some way or another - without the existence of a structure. Hence, structuralism *ante rem*: before the thing.

In particular, this means that structures that may not be instantiated in any systems, such as Euclidian geometry, still exist. Thus, the problem of "ontological poverty" that strikes the *in re* structuralist is keenly avoided. On the other hand, some of the weaknesses platonism has vis-a-vis that view fall upon the *ante rem* structuralist as well. The applicability of mathematics is still somewhat problematic, even if we can get halfway to a solution. After all, if it is not independent mathematical objects that we can apply to the world, but entire structures, then it follows that *if* we can apply such a structure, we can manipulate it mathematically and thus help us understand or manipulate the physical objects instantiating the structure. Thus, once the link between structure and physical system is made, it follows that mathematical tools are highly useful. The establishment of this link, however, is more difficult: how come physical objects can reflect a certain structure?

It is this inaccessibility of *ante rem* structures that lies behind other major challenges to this position as well. Which structures exist? How can we gain knowledge of them? The latter problem has been put forward to challenge traditional platonism as well, in particular by Paul Benacerraf: if mathematical objects are independent of the world and of our minds, then how is it possible for us to have knowledge of these objects?[35] This particular epistemological question can perhaps be answered more readily by the structuralist than by the platonist, and we will turn to it in section 1.4. For now, let us turn to the

35 See [Benacerraf 1973].

third major position within structuralism: that there are no structures at all.

1.3.2 *Eliminative structuralism*

Eliminative structuralism is one of the philosophies of mathematics that favour a strict ontological parsimony. As the nominalist rejects the existence of mathematical objects, the eliminativist structuralist denies that there are such things as structures. On this account, we make sense of mathematical expressions not as referring to structures, but as generalisations over systems expressing this structure.

Take the simple mathematical statement:

$$3 < 6 \tag{2}$$

The traditional platonist sees this as expressing something about the objects 3 and 6, and the *ante rem* or *in re* structuralist as expressing a truth regarding the natural numbers structure. The eliminativist structuralist, however, wants to avoid direct reference to both structures and objects within these structures. After all, according to this view there is no structure to directly refer to, only particular systems that can be regarded as instantiations of the structure. Thus, even a basic statement such as (2) has to be seen as an implicitly general statement, and may be interpreted as follows:[36]

> In any system S expressing the natural number structure, the S-object in the 3-place of the structur is S-smaller than the S-object in the 6-place of the structure. (4)

36 The idea of reinterpreting mathematical statements has been criticised as a radical departure from ordinary semantics. The reinterpretation (4) of (2) is, however, not a radical departure from other structuralist theories simply for reinterpreting a mathematical statement. Both *ante rem* and *in re* structuralism can be regarded as a reinterpretation of the meanings of mathematical statements as well. After all, the structuralist would see an expression regarding numbers not strictly as a statement about mathematical objects, but as expressing something about the natural number structure. The *ante rem* structuralist ought then to interpret (2) as

> "In the natural number structure N, 3 is smaller than 6." (3)

There is no implicit quantification in the *ante rem* interpretation of (2), but there is an implicit reference to the structure 3 and 6 are part of.
Some authors, notably Benacerraf in [Benacerraf 1973] and Shapiro in [Shapiro 1997], have argued against any such reinterpretations of mathematical statements, arguing that they are to be taken at face value, in as far as that their interpretation should not differ radically from the semantics of ordinary sentences. Benacerraf noted that semantic theories of mathematics seemed to be either unlike other semantic theories or epistemologically unsatisfactory. It seems difficult to regard (3) as *wrong* within a structuralist framework, though. It might be wiser to strive for a more uniform epistemology, and to argue that the way in which we gain knowledge of structures is little different from the way in which we gain knowledge of other sorts of objects.

It should be noted that this view of structuralism entails a different view of reference to mathematical objects, and a different interpretation of what Dedekind abstraction is. Reference to positions within structures as objects in and of themselves, the *places-are-objects* perspective, is eschewed completely, in favour of generalised *places-are-offices* statements, ranging over a variety of systems. Any mathematical term is seen as a role to be fulfilled by some other object.

Recall that on our previous account of Dedekind abstraction, which is consistent with an *ante rem* view of structuralism, we truly do obtain a new structure when we abstract e.g. the natural numbers from the set-theoretic universe. The first step remains the same - we go through the mathematical process of establishing that certain relations hold among a certain collection of objects in the system. In the case of the natural numbers, that means that we have to establish that they obey the axioms of a simply infinite system. However, we do not follow this up by disregarding everything in our system that is not part of our chosen collection of objects and relations; rather, we see any theorem proven on this collection as an implicitly general statement over every system in which we can perform the first step.[37]

Generalising this, let $x_1, ..., x_n$ be the objects we choose to distinguish, $X_1, ..., X_n$ sets, and $R_1, ..., R_n$ relations. We can then define $C(x_1, ..., x_n, X_1, ..., X_n, R_1, ..., R_n)$ as the conjunction of all conditions that have to hold on these objects, set, and relations for them to "be" our intended structure C (e.g. the conditions to be a simply infinite system).

Say we want to prove some statement $S(x_1, ..., x_n, X_1, ..., X_n, R_1, ..., R_n)$ that holds on C. Rather than interpreting S straightforwardly as expressing something about the objects, sets and relations it involves, we would consider it to be shorthand for the following:

$$\begin{aligned} &\text{For any } x_1, \ldots, x_n, X_1, \ldots, X_n, R_1, \ldots, R_n, \\ &\qquad \text{if } C(x_1, ..., x_n, X_1, ..., X_n, R_1, ..., R_n), \\ &\qquad \text{then } S(x_1, ..., x_n, X_1, ..., X_n, R_1, ..., R_n) \end{aligned} \tag{5}$$

This reformulation of mathematical statements successfully avoids direct reference to mathematical objects in favour of a generalised statement. Like any such statement, though, this means that we are now dealing with quantifiers of some sort. Likewise, the formulation (4) quantified over various systems. However, in order for there to be any systems for the statement to express something of, there must be some domain for the quantifier to range over. In any *places-are-offices* statement, some kind of background ontology is required.

Various such domains are available to the eliminative structuralist. The most straightforward option, referred to by Shapiro as the

37 [Parsons 1990] pp. 307

"ontological option"[38], is to accept a background ontology of objects, rich enough to allow all mathematical structures to be instantiated in some form or another. This option brings eliminative structuralism very close to *in re* structuralism, and in particular the same problem of ontological poverty rears its head. It is not clear that there are enough physical objects to instantiate all the structures the mathematician is interested in.

A second option for a background ontology, known as "Set-Theoretic Structuralism"[39] is to accept an explicitly mathematical background theory, rich enough to feature instantiations of every mathematically interesting structure. Such a background ontology is classically provided by set theory, since the set-theoretic universe V associated with the Zermelo-Fraenkel axioms with Choice (ZFC) is widely considered sufficiently rich to found mathematics, and in particular to feature all mathematically interesting structures. An infinite regress threatens this solution though. If the background theory is mathematical, it is then to be interpreted in a structuralist way, and the eliminative structuralist will require a mathematical background ontology for their mathematical background ontology, and another one for that, and so forth. One can interpret the background ontology non-structurally, and this has been proposed for set theory specifically. But it is, of course, highly unsatisfying for a structuralist to rely on a mathematical universe built up out of objects, in the end. Moreover, per above, the dependence account of structuralism seems to point in the direction of set theory being interpreted in the structuralist manner, too.

Another manner to avoid reference to structures as objects without running into the problem of the background ontology is to interpret mathematical sentences in yet another way, avoiding the problematic quantifier. Geoffrey Hellman's project of *modal structuralism* aims to do just that by interpreting mathematical sentences as preceded by a modal quantifier:[40]

> In any logically possible system S expressing the natural number structure, the S-object in the 3-place of the structure is S-smaller than the S-object in the 6-place of the structure. (6)

Thus, rather than quantifying over any particular fixed ontology, modal structuralism aims to be open-ended in nature: as long as it is logically possible for some structure to occur, the quantifier ranges over it.[41] Thus, the possibility for it to actually encompass all of mathematics is not excluded as it is in set-theoretic structuralism. That is

38 [Shapiro 1997] pp. 87

39 [Hellman 2001] pp. 185

40 [Hellman 2001] pp. 189

41 This particular rendition is inspired primarily by Shapiro's expression of modal structuralism in [Shapiro 1997].

not to say that this school of structuralism has not been argued to run into the problem of a background ontology. This is because logical possibility can be argued to require one, just as quantification does. After all, semantic truth in logic is usually treated model-theoretically. The open-endedness of the endeavour would be sabotaged as well. The obvious answer is to reject that the importance of model theory in logic amounts to a reduction of logic to set theory, and indeed Hellman has responded by taking logical possibility as a primitive.[42]

Similar problems face each of the different schools of structuralism: they all revolve around demarcating which structures there are. For *ante rem* structuralism, this is the problem straightforwardly. Without an idea of which structures there are, it seems impossible to distinguish true statements from false ones. *In re* structuralism provides a limit to which structures can and cannot exist, as does set-theoretic structuralism - and as a result, both face the problem of ontological poverty, as it is not clear that every mathematically interesting structure exists, or can even be referred to sensibly. Modal structuralism seems open-ended enough to not run into the problem of ontological poverty, but faces essentially the same problem as *ante rem* structuralism in being unable to state which structures are logically possible and which are not; and should we use mathematics to analyse this, we are left with a background ontology that might be both too poor and threatening an infinite regress.

1.4 epistemology

Mathematics concerns, *prima facie*, a different kind of object than most fields of scientific enquiry - or even ordinary day-to-day enquiry. Whereas the latter involves physical objects in some way, mathematical enquiry typically concerns purely abstract objects. Perhaps more has been made of this distinction than is warranted. It does not stretch the imagination to claim that most scientific fields deal with a great amount of abstracta in their theories, or even that ordinary human perception seems tightly bound to them. Nevertheless, there is something unique to mathematics in that there is typically no clear object of sensory perception linked with our enquiries. There may be borderline cases - one could argue that geometry has something to do with physical shapes or physical distances - but the mathematician is generally not concerned with any concrete particulars. This may be a matter of generality rather than one of abstractness *per se*: according to some philosophies - *in re* structuralism comes to mind - mathematics does concern concrete individuals, but it is certainly not aimed at any specific ones. It is possible that it is concerned with *whatever* objects fit a certain structure.

42 [Hellman 2005], pp. 557

Whichever is the case, it is clear that the epistemology of mathematics faces some problems that not every epistemological theory does. The lack of a concrete object one aims to gain knowledge about has driven philosophers since Plato to formulate different theories of knowledge regarding mathematical objects. Scepticism, too, has been fuelled by the apparently peculiar nature of mathematical objects, usually taking the form of nominalism: the rejection of abstract objects. On the subject of the epistemology of mathematics, a famous modern case for it has been made by Benacerraf in [Benacerraf 1973], arguing that since we have no causal interaction with abstract objects, we cannot gain knowledge of them. Of course, it is not a given that causal interaction is necessary for knowledge - in particular for the *a priori* knowledge that mathematical enquiry purports to give. Nevertheless, nominalist theories hold sway in both the ontology and epistemology of mathematics. Epistemological concerns are never far in the background in any philosophy of mathematics.

Structuralism, as a relatively young member of the family of philosophies of mathematics, does not have a single, canonical epistemology behind it, but multiple attempts have been made to provide it with one. Some of these epistemologies are deeply linked with human perception, others with mathematical practice. All give us valuable insight on what it really means to be a structure.

1.4.1 *Pattern recognition*

Michael Resnik, though a structuralist, prefers to speak of *patterns* rather than of structures. It suggests a stronger link with scientific research on pattern recognition and pattern cognition.[43] The epistemology of structures he proposes is strongly linked to such faculties. This epistemology consists of different stages of experience, rather than proposing a uniform solution, such as a faculty for intuiting abstract objects.

The process of knowing structures or patterns starts with experiencing everyday objects *as* structured or patterned in some way or another. We recognise objects of our senses, be they sounds or sights, as shaped in a certain way. Eventually we may recognise an equivalence between objects of our senses; they may share a shape, a colour, or there may be equally many of two collections of things. At this point, we may express statements concerning all the objects that share such a property. There are two significant leaps involved until we arrive at structures proper, though. The first is the extension of statements that have a definite basis in experience to statements that do not, but that nevertheless make sense based on the equivalences we previously discovered:

43 [Resnik 1982], pp. 96

> There is no reason to believe, however, that our knowledge that one million and three things are fewer than one million and five things has the same sort of evidential basis or genesis. This knowledge would derive, I conjecture, from a rudimentary theory of counting and principles concerning the generation and use of numerals.[44]

The second leap is then the full movement to abstraction:

> [...] the next move is to supplement predicates with names for shapes, types, and other patterns. In addition to talk of square, circular and triangular things there is now talk of squares, circles, and triangles.[45]

At this point we study the commonality between different objects as entities in their own right. With this comes the introduction of the *places-are-objects* perspective on structures.[46] Resnik points to an analogy with mathematics itself: the movement from equivalence relations to equivalence classes.[47] In mathematics in particular, we eventually cleansed our speech of reference to physical entities, and attempt to define them strictly in abstract terms. The association with instances may remain in our understanding, though, as we try to visualise a concept (say, a square) in order to understand it.

1.4.2 *Implicit definition*

The introduction of many a student to a new sort of mathematical structure does not take place in this roundabout manner, however. Typically, he is introduced to it through a direct description. This description usually consists of a few axioms; the characterisation of the natural numbers through the definition of a simple infinite structure comes to mind.

Through an implicit definition, one can gain knowledge of an abstract structure without ever gaining knowledge of an instantiation of it, or of any non-mathematical origins or relations to it. The only relations introduced are those internal to the structure itself, a subcollection of the relations that compose the structure as a whole. The definition introduces objects and relations strictly in terms of each other, in a matter that is rich enough to imply a grander structure than merely what was expressed directly in the axioms. It is the role of the mathematician to discover relations or objects within a struc-

44 [Resnik 1982], pp. 98
45 *Ibidem*
46 This does not need to imply an *ante rem* structuralism.The eliminative structuralist may very well hold that even if we do speak as if there were abstract entities, this is merely an easy method to express a complex quantified statement.
47 [Resnik 1982] pp. 98

ture that are necessarily part of the structure, even if they were not explicitly mentioned in its definition.[48]

It is noteworthy that this view of knowing abstract objects does not conflict with Resnik's multi-stage approach. Resnik's approach does not focus on the process of acquiring knowledge as an individual, as a culture, or as a child learning to use patterns and structures at all. Rather, it sets this matter aside and attempts to give a general account. In this light, we may see implicit definition as a "shortcut" mechanism to bring an individual up to speed with the abstraction process another individual - or the culture as a whole - has gone through. A culture may develop abstract arithmetic from particulars, through relations of equinumerousity, to numbers as abstract objects and strictly abstract arithmetical relations. Perhaps children go through a similar process when learning to count. An individual unfamiliar with a certain structure may learn it through such a slow process. Alternatively, if others around him *are* familiar with it, he can be instructed in the properties of the structure directly and acquire knowledge of it in this manner. Implicit definition is then the tool with which this educational goal is achieved. If the student is familiar with abstract mathematical objects in the first place, such a direct introduction may not prove too difficult for him to grasp.

Implicit definition can then be seen as a valid method for gaining knowledge of abstract structures without thereby severing the link with the concrete particulars it was abstracted from in beginning. We need not deny that the road towards new mathematical structures is long-winded, arduous, and an enormous investment by professional mathematicians. It may very well be impossible to recall which abstract structures or semi-abstract equivalence structures have all played a role on the road from the experience of concrete particulars to knowledge of fully abstract structures. This need not convince us that mathematical objects are truly "free standing", that they are created out of thin air when they are defined, or that they were suddenly discovered once the right definition was written down. The "shortcut theory of implicit definition" allows us to recognise two intuitions that are seemingly at odds: the awareness of the historical contingency and origins of a mathematical structure, and the experience of structures as a finished whole, as entities "out there" for us to grasp.

48 The various tools the mathematician has to do this, notably including deduction, fall beyond the scope of this thesis. Suffice to say that they are rightfully a subject of study on their own, and that their role in a structuralist framework *per se* is also worthy of further research.

2

CATEGORICAL STRUCTURALISM

In this chapter, we shall introduce the link category theory has been claimed to have with structuralism. Various manners in which this connection is thought to exist are brought to light. The role of category theory plays in mathematics according to a few authors shall be set out in detail, without delving too much into the manners in which category theory could be seen as foundational to mathematics; we shall turn to that matter in chapter 4.

2.1 CATEGORY THEORY AND STRUCTURALISM

Category theory is a relatively young branch of mathematics: categories were introduced in 1945 by Eilenberg and MacLane, and became objects of study in their own right only in the 1950s. It has quickly proven itself useful in many branches of mathematics, starting with topology and algebra. The particular properties of this branch of mathematics have endeared it not just to mathematicians, but to quite some structuralist philosophers of mathematics. Category theory itself has been described as the mathematical theory of structures.[49]

2.1.1 *A short introduction*

Category theory is mathematical theory of great generality, consisting only of objects and arrows, and concerned, roughly, with the composition of arrows. Without further ado, let us introduce the Eilenberg-MacLane axioms for category theory:

Definition. *A category $\mathcal{C}$ is a system with two kinds: Objects $X, Y, \ldots$ and morphisms (or "arrows") $f, g, h, \ldots$ that satisfy the following:*

1. *Each morphism has a domain and a codomain, both of which are objects; write $X \xrightarrow{f} Y$ or $f : X \longrightarrow Y$*
2. *For each pair of morphisms $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ where the codomain of f is the domain of g, there is a composite morphism gf such that $X \xrightarrow{gf} Z$*

49 [Marquis 2014]

3. *For each object X, there is an identity morphism* $1_X : X \longrightarrow X$ *such that for all* $f : X \longrightarrow Y$, $f1_X = f$ *and for all* $g : Z \longrightarrow X$, $1_X g = g$

4. *Composition is associative:* $f(gh) = (fg)h$.

A category is then anything that satisfies these axioms. For example, groups and homomorphisms on groups satisfy these conditions, as do topological spaces and continuous functions. Hence, we can work in categories whose objects are groups, and whose morphisms are group homomorphisms, or whose objects are topological spaces and morphisms are continuous functions. These are, then, the *category of groups* **Group** and the *category of topological spaces* **Top**. This wide applicability is one of main strengths of category theory, along with the fact that it is self-applicable: we can take our objects to be categories, and let our morphisms be "category homomorphisms" or *functors*: that is to say, mappings between categories that preserve domains, codomains, composition and identity morphisms. We can "move up" further in our analysis, and investigate morphisms of functors (natural transformations), and so forth.[50]

In the history of category theory, the way in which we looked at categories and morphisms has changed. Early in its history, category theory was used as a language for describing other sorts of objects. We were always concerned with the category *of* something or other - usually some topic in algebra. From the 1960s onward, a reversal in priority occurred: it became more and more commonplace to start out explicitly from the categorical perspective. The category, morphisms and objects are left uninterpreted by themselves, and are then used to *define* other notions, such as "set".[51] This approach was started by F. William Lawvere, whose *Elementary Theory of the Category of Sets* does not assume that its objects are sets, but *makes* them sets by defining elements in terms of morphisms from a specific object, defining a subset relation, and so forth.[52]

With this came a new "perspective" on mathematics. Rather than building up mathematical objects out of (usually set-theoretic) atoms, structures could now be defined from within a category. The context of a mathematical object is then no longer the set-theoretic universe, but the overarching category. This approach was first made explicit as "an alternative foundation" for mathematics in Lawvere's *Category*

50 For an introduction to category theory, see the classic textbook [MacLane 1978], [Lawvere and Schanuel 1997], or [Awodey 2010].

51 [Landry & Marquis 2005], pp. 8-9. In general, see [Landry & Marquis 2005] for a historical overview of category theory culminating in the "categorical structuralism" under investigation here, and see [Marquis 2009] for a thorough account of the history and philosophy of category theory.

52 See [Lawvere 1964]. A more accessible contemporary reiteration can be found in [Linnebo & Pettigrew 2011].

of Categories as A Foundation.[53] Category theory, and the method of defining mathematical objects from within it, have been used with success across many areas of mathematics. A while after their introduction as a foundation for mathematics in a mathematical paper, then, it is the turn for the philosophers of mathematics to turn their eyes and ears as well.

2.1.2 *Mathematical structuralism*

Steve Awodey distinguishes philosophical structuralism - structuralism as a philosophy of mathematics - from a tendency to value structures and structural properties highly in mathematics itself. The latter he dubs *mathematical structuralism*. Category theory, being the mathematical study of structure *par excellence*, is the most prominent framework in which the structural properties of a mathematical object are studied. In fact, category theory is employed to make precise the very notion of a structural property of mathematical objects.[54]

It is a common occurrence in mathematics for a particular kind of structure to be instantiated in different systems. Of course, these systems may be quite different in nature otherwise, and certain strongly differing properties may seem fundamentally linked to notions that seem analogous at first glance (recall Benacerraf's problem regarding the set-theoretic properties of numbers). Thus, it is not merely a philosophical problem to figure out when two objects have the "same" structure. It is a question of mathematical import. Leaving aside for now whether a mathematical solution could truly solve the philosophical problem involved - we are dealing with *mathematical* structuralism at the moment - there is such a mathematical solution to the problem.

Suppose some morphism f is an isomorphism from A to B; that is, there is an inverse f^{-1} of f such that $ff^{-1} = 1_B$ and $f^{-1}f = 1_A$. This categorical notion of isomorphism serves as the definition of two objects A and B having the same structure. Of course, the particulars of this isomorphism depend on the particular structure we are investigating: group isomorphisms are different creatures than isomorphisms of sets. Once we know what isomorphism looks like on a particular structure, we also know what a morphism in general looks like, and with this information we can construct a category of systems exhibiting a certain kind of structure. With this, in turn, we have a notion of what the structural properties of an object are. It is simply those properties which are invariant under isomorphism. If a property is maintained through an isomorphism in the category we

53 See [Lawvere 1965] for the original formulation of this foundation, unfortunately not free of error, and [McLarty 1991] for a more recent version avoiding the problems of the original.

54 [Awodey 1996], pp. 214.

are working in, it is structural in nature. Better yet, the very notion of being expressable in category-theoretical terms will guarantee that something is a structural property: if a property can be expressed in terms of morphisms, it will remain invariant under isomorphism.[55]

The link with philosophical structuralism can be shown to be rather strong by revisiting the issue of Dedekind abstraction. We can make precise both steps of the process. Recall that the first step in the process was one of showing that specific relations hold in the mathematical system we are working in. By defining an isomorphism over two systems sharing our proposed structure, we can make precise which relations are the ones that are emphasised: the isomorphism-invariant ones. The other relations are particulars of the system the structure is expressed in, and are abstracted away from in the process. The second step was to consider these relations, and the objects thus connected, as no longer within the original system, but as a *new* mathematical structure. But this is no more than to treat these systems as objects in a category! After all, by working within the framework of a category and expressing properties in terms of morphisms, the very properties we would like to abstract away from - those which are not isomorphism-invariant - cannot be expressed.

2.1.3 *Revisiting Benacerraf's Problem*

If expressing something categorically is indeed enough to ensure that no non-structural properties can be captured, this suggests a straightforward way out of Benacerraf's problem. If the natural numbers can be defined categorically - or more generally, if we can define sets, including the set of natural numbers, in the language of category theory, we can avoid the inconsistencies invited by multiple different "natural numbers" in ZFC. Moreover, we may do this without having to abandon the set-theoretic level.

Recall that Benacerraf's Problem is only mathematically fatal if we directly identify the natural numbers with particular sets (and hence $\{\{\emptyset\}\} = 2 = \{\emptyset, \{\emptyset\}\} \neq \{\{\emptyset\}\}$). We preserve the association of numbers with sets by regarding this "equality" not as an identification, but as a case of the use of Dedekind abstraction; what we mean by $2 = \{\{\emptyset\}\}$ or $2 = \{\emptyset, \{\emptyset\}\}$ is not that 2 and these sets they are the same entities, but that the former can be obtained from the latter by Dedekind abstraction. One can compare this "is" with the "is" of predication; "the cat is black" and "the dog is black" do not imply that the cat is the dog.

The upside of this is that we can identify natural number series in Zermelo-Fraenkel set theory, and that we can reason about these numbers *as* numbers, that is to say, at the level of the natural number structure. The downside is that we cannot do both at the same time.

55 [Awodey 1996], pp. 214.

Any set-theoretic formulation invites non-structural properties. Thus, doing arithmetic means abandoning the set-theoretic perspective, and vice versa. As Colin McLarty has a character in [McLarty 1993b] surprisedly exclaim: "So the advantage of your set theory is that mathematicians never work with your sets!"[56]

McLarty suggests we work with categorical set theory instead. If the properties we can express in the syntax of category theory are simply the structural properties, and if we can formulate sets categorically and the natural numbers in terms of sets, then we should be able to talk of numbers from a set-theoretic perspective without treating them non-structurally. We can do this by treating sets according to the axioms of Lawvere's Elementary Theory of the Category of Sets (henceforth ETCS). Herein, the sets are the objects in the category of sets, and elements of a set S are defined as morphisms $1 \to S$ where 1 is the terminal object.[57] We can then define the usual kinds of sets, such as products, subsets and disjoint unions using the category-theoretical apparatus: with products, equalisers and coproducts, respectively.

Sets, of course, take a different character all together in this kind of set theory. This difference can be characterised as the difference between concrete and abstract sets: whereas the former are defined as collections of concrete individuals, the latter are defined in terms of relations to each other.[58] Hence the element as a relation between the singleton and a set, for example. This again can be characterised as Dedekind abstraction; what we are left with is the structure of sets regardless of their elements, that is, their structure at the subset level at best. Thus, there is only one singleton set: the set 1 containing just one element. In effect, this means that this set theory abandons extensionality on the level of elements for extensionality on the level of subsets.

We can then define a *natural number object* within this set theory as follows: let N be a natural number object if it is a set with an element 0 and a successor $s : N \to N$ such that for any A with element x and a function $f : A \to A$, there is a mapping $u : N \to A$ such that $u0 = x$ and $usn = fun$ for all $n \in N$.[59] Then we have defined numbers set-theoretically in such a way that Benacerraf's problem is avoided: $0, s0, ss0, ...$ Each number is an element of N, and has only the properties it has on account of being a number in a natural number object.

56 [McLarty 1993b], pp. 496

57 That is, the object 1 such that for each object, there is exactly one morphism to 1. Compare the singleton set, to which there is only one function from any set: the function projecting each element of the set to the single element of the singleton.

58 [McLarty 1993b], pp. 489

59 [McLarty 1993b], pp. 492. To recover the Peano axioms, we need a slightly stronger definition which allows us to have a recursive definition with parameters: let P be any parameter set, $x : P \to A$ an initial condition. Then for every $f : A \times P \to A$ there must be a unique $u : N \times P \to A$ such that $u(0, p) = x(p)$ and $u(s(n), p) = f(u(n), p)$.

Recall Dedekind's definition of a number as an element of a simply infinite system, stripped of all properties but their distinguishability and the relations they are placed in by virtue of their ordering in the system. The numbers defined in this way are "really" numbers on this account, since there is nothing to strip away; there is no further level of abstraction to go to, no properties they have by virtue of something other than to be a number in this system. Those that seem "un-numberlike" at first glance - such as their being a mapping from 1 to N - are no more than a formulation of a fact that is necessary on account of their structural properties: it is an expression of the fact that they are elements of the set of natural numbers. In particular, the kind of "internal structural baggage" that the Zermelo ordinals have is avoided. The numbers stand in no relation *to each other* other than those of arithmetic; we need not ask whether $3 \in 7$ or not.

One might wonder why sets defined categorically are "purely structural", whereas sets defined in ZFC would not be. If the structuralist thesis is that *all* mathematics is about structure or pattern, then shouldn't either construction be structural? The difference lies in a matter of explicitness. In either case, the mathematical system does rely, in the end, on a structure - and thus, on a whole of relations. But only in one of these set theories, sets are also defined that way. Thus, in categorical set theory, the definitions of sets track their structural properties, whereas in "full-blown"[60] set theory, they do not. That does not mean that they cease to be structures, but it does mean that it requires more mathematical or conceptual work to identify structural properties. The difference does not come to light, of course, until we try to find some new (sub-)structure - in other words, until we move up or down a level of abstraction. And it is exactly there that Benacerraf's problem rears its head for traditional set theory, but not for the categorical variant. We will come back to this difference in section 2.2.3 with Awodey's view of categorical structuralism.

McLarty notes that one advantage to orthodox set theory is that its sets obey Leibniz' Law of indiscernibles: two set-theoretic objects that are indiscernible in their properties are, indeed, identical. Categorical set theory does not obey this law, since we can prove that there are infinitely many natural number objects, each isomorphic to all others, and stronger yet, each with exactly the same properties.[61] McLarty suggests, by word of a character in his article, that we then abandon this principle: categorical set theory falsifies it. Nothing quite so drastic is necessary. Leibniz' principle concerns metaphysics, and hence applies if we seek to identify mathematical structures metaphysically with categories, functors or objects. Whether we want to make this step depends on our view of the metaphysics of structure. The elimi-

60 From this point on, I will occasionally refer to the usual mathematical approach to set theory in this manner to distinguish it from categorical set theory.

61 See [McLarty 1993b], pp. 493 for a full proof.

nativist structuralist would deny the metaphysical existence of structures in the first place, seeing them as mere quantifications over systems expressing them - including the set-theoretic ones - and hence deny this equality in the first place. The *ante rem* structuralist might balk at the idea of there being many metaphysical versions of the same structure in the first place, regardless of Leibniz' law - to him, all the different systems expressing some structure truly do express the same single structure, defined up to isomorphism. The problem with Leibniz only arises if we adopt a straight-up metaphysical category-theoretic structuralism, wherein structures are metaphysically identified with category-theoretic constructs, and hence, for example, all set-theoretic number objects are simply to be seen as expressing some category-theoretic one. If we want to metaphysically identify structures with some kind of mathematical object, we ought to do it with a kind of mathematical theory wherein isomorphic objects are always identical if we want to obey Leibniz' law.

2.2 theories of categorical structuralism

The supposed "natural match" between category theory and a structuralist account of mathematics has not been without consequence. A number of programmes have been put forward employing category theory in conjunction with a structuralist view of mathematics in order to make a philosophical or foundational point. Prominent among these are McLarty, Landry and Awodey, whose views we introduce in this section.

2.2.1 *McLarty: Categorical foundations*

McLarty sees promise for the role of category theory in a foundational project. Mathematical foundations are not taken as a single theory on which all other mathematics is - in some way or other - based, or which justify mathematics. Rather, he views the foundational project as a continuous and ongoing process of organising mathematics. The upside of foundations is their continuing effect of allowing mathematicians in different fields to express themselves in a single framework, thus promoting understanding and error-finding. In particular, their role in making explicit the assumptions made in proofs makes it less likely that unwarranted leaps are made.[62]

Given the contemporary tendency to approach mathematical objects among isomorphism-invariant lines, often explicitly employing category-theoretical tools, category theory is a promising candidate to put in some work in this field. McLarty emphasises the ubiquity of categorical tools to argue that category theory already plays an important role in the process of making mathematics, more uniformly

62 [McLarty 2013], pp. 81-82

understandable and error-free. In particular, when set theory is employed to strictly define some notion or other, it is usually formulated in such a way as to be neutral towards an interpretation in either full-blown set theory or categorical set theory. This is not to say that categorical foundations are frequently referred to, which they are not; but given their current use, they could very well be.

The focus of McLarty's contributions to discussions on categorical structuralism, then, lie in his view of the foundations of mathematics, and in a defence of the role categorical foundations in particular can play therein. We will turn to various views on the foundations of mathematics, including McLarty's "organisational" notion of foundations, in chapter 3, and to his arguments with regards to categorical foundations in chapter 4.

2.2.2 *Landry: Semantic realism*

Elaine Landry employs category theory to find a compromise position between *ante rem* and eliminative realism.[63] Rather than aiming to establish the existence of mathematical objects or structures metaphysically, through an *ante rem* approach, or an *in re* approach in which they arise from a process of abstraction, Landry attempts to provide a linguistic basis for our talk about these structures and objects.

> ... the only reality that categories need to be taken as part of is 'linguistic reality', that is, the reality that concerns us with *what we say*. Interpreted along semantic realist lines, categories are not claimed to exist independently of their linguistic use and even when they are held as "objects", they are only taken to exist in the sense of being required to *talk about* the way things are in a given structure.[64]

This position she dubs *semantic realism*. Objects and structures are established as existing in as far as is necessary to provide our linguistic utterings with content. She aims to secure this measure of realism by analysing structures as they behave in category theory. She investigates the manner in which we refer to a singular object in a particular structure, expressed in the language of category theory, and the way in which this changes if we in turn approach the structure from a general perspective. From all this, she finds that the former analysis leads us to an *ante rem* interpretation of structures, whereas the latter is best captured on an eliminative interpretation.

63 See [Landry 1999a]. Landry herself refers to the latter as *in re* structuralism, following Shapiro's older method of classifying all non-*ante rem* variants of structuralism as *in re*. I will continue using the term "*in re*" to refer to the position arguing that mathematical structure are to be found in the world specifically.

64 [Landry 1999a], pp. 138, emphasis hers.

Consider first the case in which we work in a specific category - that is, a category defined according to the Eilenberg-MacLane axioms above, wherein the undefined terms are composition, identity and domain, with an interpretation of its terms. In other words, consider any *particular* set of objects and mappings that happen to fit the Eilenberg-MacLane axioms. In this case, we can safely run with the *ante rem* view of structures, and take statements about singular terms at face value. That is to say, when establishing what we are talking about when we refer to some object in a structure, we need not see it as an implicit generalisation over all objects of a certain kind; rather, from the perspective of the mathematician working within this structure, we can take the object at face value. We can use a model-theoretic argument, and assert that we can safely talk of a singular term referring to an object if there is a model for the structure wherein the singular term has some denotation.

However, we cannot interpret statements containing singular terms in the same way when we speak of a structure in general - that is to say, when we speak of all structures of a certain kind. Since there is no theory establishing the existence or non-existence of a structure as a whole, we cannot refer to said theory to fix the reference of our terms. It is here that we can employ category theory to fix structures as a whole as objects within some other theory, thereby fixing the reference and securing a semantic realist position not just for singular terms, but for mathematical structures as a whole. This is possible because we can "move up a level" in category theory and establish categories not by defining them with the Eilenberg-MacLane axioms, but as objects in the Category of Categories. We are still talking of the same structure, in this case: if we can express the structure as a category in the first place, then we can preserve the objects as functors $1 \to C$ and the morphisms as functors $2 \to C$, where C represents our category, 1 is the terminal category and 2 is a category with two objects, $\mathbf{0} : 1 \to 2$ and $\mathbf{1} : 1 \to 2$. Clearly, C in this system expresses the same structure as the original formulation of our category. Now, we can fix the reference of the term referring to the structure by taking its representation as an object in the category of categories as its denotation. This happens, however, at the expense of the interpretation of the original structure: whereas our category previously might have been a category of groups or of sets, it is now indistinguishable from all other categories that feature the same pattern of morphisms and objects. Thus, Landry argues, we move towards the eliminativist interpretation here when we refer to singular terms in the structure: we can establish its reference if we take the singular term to implicitly quantify over all categories of a specific kind. Using Shapiro's terminology, it would be seen as quantifying over all systems expressing the structure exhibited by the general category C. This neatly fits with the eliminativist interpretation of structuralism.

The role of category thus becomes that of "the language of mathematics". It functions as the foremost mathematical theory in which to represent our talk about mathematical structures.[65] This idea builds on Frege's position that a philosophy of mathematics should account for the content and structure of what we say. We can see model theory as providing us with a mathematical framework in which we can express what we say about the content of mathematical concepts. Category theory then performs a similar role in providing us with a mathematical framework in which to express what we say about the structure of mathematical concepts.[66] In either case, the mathematical theory is employed not to express some metaphysical fact about some mathematical concept, but to express something about their meaning. The very fact that we can separate these two is a feature of structuralism. If mathematics is not about any particular objects, but rather about their patterns, represented in axiomatically-captured structures, we can investigate the structure and content of mathematical expressions without needing to commit to a corresponding metaphysical position.

Landry's analysis highlights the possibility category theory gives us to make explicit the level of generality at which we approach a single mathematical structure. What sets it apart is that it allows us quite naturally to analyse not just mathematical structures, but the structure of mathematical structures in turn.

2.2.3 *Awodey: No foundations*

Steve Awodey, in tandem with his view on mathematical structuralism, sets out a categorical theory of philosophical structuralism.

The principal idea behind his approach to the philosophy of mathematics is that it is well-served by using a category-theoretical mathematical apparatus. Awodey suggests that set theory most naturally implies a monolithic kind of foundational thinking that is at odds with a structuralist approach to mathematics, i.e. an approach in which mathematics is concerned with various structures, each characterised by their own rules. The Bourbaki approach to structure, that is, the technical view of structures as sets with an additional structure defined upon them, is not readily adaptable for a mathematical analysis of structure. When working with structures rather than strictly within one, the mathematician will not separate set-theoretical and additional features of a structure in order to make sense of his structure, but will rather identify it in terms of mappings. The accurate description of, and separation between, different structures calls for an approach in terms of morphisms. Hence, the natural mathe-

65 [Landry 1999a], pp. 137
66 [Landry 1999b], pp. S18

matical environment for the study of structures is the category.[67] In short, mathematical structuralism is done category-theoretically, and this should be reflected in our philosophical beliefs with regards to the nature of structures, and hence the nature of mathematics in general. In other words, it should be reflected in philosophical structuralism. This is not to say that there is anything *fundamentally* category-theoretic about mathematics per se: rather, category theory is just an excellent theory to capture the structural features of mathematics, and we *do* see mathematics as being concerned with structure at a fundamental level.[68]

Consider, then, what is required for the mathematician to prove something about a structure he is interested in. Since we are dealing with a structure here, we are not concerned with specific objects, but rather with a network of relations. The array of tools that the mathematician requires to make headway in this network are then suited to exactly that purpose: they are the tools defined in the structure itself and in the axioms explicitly governing it.

> The proof of a theorem involves the structures mentioned, and perhaps many others along the way, together with some general principles of reasoning like those collected up in logic, set theory, category theory etc. But it does not involves the specific nature of the structures, or their components, in an absolute sense. That is, there is a certain degree of "analysis" or specificity required for the proof, and beyond that, it doesn't matter what the structures are supposed to be or to "consist of" - the elements of the group, the points of the space, are simply *undetermined*.[69]

We see here a reiteration of the structuralist reaction to Benacerraf's problem: rather than to look at the elements and subsets the natural numbers are supposed to consist of, we take the natural numbers as something more general than either the Zermelo ordinals or the Von Neumann ordinals. The tools we use to investigate the natural numbers are the tools that are defined on those numbers *as* numbers, or more abstractly, on some constellation of objects we mapped these numbers onto, but almost never on a specific internal structure given to these numbers through a mapping, such as internal structure of the Zermelo or the Von Neumann ordinals.

More generally, we see that the categorical account Awodey proposes harks closely to the structuralist accounts of the nature of mathematical structures. This is perhaps most clear on the abstraction account of structures. The indeterminacy of its objects beyond the structural level is a clear reflection of the idea that mathematical objects

67 [Awodey 1996], pp. 211-212
68 [Awodey 2003], pp. 9-10
69 [Awodey 2003], pp. 7, emphasis his.

are mere empty points within a structure, with no inner structure of their own.

Category theory, however, also provides us with tools to tackle an open problem in the dependence account of mathematics. Recall that structures are seen as relational wholes, and that objects, on the structuralist account, are said to depend on the structure as a whole or on the relations it consists of. Dependence on a relation, however, is difficult to make sense of. Relations, after all, are only relations on account of their relating two objects. This brings us back to relata, and hence objects, as the fundamental building block of mathematics.[70] A categorical approach avoids this problem by substituting morphisms for relations, as the former do not presuppose any objects, and can in fact be defined entirely without reference to them.

What Awodey rejects, then, is the marriage of a structuralist philosophy of mathematics with an approach to the mathematics themselves that does reduce all mathematics to objects of some sort. In particular, ZFC, taken as a foundation as mathematics, is exactly such an object-based approach. Of course, the rejection here is not one of the mathematics *per se*. There is no objection if one has a valid mathematical reason to express a number set-theoretically (e.g. in order to make an argument that requires reference to a cardinal sufficiently large to make it ill-describable outside set theory). Rather, on Awodey's account, the problem lies in the foundational claims that are linked with such a set-theoretic construction. If one wants to adhere to a structuralist philosophy and analyse philosophically interesting features of mathematics, one should not do so set-theoretically or even from within a set-theoretic mindset, as either approach irrevocably leads us back to object-based thinking. Category theory gives us a way to express mathematical ideas that does not trap us in non-structuralist terms, and in fact naturally focuses on the structural properties of various theories.[71]

Awodey describes the contrast as one between *bottom-up* and *top-down* mathematics. The former idea is of mathematics as an architectural structure, built from the ground up; we start with a large pool of objects and proceed to build mathematical structures out of them, reaching ever higher up as we use old structures to create new ones. Or rather, we have some foundational system consisting of objects and certain axioms governing these objects. If we have enough objects and if the axioms are strong enough, all of common mathematics can be done with some mix of objects and axioms to represent respectively the objects we actually want to talk about and the axioms

70 [Awodey 2003], pp. 9

71 We will see a very concrete difference in section 4.1.4, where the philosophical interpretation of a simple group-theoretic statement will be informed by either a categorical or set-theoretic background.

governing them.[72] With this approach each mathematical object carries with it an internal structure in the sort of objects that form our foundational system. This is exactly the sort of thing that the structuralist view tells us is no part of the mathematical objects proper, so why invite its presence through a bottom-up approach to mathematics?

The top-down approach to mathematics is, by contrast, thoroughly structural from the outset. When we set out to work on a particular structure, we require a determination of its properties only up to a certain degree; and on the top-down approach, we simply only determine the mathematics up to that degree. Going back to our standard example, this means that we take numbers as a simply infinite system, and stop there - unless we need some further structure for our specific proof, of course. This means that there is not always a way to express our structure in the currently prominent foundational system, and there in fact needs not be. The exact extent of our objects and the very rules that govern them are dependent on the structure we work in, and hence, might vary from field to field, from problem to problem, or even from mathematician to mathematician. We may, of course, still use further tools to create new mathematical objects and structures, but this is not seen as "building them up" from the previously created structures, let alone from some foundational stuff. It is seen as further specification, as a "going down" from a general case to a more specific kind of structure.

In order to give a serious take on Awodey's rejection of "foundational thinking", and of McLarty's "organisational" view of foundations, we need to know what foundational thinking in mathematics is. To this end, we turn our attention to various forms of foundationalism in mathematics in the next chapter.

72 Of course, there are never enough objects, and our axioms are never strong enough to capture *all* of mathematics. This does not matter at this juncture: for one, there are ways to still consider such a system as foundational, as we will see in chapter 3. For now, it is the form of the entire approach that is questioned, regardless of its formal adequacy.

3

FOUNDATIONS OF MATHEMATICS

In order to clarify how category theory, or any mathematical theory for that matter, can serve as a foundation for mathematics, we need to clarify what exactly is meant by the term "foundation". In the following chapter, we shall identify various *kinds of foundation*, that is to say, various ways in which a theory can be taken to be foundational for mathematics or part of mathematics. The view of mathematics here shall remain structuralist - implicitly so at first, and then explicitly when we investigate what kinds of foundation are sensible to ask for from a structuralist point of view.

Various mathematical theories and philosophical frameworks have been proposed as *foundations* of mathematics. What exactly is meant by such a declaration, though, is not unambiguous. Perhaps the only thing universally held as true is that a foundational theory has a special place within mathematics or alongside it. Therefore, in order to evaluate whether a category-theoretic foundation of mathematics is feasible, we need to make it clear exactly what is meant by a "foundation of mathematics" in different contexts.

There are various senses in which one theory can be said to be a foundation of another. Marquis in [Marquis 2005] and Shapiro in [Shapiro 2004] and [Shapiro 2011] attempt to give an overview, and note that these different kinds of foundations tend to be interrelated. In the following chapter, we will do a modest taxonomy of "kinds of foundations", with emphasis on the varieties most important to the discussion at hand, and relate them to the structuralist philosophy. We will see that different kinds of foundations are often difficult to distinguish from one another, and that many proposed foundations for mathematics will not be limited to a single kind of foundation.

The criterion we will use to classify foundations will be their purported *goals*. A foundation is a mathematical theory, a structure, and is thus worthy of attention out of mathematical interest - to see if we can prove something new. This goal is not served by heaving a "foundational" status upon the structure. By naming something a foundation, we must serve goals that are not strictly structural in nature. By classifying foundations according to the purpose they serve, we can identify *why* they have a claim to being a foundation of anything, and judge them by the progress they make towards this goal.[73] It also lets us judge the goal itself. Of course, certain mathematical theories will be exceptionally suited to certain foundational goals by virtue of their mathematical structure. This is exactly why these structures are taken as foundations of a certain kind - once a certain goal has been chosen. It does not make their particular mathematical structure a foundational goal in and of itself.

Thus, we will not see "logical foundations" as a separate kind of foundation, despite its presence in the taxonomy of e.g. Marquis. The goals served by giving a logical foundation to a particular area of mathematics may be classified as epistemological, cognitive, or even metaphysical. What sets logical foundations apart is their mathematical form, which, though an interesting subject on its own, does not concern us here.

3.1 ONTOLOGICAL FOUNDATIONS

Perhaps the most prominent aspect of foundations is their claim to providing mathematics with its *ontology*. The objects of mathematics, if there are such things, exist by virtue of the foundations, or are at least delimited or described by the foundation. The theory that classically plays this role, providing the field of mathematics with a large "universe" of objects, is set theory, in particular as defined by the Zermelo-Fraenkel Axioms with Choice (ZFC).

3.1.1 *Ontology as metaphysics*

Most straightforwardly, ontological foundations can be seen as describing what sort of things mathematics is about. Thus, the set-theoretic foundationalist may hold that the objects of mathematics *are* sets. Likewise, a category theorist may claim that mathematics is about categories, morphisms and objects. More abstractly, Shapiro, as an *ante rem* structuralist, holds that the ontology of mathematics consists of structures.[74] The classical platonist is most straightforward: he posits a universe of mathematical objects, consisting of numbers,

73 This approach assumes that we can distinguish between strictly mathematical and philosophical aims and arguments. Any analysis of a structure from the internal perspective of another (or the same) structure is considered mathematical. Any other sort of argument is not. In particular, this means that there are no mathematical reasons bar proof to reject a certain statement as false. Any theory that allows us to do this, then, has some philosophical import: it adds something to the strictly mathematical study of structure. For example, if we want to outright reject the Continuum Hypothesis or its negation in a structure in which it is undecidable, such as ZFC, any arguments to that end would be considered philosophical in nature. Taken to its extreme, this implies a form of logical pluralism, as we cannot reject any of a variety of internally coherent logics.

74 He makes this notion more precise with his *structure theory*, which we will discuss in more detail in section 3.1.3.

geometric objects, and so forth. The foundational claim is then a metaphysical one: the foundation describes or even provides the very subject matter of mathematics.

This is a strong claim which is generally accompanied by a claim of *exclusivity*: since *this* foundational theory describes exactly what mathematics is really about, it cannot be about anything else as well.[75] Thus, the set-theoretic foundationalist holding that e.g. ZFC describes the "mathematical universe" thereby excludes alternative theories as well as extensions of, and variations on, itself.

The exact strength of this claim may vary. One may hold that there is a single mathematical universe in the metaphysical sense, but not in mathematical practice. That is to say, there is one "universe" of mathematical objects in some way, and perhaps we may capture it or describe it accurately using set theory for example, but a mathematical theory may not fully cover all of this universe.

This view has recently been challenged by Hamkins,[76] where he proposes that our familiarity with both set-theoretical "universes" in which CH is true and ones in which it is false make it impossible for us to decide the matter either way. Thus, we cannot hold to a single-universe view of mathematics, and we must admit that there can be multiple, mutually inconsistent mathematical "universes" existing side-by-side: a mathematical "multiverse".

The crux of either position is that a Gödelian view of a mathematical universes[77] can handle plausible *extensions* of the common foundational theory by claiming that such an extension simply covers more of the true mathematical universe, but cannot handle *variations* on the commonly accepted theory lest the mathematical universe itself contain inconsistencies. On this view the mathematical universe is to be seen as static, as a landscape for us to discover, but it is only partially captured by our best present axioms. Thus, we extended Zermelo's axioms first with a replacement scheme, and later with the Axiom of Choice, so as to cover more and more of this landscape. We cannot, however, *replace* one of these axioms (e.g. by dropping well-foundedness and replacing it with an axiom allowing for non-well-founded set theory), since our previous axioms did correctly describe the landscape. At best, we might find that we were mistaken previously and work with these alternative axioms instead, but the different axiomatic systems cannot exist side-by-side: that would mean that the mathematical landscape itself allows for a statement (e.g. the well-foundedness of sets) to be both true and false. If Hamkins is right about the Continuum Hypothesis, then this interpretation is too strict: we can, and should, investigate multiple mathematical landscapes.

75 [Shapiro 2011]
76 [Hamkins 2015]
77 See [Tieszen 2005] for a thorough account of Gödel's view on the philosophy of mathematics.

The strongest position here is a conjunction of the metaphysical and practical claims. On this position, there exist exactly those mathematical objects that can be described by the foundational system. This view is conceptually tricky, since such a view with regards to ZFC would mean that CH really is true nor false. Since we are speaking in metaphysical terms, this would mean that there is no truth to the matter of the existence of sets with a size strictly in-between the cardinality of the naturals and the cardinality of the reals. One would need to employ a non-classical semantics to account for objects that neither exist nor fail to exist without having to abandon classical logic.

3.1.2 *Ontology as mathematics*

If ontological foundations were a matter of metaphysics proper, it would be the battlefield of philosophers exclusively, while the mathematicians ignore the ruckus and continue their work. This is not the case - the question of existence is a matter within mathematics as well. The structuralist can distinguish two variations of this problem: the matter of the existence of an object or relation *within* a certain structure, and the question of the existence *of* a certain mathematical structure.

The former is relatively unproblematic. Without the philosophical baggage of establishing the *metaphysical* existence of an object, say, the number 63, it is not a problem that such an object exists in $\mathbb{N}$ but not in $\mathbb{Z}/60\mathbb{Z}$. The mathematician will without further thought refer to it in the former case, and will not refer to it in the second, unless he is establishing a relation between two different structures (e.g. by linking $\mathbb{N}$ and $\mathbb{Z}/60\mathbb{Z}$ in the usual way and establishing that $63 \equiv 3 \bmod 60$). Likewise, mathematicians have learnt to live with the fact that complex structures tend to lead to unanswerable mathematical questions about them (most famously the Continuum Hypothesis in ZFC). Although controversy can arise when long-standing traditions or intuitions within a mathematical community are put into question, there tends to be no significant problem with regards to the existence of objects within a structure as a *mathematical* fact.

The existence of new structures, not in a metaphysical sense, but simply as proper objects of mathematical study, is not unproblematic, however. Mere intuition, or even the intuition of great mathematicians, does not suffice to establish that a proposed structure is not inconsistent, incoherent or otherwise trivial. This is where a new, mathematical use of foundations comes in. *Mathematical ontological foundations* serve as an arbiter in mathematical disputes of existence. The idea is simple. There are a number of structures whose coherence we are generally convinced of, either through a mathematical proof of some sort or through a long tradition of working within said structure. Some of these "safe" structures are rich enough for many other structures to be translated into. That is to say, we can give faithful representations of most interesting mathematical structures within the safe structure. As long as the objects and relations essential to the structure are represented properly in the safe structure, we can be sure that the structure we are interested in is likewise safe.[78]

This is perhaps most naturally explained through Hilbert's process of giving a *reinterpretation* of some term in another theory. In his *Grundlagen der Geometrie*,[79] Hilbert translated geometric objects into different mathematical theories, such as the reals, in order to prove things about the consistency of geometric axioms. For example, one could reinterpret a point as an ordered set of two real numbers. This allowed for us to say something not just about geometrical objects, but about their axioms. The axioms could, after all, be reinterpreted in the theory of the reals, in which they would not turn out to translate into axioms but into "mere" ordinary sentences. This means that we can establish their dependence or independence from each other, and moreover, this allowed us to give relative consistency proofs: if the theory we translated our axioms into is consistent, then so is the axiomatic system we are interested in.[80]

On a structuralist analysis, what we are doing here is expressing one and the same structure in two different ways. Thus, using two different syntaxes, we give two different systems expressing the structure. The mathematical gain is evident when one of these systems is part of a larger system, expressing some other, larger structure. Thus we employ Dedekind abstraction once more to relate two structures: we emphasise certain relations within the larger structure - the translated relations of the smaller one. Rather than then disregarding the rest of the larger structure, though, we use the tools in the larger structure to prove things about the smaller one. We have the explicit goal of allowing one structure to piggyback on the mathematical "safety" the other. We do in fact require a bit more of our abstraction process than usual, since we do not only want to guarantee the existence of a particular structure, but we want our safety to be closed under the methods employed in the structure we're interested in. It is little use to establish that the structure we're interested in can be translated to a "safe structure" if the proofs we aim to establish cannot. Hence, we require that the axioms of our "safe structure" are likewise strong enough.

Nowadays, the role of "safe structure" is usually taken by set theory, in the form of the set-theoretic universe generated by the ZFC axioms, although some alternatives have been proposed. Notable examples include alternative set theories and category-theoretical foun-

78 [Shapiro 2011], pp. 99-100
79 [Hilbert 1899]
80 [Blanchette 2014]

dations of mathematics, such as ETCS and the Category of Categories as A Foundation (henceforth CCAF).

It is noteworthy that the practice of translating a certain part of mathematics to establish its coherence presupposes that its important features, such as its coherence, are preserved in such a translation. In other words, we are assuming that structurally equivalent systems express the same structure. Thus, this very practice has been taken by some authors, notably Shapiro, as evidence of structuralism. Those seeking a single metaphysical background, such as the position that all mathematical objects are really sets, should seek to establish that the original and its translation are indeed the "same" mathematical structure in every relevant sense; a question that quickly devolves into a variant on Benacerraf's problem.

One should be aware that there is a certain circularity to this kind of argument, however. The very practice of establishing the (relative) consistency of a system through reinterpreting it in another theory was challenged when it was introduced. Gottlob Frege held that Hilbert's consistency proofs based on the tactic of reinterpretation were invalid. One could not be sure that the same thought was expressed by a sentence in some theory A and its interpretation in another theory B. Therefore, the relative consistency of the translated sentences in B said very little to nothing about their consistency in A. Hilbert's counterargument defending the validity of these proofs hinges on what we would nowadays call a structuralist interpretation of mathematical objects. To him, there was nothing more to the objects he was talking about than the logical interrelations between them, and hence, we were talking about the same objects still after translation. Hence, the justification of the reinterpretative method by Hilbert relies on a structuralist view of mathematics. Thus there is the threat of a circular argument if we in turn take the use of this method as an argument for structuralism, as Shapiro does.

Those sceptical of the structuralist thesis might therefore well regard both this method and structuralism to be false, thereby invalidating the structuralist argument either way. The structuralist merely shows that his philosophy and the validity of the process of mathematical reinterpretation imply one another, which is of no great consequence if you hold both to be strictly false to begin with. One might go as far as to say that perhaps the reason structuralism seems such a natural fit for mathematical objects is not because mathematical objects really behave in the way structuralism prescribes, but because we have (falsely) gotten used to treating mathematical objects in this way. For present purposes, though, we consider foundational issues from an explicitly structuralist background, and leave this debate for another time.

The difference between mathematical and metaphysical ontological foundations should be emphasised. A metaphysical ontological foundationalist may hold that all mathematical objects truly *are* sets, whereas his mathematical counterpart may translate all of mathematics into set theory without holding such a belief. The mathematical foundationalist may simply consider it a very convenient way to secure the coherence of many mathematical structures and the existence of many mathematical objects. Moreover, it has mathematical advantages to have many different theories present in a common playing field; it allows us to make notions precise that cross different fields of mathematics, which in turn may lead to new proofs and theorems.

From this perspective, there is no reason beyond practicality to demand that there be one mathematical "universe". It is entirely possible for multiple large structures to serve as safety nets for different displays of mathematical acrobatics. It is worth keeping this in mind when we turn to the vivid arguments for and against different kinds of mathematical foundations in the next chapter. An argument against the use of a certain foundation is necessarily "philosophical" in nature: it assumes, explicitly or implicitly, that foundations have a bigger role to play than merely serving as a *mathematical* ontology. There are metaphysical or epistemological demands in the background whenever there is a claim of exclusivity for one theory, or a claim of foundational failure for another.[81]

The mathematical naturalist position of authors such as Penelope Maddy, holding that the goal of foundations is to be found in the principles which the mathematicians employ in creating these foundations, is then obviously mathematical-ontological in nature. In fact, it is very strongly so - any other ontological concerns, e.g. metaphysical ones, are regarded as scientific non-questions. At the same time, ontological concerns are very much placed at the centre of mathematical discourse, and a foundation - in Maddy's proposal, set theory - is regarded as the final adjudicator in discussions of existence.[82]

3.1.3 *Shapiro on ontology*

Shapiro, the *ante rem* structuralist *par excellence*, has attempted to establish a theory of structures themselves. Given that we allow structures in our ontology, it is argued, we need a way to differentiate between structures, to identify structures, and to establish which struc-

81 Of course, one could also be uncertain of the consistency of the proposed foundation. This problem is, unfortunately, not to be solved mathematically. Gödel's second incompleteness theorem holds that for any structure of sufficient complexity, its consistency can only be proved internally if it is, in fact, inconsistent. We cannot be mathematically sure of the consistency of even the most used foundation, ZFC. Uncertainty regarding the consistency of any proposed foundation then simply expresses unfamiliarity with a certain kind of structure - one that is best solved by studying the structure and its consequences. Thus, this kind of argument can be little more than a warning: "Perhaps study this structure a bit more before invoking it to establish the coherence of other structures!"

82 [Maddy 2011] pp. 33-34

tures exist and which do not.[83] To this end, he developed structure theory, which amounts to a set of axioms settling the existence of structures.

Structure theory allows for an ontology of structures, relations, functions and places, with a second-order background language. The axioms of structure theory are largely analogous to those of Zermelo-Fraenkel set theory, encompassing axioms of powerstructure[84] and infinity, as well as a replacement scheme. These axioms seem to be present for mathematical-ontological reasons: they are necessary to ensure that the large systems we tend to use in mathematics, particularly in set theory, indeed exist. Further axioms establish the particular behaviour of structures: one may add or subtract functions or relations from a structure and still have a structure, and one may take any subclass of places as a structure without any relations and functions on it.

The central claim Shapiro wants to make is that there is a structure for every "good" mathematical theory.[85] A straightforward statement of this idea within structure theory is problematic, however. The principle of coherence states that whenever some formula ϕ is coherent in a second-order language, there is a structure that satisfies ϕ.

Unfortunately this is either difficult to make mathematical sense of, or it relies on a mathematical theory, be it set theory or model theory taken as primitive, thus undermining the entire project of structure theory. If we leave "coherence" as an informal notion, mathematically uninterpreted, we might as well do away with the entire quasi-mathematical format of structure theory and simply state that every coherent structure exists. The only addition the axioms of structure theory would then provide is a minimum on existence: at least roughly the structures that can be expressed in ZFC exist.

If, on the other hand, we choose to give a mathematical interpretation of coherence, by stating that a mathematical structure exists if, for example, it has a model, we are left in no better waters. This would render the principle of coherence nearly empty: if there is a structure satisfying ϕ if and only if there is a model satisfying it, all our theory says is that models are, indeed, valid mathematical structures. An alternative is formulated in terms of a reflection scheme, establishing

83 [Shapiro 1997], pp. 92-96

84 This serves as the equivalent of the powerset axiom.

85 Coherence is the term here used to describe the informal notion of a structure "being good" or "making sense". Shapiro notes the difficulty in establishing the coherence of a formula. Consistency seems the most natural mathematical fit for this, but results in some intuitively wrong results: Shapiro mentions the conjunction of the axioms of Second-Order Peano arithmetic with the statement that Peano Arithmetic is not consistent. This theory is free of inconsistencies, but not satisfiable. To avoid this pitfall, Shapiro proposes to use satisfiability as the measure of coherence. See [Shapiro 1997], pp. 95.

the existence of structures satisfying the axioms of structure theory themselves.

> **Reflection:** If ϕ, then there is a structure S that satisfies the (other) axioms and ϕ. (7)

Here ϕ is any second-order sentence in the language of structure theory.[86] This results in a "structure-theoretic universe" of increasingly large size as the complexity of ϕ increases, up to and including large cardinal structures.

Shapiro's aims here are metaphysical: since the *ante rem* structuralist allows for the existence of structures, he herein tries to delimit their ontology. The theory seems to be stilted on two thoughts: it needs to be like mathematics, in order to ensure that there are structures for our mathematical theories, and it needs to be extremely large in scope, in order to ensure that there is a structure for every such theory. The result is a quasi-set-theoretic universe that is itself structurally equivalent to second-order ZFC with a reflection principle akin to (7). This ensures that structure theory cannot serve as a mathematical ontological foundation by any standard, as the consistency of such a theory is likely to be more doubtful than that of any common mathematical structure. This is then not the goal of structure theory. As a metaphysical background, though, it is uncertain why mathematical structures should metaphysically be anything like sets. We simply seem to lack sources for such knowledge. This is a defect Shapiro readily acknowledges for metaphysical foundations in general, but does not apply to structure theory. If the aim is to make the theory sufficiently general as to allow for any mathematical theory to have an associated structure in structure theory, then there it is unclear the goal can ever be achieve. Mathematics is open-ended in nature; if we formulate some mathematical structure, no matter how broad, we can always go beyond it and formulate a mathematical object that does not fit in the structure in any way. Dismissing such objects as incoherent out-of-hand cannot have been the goal of Shapiro, but any mathematical formulation of structure theory is going to result in such situations, as there cannot be a single formalisation of all of mathematics. What we're left with, then, is an ever-incomplete copy of all of mathematics.

All in all, this leaves it uncertain what goals a project like structure theory achieves. There are no epistemological or organisational elements to this theory - aspects of foundations we will discuss in the following sections - but the ontological value of the theory seems limited as well. The ante rem structuralist may be better off following the footsteps of Bourbaki in treating structure (mostly) as an informal concept. Recalling the motto of structuralism, there is already a formal study of structure: mathematics itself.

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86 *Ibidem*

3.2 epistemological foundations

Whereas the ontological variety of foundations take a perspective removed from the mathematician, preferring to talk of "mathematics" as a whole, the brands of foundation in this section are expressly concerned with him as thinking subject. Epistemological foundations are concerned with the epistemological properties of the foundational and founded theories. This is quite evident in Frege's logicist project. We will come back to Frege's view on foundations after establishing the cognitive element foundations may have.

3.2.1 *Cognitive foundations*

Cognitive foundations concern the way in which we come to know different mathematical structures. More specifically, we are concerned with the pedagogical ordering of different theories. We call some structure or theory a cognitive foundation for another structure if we need to get to know the former structure before we can properly grasp the latter.[87] For example, we need to be familiar with the natural numbers before we can make sense of coordinate systems using them.

Cognitive foundations take an unique spot in the pantheon of foundations, in that it can be tested empirically whether a certain theory serves as a cognitive foundation for another. One may simply try to teach someone a theory without going through its supposed cognitive foundation first. Of course, the usual caveats for empirical research of this sort will apply, and there might be exceptional cases. Furthermore, cognitive foundations are relative in character. Whereas ontological foundations admit of no further foundations *for* these foundations, we are bound to find exactly such situations in the context of cognitive foundations. The reals may serve as cognitive foundations for the complex numbers, but natural numbers may play such a role for the reals. Of course, this relation is transitive: the aforementioned situation also makes knowledge of the natural numbers necessary for knowledge of the complex plane.

The exact degree of knowledge required of a proposed foundation may also differ from case to case. We may only need to have limited knowledge of the natural numbers to understand theory A, but need to be extremely well-versed in the theory of natural numbers to understand B. Given the relative character of cognitive foundations, though, this need not be a problem: we can simply acknowledge that a single theory can serve as a cognitive foundation for different structures in different ways.

There is also a difference in strength between proposed cognitive foundations. In the strongest sense, a theory can simply not be under-

87 See also [Marquis 2005], pp. 427-429

stood at all without a proper understanding of the theory founding it. In weaker senses, learning a theory may simply become more difficult or less intuitive. Marquis suggests another sense, weaker yet: a role as a heuristic foundation. In this case, the role of the foundational theory is not pedagogical. It may be very possible to come to learn, understand and be familiar with the theory without knowledge of its heuristic cognitive foundation. However, familiarity with it will serve the researcher with a wealth of pointers for further research. Application of the methods of this field on our theory, or the reformulation of this field in terms of this theory, may reveal links to the beholder that were invisible before. Consider for example the use of category theory to find useful material in mathematics by asking questions such as "What are the morphisms?" or following the dictum to "Look for adjoints". The formulation of certain fields in category-theoretic terms promotes the progress of research in those fields.

This kind of foundation may be the most relative of all, considering that the connection of most fields of mathematics may prove fruitful in the right circumstances; indeed, some successful mathematicians are known for the ease with which they switch between different fields and combine insights from within them. Thus, for us to consider some theory a cognitive foundation proper, it needs a measure of generality and consistency: if in order to do successful research in some theory A we consistently reach for the methods and syntax of theory B, we may consider A a heuristic foundation.

Even with all these caveats in mind, though, it is clear what this kinds of foundations is, relative to other types. The practicality and empirical quality of it make it stand out amongst the crowd. That is not to say it is not commonly linked with other kinds of foundations. This is especially true for the stronger kind of claims, e.g. metaphysical ones.

Consider one last variety of cognitive foundations, which we may call *internal* cognitive foundations. These have to do with the internal workings of our mind. Suppose that we consider the brain to function probabilistically: that all the operations perform when reasoning or when doing mathematics are in fact probabilistic calculations. We can then consider probability theory to cognitively found the rest of mathematics, since what we are doing when we do e.g. set-theoretic calculations is *really* probability theory, whether we are aware of it or not. Let us say in this case that probability theory is an internal cognitive foundation for (some field of) mathematics. Again, this notion of foundation may be relative to a field; our brains could be doing probability theory for one sort of problem and predicate logic for another. What's more - we cannot be sure that there even is such a thing as an internal cognitive foundation in any meaningful sense. Nevertheless, positions like these do play a role in our thinking about mathematics. Suppose one takes a strong metaphysical stance, e.g. the position that all mathematics is "really" set theory. Although it does not strictly follow, it is then tempting to hold that we internally "do set theory" whenever we perform an algebraic operation.

When considering the internal processes necessary for mathematical reasoning, we can also take the perspective of transcendental philosophy. We may consider the *a priori* necessary conditions for the possibility of gaining mathematical knowledge. We can refer to these preconditions as the *transcendental* foundations of mathematics. For example, we may consider the possibility of intuiting space a necessary precondition for the possibility of geometrical knowledge.[88] Since this kind of foundation, too, concerns our cognitive faculty of doing mathematics, it is here shared under the header of cognitive foundations, although it is arguably more suitable to transcendental argument than empirical research.

Cognitive foundations in the pedagogical or heuristic sense may not align well with more theoretical foundations, such as the metaphysical-ontological variety or even the (rather practical) mathematical-ontological brand. Since matters of mathematical existence tend to be complex when compared to simple algebra for example, it may require quite a hefty amount of education to understand the theory providing ontological foundations for the algebra. Thus, the algebra that is ontologically dependent on another theory may be cognitively prior to it. Scenarios like these are unavoidable for mathematical-ontological foundations, as the drive for a single framework to express as many mathematical fields as possible in is at odds with the drive to create small, easily-understood theories. Metaphysical-ontological foundations, especially ones with a claim of exclusivity, are bound to create the same situation.

3.2.2 *Epistemological foundations*

Following Marquis,[89] we shall refer to foundations as epistemological foundations if their purpose is to transfer some epistemological property from the foundation to the founded theory. The exact nature of such a property may vary from case to case, but certain themes are common. For example, we may see an emphasis on the self-evidence of the axioms in the foundation. The thought behind this would be that this ensures their truth; and if we then deduce some theory from these foundations using only truth-preserving operations (e.g. logical ones), we are thereby ensured of the truth of the theory thus founded. We find similar ideas whenever it is argued that the justification of some theory derives from the foundation the theory is based on. In

88 Marquis has a number of examples of transcendental foundations by Hilbert and Russell in [Marquis 2005], footnote 26.
89 [Marquis 2005], pp. 429

this case, the foundational arrangement is supposed to convince us that some theory of mathematics is justified, because its foundation is, and the theory is related to its foundation in such a way as to be justification-preserving.

We can distinguish statements of different strength whenever we invoke epistemological foundations to transfer some property x upon a theory. For some x, it is clear that mathematics as a whole, or some mathematical theory, already possesses it - for example, few would argue that geometry isn't justified as a practice. In such cases, if we invoke a foundational relation of this kind, our aim is not to dispense x on a theory that didn't have it before, but to explain why it is that the theory possesses property x. In other cases, the goal of the scheme may be exactly to show that some field possesses a property that it is not widely considered to have (e.g. analyticity). It is then simply demonstrated that x holds of the theory; it is another matter whether this foundational scheme is the ultimate reason why the field possesses x.

The fact that the exact purpose of an epistemological foundation depends on the property it aims to transfer means that the exact relation that holds between the foundation and the founded theory will not be consistent across different epistemological foundations either. Whatever relates the two theories, the only demand is that it be x-preserving. Since one can imagine quite a few properties that are preserved under logical consequence, it is not surprising that many an epistemologically founded theory simply be deduced from its foundation, but it is by no means necessary.

3.2.3 *Frege's foundational project*

Gottlob Frege concerned himself heavily with the foundations of mathematics, and in particular with the foundations of arithmetic. His aims therein are perhaps most well-known as a programme of *logicism*: the idea that mathematics can be reduced to logic. Such a summary does not suffice for our analysis of the kind of foundation he tried to provide for mathematics, as this statement alone can be interpreted as merely mathematical; as merely a matter of form. The translation of one mathematical system into the syntax of another need not be more than an enterprise of mathematical curiosity. Indeed, Frege believed that whenever a proof could be provided of some statement previously taken for granted, it should. He therein aspired to ever greater generality. This is the more modest reading of Frege's programme: as merely a mathematical programme.[90] More generous readings have Frege proclaiming extensive philosophical goals: for example, his logicist programme had the aim of showing that mathematical concepts can be defined in terms of purely logical con-

90 See [Benacerraf 1981] for this account of Frege's programme.

cepts, or it had to prove that mathematical principles can be derived from the laws of logic alone. The first of these aims may be seen as metaphysical-ontological in nature; it aims to tell us something about the nature of mathematics. The second aim points us at Frege's wish to establish an epistemological foundation for mathematics. On the strongest reading, he is truly "securing" the epistemological status of arithmetic, since its axioms may be doubtful in nature unless based on truly self-evident notions; on a weaker one, he is merely investigating the epistemology of arithmetic by establishing the epistemological status of statements of arithmetic: in this case, that they are analytic.

Robin Jeshion identifies different readings of Frege's goals.[91] She identifies three different ways of reading Frege and his intentions. According to the *Mathematical rationale*, his aim were simply mathematical: he wanted to prove statements admitting of proof, including ones that were generally taken as axioms. According to the *Logico-Cartesian* rationale, his aims were to secure arithmetical knowledge in a logical source, as only that is beyond doubt; and according to the *Knowledge-of-Sources* rationale, his aims were to describe the epistemological and ontological properties of arithmetical knowledge. These different aims ascribed to Frege encompass different kinds of foundation.[92]

Perhaps most directly, the mathematical reading of Frege's aims is not strictly foundational at all. After all, there would be no philosophical aims or claims involved.[93] In particular, as Shapiro rightly points out, different "foundational" systems could very well exist alongside each other, providing multiple different proofs for the same statements.[94] On the Logico-Cartesian and Knowledge-of-Sources readings of Frege, his project is epistemological in nature. Showing arithmetic to be analytic is clearly the job of an epistemological foundation. But from this point onwards, other kinds of foundations tend to run together in Frege. If the truth of general arithmetic is truly in doubt prior to its reduction to fully logical principles, then we are not merely

91 [Jeshion 2001], pp. 940. In general, this is an excellent overview of the different interpretations of Frege's foundational aims.

92 Frege's actual aims remain a source of debate, with e.g. Benacerraf holding that Frege had a purely mathematical rationale and Kitcher holding that the Logico-Cartesian rationale is correct. In [Jeshion 2001], Jeshion holds that none of these strict readings are broad enough. Stewart Shapiro follows Jeshion's analysis in [Shapiro 2011]. Here, we accept it as well, though not based on any specific reading of Frege. Rather, by taking as a starting point broader view of Frege's project as encompassing both mathematical and "philosophical" goals, we can aim to *characterise* these different readings. In particular, Jeshion's readings can be put into a wider context as "kinds of foundation" that are not limited in their scope to Frege's specific foundational project.

93 [Shapiro 2011], pp. 102.

94 Of course, not all kinds of foundation are incompatible with alternatives: for example, any complex area of mathematics is going to have various cognitive foundations. Ontological foundations, of course, are generally considered exclusive (although alternatives exist, see e.g. [Hamkins 2012]).

using these foundations epistemologically, to heave the property of truth upon arithmetic. On the assumption that no contradiction can be true, we are simultaneously establishing that arithmetic is free of contradiction. Thus, there is a mathematical-ontological element to these foundations as well. After all, any foundation with a goal of establishing mathematical coherence falls in this category.

Moreover, Frege took an absolute view of justifications. He was not concerned with any kind of personal justification, or with anything even vaguely reeking of cognitive foundations for mathematics. Rather, he was concerned with the objective ground of mathematical propositions. This gives a peculiar "ontological" flavour to a foundational project that seems epistemological at first glance. We are not concerned much with the thinking subject anymore, but rather with objective grounding relations holding between propositions, and hence with the metaphysics of propositions.[95]

Thus, Frege's foundations serve as an interesting case study to illustrate kinds of foundations to identify not only different foundational aims of different programmes, but multiple possible (and arguably non-exclusive) readings of a single foundational programme.

3.3 PRAGMATIC FOUNDATIONS

Whereas ontological foundations were concerned with mathematics as a whole and epistemological foundations were concerned with mathematics as knowledge, as related to the thinking subject, there is a further perspective to consider: the view of the working mathematician. The goals of ontological and epistemological foundations are analytical. We study these foundations and apply them to mathematical fields to gain an understanding of or about established mathematical practices. Of course, certain fields of mathematics, such as model theory, are themselves concerned with mathematical practice. Still, this leaves foundations apparently completely removed from those mathematicians whose primary concern lies elsewhere, from the geometer or the probability theorist. And yet, they too talk of certain areas of their field serving as foundations for another. Thus, we stumble upon yet another sense of foundation, perhaps one less "philosophical" in nature: foundations as practical preconditions for mathematical practice.

95 [Shapiro 2011], pp. 104. This builds on the rationalist reading of Frege, wherein he is concerned not with any personal justification for any statements, but with the true reasons grounding any arithmetical truth, and thereby with the dependency structure between arithmetical statements. Thus, the metaphysical and epistemological aims run together. See [Burge 1990] for this account of Frege.

3.3.1 *Methodological foundations*

Methodological foundations serve as the array of tools a mathematician in a certain field needs to develop objects and structures within this field. The relation here is one of mathematical necessity: the mathematician needs to use tools of the foundational theory to be active as a mathematician in his field. For example, we need to employ the toolkit of group theory to get anywhere in algebraic geometry.[96] Methodological foundations are explicitly relative like that: the methods useful in one branch of mathematics may be of dubious worth in another. Thus, methodological foundations will generally be foundations of a specific structure or group of structures, rather than foundations of mathematics as a whole.

This brings methodological foundations close to cognitive foundations, but methodology need not be concerned with understanding per se. Rather, these tools are needed to be active in creating new mathematical structures. It is very well possible we may reformulate a theory in such a way as to make it comprehensible without knowledge of its methodological foundations - but it is not likely we could effectively *develop* the theory in such a manner. Thus, the comparison with heuristic cognitive foundations seems more apt. Methodological foundations are not to be confused with ontological foundations either - the latter may be constructed after the fact, to settle issues of existence of some object or other, or to soothe our doubts and assure us that some strange or unintuitive new theory in fact *does* make sense. Methodological foundations, on the other hand, are employed during the development of strange and unintuitive new theories. Of course, we may see a link here as well, as it is unlikely that some theory may provide the proper methods for a fields of mathematics if it is ontologically unreliable; a methodology resulting in impossible objects is a doubtful methodology.

3.3.2 *Organisational foundations*

Finally, we can consider pragmatic foundations in a weaker, yet more general sense: as a body of truths not for creating mathematics, but for organising it.[97] An organisational foundation is a mathematical framework of sufficient generality to connect various distinct bodies of mathematical work. If effective, it boasts some of the same advantages that mathematical-ontological foundations did: a single encompassing framework makes it easier to get an overview of the interrelations between different areas of mathematics, and is conducive

96 [Marquis 2005], pp. 430-431

97 I borrow the term "organisational" from McLarty in [McLarty 2013], but with a further distinction; whereas he runs together methodological and organisational foundations, I attempt to distinguish between matters of organisation and matters of methodology.

to finding and correcting errors within theories. It is as this kind of foundation that category theory is least controversial. On the other hand, this kind of foundation, and perhaps all pragmatic foundations in general, are often not deemed worthy of the honorific "foundation" by philosophers used to using foundations for analytic purposes.

3.4 what's important?

Mathematicians and philosophers have proposed foundations for a variety of purposes - and most, if not all, of their goals seem to be lofty. Nevertheless, different approaches call for different priorities, and a structuralist philosophy puts certain questions in the foreground.

3.4.1 *On the necessity of foundations*

Recently, the necessity of foundations has been called in to question.[98] If we are to answer the question of the necessity of foundations, however, we should be sure to ask what kind of foundation we are doubting.

For certain kinds of foundations, the question is seemingly irrelevant. The question for these is not whether mathematics ought to have them - they simply *do* have them, as an empirical fact. Cognitive foundations concern our ability to understand a certain structure prior to understanding another. Unless one is really willing to hold that we can understand complex numbers before ever even becoming familiar with the natural numbers, it seems that the presence of these kinds of foundations is beyond discussion. Likewise, unless each mathematical theory henceforth provides its own unique tools for proving theorems within them, it is a given that certain theories serve as methodological foundations for others.

For others, I consider the kind of foundation to be defined broadly enough that the mathematician will always aim to have them for purely mathematical reasons. There is no reason why one would not want good heuristic methods to find new mathematical truths. Likewise, its serves the mathematician well to have organisational foundations, to help him structure his work and find errors in it. For argument sake, we can consider the possibility that mathematics will grow so fragmented that we cannot really speak of one organisational framework for all of mathematics anymore. In this case, though, we may still have limited organisational foundations, relative to a specific field. It seems difficult to imagine that mathematics will ever be so fragmented that it will truly be an agglomeration of islands with little to connect them in any meaningful way. Organisational foundations for a certain field, such as category theory is for algebraic geometry, are indispensable for the working mathematician.

98 See [Awodey 2003]

Thus, when the necessity of foundations is called in doubt, we are concerned about mathematical-ontological, metaphysical-ontological, or epistemological foundations. We should take care to differentiate between the need for any of the latter two foundations and the need for a metaphysics or an epistemology of mathematics. We could have a metaphysics or an epistemology without having a foundation serving an important role in them. This would simply mean that there is no specific mathematical theory that plays a special role in either.

For metaphysics, this means that there would be a theory about the nature of mathematical structures and objects, but this theory itself is not mathematical in nature, or cannot be reduced to a single mathematical structure.[99] Thus, the position reducing mathematics to either set theory or category theory is generally excluded. Likewise, logicist projects are to be excluded, unless we take "logic" more generally than can be captured in any mathematical theory, e.g. as a schematic approach to human reasoning in general. A metaphysics of mathematics without a mathematical, foundational level would have to provide a metaphysics of all of mathematics directly, without taking a detour through any particular structure. One can easily imagine a variant of *in re* or *ante rem* structuralism fitting the bill, or, a theory linking mathematics directly to our cognitive faculties or to our linguistic capacities. None of these philosophies are without their own problems, of course. The fact that they are sensible positions to hold, though, means that a rejection of metaphysical-ontological foundations is a valid philosophical position.

The situation is somewhat trickier when one wants to reject epistemological foundations. Up to a certain point, the situation is analogous to the one regarding metaphysical-ontological foundations. If one favours an epistemology of mathematics that does not take any particular mathematical structure as epistemologically special, one might bite the bullet and reject epistemological foundations of mathematics. The situation is not quite as straightforward, though, as the exact role of an epistemological foundation my vary from case to case. Recall that an epistemological foundation of mathematics is employed to transfer some epistemological property from a single theory upon the rest of mathematics, or part of it, at least. Recall as well that this can be taken in two ways: the epistemological foundation can be the reason why mathematics as a whole has a certain property x, or

99 Of course, the position that a certain kind of mathematics (e.g. set theory) plays a special metaphysical role, but *cannot* be fully captured in a single mathematical structure, does not immediately exclude the possibility of mathematical foundations. One could take a large theory of this kind as foundational while simultaneously seeing the foundation as temporary and incomplete in nature. An outright rejection of foundations may or may not exclude positions like these as well. One could accept such theories as being truly metaphysically foundational and thus reject them, or one may accept them as carrying some metaphysical worth but reject the honorific "foundation". We have seen a few of the subtleties positions like these hinge on in section 3.1.1.

it may have the property x anyway, and the foundation just serves as a convenient method to show the reader that it holds for all of mathematics. For example, one reading of Frege's logicist programme is as a case of the former: one attempts to show that all of mathematics is analytic by providing a logical foundation of mathematics. The property of analyticity transfers because this would establish mathematics as a logical enterprise, and because classical logic is in an analytic enterprise if there is any. Thus, the foundation in this case truly is the reason why all of mathematics would be analytic. Imagine, though, that some clever philosopher has an argument for the analyticity of mathematics that should hold for all of mathematics equally. She happens to have shown particularly convincingly that some number theory is analytic - not because number theory plays a role in ensuring the analyticity of mathematics, but because it happened to be an easy and convincing case to make. She then establishes this theory as an epistemological foundation for mathematics by translating most mathematical fields into number theory in such a way that the reader is convinced that the property of analyticity is transferred. In this example, the status of number theory as an epistemological foundation for mathematics would be clear. It is not clear, however, that the rejection of this sort of foundation for mathematics is a sensible position to hold. One can imagine opposition to taking any particular part of mathematics to be epistemologically special - and hence opposition to e.g. Frege's logicist project. It is difficult to imagine any sensible opposition to the kind of epistemological foundation number theory is in our example, though. It would amount to opposition to the fact that a particular mathematical structure may be more convenient than others to establish a certain epistemological property of. This is simply a fact; historical events and current knowledge make us more readily able to talk of epistemological properties of certain mathematical structures than others. We have a lot more to say epistemologically about first-order logic than we do about some new number-theoretical construct, for example.

The necessity of mathematical-ontological foundations is a particularly tricky subject, since it does not concern just philosophical necessity, but claims to a need for this kind of structure in mathematics - much like methodological foundations, for example. At the same time, it invites questions of a philosophical nature because of its ontological form, because it describes mathematics in terms of what exists and what does not. We shall turn to mathematical-ontological foundations at length in the next section. Suffice it for now to say that opposition to the very idea of mathematical foundations seems to be directed at this kind of foundation in particular.

3.4.2 *On mathematical-ontological foundations*

First and foremost, the functions of proposed foundations that are important to the mathematician rather than the philosopher cannot be dispensed with. Now, of course, no one is suggesting that mathematical theories *are* dispensed with because they are the wrong species of foundations. Those opposed to e.g. category theory as a foundation rather see a lack of qualities in it that they feel necessary for anything worthy of the honorific "foundation" - typically, a foundation should be epistemological, cognitive or ontological. But the importance of pragmatic foundations does relativise the claim of mathematical necessity that comes with mathematical-ontological foundations. The need for the latter kind of foundations is invoked regularly with a claim to prevent mathematical error. The frameworks mathematicians themselves use just to prevent error and link up their works are probably more well-suited to error-finding than a single ontological framework, however. For these purposes, the mathematician may employ methodological or organisational foundations. In spite of claims to the contrary[100], the mathematician does not tend to define his objects in a single ontological framework to ensure the well-behavedness of his proposed structure. In a textbook, set-theoretic interpretations of common tools and structures may be absent or cursory, and may in most cases not be identified as grounding the field in any specific set-theoretic framework.[101] Without such a framework, we can hardly claim that the mathematician relies on it to ensure the well-behavedness of his objects. For the mathematically necessary purpose of error-finding, perhaps ontological foundations are not quite as indispensable as they purport to be.

The mathematical-ontological perspective equates matters of coherence with matters of existence: a theory is coherent if and only if it exists within some theory. This brings us to a second concern with regards to ontological foundations. Questions of ontology take on a different guise in a structuralist framework. As far as mathematics is concerned, the question of the existence of some structure can only be considered from the perspective of another structure, wherein the former plays the role of substructure. Thus, mathematical ontology itself is always relative to some mathematical structure, lest we try to

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100 e.g. Penelope Maddy in [Maddy 2011], pp. 33-34:

> [...] set theory has solidified its role as the backdrop for classical mathematics. Questions of the form - is there a structure or a mathematical object like this? - are answered by finding an instance or a surrogate within the set-theoretic hierarchy. Questions of the form - can such-and-such be proved or disproved? - are answered by investigating what follows or doesn't follow from the axioms of set theory.

101 See [McLarty 2012] for a number of examples of this kind. McLarty notes that set theory might not usually be absent completely, but can not commonly be identified as some foundational system or other.

stuff all of mathematics into one superstructure. This puts ontological foundations in an awkward spot, however, as those aim to establish, once and for all, the existence or non-existence of mathematical objects and structures. If the answer to the question "whether x exists" is always simply the counter-question "That depends - what structure are you working in?", mathematical-ontological foundations lose most of their sting.

The combined force of these two arguments should serve to "demote" mathematical-ontological foundations from the position as all-important judges of mathematical coherence that some philosophers would assign to them. Rather, matters of mathematical error and coherence, if they play a role in foundations, need not favour ontological foundations over other varieties.

Frege's foundational programme had an air of mathematical necessity around it. Without his extensive programme, one could argue that arithmetic was messy and error-prone, and that arithmetic knowledge could be seen as uncertain. This served as a justification for an extensive epistemological and ontological programme. Nowadays, mathematics is still in need of clarity and error-finding. But it can fend for itself better. Clarity and formal thoroughness have become accepted as mathematical virtues. It is seen as good practice to specify which particular structure the mathematician is working in, what axioms he takes to be true, and perhaps with most difficulty, what assumptions he is willing to make. With "merely" good heuristic and organisational foundations, the mathematician can provide us with knowledge unmarred by doubt and vagueness.

3.5 examining contemporary foundations

3.5.1 *The status of ZFC*

The Zermelo-Fraenkel axioms with Choice are perhaps most widely used explicitly as "foundation of mathematics". Indeed, we have referred to ZFC for purposes of example or comparison a number of times. Given our taxonomic enterprise above, we may wonder what kind of foundation ZFC indeed is. We cannot expect it to play every role at once.

Let us start with a rather obvious exclusion: ZFC is not a cognitive foundation in the pedagogical sense. We need no prior knowledge of set theory to understand algebra, arithmetic or calculus. Of course, it is possible to teach someone set theory first (and it has been attempted on a rather grand scale in the United States with the *New Maths* programme), but such an ordering is exceptional. We are certainly able to understand many fields of mathematics without any knowledge of set theory, as evidenced by the long history of mathematics before the advent of set theory.

There is an argument due to Feferman claiming that set-theoretical notions such as ZFC do indirectly serve as cognitive foundations. This is not through a cognitive priority of the set-theoretic axioms of ZFC themselves, but rather because they capture certain ideas that the subject must be able to conceive of before engaging in any other mathematical activity. These are the ideas of collection and operation. In order to understand, for example, a group, we need to be able to conceive of a collection first, as to understand the collection of elements in the group, and we need to be able to understand the concept of operation, so as to understand the functions defined on this collection that make it a group.[102] Thus, by making explicit the assumptions we make about our ability to collect and our ability to relate objects, ZFC stays close enough to these cognitive foundations to function as a believable foundation of mathematics. It is difficult to put our finger on the exact sense in which this makes ZFC a foundation for the rest of mathematics. Feferman's argument harks close to a transcendental argument, noting necessary presuppositions for our ability to reason in mathematics. Perhaps the argument is to be taken as a staged one: the ideas of operation and collection are transcendental foundations for ZFC, and ZFC are in turn found the rest of mathematics. Using non-set-theoretical foundations then amounts to skipping a step: by omitting the set-theoretic level, the transcendental foundation that the ideas of operation and collection provide are not transferred onto the rest of mathematics.

The main claim to fame of ZFC, however, is its position as the go-to mathematical-ontological foundation. This is due to two mathematical virtues. Firstly, its axioms speak almost exclusively of the existence and nonexistence of sets. Hence, in the language of set theory, every mathematical question is an ontological question. The existence of a relation is as much governed by the axioms of ZFC as the existence of a group or a hypercube. All of these are simply sets; they occur at some point in the set-theoretic hierarchy if they exist, and otherwise they do not. Second, ZFC is extremely rich. Most of conventional mathematics can be given a set-theoretic interpretation in ZFC. In particular, a part of mathematics that has been historically tricky for mathematicians and philosophers, the study of infinity, has found a fruitful basis in set-theoretic study, most commonly within the context of ZFC. The structural nature of mathematics ensures that the set-theoretic system expresses the exact same structure that was investigated in the original formulation, provided that the essential properties of the structure were preserved - in other words, provided that the set-theoretic formulation is indeed isomorphic with the original formulation. The combination of these two virtues makes it a prime candidate as a mathematical-ontogical foundation, since most structures the mathematician encounters in his research can be ex-

102 [Feferman 1977], pp.150

pressed set-theoretically and consequently investigated for their existence within Zermelo-Fraenkel set theory.

It is this very call for a translation of various kinds of mathematical systems into set-theoretic terms that makes it less successful as a methodological or organisational foundation, however. These sorts of foundations are expressly not syntax-neutral. Certain formulations of the same structure may be more conducive to creating or organising mathematics than others. And although many structures often can be expressed in set-theoretic terms, they often are not - at least not beyond a definition of primitive concepts as sets. This generally means that all is well on the mathematical-ontological front, bar operations that turn out to be untranslatable into a succession of applied axioms of ZFC. But such a translation does not make ZFC, or any form of set theory, a methodological foundation for mathematics. The methods employed are, outside of set theory, typically not set-theoretical. In fact, when it comes to complex proofs, it can even turn out to be very difficult, not to mention a daunting task, to translate them into a purely set-theoretic proof on the basis of ZFC. This alone means that it does not function as a methodological foundation for many of the fields it purports to found. After all, a methodological foundation provides the very tools that the mathematician uses in his proofs. Had ZFC functioned as a methodological foundation in these situations, the proofs would have been expressed set-theoretically, or at least easily understood for the most part in set-theoretic terms. The fact that it is difficult to express common proofs set-theoretically means ZFC does not serve as a methodological foundation commonly.

Another consequence is that it is of limited use as an organisational foundation. Links between different parts of mathematics are rarely found through translation of all the systems involved into set-theoretic terms. Error-finding, likewise, is not done through a set-theoretic lens. Long, complex proofs that are subject to extensive error-finding sessions (such as, famously, Andrew Wiles' proof of Fermat's last theorem) are not generally submitted to error-finding through set-theoretic methods - and, perhaps more ominously for ZFC's role as an organisational foundation, may be accepted as valid before such a translation is finished or even attempted.

The discussion with regards to the status of ZFC as a metaphysical-ontological foundation is ongoing. The trouble with this particular kind of foundation is that there are few methods if any for us to acquire the kind of metaphysical knowledge required to settle this debate. What kind of argument could convince us that mathematical objects "really are" sets or categories? Thus, this debate is often held through connections with other manners in which ZFC claims to be a foundation. Those that greatly value a status as mathematical-ontological foundations may tend towards accepting it as a metaphysical one, while those who feel that the behaviour of mathematical objects in practice tells us more about the nature of mathematical objects may oppose, on the grounds that it is a mediocre methodological foundation. At present, we would like to note that structuralism does not in particular point us to a single mathematical universe, which has to be captured through metaphysical foundations, in a way that platonism arguably does. Structures are relatively self-standing: we can investigate the natural numbers independently of group theory just as easily as we can link the two. Hence, ZFC, if taken as a metaphysical foundation for mathematics, would place mathematical structures into a single framework, which is at best arbitrary in its portrayal of mathematical structures, and at worst contrary to our understanding of mathematical structures as self-standing.

A link between cognitive and metaphysical foundations of mathematics, however, seems hard to deny. It is difficult to imagine that we could get to know mathematical structures in such a way that it would be completely independent from the very nature of these structures. Thus, if we accept Feferman's claim of the cognitive priority of the ideas of collection and operation for all of mathematics, set theory may metaphysically found most of mathematics in the same staged manner in which it cognitively does. One might suggest that the reason it may appear to us as arbitrary is because of this staged approach; in fact, it may *be* an arbitrary, yet sufficient way to capture these ideas and hence found mathematics. Indeed, Feferman did not intend his work as a defense for ZFC or any current set-theoretical foundations of mathematics.[103] The exact manner in which these two ideas might be said to found mathematics, via set theory or otherwise, lies beyond the scope of this thesis, however.

3.5.2 *The status of category-theoretic foundations*

Category theory, when taken as foundational for mathematics, tends to be considered such in a different manner than set theory is. It is controversial as "a foundation for mathematics" as classically conceived, that is to say, as an epistemological, cognitive or mathematical-ontological foundation. We will come back to this at length in chapter 4. In particular, category theoretic mathematical-ontological foundations and its criticism will be treated in section 4.1.1.

For now, suffice it to say that Feferman's argument in the previous section was aimed primarily at category theory. The primitives introduced in the Eilenberg-MacLane axioms for category theory - that is, objects and morphisms - can indeed be argued to be cognitively dependent on a notion of collection or operation. There are two avenues for response to this; the first is to accept it, but deny that it is a problem. This amounts to letting go of any category-theoretical

103 [Feferman 1977], pp. 154

foundations as cognitive foundations, but sticking with it as another kind of foundation. The other is to argue that category theory simply captures these concepts at the right level of generality in its objects and morphisms,[104] and that there is nothing more cognitively "remote" from these ideas than there is in the case of set theory. If the rather natural fit of category theory with structuralism reflects a closeness with our capacity for understanding mathematics, then we might make the latter case. In practice, however, we know too little of the cognitive features involved in mathematics to argue either way.

Category theory as a whole[105] is perhaps most uncontroversial in its role as a heuristic cognitive foundation. Category-theoretic tools are commonly used as tools in mathematical research to indicate promising venues for further work. Identifying the morphisms in a given structure is often a fruitful enterprise, as is looking for adjoints. Likewise uncontroversial is its status as a methodological foundation for many fields of mathematics - in particular, those in algebra or closely related fields.

Moreover, as we saw reflected in the position held by McLarty, category theory has a good claim to being an organisational foundation for mathematics. The links between different fields can often accurately be reflected through a categorical framework. A formulation in category-theoretic terms tends not to obfuscate the meaning of the terms involved as much as the translation of a proof in set-theoretic terms might.

Thus, category theory shines as a foundation of the kinds that set theory was weak in. At first glance, however, it seems that the converse might also hold. As such, the status of category-theoretic foundations stirred up some debate, which we will turn to in chapter 4.

104 This view is held by Marquis; see [Marquis 2014] pp. 436

105 That is to say, categories as simply conceived by any traditional axiomatisation of categories, not as in any specific proposed foundation of mathematics, such as the ETCS or CCAF.

4

CATEGORICAL FOUNDATIONS OR FRAMEWORKS

In this chapter, we turn towards the contemporary debate regarding categorical structuralism and category-theoretic foundations of mathematics. We shall examine the criticism of category theoretic foundations and the rejoinders from the categorical structuralists. Using the structuralist view of mathematics from chapter 1 and the analysis of mathematical foundations from chapter 3, we can analyse the arguments on both sides of the debate, and come to a view of mathematics, and the role of category theory therein, that avoids the pitfalls of either side.

The project of founding mathematics - in one sense or another - in a category-theoretic framework quickly became a target of criticism, usually following the lines of Feferman's [Feferman 1977], which we discussed before in section 3.5. Further criticism mostly concerned the mathematical form that categorical foundations necessarily take just by being categorical. Nevertheless, the criticisms are quite varied in nature. To bring some order to these concerns, we deal with them according to the *kind of foundation* they see categorical foundations failing in, following our taxonomy from chapter 3.

4.1 ONTOLOGICAL CONCERNS

First on the agenda is, again, the ontological aspect of foundations. Whereas there is an established (if informal) ontological background to full-blown set theory, the "set-theoretic universe", there is no categorical equivalent quite as established. Moreover, there are concerns that category theory *cannot* give any satisfactory ontology *a priori*. The chief proponent of this line of thinking is Geoffrey Hellman.[106]

4.1.1 *Assertory versus algebraic foundations*

To treat Hellman's criticism properly, we need to introduce the exact difference in mathematical form that his argument hinges on. Using modern terminology for a distinction going back to the early twen-

106 See section 1.3.2 for his alternative account of modal structuralism

tieth century, we can distinguish between *assertory* statements and *algebraic* ones.[107]

"Algebraic" axiomatic statements go back at least to Hilbert. As a mathematician in the late nineteenth century, he was faced with the flourishing of many different geometries besides the traditional Euclidean geometry. It had grown increasingly obvious that these geometries, while mutually inconsistent, were more than passing fads of the mathematician's fancy. With geometry becoming less obviously a study of space but rather the study of a certain range of structures, geometry lost its status as the example of epistemic certainty.[108] The role of physical space in geometry became that of an application of a mathematical theory that has a status of its own. Shapiro characterises Hilbert's *Grundlagen der Geometrie* as the culmination of this process of abstraction:

> Issues concerning the proper application of geometry to physics were being separated from the status of pure geometry, the branch of mathematics. Hilbert's *Grundlagen der Geometrie* [1899] represents the culmination of this development, delivering a death blow to a role for intuition or perception in the practice of geometry. Although intuition or observation may be the source of axioms, it plays no role in the actual pursuit of the subject.[109]

Hilbert's axiomatic system of geometry did not set out to capture a pre-mathematical concept of space, but rather to provide a schematic set of axioms, which, taken together, describe a geometric pattern on any possible interpretation of its terms. The undefined primitives in Hilbert's axioms could be filled by anything at all as long as the axioms are satisfied. In this way, they describe a schema of concepts and their interrelations, not a particular chunk of reality. What it means to be a geometric object, such as a line, then, is defined by these axioms: anything that satisfies the axiom will fulfil the schematic requirements for being a line, and thus be one. Hellman and Shapiro call axioms formulated in this way *algebraic*, based on an analogy with elementary algebra: a group or ring is anything satisfying the axioms of a group or a ring, rather than any specific unitary object.

Hilbert's formulation of axioms was attacked by Frege, on whose account axioms, like other sentences, should express a proposition that can be grasped by the listener.[110] In particular, axioms should be true, and for them to be true, the terms expressed in them should have a definite sense. Hilbert's schematic axioms seemingly lack this

107 [Shapiro 2005], pp. 67. The term "schematic" is sometimes used synonymously with "algebraic" in the literature.

108 [Torretti 1999]

109 [Shapiro 2005] pp. 63

110 See [Blanchette 2014] for a full account of the Frege-Hilbert controversy, and [Shapiro 2005] for a summary aimed at defining the algebraic/assertory dichotomy.

property. Should, however, they be seen as having meaning beforehand, they then cannot be said to define what it means to be a geometrical object of some sort or other. Hence, these proposed axioms fail either as defining what it is to be a mathematical object of some sort, or fail as being meaningful axioms. On Frege's account, axioms should express a definite and true proposition about concepts that have been defined already, or are otherwise known already. Hence, the axioms of arithmetic express truths regarding the realm of natural numbers, and the axioms of geometry express truths regarding physical space. As a result, the axioms will have a definite truth value. Reverting to Hellman and Shapiro's terminology, we shall refer to statements formulated in this way as *assertory*.

Turning back to the matter of categorical foundations, then, it is critical to note that the axioms of category theory itself are algebraic.[111] A category, after all, is anything satisfying the Eilenberg-MacLane axioms. Categories are presented as schematic: there is no obvious external referent involved in the understanding of these terms. Rather, the structure is stated by defining some mathematical terms in terms of their interrelations.

Hellman's criticism now resides in the inappropriateness of algebraic axioms as a foundation of mathematics.

> [...] somehow we need to make sense of talk of *structures satisfying* the axioms of category theory, i.e. *being categories* or *topoi*, in a general sense, and it is at this level that an appeal to "collection" and "operation" in *some* form seems unavoidable.[112]

Or, more recently:

> Of course we know what the primitives of the first-order CT axioms are; however, the question is not about the definition of "category," but rather about the primitives of the background (informal) substantive mathematical-foundational (meta-)theory, which, as Feferman observed, employs notions of *collection* and *operation* and *functor*.[113]

Thus, the problem lies a conceptual level below the actual Eilenberg-MacLane axioms. In order for there to be any *content* to these axioms, we need something more, and it is this "something more" that we cannot deal with categorically.

We can analyse the idea behind this criticism by investigating the purported logical form of their axiom. The idea is that, being algebraic, they feature a "silent" quantifier ranging over possible systems exemplifying the categorical structure:

111 Recall the definition of a category on page 34.
112 [Hellman 2003], pp. 135
113 [Hellman 2005], pp. 550

> For all systems A, if A satisfies the Eilenberg-MacLane axioms, then it is a category.

and by extension:

> For all systems A, if A satisfies the Eilenberg-MacLane axioms, the following mathematical statements hold of it...

As such, mathematics requires the existence of such an A, which would in turn require a proof likewise based on category theory. It was this very same weakness that haunted the *in re* structuralist trying to reinterpret the meaning of mathematical statements, and the *ante rem* structuralist trying to give an account of which structures truly exist: the problem of the background ontology. There needs to be a domain for the quantifier "any" to range over in their interpretation of even simple mathematical statements. But on Hellman's account, such a domain was exactly what the foundation was supposed to provide! Thus, a non-assertory theory cannot play the role of foundation itself.

This informs what Hellman calls the "problem of the home address" - where do all these categories live? If the axioms do not establish a universe of structures, how can we be sure what domain these axioms range over? As Hellman puts it:

> [...] just as in the cases of more familiar algebraic theories, the question about mathematical existence can be put: *what categories or topoi exist?*[114]

We can see that the problem here is quite straightforwardly failure as *mathematical-ontological foundations*. What is lacking is a mathematical theory establishing the existence or non-existence of certain structures or other. In the quotes above, of course, Hellman directly referred to the concepts of collection and function, and thus to Feferman's famous argument. The idea is that we cannot conceive of the *background theory* in terms other than those. As this is rather clearly an epistemological matter, we will return to this issue in section 4.2.

Moreover, if we employ algebraic axioms, it would leave mathematical statements with an issue of modality: whereas these statements are supposed to be categorical (in the ordinary sense of the word), they would take the form of a generalised hypothetical. Their truth would hinge on the existence of some sort of structure in the background ontology. In this way, the mathematician working with algebraic statements and without an assertory foundation for them, will see his mathematics reduced to a variation on hypothetico-deductivism. Any theorem proved will not be a truth, but rather a hypothetical dependent on the structure it is supposed to clarify.

114 [Hellman 2003], pp. 137

All these problems are made possible directly by the algebraic nature of these axioms. Because they seem to quantify some domain of background objects, there is uncertainty over the extent of these axioms as long as there is uncertainty about the existence of the objects they are about. Likewise, we can ask questions about other properties of the background ontology - cognitive, epistemological, and so forth. By contrast, the standard reading of the axioms of full-blown set theory is in an assertory sense: they are seen as expressing truths concerning the set-theoretic universe. The set-theoretic axioms of ZFC simultaneously define the behaviour of sets and the extent of existing sets. Hence, there is no ontological uncertainty even if there *were* a problem of background ontology for full-blown set theory.

According to Hellman then, to avoid any of these problems without appealing to set theory, mathematics needs not be assertory per se, but its foundation should be. Only this can ensure the existence of mathematical objects for the statements to be true of, and avoid the problem of the background ontology. Thus there is, on Hellman's account, a fundamental mismatch between the assertory nature of proper foundations and the algebraic nature of category-theoretic approaches.

4.1.2 *Responding to Hellman*

Now first and foremost, let us refer back to section 3.4.2 on mathematical-ontological foundations in response. We do not need this kind of foundations *per se* for the strictly mathematical purpose of error-finding. We can distinguish between organisational and ontological kinds of foundations and rely on the latter to avoid impossible or trivial structures. Nevertheless, distinguishing between coherent and incoherent structures is something that we must be able to do, one way or another. We will come back to the matter of coherence in section 4.3.1.

For now, let us recall that on the structuralist account, mathematical existence can only be defined from within another mathematical structure. Thus, what we can take from Hellman's criticism is that, if we want to get anywhere ontologically on a categorical account of mathematics - or at least if we want to get "as far" as full-blown set theory does - then we need some category-theoretic theory asserting the extent of the existence of mathematical objects.[115] Of course, the Eilenberg-MacLane axioms themselves do not meet this demand in

115 It is important to note that it is *not* necessary to have one once-and-for-all categorical account of all of mathematics, or even in principle one framework that could be extended indefinitely (as full-blown set theory on some accounts purports to be - see e.g. Gödel's views as set out in [Tieszen 2005]). Rather, one can see this as meeting the demand for mathematical-ontological foundations halfway: such foundations might not be necessary, or even reflective of the nature of mathematics on a structuralist account, but given their *mathematical* usefulness, i.e. their way of serving si-

any way. What is needed for a categorical, ontological account then, it some sort of assertory mathematical foundation.

Now as a matter of fact, there are such foundations. In fact, one of the most prominent categorical foundations for mathematics, Lawvere's ETCS, is assertory in nature.[116] It features extensive ontological claims: for example, it has an axiom establishing the existence of a terminal object 1, and an axiom establishing the existence of a natural number object (analogous with ZFC's axiom of infinity.) There is no need to wonder about the background ontology, then: it is carefully delineated by our axiomatic system. We do not invite these problems back in the long way around simply by using category theory, either. The algebraic character of the Eilenberg-MacLane axioms never enters play, as we are concerned from the start with a single, concrete category. Thus, we never have to consider the axioms as defining some indeterminate number of categories, raising questions on the level of the background ontology. The axioms concern, in this case, the category of sets, whose behaviour we further establish through our axioms.[117]

Alternatively, one may reject Hellman's analysis that there is something wrong with algebraically-formulated axioms in the first place. This line of thinking is due to the antifoundationalist Awodey.

4.1.3 *Revisiting Awodey*

Using our analysis in terms of *kinds of foundations* from chapter 3, we can make further sense of Awodey's straight-up rejection of "foundations". What is clear is that his top-down view of mathematics precludes any kind of foundation that aims to provide the "building blocks" of mathematics, or that aims to define the universe of mathematical objects. As such, he rejects ontological foundations, both mathematical and metaphysical, almost explicitly. More strongly, his rejection of "foundationalism" in general implies a rejection of any kind of foundation that is not strictly relative to some structure or to some field of mathematics - that is to say, any kind of foundation of "all of mathematics" rather than of a specific structure or class of structures. This then includes most, if not all, epistemological foundations. After all, if we wish to transfer some epistemological property from one specific structure to all of mathematics, that structure needs to find its way to all of mathematics in such a way as to transfer the property. Such a situation is unlikely at best if we do not see all of mathematics as somehow derived from a particular structure. Hence,

multaneously as an organisational foundation for mathematics, a category-theoretic account is useful and must then be assertory in nature.

116 See [Linnebo & Pettigrew 2011], pp. 233

117 One might ask whether we need not *understand* the Eilenberg-MacLane axioms beforehand in order to make sense of these theories. That way, we might run into the cognitive aspect of Feferman's criticism. We shall come back to this in section 4.2.1.

if we want to make an argument about e.g. the cognitive accessibility of mathematical structures, we have to start *from* a structuralist analysis in order to "transfer" any kind of property over all of mathematics - no single mathematical structure in particular will do. In other words, an ontology or epistemology of foundations will need to be explicitly philosophical in nature rather than mathematical. We might get pretty far still if we use category theory, since it allows us to express things on a structural level easily - but since Awodey does not identify mathematics with category theory (i.e. as metaphysical foundation), but rather sees it as just a good way to express structural mathematics, it is unlikely that this method will get us any categorical (in the ordinary sense of the word) epistemological knowledge.

Turning back, then, to the issue of the correct form of foundations, Awodey is perhaps most direct in answering Hellman's problems with algebraic foundations: he rejects the problems outright. He does not share the interpretation of any algebraically formulated sentence as featuring a silent universal quantifier, and does not put it shyly:

> This lack of specificity or determination is not an accidental feature of mathematics, to be described as universal quantification over all particular instances in a specific foundational system as the foundationalist would have it - a contrived and fantastic interpretation of actual mathematical practice (and even more so of historical mathematics!).[118]

From the perspective of *top-down* mathematics, this is indeed clear. Awodey does not see a specific instance of a general theorem as that theorem *applying* to something more basic, or as expressing a truth about an infinity of basic building blocks. Rather, by proving the theorem, we have established it as true of a generic structure "higher up", and when we "apply" it, we are only *adding more structure* to it to bring it down to a lower, more detailed, more specific level. Thus, a truth established of groups in general can be seen as still true of Abelian groups by *adding* commutativity.

Even an algebraic axiom or theorem then simply *establishes* a truth. It does not rely on a further level to quantify over - until we specifically want to quantify over something, there is no such level needed in our analysis. The seeming indeterminacy of algebraic axioms is then no mistake to be avoided, but is reflective of the very nature of mathematics. Mathematical truths are expressed on a structural level - that is, they are expressed regardless of any inner structure to the objects in the structure. As any mathematical statement concerns structure, not objects, this invites a different reading of mathematical statements.

118 [Awodey 2003], pp. 7

4.1.4 *Interpreting mathematics*

To close the gap between Awodey and Hellman, we can formulate Awodey's interpretation of mathematical statements, much like Hellman does in his account of modal-structuralism. Let us take a simple example:

In every group, the unit e is unique. (8)

Clearly, this statement is algebraic in nature. On the "standard reading" - perhaps more accurately the *foundationalist* reading - we would see this sentence as implicitly quantifying over some ontological domain:

For all G, if G is a group, then its unit e is unique. (9)

Of course, this formulation harks closer to platonism than to any kind of structuralism. We can bring this formulation in line with a generic account of structuralism, be it *ante rem*, *in re* or eliminative, as follows:

For all systems G, if G expresses the group structure, then the unit e of G is unique. (10)

For completeness sake, recall that, on Hellman's modal-structuralist account, we would interpret the sentence thus:

For all logically possible systems G, if G expresses the group structure, then the unit e of G is unique. (11)

On an "algebraico-structuralist" reading, though, we can also give an interpretation that does not imply a background ontology, let alone an implicit quantification over it. Rather, we want to read (8) as stating a truth about the structure of a group:

Groups are such that the unit e is unique. (12)

The sentence then expresses something at the level of a structure, and only at that level. Thus, the problems invited by Hellman's reading of algebraic sentences are avoided. This does not mean that the *indeterminacy* involved is lost - and with it, the strength in variation that category theory has (by taking its objects to be groups, categories, rings, sets, *etc.*, and morphisms to be the associated homomorphisms). Rather, we can avoid a reading in which we have to fix a background domain and determine the extent of our algebraic statement rather than taking it at face value.[119]

Note that our aim in giving this alternative interpretation of a mathematical statement is *not* meant to, once-and-for-all, give a correct account of "how to read" mathematical statements on a structuralist view. Rather the aim is to show, by providing an alternative, that the universal quantifier reading is not the only one. In fact, on a structuralist account, both readings - or something akin to them - are necessary. We need to be able to see this rule *as* quantifying over systems, i.e. as in (10), in order to identify some particular structure as an instance where the rule expressed in (8) holds. For example, we want to be able to apply this rule to say that $\mathbb{Z}/60\mathbb{Z}$ has a unique unit, and so do the quarternions. At the very least, we want to be able to quantify over some given collection of groups. At the same time, though, we can work within the structure of a group, determined *only* by the axioms of group theory, and *assert* the rule from a top-down perspective. For this, we need a reading akin to (12). We can see a mathematical statement either as a rule to be applied or as constituting some truth, as implicitly defining something. It is worth noting that the problem of modality is sidestepped completely, as there is nothing hypothetical about such a statement.

It is crucial to note here that on Hellman's account, we are forced by the *mathematical* structure of an algebraic statement into a certain *philosophical* interpretation of the statement. Simply because it is not assertory in form, we are to read it as if it had a quantifier over some background domain. What the above example shows, then, is that this is a misunderstanding. Mathematical statements, including ones algebraic in form, can be read at face value.

Of course, one may still have philosophical objections to such a reading, or to any particular reading. It is such concerns that led Hellman to his modal account of structuralism, for example. But it is important to note that the face-value reading of algebraic statements is entirely in line with the nature of mathematical structures. After all, these have a certain indeterminacy: they are only determined up to isomorphism. Thus, there is *always* room for further interpretation, and yet, such a interpretation would go beyond the structure as described. Thus, a face-value interpretation of many mathematical statements is necessary to neutrally describe any particular structure - that is, to describe it without describing it from the perspective of a richer structure.[120]

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119 One can see this "face value reading" as an extension of Shapiro's *places-are-objects* perspective in section 1.1.1, allowing us to refer to structures as a whole at face value "as objects", as well as to places in the structure.

120 Of course, a "face value" reading presupposes that it is clear what structure we are talking about; if it is not, multiple interpretations involving various structures are possible. This would lead us to the kind of interpretation of (quasi)mathematical statements we saw in the footnote on page 27.

The interpretation of even simple mathematical statements such as (8) is not set in stone. We may even switch freely between these interpretations as mathematical need arises.[121] We are not, then, forced into any particular philosophical view of mathematics due to the mathematical form of an expression, despite Hellman's gripes with algebraic statements.

What is happening on the foundationalist reading of (8) is inadvertent non-structural thinking, invited by a certain interpretation of mathematics rather than by mathematical fact. By interpreting the sentence as quantifying over some universe of objects, we are inviting an object-based view of mathematics. Hellman then tries to draw a philosophical conclusion with regards to the categorical-structural approach by inviting a nonstructural view of mathematics in the first place. That is simply begging the question; of course we cannot express mathematics categorically in a structuralist manner if we explicitly ask for a non-categorical-structuralist interpretation of those axioms.

On Awodey's account, the only thing we need check to be sure of an algebraic statement is whether the antecedent of a mathematical statement is *ever* filled - in other words, whether there is a system expressing the proposed structure *at all* - or in other words yet, whether the structure is *coherent*.[122] This, he notes, is done simply by investigating the structure itself and the consequences of the statement whose antecedent we doubt.[123] We do not, then, invite any particular new sort of doubt on the coherence of our mathematical structures by employing a non-foundationalist point of view.

4.2 epistemological concerns

We saw that the bulk of Hellman's argument is mathematical-ontological in nature. Through a specific reading of mathematical statements, we invite questions that are ontological in nature - questions of the kind that the categorical structuralist wants to avoid simply by not aiming for such a foundation at all. There are further concerns though, regarding the associated epistemology.

121 Of course, this freedom may be limited by philosophical concerns; for example to ensure reference to structures is avoided if one is an eliminative structuralist, or to ensure a background ontology is avoided if one follows Awodey.

122 By asking "in cases where it is not sure whether the conditions at issue are ever satisfied" ([Awodey 2003] pp. 9), Awodey uses language inadvertently suggestive of the "quantifier reading" of mathematical statements. It is clear that the only thing he means is that we need to ensure the consistency of the antecedent. Hence, we are speaking of matters of coherence, and thus of *organisational* foundations if we speak of foundations at all, and not of ontological foundations "the long way around".

123 [Awodey 2003], pp. 9

4.2.1 *The matter of autonomy*

Suppose that we aim to establish a categorical foundation in much of the same way that ZFC can be a foundation for mathematics. We grant, for the sake of argument, the assertory/algebraic distinction, follow Hellman's advice to avoid any algebraically formulated axioms, and seek to adopt an assertory categorical theory. Even though we've then dealt with the mathematical-ontological concerns, we are not out of the woods yet. Our foundations are expressed in the language of category theory: in terms of objects and morphisms. Hence, in order to *understand* our foundations, we need to understand these category-theoretic basics first. But this means that we risk running afoul of Feferman's argument the long way around. For if we need to understand concepts of collection and function, or even the full-blown concept of set, in order to understand the Eilenberg-MacLane arguments, and thus to understand categories at all, there is indeed a cognitive dependence in play.

The notion appealed to here is one of the *foundational autonomy*. In order for any theory to take a role as a non-relative foundation of mathematics, for example as a mathematical-ontological foundation, it must be autonomous: it cannot depend in any way on another mathematical theory external to the proposed foundation.[124] To fail this demand of autonomy would be to fail as a foundation: how is some other part of mathematics to be described as depending on the proposed foundation if the foundation itself cannot be expressed without referring to these mathematics? It is important to note that likewise, the tools employed by the antifoundationalist cannot be dependent on some foundation either, lest the entire enterprise reduces to some foundational system the long way around.

Linnebo and Pettigrew, in their investigation whether category theory can provide a foundation of mathematics, make a distinction between three different types of autonomy which any foundation must have.[125] The *Logical Autonomy* requirement is straightforward: if a proposed foundation depends logically upon another foundation, such as the orthodox one, it cannot itself function as a foundation. This is a syntactic, mathematical matter: if we were to explicitly need full-blown set theory to formulate our categorical set theory, for example, we simply cannot say to be doing categorical set theory proper.[126] Another requirement is that of *justificatory autonomy*: can the existence

124 Foundations that are relative a certain mathematical structure by nature, such as methodological or cognitive foundations, of course need not be autonomous - although we need to start somewhere with understanding mathematics. Thus, a proper thorough account of the cognitive foundations of *all* mathematics has to start with something as basic as learning to count.

125 [Linnebo & Pettigrew 2011], pp. 227

126 It should be clear that this is not the case. The sceptical reader is invited to read an account of ETCS in [Linnebo & Pettigrew 2011] or the original [Lawvere 1964] and find a full-blown set-theoretic term in the axioms.

of the objects of a certain foundational scheme be argued without relying on a justification belonging to some other theory? We leave this matter of coherence to section 4.3. The *conceptual autonomy* requirement demands that a foundation can be fully understood by itself; there should be no need to refer to concepts belonging to another foundation to explicate the concepts at work here. It is this latter form of autonomy that concerns us here. For if we truly need to understand full-blown set theory in order to make sense of categorical set theory, for example, then the latter cannot serve as a cognitive foundation for mathematics.[127]

This leads us straight to the primary weakness of Feferman and Hellman's argument with regards to cognitive matters. The structure of both their arguments is roughly as follows: There is a reason why set theory is cognitively prior to category theory. Thus, category-theoretic theories cannot be cognitive foundations. Thus, they cannot be foundations at all. Now the latter simply does not follow. At the least, what we lack is a substantive argument why a theory that is cognitively prior to another also has to found it ontologically - or has a greater claim to being an organisational or methodological foundation to mathematics. There is simply no such argument made. The implicit assumption is that a "proper foundation" needs not to rely cognitively on another theory. This might conceivably have been acceptable if what proper foundations are were not the very thing contested, but that is the case. Moreover, the idea that an ontological foundation of mathematics also has to be a cognitive foundation for all of it is frankly preposterous. Recall that cognitive founding, as in 3.2.1, is a relation of pedagogical priority: some theory A cognitively founds B if we need to understand A in order to understand B. But we can all certainly understand the natural numbers without being familiar with the axioms of Zermelo-Fraenkel set theory. In fact, lots of mathematical research is done by people who are hardly familiar with ZFC (or ETCS for that matter). None of the currently proposed foundations for mathematics have a decent claim to cognitively founding mathematics. And so they shouldn't. Finding our way back from algebraic geometry or set theory to *counting* is perhaps an epistemologically or cognitively interesting project,[128] but it is unlikely it will help the mathematician organise his many theories and find errors in his work.

Now of course, Feferman, and Hellman following him, do not claim something quite so radical. Rather, what they claim is that

127 Linnebo and Pettigrew make the same mistake as Hellman and Feferman by assuming that any foundation of mathematics must necessarily serve as various kinds of foundation when they argue that any foundation *needs* all these kinds of autonomy at the same time. We proceed with their work, though, as an analysis of these different kinds of autonomy.

128 Compare [Resnik 1982] for an attempt to build up to mathematics from an account of pattern recognition.

there are certain informal notions, the ideas of *collection* and *function*, that we cannot dispense with when talking about categories. This again happens at the supposed level of the background ontology.[129] The idea is that the very notions of category and morphism themselves are not fully determinate, and implicitly quantify over various possible intended interpretations (e.g. morphisms can be group homomorphisms, functors, etc). Now to understand all these structures that are quantified over, we need notions of collection or function. Hellman rejects[130] attempts to give content to the category theory,[131] as those implicitly reject the algebraic nature of the axioms, which are vital to the multiple interpretability of category theory. Hence, on Hellman's account, category theory necessarily invites the very indeterminacy that makes it unsuitable as a cognitive foundation.

This is to place categorical structuralism before a dilemma it need not answer. For if we reject the "implicit quantifier" reading of the axioms of category theory, we then need not worry about a reliance on notions of collection in an ontological domain we do *not* quantify over. Likewise, taking the axioms at face value, there is nothing wrong with giving "common-sense content" to the axioms - it is fine for "straight up" category theory to be about composition, or composing functions.

The possibility remaining open is that category theory somehow requires set-theoretic concepts not in its very definition, but in the process of founding mathematics. Linnebo and Pettigrew use ETCS as a case study in their investigation of categorical foundations. They recognise that the axioms of ETCS are assertory. Clearly, there is no logical dependence on the traditional foundational scheme, since we can define our categories "from scratch" without referring to sets or functions directly: morphisms can function as completely autonomous concepts. The demand of conceptual autonomy is trickier, since we are presented with a theory of sets. The question then becomes whether we can understand sets based on their presentation in this foundational system. The answer is yes: although ZFC has the advantage of being the traditionally accepted axiomatisation of set theory, that does not mean that ETCS cannot have an equal claim to the concept. There is nothing in particular about the way it is presented that suggests that we require some kind of full-blown set-theoretical intuition of the objects in question in order to understand the theory.[132]

129 [Hellman 2003] pp. 134-135

130 [Hellman 2005] pp. 549

131 See for example [Logan 2015] who gives content to the notion of morphism as "combining two things to make a third". Specifically, it is about doing this in a specific way: through composition. Thus, category theory is about "combining two things to make a third" in much the same way set theory is about "collections of things".

132 [Linnebo & Pettigrew 2011] pp. 242-244

4.2.2 *Revisiting McLarty*

McLarty views the entire matter of "foundations" as an ongoing project, starting with the sense of "foundation" used in common parlance among mathematicians - that is, meaning very little compared to the foundations philosophers often have in mind, thick with interpretations and meanings. Rather, the starting point for research on foundations is the methodological foundation the mathematician uses relative to some mathematical theory. It is then a mathematical task to investigate and generalise over all these foundations. The philosophical investigation is a third step, investigating the epistemology of that foundation.[133] McLarty's aim is then combine methodological, organisational and epistemological kinds of foundations. This arguably puts him in a position where he does need to answer Hellman's criticism - which he does.

First, it is important to note that most of the time, use of a category is not done in an algebraic sense, but with a definite intended interpretation.[134] This allows us to partially answer the problem of cognitive foundations: we can, and do, gain access to category theory in a nonalgebraic sense before investigating it abstractly. Thus, we can come to know categories and how they work before needing to make sense of the Eilenberg-MacLane axioms and the implied level of the ontological background. This reflects a development in the history of category theory, where it took quite a few years until the theory became a subject of research without an intended interpretation in algebra.[135]

More straightforwardly, though, McLarty too rejects the idea that we need to make sense of background level that the Eilenberg-Maclane axioms are about:

> All categorical foundations begin with the Eilenberg-MacLane category axioms, but *not* by saying anything is a model of them. Rather, we *affirm* them.[136]

McLarty's theory, then, is extremely close to Awodey's. This is despite the seemingly large gap between McLarty as proponent of categorical foundations and Awodey as antifoundationalist. This lies in the fact that there is barely any difference between the kinds of foundation either proposes. It is hardly possible to oppose organisational and methodological foundations, as these are indispensible to mathematical practice - and hence Awodey doesn't, focusing instead on mathematical-ontological foundations. The main difference lies in

133 [McLarty 2013], pp. 80-81.
134 [McLarty 2005], pp. 50: "forty-one of the latest fifty references to "category" in *Mathematical Reviews* were to specific categories, *i.e.* they had intended interpretations".
135 [Landry & Marquis 2005] pp. 4-6
136 [McLarty 2013] pp. 83

an interest in cognitive foundations from McLarty's side, and perhaps in how broadly either takes the concept of "foundation" to be.

McLarty rightly notices that Awodey skips a few steps when he asserts that all we need to do to establish the coherence of a theory is to investigate its consequences to see if it is consistent. After all, whether it is or not depends on the ambient structure we are working in, or at the very least on the logic we employ.[137] It is this notion of "ambient structure" then, that deserves more attention. Even on an antifoundationalist view such as Awodey's, we cannot get very far mathematically without placing the structure we are interested in in a larger environment. To Awodey, this is just "specifying more of the ambient structure to be taken into account".[138] But McLarty rightly emphasises that therein lies another foundational task: to specify and clarify these ambient structures. Again, this is largely in line with the project of finding organisational foundations for mathematics.

4.3 pragmatic concerns

The matter of establishing the coherence of a theory, then, brings us to the last kinds of foundation at issue. In spite of earlier claims that this is a strength of categorical foundations, there are concerns as to their functioning as organisational foundations - that is, as the kind of framework we employ to find mathematical errors. If we cannot establish the existence, and thereby the coherence of mathematical structures in a single framework, since we lack mathematical-ontological foundations, how else are we to ensure the coherence of our theories?

4.3.1 *The matter of coherence and consistency*

Another function of foundations, and a mathematically prominent one, is as a guarantee that whatever mathematical structure we are describing is not an impossible one. If we set out to research a system X and it will turn out to be incoherent, the mathematician is to recognise the error. Preferably, he should know in advance whether his system is coherent, so as not to waste time proving theorems in an impossible or trivial system. Traditionally, this question is linked with one of existence: as long as we are researching objects which exist, we are researching a coherent system, and hence all is going well. Any framework for mathematics that does not govern existence, then, should at least allow for the mathematician to secure coherence.

Hilbert suscribed to the view that as long as a theory is consistent, the mathematician is in safe waters, exploring a coherent sys-

137 Borrowing an example from McLarty: $x \in x$ is inconsistent in ZFC, but consistent in various non-wellfounded set theories.

138 [McLarty 2005], pp. 53

tem. While a stronger statement than the more generally accepted converse, that something has gone wrong if one is exploring an inconsistent system, this too is commonly accepted. Hilbert went as far as saying that the objects of a consistent system must then exist;[139] for the moment, we can accept that he effectively reduced the philosophical question of existence to a mathematical question of consistency for the working mathematician.

However, the question of consistency itself is metamathematical in nature. But meta-mathematics, too, faces the assertory/algebraic dichotomy. Meta-mathematical statements concern mathematical systems, or express a property of any system of such-and-such sort. Shapiro claims they must be assertory even to the algebraically-inclined structuralist, since the philosopher or mathematician must be asserting something when stating the algebraic position.[140] Moreover, if we employ algebraic terms, quantifiers in meta-mathematical statements would have to range over some sort of structure themselves. If we define this background structure algebraically as well, we would be stuck without assertory metamathematical statements about *this* structure in turn, leaving us with an infinite regress.

Nevertheless, metamathematical questions of the coherence of certain mathematical structure are needed beyond mere intuition. After all, even great mathematicians have worked with "intuitively sound" objects that turned out to be impossible. Thus, we need some guarantee of the coherence of the structures we are researching. The standard solution for this is to find a way to reduce a structure to a more traditional one, allowing it to piggyback on the arguments supporting the existence of the traditional mathematical structure.[141] In contemporary mathematics, this usually involves expressing the structure in set-theoretic terms. Hellman's criticism of the category-theoretic approach can be seen in this last light: the algebraically expressed structures the mathematician pertains to work with need some assertory basis to ensure their coherence, if not just their consistency.

Now we could, of course, employ an assertory, categorical theory such as ETCS and be done with it. However, if we prefer a non-foundational approach in the style of Awodey, we must defend the claim that everything in mathematics is algebraic, up to and including meta-mathematical statements. If metamathematics needs to be assertory, then any questions of coherence will be seen in a non-mathematical context. Shapiro notes that this is difficult to accept, given the existence of some mathematical answers to questions of existence. Relative consistency proofs are of course still obtainable, and may be seen as algebraic in themselves, lacking a real subject matter.

139 [Shapiro 2005], pp. 69
140 [Shapiro 2005], pp. 68
141 [Shapiro 2005], pp. 72

Suppose that the meta-mathematical framework we are using for such a proof is algebraic in nature as well. We are then faced with the question whether this structure itself is coherent, lest our consistency proof be void. If the only way we can prove its consistency is through another meta-mathematical algebraic system, we are left in an infinite regress. The alternative is to move meta-mathematical questions out of the domain of mathematics and into the domain of philosophy, which seems awkward to Shapiro given the wealth of mathematical results in the field of meta-mathematics. Taking into account this infinite regress, Shapiro argues that we do indeed need an assertory statement of meta-mathematics to ensure the coherence of a mathematical system, and emphasises that we should see Hellman's critique of non-assertory theories in this light.

Let us take a step back and look at full-blown set theory once more. The hierarchy of sets can be seen as a solution to two distinct problems. On one hand, it eliminates a particular challenge to the coherence of set theory by disallowing self-referential sets such as the one used in Russell's Paradox. On the other, because it is formulated in an assertory manner, it solves a central semantic problem of mathematics, by providing subject matter to it in the form of a specific collection of sets that exist. The incoherent sets can be claimed to simply not exist by virtue of the way the Zermelo-Fraenkel axioms are set up. It is worth noting here that by making this manoeuvre, the existence of mathematical objects is in a sense bound to their coherence simply because of the axioms we chose. In particular, there is an obvious parallel with Hilbert's belief that a set of axioms, if consistent, describes objects which therefore exist.

Thus, if we abandon the set-theoretic approach and in fact any assertory theory of mathematics, we need to tackle but two problems: the coherence of our mathematical structures, and the semantic sense that mathematical statements should make. If both these goals can be achieved without resorting to an assertory theory, then we have a mathematical foundation on par with set theory. Further philosophical questions, for example on the nature of mathematical objects, should then be tackled by the philosopher from this point of view. It is not the role of the philosopher to dictate which mathematics are valid and which are not. He must try to answer why and how we can understand and communicate mathematical facts. As such, he should attempt to discover what the epistemology and ontology of mathematics must be, without being bound to a single mathematical framework.

Shapiro acknowledges the role of a philosophical defence for an assertory meta-mathematical theory, to explain the existence of the objects asserted by the theory. We have no other guarantee of the coherence of even commonly used large mathematical systems such as ZFC. In this light, his "infinite regress" argument against algebraic meta-mathematics misses the mark. If we allow for such a philosophical defence of assertory theories, we should certainly allow a philosophical defence of an algebraic meta-mathematical theory with much weaker aims - merely to indicate the coherence of our theory, rather than to establish the existence of a set-theoretic universe.[142]

Should we accept that providing an account of coherence for an algebraic theory is a valid philosophical enterprise, much like accounting for the existence of the objects asserted by an assertory theory, then there is no reason why we could not admit of algebraic, mathematical meta-mathematics before grasping for philosophical methods to defend these methods in turn. The very same manoeuvre is practised routinely for assertory theories. Should Shapiro worry that there can be no proper philosophical defence for non-assertory theories, then we refer back to our above observation that the two main questions to be answered are those of mathematical coherence and semantic sense. Perhaps unlike the call for mathematical existence, there is no immediate reason why these cannot be answered in an algebraic framework.

4.3.2 *Revisiting Landry*

Landry's *semantic realism* takes an interesting spot in light of the above discussion and our foundational taxonomy. The label "realism" leads us to ontological associations, but Landry explicitly notes that these are to be avoided - we are talking "existence" only in as far as as is necessary to fix a denotation.[143] What then, is the goal of such a mathematical project? If the goal is to answer Frege's demand for a fixed denotation, then we must ask - what is the purpose served by that in turn? Surely it is not patch a leak in Frege's theory.

The answer is dual in nature. On one hand, by establishing the denotation of our mathematical objects, we gain a measure of epistemological certainty. It allows us to answer the epistemological question of whether we can know what we are referring to when we assert a mathematical position. In the terms of "property-transfer" we have characterised epistemological foundations as, what we are transferring is a notion of semantic accessibility or knowability. On the other, providing a context in which we can give an interpretation of a certain structure is a method to give us confidence in the coherence of said structure. For example, non-Euclidian geometries gained significantly in status when they were first given an Euclidian model.[144]

142 Compare [Landry 2011], pp. 25

143 [Landry 1999a], pp. 81-82

144 Nevertheless, we are not "really talking about Euclidian geometry" when we talk about non-Euclidian geometry. Here, we see again the possibility to vary in our hermeneutic approach to a certain concept: we can see it as interpreted within one structure, within another, or we can take it at face value. One can see the analogy with the role of set theory in mathematics: one can give a set-theoretic interpretation

Thus, the semantic project also has a role to play as an organisational kind of foundation. It allows us to move our mathematical knowledge to a more general level, thereby allowing us to gain confidence in the coherence of our structure, whether that be through increased familiarity with our structure, or through "piggybacking" on an established structure.

In response to Shapiro's criticism, Landry defends the position that we can remain a structuralist "all the way down" - that is, we can interpret mathematics, meta-mathematics, and so forth, all algebraically, without ever "hitting bedrock" through either an assertory metamathematical theory or a turn to philosophy.[145] This is pictured as a generalisation of Hilbert's programme employing category theory. Hilbert worked algebraically, up to a degree - that is to say, ordinary mathematics employs algebraic terms, and becomes known to us by implicit definition. But the meta-mathematical project was not entirely in the algebraic form - rather, relative consistency proofs were sought grounding mathematics in finitary proof theory, which was taken in turn as fully assertory and contentful, and as grounded in concrete sign configurations. What the categorical structuralist can do, then, is largely analogous. On the mathematical level, set-theoretic and category-theoretic objects can be investigated using ETCS or CCAF, respectively, or abstractly using the Eilenberg-MacLane axioms. On the meta-mathematical level, categorical logics, topos theory, and category-structured proof theory all suffice to say what we want to say about the logical and proof-theoretic aspects of our mathematics.[146] Meta-mathematics thus needs no assertory theory for any structural reasons. What remains is the naked claim that a metamathematical statement itself must be assertory in order to state anything about the theory it is about. But this betrays an underestimation of the mathematical tools at our disposal. We can take the statement "as assertory in the theory"[147] - that is to say, as given an interpretation in some further level of (algebraically-formulated) analysis. As before, this fixes an interpretation and thereby a reference, allowing us to state a meaningful position when asserting the metamathematical statement. The threat of infinite regress is handwaved with an *et tu*; statements asserting the truth of a set-theoretic position, too, must "turn to philosophy" at some point, or risk an infinite regress.

Though an impressive display of the power of algebraic mathematics, this conflates our ability to give a mathematical account of semantic reference with our ability to understand a mathematical structure

of some concept in algebraic geometry if one really wants to, but this does not mean we are "really" talking about a set theoretic universe when we do algebraic geometry, any more than we "really" do Euclidian geometry if we do non-Euclidian geometry.

145 [Landry 2011], pp. 24
146 [Landry 2011], pp. 21-23.
147 [Landry 2011], pp. 24

per se. Though the ability to fix the reference of mathematical terms by moving up a level in abstraction is interesting, such a move necessarily introduces the mathematics of the superstructure. For example, we may understand a simple category just fine, but to fix the denotation of that category on Landry's account, we would have to see it as a general category *in* the category of categories in e.g. CCAF - and we may have little in the way of understanding with regards to this theory. And if we take Landry's goals to be merely organisational rather than epistemological in nature as well, the questions are whether this move helps us in organising mathematics and in error-finding - to which the answers are respectively "yes" and "maybe", because of course, we may have less confidence in the coherence of the entire CCAF than in some simple category.

Thus, while Landry's approach is valuable in showing how algebraically formulated mathematics can contribute to the larger metamathematical project, it does not settle anything epistemologically. In particular, what is lacking is an account of our understanding of algebraic structure "at face value". There is nothing in the structuralist view of mathematics that should make us believe that we can only grasp a structure from within some larger structure we can place it in.[148] Quite to the contrary, we can understand many mathematical structures without an inkling of mathematical interpretation. To convince oneself of this, one only needs to compare the number of people who understand the natural numbers with the number of people who know model theory.

4.3.3 *On mathematics and philosophy*

It is true that full-blown set theory has access to a philosophical argument for the truth and existence of its theorems and objects, in a way that categorical theories do not (yet).[149] It is for this reason that Linnebo and Pettigrew argued that the Elementary Theory of the Category of Sets has no *justificationary autonomy* just yet, unless we take mathematical practice as "sufficient justification". I wholeheartedly follow McLarty's answer[150] that, while we indeed should search for philosophical justifications if we can, it is not reasonable that we should decide it is *not* a justified practice without.

Nevertheless, it is true that for now, there is no positive argument explicitly for the coherence of categorical foundations. Given the growing literature on mathematics as the science of structure, and of category theory as the mathematical study of structure, one might hope that it is not far off. But for now, at least a negative result can

148 By contrast, Landry explicitly holds that categories do not exist independently of a system that exemplifies them, but by virtue of being exemplified in some other category. See [Landry 1999a], pp. 136

149 See for example Boolos' seminal [Boolos 1971].

150 [McLarty 2012], pp. 113

be set aside - we can reject Hellman's and Shapiro's argument for the *a priori* failure in philosophical aspects of categorical theories.

What I want to suggest, then, is that the distinction between algebraic and assertory statements is not just a matter of mathematical form, but of philosophical interpretation. We are free to take any statement formulated in an algebraic manner *as* algebraic, that is, as waiting for interpretation to be contentful or as staying without one to be general - but we may likewise take it as contentful at face value. Thus, we can read a sentence such as (8) in the contentful manner given by (12). Likewise, we may take any meta-mathematical statement formulated in an algebraic manner as nevertheless asserting something regarding the mathematics it is about, without inviting an infinite regress. If we can understand the statement at face value, and if we have good reason to trust in its coherence, then there is no reason to shun algebraically formulated metamathematics - including categorical ones.

On the structuralist account, all mathematics is structure. This includes, then, assertory theories such as full-blown set theory. If there is a certain indeterminacy to all mathematics then - as the abstraction account of structure would suggest there is, as objects are merely to be taken as empty points - then all mathematical statements can likewise be taken in an algebraic manner. There is nothing, in principle, precluding us from re-interpreting full-blown set theory by giving further structure to the $\in$-relation.[151] On such a view, the difference between assertory and algebraic form evaporates. We can take full-blown set theory as implicitly quantifying over interpretations of $\in$ just as easily as we can take the axioms of category theory to quantify over different interpretations of the morphisms and objects. Likewise, we can take both in an assertory sense. The ease with which we can take category theory algebraically is then not a philosophical downside, but merely a sign of its wide applicability.

4.4 SCHEMATIC MATHEMATICS

We have everything we need then, to paint a picture of mathematics on a categorical-structuralist view. Let us first recall the main points of the structuralist philosophy. Mathematics concerns structures; that is to say, it concerns wholes of relations between objects, concrete or abstract, regardless of any further structure to these objects. There is then always an amount of indeterminacy to a structure - we are talking of some arrangement of objects merely in terms of the arrangement, thus leaving out the particulars, if any. Finally, we can relate different structures by placing one within another through a process of Dedekind abstraction.

151 Consider, for example, a set theory restricted to trees, or ordered pairs.

Appropriating a term of Awodey's, used only vaguely so far, let me give a description of **schematic mathematics**. On this view, all of mathematics is schematic, or algebraic "in form", so to speak. The difference between algebraic and assertory statements is a hermeneutic one: it depends on how we wish to interpret a particular statement. If we see it as generalising over other objects, it is algebraic; and if we take it as having a definite interpretation, or as contentful at face value, it is assertory. Using Landry's work, we can provide a mathematical semantic interpretation of our statement either way, should we be so inclined.

Schematic mathematics implies a rejection of ontological foundations. This is because any mathematical foundation implies that there is a fixed interpretation for our mathematical statements. Moreover, the interpretation would itself *not* admit of any further interpretation, contrary to its structural nature. On the schematic view, interpretation is a business that can be done independently from any particular mathematical theory - or from all of them, if we so wish. Of course, the rejection of ontological foundations does not mean the rejection of mathematical ontology per se. Rather, it means that if we are to formulate an ontology, the starting point would be philosophical rather than mathematical in nature - for example, by starting with an analysis of structure. As such, the schematic view is consistent with *in re* and *ante rem* interpretations of structuralism.

Likewise, if we wish to formulate an epistemology of mathematics, it should not be through the epistemology of one particular theory of mathematics. There is nothing to guarantee that this theory is always in the picture when we know some mathematical structure. Thus, there is no room for epistemological foundations for all of mathematics either. Of course, should one find an epistemological peculiarity to a certain field, there is nothing preventing foundation relative to that field. The more promising avenue for an epistemology seems to be within a general theory of knowing structures or patterns. We are free, however, to provide a philosophical interpretation of some structure, including "algebraic" ones. Cognitive foundations, on the other hand, are part and parcel of mathematics, and always will be - it is a simple fact of human cognition that we need to understand certain fields of mathematics before we can understand certain others.

Turning to organisational matters next, we can recognise the wide applicability of category theory in providing links between different parts of mathematics. I have spoken little of category theory in my description of schematic maths so far. Category theory takes a special place with regards to a structural approach because it allows us to describe things purely in terms of relations - something that falls neatly in line with the structuralist account of the very nature of mathematics. This does not, however, imply that mathematics is about categories in any metaphysical sense. It merely means that category theory is a good way to describe mathematical structures. There might very well be better ones yet!

The boon of category theory to the philosopher is that it allows us to avoid implicitly "unstructural" notions. It is tempting to see structures as "the new objects" and reason about them as if they were,[152] but they are not. Mathematical structures are schematic in the sense that they lack content unless we explicitly choose to give an interpretation. It is this very feature that allows us to make sense of them semantically, by giving a mathematical interpretation, without making metaphysical sense of them first.

What schematic mathematics is not, is a rejection of the role of sets in mathematics. The concept of a set is a powerful one without doubt, and one that will continue to carry weight in mathematics. Rather, sets, as put forward by full-blown set theory, are not indispensable in a foundational sense. Of course, sets as a structure are still important, and will continue do be found all over mathematics, though not necessarily always as defined by the axioms of a full-blown set theory. Set theory itself, as a field of mathematics, likewise is still of mathematical interest - especially since sometimes it is exactly the behaviour of elements taken as atoms that we are interested in. But it should not form the ontological background for all of mathematics, if only because no theory should. The upside is that mathematics doesn't need it to. As long as we can generalise when we need to and give an intended interpretation when we need to, we can explore mathematical structures as much as we like. We can guarantee the coherence of our theories not by defending the coherence of a single mathematical structure or of a single idea,[153] but by giving an interpretation in any ambient structure we find trustworthy.

Let me conclude with a note on philosophy. If we want to further philosophical arguments in debates regarding the foundations of mathematics, different foundational goals can and should be separated to make sense of the different ways one can employ mathematical structures. Only by careful communication in this regard can we prevent misunderstanding, sharpen the battle lines, and investigate the viability of new proposed foundations from a neutral perspective.

152 Shapiro's structure theory harks dangerous close to this when attempts to give a quasi-mathematical account of the existence of structures.

153 e.g. the iterative concept of set

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