Sheaf Models for Intuitionistic Non-Standard Arithmetic

MSc Thesis *(Afstudeerscriptie)*

written by

Maaike Annebeth Zwart
(born 6th May, 1989 in Nijmegen)

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Date of the public defense:	**Members of the Thesis Committee:**
4th September, 2015	Dr. Alexandru Baltag
	Dr. Benno van den Berg
	Dr. Nick Bezhanishvili
	Prof. Dr. Ieke Moerdijk
	Dr. Jaap van Oosten
	Dr. Benjamin Rin
	Prof. Dr. Ronald de Wolf

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Abstract

The aim of this thesis is twofold. Firstly, to find and analyse models for non-standard natural arithmetic in a category of sheaves on a site. Secondly, to give an introduction in this area of research.

In the introduction we take the reader from the basics of category theory to sheaves and sheaf semantics. We purely focus on the category theory needed for sheaf models of non-standard arithmetic. To keep the introduction as brief as possible while still serving its purpose, we give numerous examples but refer to the standard literature for proofs.

In the remainder of the thesis, we present two sheaf models for intuitionistic non-standard arithmetic. Our sheaf models are inspired by the model I. Moerdijk describes in A model for intuitionistic non-standard arithmetic [Moerdijk95].

The first model we construct is a sheaf in the category of sheaves over a very elementary site. The category of this site is a poset of the infinite subsets of the natural numbers. Apart from the Peano axioms, our sheaf models the non-standard principles overspill, underspill, transfer, idealisation and realisation. Many of our results depend on a classical meta-theory. Moerdijk's proofs are fully constructive, which is why we improve our site for our second model.

For the second model, we use a site with more structure. In the category of sheaves on this second site, we find a non-standard model that much resembles our first model. We get the same results for this model and are able to prove some of the results that previously needed classical meta-theory, constructively. However, there remain principles of which we can only show validity in our model using classical logic in the meta-theory.

Lastly, we try to construct a non-standard model using a categorical version of the ultrafilter construction on the natural numbers object of the category of sheaves on our first site. This yields a sheaf which has both the natural numbers object and our first model as subsheaves.

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Chapter 1

Introduction

1.1 A short history of the (non-standard) natural numbers

Natural numbers are, and always have been, a central concept of mathematics. The abstraction from 3 bananas, 3 humans and 3 moon cycles to the number 3 is possibly what started mathematics: finding patterns, abstracting away from unimportant details and thereby inventing new, more abstract, concepts. But at some point, people started to examine what is left after this abstraction: *What is the number 3?* This question dates back to at least the ancient Greeks: Pythagoras treated numbers (especially 1,2,3 and 4) religiously, as being the source of all wisdom[Standford-P]. Aristotle was not satisfied with such a divine explanation, and wanted a better understanding of numbers; are they something physical or purely made up by the human mind? As the Stanford Encyclopedia of Philosophy[Stanford-A]) puts it:

> "The **unity problem of numbers**: This problem bedevils philosophy of mathematics from Plato to Husserl. What makes a collection of units a unity which we identify as a number? It cannot be physical juxtaposition of units. Is it merely mental stipulation?"

In the 19^{th} century, prominent mathematicians were again engaging in philosophical discussions about the foundations of mathematics. And again, they were trying to find an answer to the question *What are numbers?* Kronecker famously proclaimed:

> "God made the integers, all else is the work of man."[1]

With him, many mathematicians agreed that the numbers were just there, and they could be used to build the rest of mathematics on. For some, however, this

[1] Although much quoted, the source of these words is not totally clear. Jeremy Gray attributes the quote to "Weber 1891/92, 19, quoting from a lecture of Kronecker's of 1886"[Gray08, page 153]

was not good enough. In the second half of the 19^{th} century, several attempts were made to define clearly what natural numbers are, among which were attempts from Frege, Dedekind and Peano. Frege focussed on cardinality: he defined a number as the class of all sets that are equinumerous to each other. That is, a number is the set of all sets that can be put in a one-to-one correspondence with each other[Frege1883, §68 and §73]. This definition stays very close to the way the number 3 was abstracted from 3 bananas, 3 humans and 3 moon cycles. Dedekind chose a more abstract route, basing his definition on ordinality: The natural numbers are that what is left after taking any infinite set which can be ordered by a starting element and a successor function and forgetting about all other properties of the individual elements of that set[Dedekind61, §6, 73]. Peano chose the same approach as Dedekind did, but formulated the idea into a set of axioms in a very comprehensive and precise logical language (see fig 1.1 below). Although Peano's axioms are equivalent to both Dedekind's and Frege's formulations (ignoring a slight foundational problem with Frege's original approach), the simplicity of the axioms made them the most popular definition of natural numbers.

§ 1. De numeris et de additione.

Explicationes.

Signo N significatur *numerus (integer positivus).*
» 1 » *unitas.*
» $a+1$ » *sequens a*, sive *a plus* 1.
» $=$ » *est aequalis.* Hoc ut novum signum considerandum est, etsi logicae signi figuram habeat.

Axiomata.

1. $1 \in N.$
2. $a \in N . \supset . a = a.$
3. $a, b, c \in N . \supset : a = b . = . b = a.$
4. $a, b \in N . \supset \therefore a = b . b = c : \supset . a = c.$
5. $a = b . b \in N : \supset . a \in N.$
6. $a \in N . \supset . a + 1 \in N.$
7. $a, b \in N . \supset : a = b . = . a + 1 = b + 1.$
8. $a \in N . \supset . a + 1 -= 1.$
9. $k \in K \therefore 1 \in k \therefore x \in N . x \in k : \supset_x . x + 1 \in k :: \supset . N \supset k.$

Figure 1.1: Fragment of Arithmetices Principia Novo Methodo Exposita[Peano1889], where Peano introduces the now well known Peano axioms. These are axioms number 1 (1 is number), 6 (the successor of a number is also a number), 7 (two numbers are equal if and only if their successors are equal), 8 (1 is not the successor of a number) and 9 (axiom of induction).

It seemed that due to the efforts of these mathematicians, there was finally an answer to the the question *What are numbers?*: Numbers are those things that fit the description given by the Peano axioms. However, in 1934, Skolem showed that there are more mathematical structures in which the Peano axioms are valid than just the $1, 2, 3, \ldots$ everyone has in mind[Skolem34]. Such structures are referred to as non-standard models of Peano's axioms, because they are not the model 'meant' by the definition (the standard model $\mathbb{N}$).

Skolem's paper shows that the Peano axioms, written in first order logic, are 'incomplete': they fail to uniquely define what we call natural numbers. Still, they cannot be improved; every set of first order sentences trying to define the natural numbers allows for non-standard models. Thus ends the quest to uniquely define the natural numbers.

In the next section, we will see a short proof of the fact that every set of first order sentences trying to define the natural numbers allows for non-standard models. Also, we shortly discuss some properties of non-standard models.

1.2 Non-standard models for the Peano axioms

It is not very hard to see that first order logic will always permit non-standard models of the natural numbers: Suppose that P is a set of logical sentences in the language of Peano arithmetic trying to define the natural numbers. We add a constant symbol $\mathfrak{c}$ to the language, and the following sentences: (here is s the successor function):

$$\begin{aligned}
&\mathfrak{c} > 0 \\
&\mathfrak{c} > s(0) \\
&\mathfrak{c} > s(s(0)) \\
&\vdots
\end{aligned}$$

Let P' be the set of sentences in P, together with the infinitely many sentences described above. Then every finite subset of P' is modeled by the (standard) natural numbers: interpret $\mathfrak{c}$ as some natural number which is large enough. Therefore P' also has a model (*compactness theorem*). The natural numbers are not a good model for P', as all the sentences above together require the existence of an element that is larger than every natural number: the model for P' must be a non-standard model.

Every non-standard model has an isomorphic copy of the standard natural numbers as a submodel. This is a direct consequence of the fact that the Peano axioms hold in the non-standard model. The elements of the non-standard model that are part of this isomorphic copy are called standard elements of the non-standard model. All other elements of the model are called non-standard elements.

First order logic cannot distinguish between a non-standard model and a standard model. Therefore, if a first order formula is true for all standard elements of the model, then it must also be true for some non-standard element. Conversely, if a formula is true for all non-standard elements, then there must also be a standard element for which it is true. These principles are called *overspill* and *underspill*.

1.3 Intuitionism and Heyting arithmetic

As mentioned before, from the mid 19^{th} century onwards mathematicians were vividly discussing the foundations of mathematics. Many attempts were made to rigorously define various mathematical concepts. There were some who strongly opposed to the emerging logical rules. One of these opposers was Brouwer. He saw mathematics as *constructions* purely taking place in one's mind. Brouwer's ideas were quite extreme, as he distrusted any language to formulate mathematics in: words or logical symbols could never give a fully accurate description of the mental image he created in his mind. One of the students of Brouwer, Heyting, did not fear logical language. He developed a formal system of intuitionistic logic to capture Brouwer's ideas. He gives the logical connectives and quantifiers a new (stricter) interpretation, based on the idea that:

> ... a mathematical proposition p always demands a mathematical construction with certain given properties; it can be asserted as soon as such a construction has been carried out. We say in this case that the construction *proves* the proposition p and call it a proof of p. [Heyting56, section 7.1.1.]

For the full description, we refer to Heyting's book Intuitionism, an Introduction [Heyting56].

The most famous intuitionistic principles are the rejection of the law of the excluded middle and the elimination of double negation. These laws cannot be deduced in intuitionistic logic because of the stricter interpretations of the connectives and quantifiers.

When the Peano axioms are interpreted in intuitionistic logic, the resulting theory is Heyting arithmetic. The non-standard model described by Moerdijk in [Moerdijk95] is a model in a category of sheaves. The internal logic of a sheaf is intuitionistic and therefore Moerdijk's model is a model for Heyting arithmetic. In this thesis, we also use sheaves as models, so whenever we say 'natural arithmetic' we mean Heyting arithmetic.

1.4 An overview of this thesis

In this thesis, we present two sheaves that are non-standard models for natural arithmetic and describe one sheaf which might be. In the preliminaries, we explain all the basics of category theory and sheaf semantics that are needed to understand the construction of these models and the proofs in this thesis. In the last section of the preliminaries, we give a summary of Moerdijk's model (which he describes fully in [Moerdijk95]), which has been the inspiration for this work.

In chapter 3, we present our first model. The category of sheaves in which we construct this model is based on a very basic site, consisting of a poset with a simple Grothendieck topology. The price we have to pay is that the meta-theory we use is classical instead of intuitionistic.

In chapter 4, we use a site that is richer than the poset from chapter 3, but still not as extensive as the site Moerdijk uses in [Moerdijk95]. We find our second model in the category of sheaves on this site and we are able to get some of the results without using non-constructive arguments. We still need classical logic to recover all of the results presented in chapter 3 for this second model.

For our last sheaf, we tried a categorical version of the ultrafilter construction for non-standard models of natural arithmetic. We used the same category of sheaves for this construction as we used for our first model. The resulting sheaf has some nice relations with the natural numbers object and our first model. We describe the construction of our third sheaf and the mentioned relations in Appendix A.

Chapter 2

Preliminaries

The following sections explain all the basics of category theory and sheaf semantics that are needed to understand the content of this thesis. Starting from the definition of a category, we discuss the notion of a limit in a category and give examples of limits that we encounter later on in this thesis (product, terminal object and pullback). Then, we cover some constructions that produce new categories from old ones: the opposite category, the category of all categories and functor categories, finally arriving at the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. The Yoneda embedding, together with the Yoneda lemma, show why this particular category is so popular in category theory. Via the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ we then slowly but steadily built towards the definition of sheaves, encountering Grothendieck topologies, sites, sieves and matching families on our way. Once we arrived at sheaves, we show that by using sheaf semantics, a sheaf can be used as a model for the Peano axioms.

To conclude the preliminaries, we give a step-by-step guide on how to build a (non-standard) sheaf model for the Peano axioms and we summarise the approach of Moerdijk in [Moerdijk95].

2.1 Category theory: a brief introduction

Category theory views mathematics from a new perspective. It pins down the basic structure of a mathematical object, ignoring unnecessary details. In doing so, it reveals unexpected connections between different mathematical fields and hence deepens our understanding of these fields.

In the following few pages, we briefly review the basic notions of category theory. We only give the definitions and some examples, leaving the proofs behind. For a more thorough treatment of basic category theory, we refer to the book Category Theory by S. Awodey [Awodey06].

2.1.1 Categories

Definition 2.1.1. *A Category*
A category C is a mathematical structure consisting of objects and morphisms (sometimes called arrows) between objects, with the following three properties:

- For each object $c \in C$, there exists a morphism $\mathrm{Id}_c : c \to c$ in C called the *identity morphism*.
- If $f : c \to d$ and $g : d \to e$ are morphisms in C, then there exists a morphism $g \circ f : c \to e$ in C, which is the *composite* of f and g.
- Composition is associative, that is: $(h \circ g) \circ f$ is the same morphism as $h \circ (g \circ f)$.

The collection of all morphisms between two objects c and d in C is denoted by $\mathrm{Hom}_C(c, d)$ (short for *homomorphisms*). The subscript C is left out whenever this does not lead to confusion. In most everyday categories, $\mathrm{Hom}(c, d)$ is a set, but there are cases for which it is a proper class. Categories for which $\mathrm{Hom}(c, d)$ is a set are called *locally small*. In this thesis, we will only consider locally small categories.

Example 2.1.1. We give some examples of categories. The last example, Set, is the most important; this category will be used extensively in this thesis. Exercise: convince yourself that the examples are indeed categories.

The category consisting of a single object, $*$, and for each natural number n a morphism $n : * \to *$. Composition is given by addition: $3 \circ 2 = 5$. In this category, $\mathrm{Hom}(*, *) = \mathbb{N}$.

$$\begin{tikzcd} * \arrow[loop above, "n"] \end{tikzcd}$$

The category of whole numbers and the $<$ relation. There is a morphism from n to m if and only if $n < m$. Here, the Homset is either empty or it contains a unique element.

$$-2 \xrightarrow{<} 7$$

An example from logic (taken from Awodey [Awodey06]): given a deductive system of logic, you can form the category of proofs: the objects are the formulas of the language, and a morphism $\phi \to \psi$ is a deduction that takes ϕ as premiss and has ψ as conclusion. $\mathrm{Hom}\left(\phi, \psi\right)$ contains all deductions of ϕ from ψ.

$$\begin{tikzcd} \phi \arrow[d, dotted] \\ \psi \end{tikzcd}$$

Set, the category of sets, has sets as objects and functions between sets as morphisms: $\mathrm{Hom}_{\mathrm{Set}}(c, d)$ contains all the functions $f : c \to d$.

$$\{1, 2\} \xrightarrow{f} \{37\}$$

From a categorical point of view, objects in a category might behave exactly the same, even though we think they might be distinct objects. For instance:

every singleton set in Set. For the practice of category theory, it makes no difference whether we consider {1} or {57}, they are considered to be *isomorphic*.

Definition 2.1.2. *Isomorphic objects*
Two objects c and d in a category C are isomorphic iff there exists two morphisms $f : d \to c$ and $g : c \to d$ such that their composition always yields the identity:

$$f \circ g = \mathrm{Id}_c$$
$$g \circ f = \mathrm{Id}_d$$

2.1.2 Limits in a category

In set theory, the cartesian product is a well known construction. From a categorical point of view, a product is a special instance of a construction known as the *limit of a diagram*. Limits are useful in many proofs and we will see them frequently in this thesis.

Given some objects from a certain category C and some morphisms between them, we visualize them as anonymous points or object names and arrows, such as the illustration below. Such a (part of a) category is called a *diagram*.

$$\begin{tikzcd}
& \cdot \arrow[loop above] & \cdot \arrow[loop above] & \\
\cdot \arrow[rrr, bend right] & & & \cdot
\end{tikzcd}$$

If two drawn arrows in a diagram represent the same morphism, then we say that the diagram *commutes*. In the diagram below, if $f \circ g = h$, then the triangle commutes.

$$\begin{tikzcd}
\cdot \arrow[r, "g"] \arrow[dr, "h"'] & \cdot \arrow[d, "f"] \\
& \cdot
\end{tikzcd}$$

Definition 2.1.3. *Limits*
Given any diagram, consisting of objects d_i (i in some index set I) and possibly some morphisms between them, for example:

$$d_1 \xrightarrow{f} d_2,$$

then limit of this diagram is an object c together with a set of morphisms $g_i : c \to d_i$ from c to each of the d_i, such that any triangle commutes (that is, in the example below: $f \circ g_1 = g_2$):

$$\begin{tikzcd}
 & c \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\
d_1 \arrow[rr, "f"] & & d_2
\end{tikzcd}$$

In addition, the limit should have the property that for any other object c' and set of morphisms $h_i : c' \to d_i$ (such that any formed triangle commutes), there exists a unique arrow $u : c' \to c$ such that $g_i \circ u = h_i$ (the limit of a diagram is therefore in a sense really the 'limit' of all possible c' and $h_i : c' \to d_i$). This is called the *universal mapping property*, or UMP:

$$\begin{tikzcd}
 & c' \arrow[d, dotted, "u"] \arrow[ddl, bend right, "h1"'] \arrow[ddr, bend left, "h_2"] & \\
 & c \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\
d_1 \arrow[rr, "f"] & & d_2
\end{tikzcd}$$

Because of the UMP, all limits are unique up to isomorphism. We give some examples of limits, all of which play important roles in this thesis: products, terminal objects and pullbacks. We also give an example of a category that has no terminal object, illustrating that not all limits need to exist in a category.

Example 2.1.2. As we promised, the cartesian product in Set is a limit. It is the limit of the following diagram:

$$c \qquad\qquad\qquad d$$

The limit of this diagram is indeed the cartesian product of c and d:

$$c \xleftarrow{\pi_1} c \times d \xrightarrow{\pi_2} d$$

Where π_1 and π_2 are the projection functions. We check the UMP: if $g_1 : e \to c$ and $g_2 : e \to d$ would be any other pair of functions into c and d, then the function $\langle g_1, g_2 \rangle$, mapping an element $n \in e$ to the pair $(g_1(n), g_2(n))$ is the unique function needed for the UMP.

Definition 2.1.4. *Products*
The limit of a diagram consisting of two objects and no morphisms is called the product of those two objects.

Another diagram of which we can take the limit is the empty diagram. The limit of this diagram is an object (usually denoted as 1), together with a set of morphisms, one for each object in the diagram (so none). The UMP states that for any other object c, there exists a unique morphism $u : c \to 1$. The object 1 is called the *terminal* or *final* object of the category.

Definition 2.1.5. *Terminal object*
An object 1 is terminal in its category if for any object c in C, there exists a unique morphism $u : c \to 1$.

Example 2.1.3. We give two examples: one example of a terminal object, and one example of a category that has no terminal object.

- In Set, every singleton set is terminal.
- The natural numbers seen as a poset with the usual order, does not have a terminal object.

The third and last important limit we will discuss is the *pullback*.

Definition 2.1.6. *Pullbacks*
A pullback in a category C is the limit of the following diagram:

$$\begin{tikzcd} & d_2 \arrow[d, "g"] \\ d_1 \arrow[r, "f"'] & c \end{tikzcd}$$

Spelling out the definition of a limit, it is an object e, together with two morphisms $h_1 : e \to d_1$ and $h_2 : e \to d_2$, such that the following diagram commutes[1]:

$$\begin{tikzcd} e \arrow[r, "h_2"] \arrow[d, "h_1"'] & d_2 \arrow[d, "g"] \\ d_1 \arrow[r, "f"'] & c \end{tikzcd}$$

So when taking the pullback, you 'pull' the two morphisms 'back' to give them the same domain. The UMP is formulated as: for any other pair $h_1' : e' \to d_1$ and $h_2' : e' \to d_2$, there is a unique morphism $u : e' \to e$, such that we have the commuting diagram below:

$$\begin{tikzcd} e' \arrow[dr, dotted, "u"] \arrow[drr, bend left, "h_2'"] \arrow[ddr, bend right, "h_1'"'] & & \\ & e \arrow[r, "h_2"] \arrow[d, "h_1"'] & d_2 \arrow[d, "g"] \\ & d_1 \arrow[r, "f"'] & c \end{tikzcd}$$

To emphasise the origin of the pullback, the object e above is often denoted as $d_1 \times_c d_2$. We again look at an example in Set:

[1] the observant reader sees that we neglect a third morphism that should be part of the limit: $h_3 : e \to c$. However, due to the commuting relations, this morphism is both equal to both $f \circ h_1$ and $g \circ h_2$, therefore it is enough to require that $f \circ h_1 = g \circ h_2$, which yields a prettier diagram

Example 2.1.4. Given the following diagram in Set:

$$\begin{tikzcd} & d_2 \arrow[d, "g"] \\ d_1 \arrow[r, "f"'] & c \end{tikzcd}$$

The pullback of f and g is given by:

$$\begin{tikzcd} \{(x, y) \in d_1 \times d_2 \mid f(x) = g(y)\} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & d_2 \arrow[d, "g"] \\ d_1 \arrow[r, "f"'] & c \end{tikzcd}$$

Where π_1 and π_2 are the two projection arrows. Exercise: convince yourself that π_1 and π_2 have the UMP (compare to the example of the cartesian product in Set).

Products, the terminal object and pullbacks are the limits that we will encounter in the coming chapters. We will now move one level higher, and discuss constructions on categories instead of within categories (although it will turn out that these constructions, too, are actually constructions within a very special category).

2.1.3 Categories from categories

There are various ways to construct new categories from old ones. Many examples (such as product categories, slice categories, etc) can be found in Awodey [Awodey06]). We will discuss the constructions that are needed to understand the category of *presheaves*. Then in the next section, we will see how to upgrade presheaves to *sheaves*. The category of sheaves is the category we will be working with in this thesis.

We start with the *opposite* category. The construction is very straightforward: given a category C, we reverse all its morphisms. The result is again a category.

Definition 2.1.7. *The opposite of a category*
Given a category C, the opposite category C^{op} has the same objects as C, but all the morphisms are reversed: a morphism $f : d \to c$ in C becomes a morphism $f^* : c \to d$ in C^{op}.

$$C : \quad \begin{tikzcd} d \arrow[r, "f"] & c \\ e \arrow[u, "g"] \arrow[ur, "h"'] & \end{tikzcd} \qquad\qquad C^{\mathrm{op}} : \quad \begin{tikzcd} d \arrow[d, "g^*"'] & c \arrow[l, "f^*"'] \arrow[dl, "h^*"] \\ e & \end{tikzcd}$$

It may not always be obvious what kind of morphism such a reversed morphism should be. We give an example to get some feeling for the mechanism.

Example 2.1.5. We consider Set. Recall that the morphisms are functions between sets. $\mathrm{Set}^{\mathrm{op}}$ then consists of sets and certain relations. When d and c are sets, and $f : d \to c$ is a function in the category Set, we can view f as a set of pairs: $f = \{(x, y) \in d \times c \mid f(x) = y\}$. Then in the opposite category, f^* is a morphism from c to d, given by the following relation: $\{(y, x) \in c \times d \mid f(x) = y\}$.

Set :

$$d \xrightarrow{f = \{(x,y) \mid f(x) = y\}} c$$

$\mathrm{Set}^{\mathrm{op}}$:

$$d \xleftarrow{f^* = \{(y,x) \mid f(x) = y\}} c$$

As the opposite category reverses all the morphisms, a limit in the category $\mathcal{C}$ becomes a *co-limit* in the category $\mathcal{C}^{\mathrm{op}}$. A terminal object 1 for instance (there is a unique morphism from each c in $\mathcal{C}$ to 1), becomes an *initial object* 0 in $\mathcal{C}^{\mathrm{op}}$: there is a unique morphism from 0 to each c in $\mathcal{C}^{\mathrm{op}}$. Products become *co-products* and pullbacks become *pushouts*. For more information and illustrations about co-limits, see for instance Awodey [Awodey06, chapter 3: Duality].

The strength of category theory is in the way it is able to link different fields of mathematics together. In order to do so, we need to be able to compare different categories to each other. This is done in the *category of all categories*. The objects of this category are of course categories. The morphisms are special functions, that map one category into another. These functions are called *functors*, and they are defined below. Of course we should be very careful when speaking about such large categories. However, for the purposes of this thesis, we do not worry about possible paradoxes arising from the existence of this category.

Definition 2.1.8. *Functors*
Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A functor $F : \mathcal{C} \to \mathcal{D}$ between these categories is a function that:

- maps objects c in $\mathcal{C}$ to objects $F(c)$ in $\mathcal{D}$
- maps a morphism $f : c \to d$ in $\mathcal{C}$ to a morphism $F(f) : F(c) \to F(d)$ in $\mathcal{D}$.
- maps identity morphisms to identity morphisms
- preserves compositions: $F(g \circ f) = F(g) \circ F(f)$

$$
\begin{tikzcd}
\mathcal{C} \arrow[rrr, "F"] & & & \mathcal{D}
\end{tikzcd}
$$

$$
\begin{tikzcd}
c \arrow[loop above, "\mathrm{Id}"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & d \arrow[d, "g"] & & F(c) \arrow[loop above, "F(\mathrm{Id})"] \arrow[r, "F(f)"] \arrow[dr, "F(g \circ f)"'] & F(d) \arrow[d, "F(g)"] \\
& e & & & F(e)
\end{tikzcd}
$$

Many functors in this thesis are functors with codomain Set. We give one example:

Example 2.1.6. *The powerset functor* $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$
Let $\mathcal{P}$ be the functor that sends a set c to its powerset $\mathcal{P}(c)$. It sends a function $f : c \to d$ to a function on powersets, sending a subset $c' \subseteq c$ to the set $d' = \{f(x) \mid x \in c'\} \subseteq d$:

$$
\begin{aligned}
\mathcal{P}(c) &= \mathcal{P}(c), \text{ the powerset of } c \\
\mathcal{P}(f) &: \mathcal{P}(c) \to \mathcal{P}(d) \\
\mathcal{P}(f)(c') &= \{f(x) \mid x \in c'\}
\end{aligned}
$$

This functor sends sets to sets and functions to functions, the identity is sent to identity and composition is preserved (convince yourself that this is true).

Instead of morphisms in the category of categories, functors can also serve as objects in yet another category. Fixing two categories $\mathcal{C}$ and $\mathcal{D}$, we can form the category *functors between* $\mathcal{C}$ *and* $\mathcal{D}$, denoted by $\mathcal{D}^{\mathcal{C}}$. In this category, the morphisms are *natural transformations*:

Definition 2.1.9. *Natural transformations*
A natural transformation $\eta : F \to G$ between two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\eta_c : F(c) \to G(c)$, one for each c in $\mathcal{C}$, such that the following diagram commutes:

$$
\begin{tikzcd}
F(c) \arrow[r, "\eta_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\
F(d) \arrow[r, "\eta_d"'] & G(d)
\end{tikzcd}
$$

Example 2.1.7. Continuing example 2.1.6, we can define a natural transformation $setify$ from the identity functor $\mathrm{Id}_{\mathrm{Set}}$ to the powerset functor. The components of this natural transformation are functions mapping an element $x \in c$ to the singleton $\{x\} \in \mathcal{P}(c)$:

$$
\begin{tikzcd}
\mathrm{Id}_{\mathrm{Set}}(c) \arrow[r, "setify_c"] \arrow[d, "\mathrm{Id}_{\mathrm{Set}}(f)"'] & \mathcal{P}(c) \arrow[d, "\mathcal{P}(f)"] \\
\mathrm{Id}_{\mathrm{Set}}(d) \arrow[r, "setify_d"'] & \mathcal{P}(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto, "setify_c"] \arrow[d, mapsto, "\mathrm{Id}_{\mathrm{Set}}(f)"'] & \{x\} \arrow[d, mapsto, "\mathcal{P}(f)"] \\
f(x) \arrow[r, mapsto, "setify_d"'] & \{f(x)\}
\end{tikzcd}
$$

So we now have three 'layers' of categories:

1. Ordinary categories, such as Set, which has sets as objects and functions as morphisms, or the opposite $\mathrm{Set}^{\mathrm{op}}$, with reversed morphisms.
2. The category of categories, with categories as objects and functors as morphisms.
3. The category of functors between two categories, which has functors between categories as objects and natural transformations as morphisms.

Putting all the information together, we can form, for each category C, this special functor category:

$$\mathrm{Set}^{C^{\mathrm{op}}}$$

This is the category of *presheaves* on C, sometimes also denoted as $\mathrm{PSh}\,(C)$. Apart from being the basis of sheaves, $\mathrm{Set}^{C^{\mathrm{op}}}$ is in itself a well-known category in all branches of category theory. We pause here for a moment to explain its significance. Also, we hope to shed some light on why the opposite of C is in the exponent instead of just C itself.

2.1.4 Yoneda embedding and Yoneda lemma

$\mathrm{Set}^{C^{\mathrm{op}}}$ is the category that consists of all functors between C^{op} and Set. Among these functors, there are functors called $\mathrm{Hom}_C\,(-,c)$. Recall that $\mathrm{Hom}_C\,(d,c)$ is the set of all morphisms in C from d to c. Leaving one spot blank turns $\mathrm{Hom}_C\,(d,c)$ into a functor: either $\mathrm{Hom}_C\,(-,c)$ or $\mathrm{Hom}_C\,(c,-)$, mapping an object d to the set of morphisms from d to c or from c to d respectively. To understand why the domain of the functor $\mathrm{Hom}_C\,(-,c)$ is C^{op} rather than C, we first consider $\mathrm{Hom}_C\,(c,-)$ in some more detail:

$\mathrm{Hom}_C\,(c,-)$ maps objects $d \in C$ to the set of morphisms from c to d in C. A morphism $h : e \to d$ is mapped to the function 'composition with h' which maps a function $g \in \mathrm{Hom}_C\,(c,e)$ to the function $h \circ g \in \mathrm{Hom}_C\,(c,d)$

$$
\begin{aligned}
&\mathrm{Hom}_C\,(c,-) : C \to \mathrm{Set}\\
&\mathrm{Hom}_C\,(c,d) = \{f : c \to d\}\\
&\mathrm{Hom}_C\,(c,h) : \mathrm{Hom}_C\,(c,e) \to \mathrm{Hom}_C\,(c,d)\\
&\mathrm{Hom}_C\,(c,h)\,(g : c \to e) = h \circ g : c \to d
\end{aligned}
\qquad
\begin{tikzcd}
c \arrow[d, "g"] \arrow[dr, "\mathrm{Hom}\,(c,h)\,(g) = h \circ g"] & \\
e \arrow[r, "h"'] & d
\end{tikzcd}
$$

This functor is a functor from C to Set, just like you would expect. Now we look at $\mathrm{Hom}_C\,(-,c)$.

$\mathrm{Hom}_C(-,c)$ maps objects d to the set of morphisms from d to c in C. It maps morphisms $h : e \to d$ to the function *'pre*-composition with h', which maps a function $g \in \mathrm{Hom}_C(c,d)$ to the function $g \circ h \in \mathrm{Hom}_C(c,e)$. Notice that this functor reverses the direction of morphisms: a morphism h from e to d in C becomes a morphism from $\mathrm{Hom}(d,c)$ to $\mathrm{Hom}(e,c)$. That is not in agreement with definition 2.1.8. The second bullet clearly states that the direction of morphisms should be preserved. Therefore, the proper domain of $\mathrm{Hom}_C(-,c)$ is C^{op}, which has the morphisms already reversed. Then $\mathrm{Hom}_C(-,c)$ maps h, which is a morphism from d to e in C^{op}, to a morphism from $\mathrm{Hom}(d,c)$ to $\mathrm{Hom}(e,c)$, preserving its direction. Keep in mind that the functor itself still maps objects to a set of morphisms in C, not in C^{op}, so that the diagram below is still entirely in C.

$$
\begin{aligned}
&\mathrm{Hom}_C(-,c) : C^{\mathrm{op}} \to \mathrm{Set}\\
&\mathrm{Hom}_C(d,c) = \{f : d \to c\}\\
&\mathrm{Hom}_C(h,c) : \mathrm{Hom}_C(c,d) \to \mathrm{Hom}_C(c,e)\\
&\mathrm{Hom}_C(h,c)(g : d \to c) = g \circ h : e \to c
\end{aligned}
$$

$$
\begin{tikzcd}
c & \\
e \arrow[u, "\mathrm{Hom}(c,h)(g) = g \circ h"] \arrow[r, "h"'] & d \arrow[ul, "g"']
\end{tikzcd}
$$

Taking things one step further, we can leave both spots blank: $\mathrm{Hom}_C(-,-)$. This can be either be defined as a functor from C to $\mathrm{Set}^{C^{\mathrm{op}}}$, mapping c to $\mathrm{Hom}_C(-,c)$, or as a functor from C^{op} to Set^{C}, mapping c to $\mathrm{Hom}_C(c,-)$:

$$
\begin{aligned}
\mathrm{Hom}_C(-,-) &: C \to \mathrm{Set}^{C^{\mathrm{op}}}\\
&: c \mapsto \mathrm{Hom}_C(-,c)\\
\mathrm{Hom}_C(-,-) &: C^{\mathrm{op}} \to \mathrm{Set}^{C}\\
&: c \mapsto \mathrm{Hom}_C(c,-)
\end{aligned}
$$

A similar argument as presented above shows that the domain of the second functor has to be C^{op} instead of C. So the opposite category is always needed when considering the functor $\mathrm{Hom}_C(-,-)$. The first formulation, where the domain of the functor $\mathrm{Hom}_C(-,-)$ is C and not the opposite category, is usually preferred over the second. This is where the opposite category in $\mathrm{Set}^{C^{\mathrm{op}}}$ comes from. The functor $\mathrm{Hom}_C(-,-)$ sending c to $\mathrm{Hom}_C(-,c)$ is called the *Yoneda Embedding*, and usually denoted as y. It has some very nice properties, which follow from the Yoneda lemma.

Definition 2.1.10. *Yoneda Embedding*
The Yoneda embedding is a functor mapping objects c in C to the functor $yc = \mathrm{Hom}(-,c)$, and mapping morphisms $f : c \to d$ to the natural transformation $yf = \mathrm{Hom}(-,f)$, which has components $yf_e = \mathrm{Hom}(e,f)$, mapping a

morphism $g : e \to c$ in $\mathrm{Hom}\,(e, c)$ to $f \circ g : e \to d$ in $\mathrm{Hom}\,(e, d)$:

$$\begin{aligned}
y &: C \to \mathrm{Set}^{C^{\mathrm{op}}} \\
yc &= \mathrm{Hom}\,(-, c) \\
yf &= \mathrm{Hom}\,(-, f) : \mathrm{Hom}\,(-, c) \to \mathrm{Hom}\,(-, d) \\
yf_e &= \mathrm{Hom}\,(e, f) : \mathrm{Hom}\,(e, c) \to \mathrm{Hom}\,(e, d) \\
&\quad \mathrm{Hom}\,(e, f)\,(g : e \to c) = f \circ g : e \to d
\end{aligned}$$

The Yoneda embedding is full and faithful, which means that it is bijective on morphisms: for any natural transformation η in $\mathrm{Set}^{C^{\mathrm{op}}}$ between yc and yd, there is a unique morphism $f : c \to d$ in C such that $\eta = yf$. That is, the Yoneda embedding finds a copy of C inside $\mathrm{Set}^{C^{\mathrm{op}}}$. This is a direct corollary of the Yoneda Lemma, one of the most famous results of category theory. We merely state the lemma here for future reference. If the reader wants a proof or a better understanding of scope and meaning of the lemma, we refer to Awodey [Awodey06].

Lemma 2.1.1. Yoneda Lemma
For any (locally small) category C, and any functor $F \in \mathrm{Set}^{C^{\mathrm{op}}}$, we have the following isomorphism:

$$\mathrm{Hom}_{\mathrm{Set}^{C^{\mathrm{op}}}}\,(yc, F) \cong F(c)$$

That is, the set of natural transformations between the representation yc of c and F is isomorphic to the set $F(c)$.

This lemma is more often used in the following form, which demonstrates the usefulness of the Yoneda embedding:

Corollary. *For any (locally small) category C, we have:*

$$c \cong d \iff yc \cong yd$$

That is, two objects in C are isomorphic if and only if there is a one-to-one correspondence between morphisms into c and morphisms into d. The image yc of an object c in C is called the *representation* of c. A functor in $\mathrm{Set}^{C^{\mathrm{op}}}$ is called a *representable* functor if it is isomorphic to yc for some c.

2.2 Sheaves and sheaf semantics

In the previous section, we looked at representable presheaves, which were special objects in the category $\mathrm{Set}^{C^{op}}$, coming from the Yoneda embedding. As the name presheaf suggests, these are not the only interesting objects in the category $\mathrm{Set}^{C^{op}}$. When a *Grothendieck topology* is imposed on the category C, there are certain presheaves that have a *local character*, these presheaves are called *sheaves*.

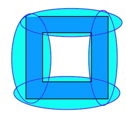


Figure 2.1: The local character of sheaves: it is enough to check a property locally to know that it is globally true.

In the next few pages, we slowly uncover the formal definition of sheaves, and we will see how we can turn a presheaf into a sheaf. We first explain the notion of a *Grothendieck Topology*, which defines when something is a *cover*. After that, we use the topology to define certain sets of elements, called *matching families*. These matching families are vital to sheaves: A presheaf is called a sheaf if every matching family has a unique amalgamation. When defining a sheaf as candidate non-standard model, we will come across all of these notions (see section 2.3, definition 3.0.1 and proposition 3.2.1, for example).

For a deeper treatment of all of the notions treated in this section we recommend the book Sheaves in Geometry and Logic, by S. Mac Lane and I. Moerdijk [MacLane&Mo92]. For a nice motivation of sheaves, we also recommend this article of the NLab: `http://ncatlab.org/nlab/show/motivation+for+sheaves,+cohomology+and+higher+stacks`.

2.2.1 Grothendieck topologies and sites

A Grothendieck topology on a category C defines, for each object $c \in C$, when a family of morphisms $\{f_i : d_i \to c \mid i \in I\}$ is a *cover* of c. We will give two definitions, one in terms of *covering sieves*, which is the more elaborate one, and one just in terms of *covering families*.

Definition 2.2.1. *Sieves*
For an object c in category C, a sieve S on c is a set of morphisms with codomain c that is closed under pre-composition. That is, for all $f : d \to c \in S$ and all g with $\mathrm{cod}(g) = d$, $f \circ g \in S$.

It is useful to actually have a sieve in mind: whenever a grain of sand (a morphism $f : d \to c$) goes through a hole in the sieve (is in S), then all smaller sand grains (morphisms of the form $f \circ g$) go through the hole as well (are also in S).

Definition 2.2.2. *Grothendieck Topology in terms of covering sieves.*
A Grothendieck topology on a category C is a function J, which assigns to each object $c \in C$ a set of sieves on c such that:

- The maximal sieve $\{f \mid \mathrm{cod}(f) = c\}$ is in $J(c)$.
- Stability: If S is in $J(c)$, and $h : d \to c$ is any morphism, then the set $R = \{g \mid \mathrm{cod}(g) = d \text{ and } h \circ g \in S\}$ should be in $J(d)$.

$$\begin{tikzcd}
& c & \\
e_1 \arrow[ur, "h \circ g_1 \in S"] \arrow[r, "g_1 \in R"'] & d \arrow[u, "h"] & e_2 \arrow[ul, "h \circ g_2 \notin S"'] \arrow[l, "g_2 \notin R"]
\end{tikzcd}$$

g_1 is in R, because $h \circ g_1$ is in S.
If S in $J(c)$, then we must have $R \in J(d)$.

- Transitivity: If S is in $J(c)$, and R is any sieve on c with the property that for all $h \in S$, the set $R_h = \{g \mid \mathrm{cod}(g) = d \text{ and } h \circ g \in R\}$ is in $J(d)$, then also $R \in J(c)$.

$$\begin{tikzcd}
d_1 \arrow[r, "h_1 \in S"] & c & d_2 \arrow[l, "h_2 \in S"'] \\
e_2 \arrow[u, "g_1 \in R_{h_1}"] \arrow[ur, "g_1 \circ h_1 \in R"] & & e_2 \arrow[u, "g_2 \notin R_{h_2}"'] \arrow[ul, "g_2 \circ h_2 \notin R"']
\end{tikzcd}$$

g_1 is in R_{h_1} because $h_1 \circ g_1$ is in R.
If S is in $J(c)$ and R_h is in $J(d)$
for all $h \in S$, then R is also in $J(c)$.

The sieves in $J(c)$ are called *covering sieves*

Sometimes, it is not strictly needed and even a bit cumbersome to define J in terms of sieves. If C has pullbacks, then it is enough to define a *basis* for a Grothendieck topology:

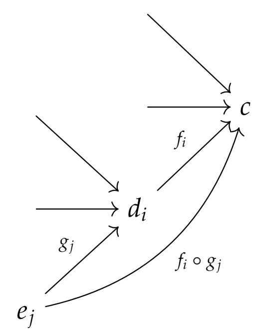
Definition 2.2.3. *Grothendieck Topology in terms of covering families or covers.*
A basis for a Grothendieck Topology on a category C with pullbacks is a function K which assigns to each object $c \in C$ a set of families of morphisms with codomain c such that:

- If $f : d \to c$ is an isomorphism, then $\{f\} \in K(c)$.
- Stability axiom: if $\{f_i : d_i \to c \mid i \in I\} \in K(c)$, then for any morphism $f : d \to c$, the family of pullbacks $\{\pi_{i2} : d_i \times_c d \to d \mid i \in I\} \in K(d)$.

$$
\begin{tikzcd}
d_i \times_c d \arrow[r, "\pi_{i2}"] \arrow[d, "\pi_{i1}"'] & d \arrow[d, "f"] \\
d_i \arrow[r, "f_i"'] & c
\end{tikzcd}
$$

When $\{f_i \mid i \in I\}$ is in $K(c)$,
the collection of morphisms
consisting of π_{i2}, $i \in I$ is in $K(d)$

- Transitivity axiom: if $\{f_i : d_i \to c \mid i \in I\} \in K(c)$, and for each $i \in I$ there is a family $\{g_{ij} : d_{ij} \to d_i \mid j \in I_i\} \in K(d_i)$, then the family of composites $\{f_i \circ g_{ij} : d_{ij} \to c \mid i \in I, j \in I_i\} \in K(c)$.



When $\{f_i \mid i \in I\}$ is in $K(c)$
and $\{g_j \mid j \in I_i\}$ is in $K(d)$
then $\{f_i \circ g_j \mid i \in I, j \in I_i\}$ is in $K(c)$

The families of morphisms in $K(c)$ are called *covering families* or just *covers*.

If a category has all pullbacks, the two definitions of Grothendieck topology are equivalent. A basis K can be extended to a Grothendieck topology J: a sieve S is in $J(c)$ iff there is a covering family F in $K(c)$ that is a subset of the sieve: $F \subseteq S$. We say that K *generates* J. Conversely, given a Grothendieck topology J, we can find a basis K that generates it: F is in $K(c)$ iff the closure of F under pre-composition is in $J(c)$.

Definition 2.2.4. *A site*
A category equipped with a Grothendieck topology is called a *site*. It is often denoted as a tuple (C, J) of the category C and the topology J on it.

2.2.2 Matching families, and amalgamating them

Given a site (C, J) and a presheaf $P \in \mathrm{Set}^{C^{op}}$, we can define a *matching family* for each $c \in C$ and cover S of c. We have again two definitions, one in terms of sieves and one in terms of covering families.

Definition 2.2.5. *Matching families and their amalgamation in terms of sieves*
When P is a presheaf and S is a covering sieve, then a matching family is a function that assigns to each element $f : d \to c \in S$ an element $x_f \in P(d)$ such that for all $g : e \to d$:

$$P(g)(x_f) = x_{f \circ g}$$

Note that $f \circ g \in S$, because S is a sieve, and hence $x_{f \circ g}$ is indeed a member of the matching family.

$$\begin{tikzcd}
e \arrow[r, "g"'] \arrow[rr, bend left, "f \circ g"] \arrow[d, dotted, "P"] & d \arrow[r, "f"'] \arrow[d, dotted, "P"] & c \\
P(e) & P(d) \arrow[l, "P(g)"] & \\
x_{f \circ g} \arrow[u, phantom, "\in" rotate=90] & x_f \arrow[l, mapsto, "P(g)"] \arrow[u, phantom, "\in" rotate=90] &
\end{tikzcd}$$

We will often denote a matching family as a set of tuples, to emphasize that the element x_f belongs to the morphism f:

$$\{(f, x_f) \mid f \in S\}$$

An amalgamation of such a matching family is an element $x \in P(c)$, such that: for each $f : d \to c \in S$:

$$P(f)(x) = x_f$$

$$\begin{tikzcd}
e \arrow[r, "g"'] \arrow[rr, bend left, "f \circ g"] \arrow[d, dotted, "P"] & d \arrow[r, "f"'] \arrow[d, dotted, "P"] & c \arrow[d, dotted, "P"] \\
P(e) & P(d) \arrow[l, "P(g)"] & P(c) \arrow[l, "P(f)"] \\
x_{f \circ g} \arrow[u, phantom, "\in" rotate=90] & x_f \arrow[l, mapsto, "P(g)"'] \arrow[u, phantom, "\in" rotate=90] & x \arrow[l, mapsto, "P(f)"'] \arrow[ll, mapsto, bend left, "P(g \circ f)"] \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

From now on, we will denote $P(f)(x)$ by $x \cdot f$ for all morphisms $f \in C$ and elements $x \in P(c)$.

Definition 2.2.6. *Matching families and their amalgamation in terms of covering families*
When P is a presheaf and $\{f_i : d_i \to c \mid i \in I\}$ is a covering family, then a matching

family is a function that assigns to each element $f_i : d_i \to c$ an element $x_i \in P(d_i)$ such that for all $i, j \in I$:

$$x_i \cdot \pi^1_{ij} = x_j \cdot \pi^2_{ij}$$

Where π^1_{ij} and π^2_{ij} are the projections from the following pullback:

$$\begin{tikzcd}
d_i \times_c d_j \arrow[r, "\pi^2_{ij}"] \arrow[d, "\pi^1_{ij}"'] & d_j \arrow[d, "f_j"] \\
d_i \arrow[r, "f_i"'] & c
\end{tikzcd}$$

The matching family shown diagrammatically (compare to the diagram shown in definition 2.2.5):

$$\begin{tikzcd}
d_i \times_c d_j \arrow[r, "\pi^1_{ij}"'] \arrow[rr, bend left, "\pi^2_{ij}"] \arrow[d, dotted, "P"] & d_i \arrow[rr, bend left, "f_i"] \arrow[d, dotted, "P"] & d_j \arrow[r, "f_j"'] \arrow[d, dotted, "P"] & c \\
P(d_i \times_c d_j) & P(d_j) \arrow[l, "P(\pi^1_{ij})"] & P(d_j) \arrow[ll, bend right, "P(\pi^2_{ij})"'] & \\
\Cup & \Cup & \Cup & \\
y & x_{f_i} \arrow[l, mapsto, "P(\pi^1_{ij})"'] & x_{f_j} \arrow[ll, bend left, mapsto, "P(\pi^2_{ij})"] &
\end{tikzcd}$$

An amalgamation of such a matching family is an element $x \in P(c)$, such that: for each $f_i : d_i \to c$:

$$x \cdot f_i = x_i$$

In the diagram:

$$\begin{tikzcd}
d_i \times_c d_j \arrow[r, "\pi^1_{ij}"'] \arrow[rr, bend left, "\pi^2_{ij}"] \arrow[d, dotted, "P"] & d_i \arrow[rr, bend left, "f_i"] \arrow[d, dotted, "P"] & d_j \arrow[r, "f_j"'] \arrow[d, dotted, "P"] & c \arrow[d, dotted, "P"] \\
P(d_i \times_c d_j) & P(d_j) \arrow[l, "P(\pi^1_{ij})"] & P(d_j) \arrow[ll, bend right, "P(\pi^2_{ij})"'] & P(c) \arrow[l, "P(f_j)"] \arrow[ll, bend right, "P(f_i)"'] \\
\Cup & \Cup & \Cup & \Cup \\
y & x_{f_i} \arrow[l, mapsto, "P(\pi^1_{ij})"'] & x_{f_j} \arrow[ll, bend left, mapsto, "P(\pi^2_{ij})"] & x \arrow[l, mapsto, "P(f_j)"'] \arrow[ll, bend left, mapsto, "P(f_j)"]
\end{tikzcd}$$

2.2.3 Sheaves

As promised, a presheaf is a sheaf if and only if every matching family has a unique amalgamation.

Definition 2.2.7. *Sheaves*
Given a site (C, J), then a presheaf $P \in \mathrm{PSh}(C)$ is a sheaf if and only if any matching family for any cover has a unique amalgamation. The category of sheaves $\mathrm{Sh}(C, J)$ is the full subcategory of $\mathrm{PSh}(C)$ having sheaves as objects and natural transformation between them as morphisms.

Both the category of sheaves and the category of presheaves have all limits and colimits, this makes them very nice to work in. For those interested in topos theory, a category of sheaves on a site is a topos. In Sketches of an Elephant [Johnstone02], Johnstone mentions the category of sheaves on a site as one of the many descriptions of 'what a topos is like'.

Clearly, not every presheaf is a sheaf. But there is a way to construct a sheaf out of every presheaf. There exists a functor $(-)^+ : \mathrm{PSh}(C) \to \mathrm{PSh}(C)$ which, when applied twice to a presheaf, yields a sheaf. This functor is called the *plus construction*.

Definition 2.2.8. *Plus construction*
$(-)^+$ is a functor mapping the presheaf P to the presheaf P^+, which consists of equivalence classes of pairs of sieves and matching families:

$$P^+(c) = \Big\{ \, [(S, \; \{(f, x_f) \mid f \in S \})] \, \Big\}$$

Where S is a covering sieve, $f : d \to c \in S$ and $x_f \in P(d)$.

Two such pairs $(S, \{f, x_f\})$ and $(R, \{g, y_g\})$ are equivalent if there exists a common refinement $T \subseteq R \cap S$ of the covers R and S such that for all $h \in T$: $x_h = y_h$.

On morphisms, P^+ acts as follows: if $f : d \to c$ in C (that is, $f : c \to d$ in C^{op}), then:

$$P^+(f) : P^+(c) \to P^+(d)$$
$$P^+(f)[(S, \{g, x_g\})] = [(R, \{(g_j, x_{f \circ g_j})\}]$$

Where $R = \{g_j \mid \mathrm{cod}(g_j) = d \text{ and } f \circ g_j \in S\}$. As $f \circ g_j \in S$ for all $g_j \in R$, $(f \circ g_j, x_{f \circ g_j})$ is an element of the matching family $\{(g, x_g) \mid g \in S\}$. It is this $x_{f \circ g_j}$ that we add to g_j in the new matching family.

Applying the plus construction once to a presheaf yields a *separated presheaf*.

Definition 2.2.9. A presheaf P is separated if every matching family has at most one amalgamation. Any amalgamation x for a matching family $\{(f_i : d_i \to c, x_{f_i}) \mid f_i \in S, x_{f_i} \in P(d_i)\}$ must satisfy $x \cdot f_i = x_{f_i}$, hence when x and y are two amalgamations, then P is separated if

$$\forall f_i \in S \; [x \cdot f_i = y \cdot f_i] \quad \text{implies} \quad x = y.$$

A separated presheaf is 'almost' a sheaf: if a matching family has an amalgamation, then it is unique, but not every matching family has an amalgamation. Applying the plus construction to a separated presheaf yields a sheaf. We hence have a functor which maps every presheaf to a sheaf: the sheafification functor.

Definition 2.2.10. *Sheafification*
The sheafification functor, $\mathbf{a} : \mathrm{PSh}\,(C) \to \mathrm{Sh}\,(C, J)$, applies the plus construction twice to a presheaf, yielding a sheaf:

$$\mathbf{a}P = (P^+)^+$$

This functor is the left-adjoint to the inclusion functor $\imath : \mathrm{PSh}\,(C) \to \mathrm{Sh}\,(C, J)$. When a presheaf is separated, it suffices to apply the plus construction only once: the plus construction applied to a sheaf yields an isomorphic copy of that sheaf.

2.2.4 Sheaf semantics: sheaves as models

In this thesis, we use sheaves as models for natural arithmetic. In order to do so, we need to interpret sentences from first order logic in sheaves: we need sheaf semantics. Given a sheaf $P : C^{op} \to \mathrm{Set}$, an object c in C and an element $x \in P(c)$, there is a forcing relation $c \Vdash \phi(x)$, stating that 'c believes $\phi(x)$ to be true'. The definition we give is not the original Kripke-Joyal semantics, but it is an equivalent formulation. Notice that the logic of sheaves is intuitionistic.

Definition 2.2.11. *Sheaf semantics / Kripke-Joyal semantics* (see Theorem 1 from [MacLane&Mo92, section 7, chapter VI].)
Let $P : C^{op} \to \mathrm{Set}$ be a sheaf, c an object in C and $x \in P(c)$. Then we define $c \Vdash \phi(x)$ inductively:

- Atomic formulas: $c \Vdash x = y$ iff $x = y$.
- Conjunction: $c \Vdash \phi(x) \wedge \psi(x)$ iff $c \Vdash \phi(x)$ and $c \Vdash \psi(x)$.
- Disjunction: $c \Vdash \phi(x) \vee \psi(x)$ iff there is a cover $S = \{f_i : d_i \to c\}$ of c such that for each $f_i \in S$, either $d_i \Vdash \phi(x \cdot f_i)$ or $d_i \Vdash \psi(x \cdot f_i)$.
- Implication: $c \Vdash \phi(x) \to \psi(x)$ iff for all $f : d \to c$, $d \Vdash \phi(x \cdot f)$ implies $d \Vdash \psi(x \cdot f)$.
- Negation: $c \Vdash \neg\phi(x)$ iff for all $f : d \to c$, if $d \Vdash \phi(x \cdot f)$, then the empty family is a cover of d.
- Existential quantifier: $c \Vdash \exists x\ [\phi(x, y)]$ iff there exists a cover $S = \{f_i : d_i \to c\}$ of c and for each f_i in S there exists a $z \in P(d_i)$ such that $d_i \Vdash \phi(z, y \cdot f)$.
- Universal quantifier: $c \Vdash \forall x\ [\phi(x, y)]$ iff for all $f : d \to c$ and all $z \in P(d_i)$: $d_i \Vdash \phi(z, y \cdot f)$.

Furthermore, there are two important principles of sheaf semantics:

- Monotonicity: if $c \Vdash \phi(x)$ and $f : d \to c$ is a morphism in C, then $d \Vdash \phi(x \cdot f)$.
- Local Character: if $S = \{f_i : d_i \to c\}$ is a cover of c and for all $f_i \in S$ $d_i \Vdash \phi(x \cdot f_i)$, then $c \Vdash \phi(x)$.

Sheaf models for (standard) natural numbers

In any category, there might be objects that behave 'like the natural numbers' and could be considered as the standard natural numbers in that category. Sheaf categories always have a natural numbers object, which makes them nice environments to look for non-standard models. We first give the general definition of a *natural numbers object*, and then say how to find the natural numbers object in a sheaf category.

Definition 2.2.12. *Natural numbers object*
In any category C, an object N together with morphisms $0 : 1 \to N$ and $s : N \to N$ is called a natural numbers object if, for each object A, together with morphisms $0' : 1 \to A$ and $s' : A \to A$, there exists a unique morphism $u : N \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "0'"'] & N \arrow[r, "s"] \arrow[d, dotted, "u"] & N \arrow[d, dotted, "u"] \\
& A \arrow[r, "s'"'] & A
\end{tikzcd}$$

Every category of sheaves based on a Grothendieck topology has a natural numbers object, and it is (isomorphic to) the sheafification $\mathbf{a}\mathbb{N}$ of the constant presheaf $\mathbb{N}$. This presheaf maps every c in C to the set of natural numbers, and every morphism to the identity morphism.

There are hence two ways to prove that a sheaf is the natural numbers object of its category: either by showing that it has the properties of the natural numbers object, or that it is isomorphic to $\mathbf{a}\mathbb{N}$.

2.2.5 A short summary of sheaves

We have seen that sheaves are presheaves (elements of the category $\mathrm{Set}^{C^{op}}$), which have the special property that every matching family (with respect to a site) has a unique amalgamation. A matching family for a presheaf P and object c in C consists of pairs of morphisms $f : d \to c$ in C and elements of $P(d)$, that 'behave well' under composition of morphisms. The morphisms in a matching family come from a cover S of c. Which sets of morphisms cover c is defined by a Grothendieck topology J on C, which is given either in terms of covering sieves of covering families. The pair (C, J) is called a site.

The sheafification functor maps every presheaf to a sheaf. The sheafification of the constant presheaf $\mathbb{N}$ yields the natural numbers object of the sheaf category.

Sheaves can be used as models for logical theories via sheaf semantics. The internal logic of a sheaf is constructive.

2.3 Sheaf models for the Peano axioms a step by step guide to obtain them

Building a sheaf model for the Peano axioms consists of four steps:

1. Choosing a category C as basis for the site.
2. Choosing a Grothendieck topology J on C, which defines which sets of morphisms are covering.
3. Choosing or constructing a sheaf, either by choosing a presheaf and proving that it is a sheaf, or by choosing a presheaf and taking its sheafification.
4. Using sheaf semantics to check whether the sheaf models the Peano axioms.

If all of these steps have succeeded, the result is either an isomorphic copy of the natural numbers object, or a candidate non-standard model. In the latter case, the natural numbers object should be isomorphic to a strict subsheaf of the resulting sheaf. This follows quite directly from the definition of the natural numbers object and the fact that the successor function in the non-standard model is injective.

In a non-standard model, it should be impossible to discriminate between standard elements (those in the isomorphic copy of the natural numbers object) and non-standard elements (those that are not), using only first order logic. Therefore, the principles overspill and underspill should be valid in the non-standard model:

5. Checking non-standard principles: overspill and underspill.

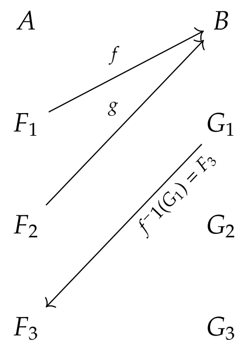
Overspill states that anything that is true for all standard numbers, must also be true for some non-standard number. Underspill is the dual of overspill, stating that anything that is true for all non-standard numbers must also hold for some standard number.

There are some other principles which are often considered for non-standard models, some of which we will encounter later in this thesis. We refer to [Berg12, the introduction of section 3: Nonstandard principles] for a more detailed explanation of overspill, underspill, and other principles, as well as their relevance in non-standard models.

We will now take a close look at the model Moerdijk describes in [Moerdijk95]. We will mention the choices he makes in each of the steps he takes in constructing his model. Then in the following chapters, we will propose simplifications of Moerdijk's choices, thereby constructing our own non-standard models.

2.3.1 Moerdijk's model

Moerdijk uses a filter category as basis for his site. The objects in this category are pairs $(A, \mathscr{F})$ of a subset $A \subseteq \mathbb{N}^k$ and a filter $\mathscr{F}$ of subsets of A. The morphisms in this category are 'equivalence classes of continuous partial functions' between two subsets A and B of $\mathbb{N}^k$. The functions are partial because they only need to be defined on a set in the filter $\mathscr{F}$, not on the whole of A. The functions are continuous in the sense that they need to have the property that the inverse image of a set in the filter belonging to B is a set in the filter belonging to A. The equivalence relation is: 'two functions are equivalent if they are equal on some set in the filter' (see the illustration below).



f is a morphism because f is defined on F_1 and has the property that the inverse of any G_i is some F_j. $[f] = [g]$ iff there exists an F_i such that $f \upharpoonright F_i = g \upharpoonright F_i$.

The covering condition is formulated as: "an arrow $\alpha : (A, \mathscr{F}) \to (B, \mathscr{G})$ is covering if $\alpha(F) \in \mathscr{G}$ for any $F \in \mathscr{F}$". A covering family then consists of a finite collection of morpisms $\{A_i \to B \mid 1 \leq i \leq k\}$ with the property that the induced morphism from the co-product $A_1 + \ldots + A_k$ to B is a covering arrow.

This site has the special property that all representable presheaves are sheaves. The sheaf that Moerdijk chooses as candidate for a non-standard model of natural numbers is the representable sheaf $y(\mathbb{N}, \{N\})$, the Yoneda embedding of the set $\mathbb{N}$ together with the trivial filter consisting only of $\mathbb{N}$.

In short, Moerdijk's construction consists of:

- The base category: a category of filters
- The topology: filters are mapped to (sub)filters
- The sheaf: the representable sheaf $y(\mathbb{N}, \{\mathbb{N}\})$

The sheaf does indeed model the Peano axioms, as well as the overspill principle. Transfer and the axiom of choice are also valid, but those proofs need classical logic in the meta-theory. The standard natural numbers are found as subsheaf of $y(\mathbb{N}, \{\mathbb{N}\})$; they are the equivalence classes of bounded partial functions.

Chapter 3

Functions as numbers

We present a non-standard model for natural arithmetic in a category of sheaves on a site. Our model is based on the construction presented by Moerdijk in A model for intuitionistic non-standard arithmetic [Moerdijk95]. We replicate almost all his results, including having the bounded functions as standard natural numbers. Furthermore, the non-standard principles overspill, underspill, transfer, idealisation and realisation are valid in our model. However, because we use a different site than Moerdijk does, we get our results using different arguments.

The structure of this chapter is as follows: we first introduce the site we'll be working with, and look for the natural numbers object in the category of sheaves on this site. Then, we define the sheaf we use for our candidate non-standard model. We show that the natural numbers object is a strict subsheaf of this sheaf and that our sheaf does indeed model the Peano axioms. After considering several non-standard principles, such as overspill and underspill, we finish by comparing our model and proof methods to those of Moerdijk, and comment on our findings.

In defining our model, we follow the steps mentioned in section 2.3, starting with the site. We simplify the site of Moerdijk as much as possible.

Step 1: The objects of Moerdijk's category were pairs of subsets of $\mathbb{N}^k$ and filters. We start by making a radical simplification: we take only the infinite subsets of natural numbers as objects. The morphisms in our category are the inclusion functions $d \leq c$ (we see this category as a poset). We deliberately leave out the finite subsets: this eventually causes the constant presheaf $\mathbb{N}$ to separated.

Step 2: As a covering condition, we say that a (finite) collection of subsets of c covers c if their union excludes only finitely many elements of c. That is, if their union is cofinite in c. Moerdijk's covering condition becomes ours when when only the Frechet filter (the filter of all cofinite subsets) is allowed in his category of filters: A union of sets covers if and only if this union is a member of the Frechet filter. This completes our site as simplification of the one used by Moerdijk.

Step 3: For the candidate non-standard model, we again look at Moerdijk's choice: $y(\mathbb{N}, \{\mathbb{N}\})$, mapping an object $A, \mathcal{F}$ to all continuous partial functions from A to $\mathbb{N}$. This time, the connection to our choice is not that direct. A very hand waving description of $y(\mathbb{N}, \{\mathbb{N}\})$ would be 'functions into $\mathbb{N}$'. Inspired by that thought, we came up with two candidate non-standard models:

(a) The presheaf $\mathbb{N}^{\mathbb{N}}$, which we prove consists precisely of functions from $\mathbb{N}$ to $\mathbb{N}$.

(b) The sheaf $\mathbf{a}\mathbb{N}^{\mathbf{a}\mathbb{N}}$, for which we might be able to use the ultrafilter construction to obtain a non-standard model.

The first candidate model is the one we fully examine in this chapter. For the sake of future research, we present our work on the second candidate model in appendix A. There, we also prove some interesting relations between the natural numbers object $\mathbf{a}\mathbb{N}$, and both candidate non-standard models and propose some questions for future research.

Formalising the ideas presented in step 1 and 2, the site on which we build our category of sheaves is defined as follows:

Definition 3.0.1. *The site $(\mathbb{P}, J)$ and the category of sheaves $\mathcal{E}$:*
Let $\mathbb{P}$ be the set of all infinite subsets of $\mathbb{N}$, and $\mathbb{I}$ the set of all finite subsets of $\mathbb{N}$. We consider $\mathbb{P}$ as a poset ordered by inclusion and define a Grothendieck topology J on $\mathbb{P}$ by saying that a sieve $S \subseteq \{d \mid d \leq c\}$ is covering $c \in \mathbb{P}$ if there exist (disjoint[1]) $d_1, \ldots, d_n \in S$ such that their union is, up to finitely many elements, equal to c. More precisely[2]:

$$S \text{ covers } c \iff \exists w \in \mathbb{I}\ \exists d_1, \ldots, d_n \in S\ [(i \neq j \to d_i \cap d_j = \emptyset) \wedge d_1 \cup \ldots \cup d_n \cup w = c]$$

We then define $\mathcal{E} := \mathrm{Sh}(\mathbb{P}, J)$ to be the category of sheaves on the site $(\mathbb{P}, J)$.

$\mathcal{E}$ has a natural numbers object. We start by analysing this object closely.

3.1 The natural numbers object of $\mathcal{E}$

We define the functor $\mathrm{FinIm} \in \mathrm{Set}^{\mathbb{P}^{\mathrm{op}}}$, mapping objects c in $\mathbb{P}$ to the set of functions $c \to \mathbb{N}$ with finite image. Taking an appropriate quotient yields an isomorphic copy of the natural numbers object.

[1]This requirement is not strictly needed. Classically, we can always make a given finite set of subsets of c disjoint. It is a nice property to have and therefore we add it to the definition.

[2]notice that we define the covering condition in terms of objects instead of morphisms. The only existing morphisms in this category are inclusion functions. To avoid confusion, we therefore speak of their domain rather than the inclusion functions themselves.

Definition 3.1.1. *The functor* FinIm
For $c \in \mathbb{P}$, we define $\mathrm{FinIm}\,(c)$ as:

$$\mathrm{FinIm}\,(c) = \{f : c \to \mathbb{N} \mid \mathrm{Im}(f) \text{ is finite }\}.$$

We define an equivalence relation on $\mathrm{FinIm}\,(c)$ by:

$$f \sim g \iff \{n \mid f(n) \neq g(n)\} \text{ is finite.}$$

Proposition 3.1.1. *For each $c \in \mathbb{P}$: $\mathbf{a}\mathbb{N}(c) \cong \mathrm{FinIm}\,(c)\,/_{\sim}$*

Proof. The presheaf $\mathbb{N}$ is separated, which we ensured by using $\mathbb{P}$ instead of the entire powerset of the natural numbers as objects for our site. To get the sheafification of $\mathbb{N}$ it therefore suffices to apply the +-operator once. This results in:

$$\mathbf{a}\mathbb{N}(c) = \mathbb{N}^{+}(c) = \{\ [(\ S,\ \{(d, x_d) \mid d \in S\,\}\,)]\ \}$$

The elements x_d of the matching family $\{(d, x_d) \mid d \in S\}$ are just natural numbers, such that for every $d, e \in S$ with $e \leq d$: $x_e = x_d$.

To prove that $\mathbf{a}\mathbb{N}(c)$ is isomorphic to $\mathrm{FinIm}\,(c)\,/_{\sim}$ it is enough to find a function from one to the other that is both injective and surjective. Given an equivalence class $X = [(S, \{d, x_d\})]$ of a cover S and a matching family $\{d, x_d \mid d \in S, x_d \in \mathbb{N}\}$, define a function with finite image f_X by:

$$f_X(n) = \begin{cases} x_d & \text{if } n \in d \text{ for some } d \in S, \\ 0 & \text{else.} \end{cases}$$

Note that the image of f_X is indeed finite: S contains $d_1, \ldots d_n$ such that their union is almost c. The properties of matching families ensure that the image of f_X consists only of the x_{d_i} belonging to one of these finitely many d_i (and then there are finitely many elements not in the union of the d_i, for which f_X can have other values). Claim: the function sending X to $[f_X]$ is an isomorphism between the sets $\mathbf{a}\mathbb{N}(c)$ and $\mathrm{FinIm}\,(c)\,/_{\sim}$. First of all, note that it is well defined: two representatives of the equivalence class X are mapped to the same equivalence class of functions. Also, it is injective ($f_X \sim f_Y$ implies $X = Y$). It is also surjective: given a function f with finite image, we can construct a cover S_f and a matching family x^{S_f}:

$$\begin{aligned} S_f &= \{d \leq c \mid f \text{ is constant on } d\}, \\ x^{S_f} &= \{(d, x_d) \mid d \in S_f \text{ and } x_d = f(n) \text{ for some } n \in d\}. \end{aligned}$$

It is straightforward to verify that S_f is indeed a cover of c. x^{S_f} is well defined as f is constant on all $d \in S$, so that is does not matter which $n \in d$ is chosen to compute $x_d = f(n)$ Moreover: $[(S, x^{S_f})]$ is mapped to $[f]$, that is: $f_{[(S, x^{S_f})]} \sim f$, which follows immediately from the definitions. □

So we have a nice description of the natural numbers object $\mathbf{a}\mathbb{N}$:

$$\mathbf{a}\mathbb{N}(c) \cong \{[f : c \to \mathbb{N}] \mid \mathrm{Im}(f) \text{ is finite}\}. \tag{3.1}$$

Where

$$f \sim g \iff \{n \mid f(n) \neq g(n)\} \text{ is finite}.$$

We complete this description by identifying zero $0 : 1 \to \mathbf{a}\mathbb{N}$ and the successor function $s : \mathbf{a}\mathbb{N} \to \mathbf{a}\mathbb{N}$. For each $c \in \mathbb{P}$, the component 0_c is just the equivalence class of the constant function $const_0 : c \to \mathbb{N}$, mapping each element of c to 0. The successor function is the following natural transformation:

$$s_c([f]) = [f_{+1}] \tag{3.2}$$

$$f_{+1}(n) = f(n) + 1 \tag{3.3}$$

We now look for a non-standard model. Inspired by Moerdijk's choice of $y(\mathbb{N}, \{\mathbb{N}\})$ as non-standard model, we consider the presheaf $\mathbb{N}^{\mathbb{N}}$, and prove that this presheaf can be seen as the presheaf 'functions from $\mathbb{N}$ to $\mathbb{N}$'. Taking equivalence classes yields a sheaf.

3.2 Functions from $\mathbb{N}$ to $\mathbb{N}$

We consider the presheaf $\mathbb{N}^{\mathbb{N}}$. By definition:

$$\mathbb{N}^{\mathbb{N}}(c) = \mathrm{Hom}\,(yc \times \mathbb{N}, \mathbb{N})$$

That is, $\mathbb{N}^{\mathbb{N}}(c)$ is the set of natural transformations between $yc \times \mathbb{N}$ and $\mathbb{N}$. Let τ be such a natural transformation. Its components, τ_d, are functions that take an element of $(yc \times \mathbb{N})(d)$ and map it to an element of $\mathbb{N}(d)$. That is

$$\begin{aligned} \tau_d &: \mathrm{Hom}_{\mathbb{P}}\,(d, c) \times \mathbb{N} \to \mathbb{N} \\ &: (\leq, n) \mapsto m \quad \text{if } d \leq c, \end{aligned}$$

where n and m are any two natural numbers. So actually, τ_d is a map from $\mathbb{N}$ to $\mathbb{N}$, for each $d \leq c$. Naturality of τ means that the following diagram commutes for $d \leq c$ and $e \leq d$:

$$\begin{tikzcd} (yc \times \mathbb{N})(d) \arrow[r, "\mathrm{Id}"] \arrow[d, "\tau_d"'] & (yc \times \mathbb{N})(e) \arrow[d, "\tau_e"] \\ \mathbb{N}(d) \arrow[r, "\mathrm{Id}"'] & \mathbb{N}(e) \end{tikzcd}$$

Which becomes a rather simple but useful diagram when we use that τ_d can be seen as a function from $\mathbb{N}$ to $\mathbb{N}$:

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "\mathrm{Id}"] \arrow[d, "\tau_d"'] & \mathbb{N} \arrow[d, "\tau_e"] \\
\mathbb{N} \arrow[r, "\mathrm{Id}"'] & \mathbb{N}
\end{tikzcd}$$

We hence must have $\tau_d = \tau_e$, for all $d, e \leq c$, so that a natural transformation $\tau : yc \times \mathbb{N} \to \mathbb{N}$ actually corresponds to a single function $f : \mathbb{N} \to \mathbb{N}$:

$$\mathbb{N}^{\mathbb{N}}(c) = \mathrm{Hom}\,(yc \times \mathbb{N}, \mathbb{N}) \cong \{f : \mathbb{N} \to \mathbb{N}\}. \tag{3.4}$$

We take appropriate equivalence classes of these functions to get our final candidate non-standard model: $\mathbb{N}^{\mathbb{N}}/_{\sim}$.

Definition 3.2.1. $\mathbb{N}^{\mathbb{N}}/_{\sim}$
Recall that $\mathbb{N}^{\mathbb{N}}(c) = \{f : \mathbb{N} \to \mathbb{N}\}$. We define an equivalence relation on $\mathbb{N}^{\mathbb{N}}(c)$ by saying that f and g in $\mathbb{N}^{\mathbb{N}}(c)$ are equivalent if

$$f \sim g \iff \{n \in c \mid f(n) \neq g(n)\} \text{ is finite.}$$

We then have:

$$\begin{aligned} \mathbb{N}^{\mathbb{N}}(c)/_{\sim} &= \{[f] \mid f : \mathbb{N} \to \mathbb{N}\} \\ [f]_c \cdot d &= [f]_d \end{aligned}$$

That is, $\mathbb{N}^{\mathbb{N}}/_{\sim}$ maps the equivalence of f (equivalence in the eyes of c! hence the subscript for clarity) to the equivalence class (in the eyes of d!) of f.

Proposition 3.2.1. *$\mathbb{N}^{\mathbb{N}}/_{\sim}$ is a sheaf.*

Proof. $\mathbb{N}^{\mathbb{N}}/_{\sim}$ is a sheaf iff every matching family for any covering sieve of any $c \in \mathbb{P}$ has a unique amalgamation. So pick any c, covering sieve $S = \{d_i \mid i \in I\}$ and matching family $\{(d_i, [f_i]) \mid d_i \in S\}$. We construct an amalgamation $[f]$ as follows:

$$f(n) = \begin{cases} f_i(n) & \text{if } n \in d_i \\ 0 & \text{if } n \notin \bigcup_{d_i \in S} d_i \end{cases}$$

Then take the equivalence class $[f]$ as amalgamation. Note that this definition is sound. If n belongs to both d_i and d_j, the properties of matching families ensure that $f_i(n) = f_j(n)$. Furthermore, we have by definition that for each $d_i \in S$, for all $n \in d_i$, $f(n) = f_i(n)$, hence $[f] \cdot d_i = [f_i]$ for each $d_i \in S$, proving $[f]$ is indeed an amalgamation.

Lastly, we need the amalgamation to be unique. Suppose $[g]$ is any amalgamation of $\{(d_i, [f_i]) \mid d_i \in S\}$. We need to show that $g \sim f$, that is, $g(n) = f(n)$ for all but finitely many $n \in c$. Since S is covering, there are $d_1, \ldots, d_k \in S$ whose union excludes only finitely many elements of c. We limit our attention to these for a moment. As g is an amalgamation, $[g] \cdot d_i = [f_i]$ by definition. So $[g]_{d_i} = [f]_{d_i}$ for all $1 \leq i \leq k$. That is, g can only differ from f on:

- at most finitely many n in d_1 (same for $d_2, \ldots, d_k$).
- possibly every n that is not in the union $d_1 \cup \ldots \cup d_k$, which are at most finitely many n.

So in total, $\{n \in c \mid g(n) \neq f(n)\}$ is finite. Hence $g \sim f$, proving uniqueness of the amalgamation.

So $\mathbb{N}^{\mathbb{N}}/_{\sim}$ is indeed a sheaf. □

The connection between the natural numbers object and $\mathbb{N}^{\mathbb{N}}/_{\sim}$ is quite obvious. We make it explicit in the following proposition:

Proposition 3.2.2. *The presheaf sending every $c \in \mathbb{P}$ to*

$$\{[f] \in \mathbb{N}^{\mathbb{N}}/_{\sim}(c) \mid \operatorname{Im}(f \restriction c) \text{ is finite } \}$$

forms a subsheaf of $\mathbb{N}^{\mathbb{N}}/_{\sim}$. This subsheaf, which we call $\mathrm{St}_{\mathbb{N}}$ (short for 'standard natural numbers'), is isomorphic to the natural numbers object.

Proof. The fact that St_N is a subsheaf of $\mathbb{N}^{\mathbb{N}}/_{\sim}$ follows trivially from the definitions. St_N and $\mathbf{a}\mathbb{N}$ are isomorphic if for all c, $\mathbf{a}\mathbb{N}(c) \cong \mathrm{St}_{\mathbb{N}}(c)$. As we saw before (equation 3.1):

$$\mathbf{a}\mathbb{N}(c) = \{[f : c \to \mathbb{N}] \mid \operatorname{Im}(f) \text{ is finite}\}.$$

Note that the equivalence relation for $\mathbf{a}\mathbb{N}$ and $\mathrm{St}_{\mathbb{N}}$ is the same:

$$f \sim g \iff \{n \in c \mid f(n) \neq g(n)\} \text{is finite.}$$

Under this equivalence relation, $\{[f : c \to \mathbb{N}] \mid \operatorname{Im}(f) \text{ is finite}\}$ is isomorphic to $\{[f : \mathbb{N} \to \mathbb{N}] \mid \operatorname{Im}(f \restriction c) \text{ is finite}\}$, resulting in:

$$\begin{aligned} \mathbf{a}\mathbb{N}(c) &= \{[f : c \to \mathbb{N}] \mid \operatorname{Im}(f) \text{ is finite}\} \\ &\cong \{[f : \mathbb{N} \to \mathbb{N}] \mid f \restriction c \text{ is finite}\} \\ &= \mathrm{St}_{\mathbb{N}}(c) \end{aligned}$$ □

The natural numbers object is hence a subsheaf of our candidate non-standard model. We continue to verify if our sheaf is indeed a non-standard model for natural arithmetic.

3.2.1 $\mathbb{N}^{\mathbb{N}}/_{\sim}$ as a non-standard model for natural arithmetic

We explore $\mathbb{N}^{\mathbb{N}}/_{\sim}$ as a non-standard model for natural arithmetic using sheaf semantics. We define some structure on our sheaf so that we can interpret various kinds of axioms in it, such as the Peano axioms, overspill and underspill. The definition of equality is already included in sheaf semantics, but we mention it here just as a reminder.

Definition 3.2.2. *Successor function, Equality, Order, Standard and Infinite numbers*

- The successor function, $s : \mathbb{N}^{\mathbb{N}}/_{\sim} \to \mathbb{N}^{\mathbb{N}}/_{\sim}$ is defined similar to equation 3.3):
$$s_c([f]) = [f_{+1}]$$
- Equality:
$$c \Vdash [f] = [g] \iff \text{for all but finitely many } n \in c : f(n) = g(n)$$
- Order:
$$c \Vdash [f] \le [g] \iff \text{for all but finitely many } n \in c : f(n) \le g(n)$$
- The predicate $\mathrm{St}(\cdot)$: We say that $[f]$ is a *standard natural number* in the eyes of c, if the image of f restricted to c is finite.
$$c \Vdash \mathrm{St}([f]) \iff \mathrm{Im}(f \restriction c) \text{ is finite} \tag{3.5}$$
- The predicate $\mathrm{Inf}(\cdot)$: We say that $[f]$ is an *infinite number* in the eyes of c, if it is larger than all standard natural numbers:
$$c \Vdash \mathrm{Inf}([f]) \iff c \Vdash \forall x[\mathrm{St}(x) \to x \le [f]] \tag{3.6}$$

Notice that the predicate St coincides with the subsheaf $\mathrm{St}_{\mathbb{N}}$ defined previously. This is of course no coincidence.

To distinguish between first order (internal) formulas and (external) formulas that could contain the newly introduced propositions St and Inf, we will use the notation introduced by Nelson [Nelson77], denoting internal formulas with small case Greek letters and external formulas with capital Greek letters. We also add the following notation: $\forall^{\mathrm{St}}x, \exists^{\mathrm{St}}x, \forall^{\mathrm{Inf}}x, \exists^{\mathrm{Inf}}x$, which is shorthand for $\forall x(\mathrm{St}(x) \to \ldots), \exists x(\mathrm{St}(x) \wedge \ldots)$ etc.

Before turning our attention to the Peano axioms and other nice properties our sheaf might have, we verify that the order is linear.

Proposition 3.2.3. *$\mathbb{N}^{\mathbb{N}}/_{\sim}$ is linearly ordered by $\le$*

Proof. Antisymmetry and transitivity follow immediately from the same properties of the order on the natural numbers, so we only prove totality:
$$c \Vdash \forall x \forall y[x \le y \vee y \le x].$$
Let $[f]$ and $[g]$ be any two elements of $\mathbb{N}^{\mathbb{N}}/_{\sim}(c)$. It is enough to find a cover $S = \{d_1, \ldots, d_k\}$ such that for each $d_i \in S$
$$d_i \Vdash [f] \le [g] \quad \text{or} \quad d_i \Vdash [g] \le [f].$$
Consider the following subsets of c:
$$d_1 = \{n \in c \mid f(n) \le g(n)\}$$
$$d_2 = \{n \in c \mid f(n) > g(n)\}$$
Then $d_1 \cup d_2 = c$. There are three possible cases:

- d_1 is finite. Then d_2 is infinite and covers c. Since $d_2 \Vdash [g] \leq [f]$, we conclude that $c \Vdash [f] \leq [g] \vee [g] \leq [f]$.
- d_2 is finite. Then d_1 is infinite and covers c. Since $d_1 \Vdash [f] \leq [g]$, we conclude that $c \Vdash [f] \leq [g] \vee [g] \leq [f]$.
- d_1 and d_2 are both infinite. In this case, $\{d_1, d_2\}$ covers c, and $d_1 \Vdash [f] \leq [g]$ and $d_2 \Vdash [g] \leq [f]$. We conclude that $c \Vdash [f] \leq [g] \vee [g] \leq [f]$.

So we always have $c \Vdash [f] \leq [g] \vee [g] \leq [f]$. Hence $c \Vdash \forall x \forall y[x \leq y \vee y \leq x]$. □

In [Moerdijk95, Lemma 2.1, Proposition 2.2], Moerdijk proves the Peano axioms rather easily by linking truth in his model to truth in the ordinary natural numbers. We do the same here.

Lemma 3.2.4. *For every $c \in \mathbb{P}$, we have:*
$c \Vdash \phi([f_1], \ldots, [f_k])$ iff for all but finitely many $n \in c$, $\phi(f_1(n), \ldots, f_k(n))$ is true in the ordinary natural numbers.

Proof. We prove this by induction on the complexity of formulas.

- For atomic formulas, this is the very definition (see definition 3.2.2)
- Conjunction: follows immediately.
- Disjunction:

($\Rightarrow$) : Suppose $c \Vdash \phi([f_1], \ldots, [f_k]) \vee \psi([f_1], \ldots, [f_k]$. Then by sheaf semantics, there is a cover $S = \{d_i \mid i \in I\}$ of c such that for every $d_i \in S$

$$d_i \Vdash \phi([f_1], \ldots, [f_k]) \quad \text{or} \quad d_i \Vdash \psi([f_1], \ldots, [f_k]).$$

By the induction hypothesis, we know that this is the case iff for all but finitely many $n \in d_i$ respectively

$$\phi(f_1(n), \ldots, f_k(n)) \quad \text{or} \quad \psi(f_1(n), \ldots, f_k(n)).$$

But then (in both cases) also for all but finitely many $n \in d_i$

$$\phi(f_1(n), \ldots, f_k(n)) \vee \psi(f_1(n), \ldots, f_k(n)).$$

As S is a cover of c, leaving out only finitely many $n \in c$, we may conclude that for all but finitely many $n \in c$, $\phi(f_1(n), \ldots, f_k(n)) \vee \psi(f_1(n), \ldots, f_k(n))$.

($\Leftarrow$) : Suppose that for all but finitely many $n \in c$, $\phi(f_1(n), \ldots, f_k(n)) \vee \psi(f_1(n), \ldots, f_k(n))$. Define d_1 and d_2 as follows:

$$d_1 = \{n \in c \mid \phi(f_1(n), \ldots, f_k(n))\}$$
$$d_2 = \{n \in c \mid \psi(f_1(n), \ldots, f_k(n))\}$$

Then d_1 and d_2 are not both finite and

$$\{d_i \in \{d_1, d_2\} \mid d_i \text{ is infinite}\}$$

covers c. By definition of d_1 and d_2 we have for all but finitely many $n \in d_1$

$$\phi(f_1(n), \ldots, f_k(n)),$$

and for all but finitely many $n \in d_2$

$$\psi(f_1(n), \ldots, f_k(n)).$$

(Actually, in both cases even 'for all n'.) By the induction hypothesis, we then have:

$$d_1 \Vdash \phi([f_1], \ldots, [f_k]) \quad \text{and} \quad d_2 \Vdash \psi([f_1], \ldots, [f_k]).$$

As $\{d_i \in \{d_1, d_2\} \mid d_i \text{ is infinite}\}$ covers c, we then also know (because of sheaf semantics) that $c \Vdash \phi([f_1], \ldots, [f_k]) \vee \psi([f_1], \ldots, [f_k])$.

- Implication:

($\Rightarrow$) : Suppose $c \Vdash \phi([f_1], \ldots, [f_k]) \to \psi([f_1], \ldots, [f_k])$. Then by sheaf semantics, for all $d \leq c$

$$d \Vdash \phi([f_1], \ldots, [f_k]) \quad \text{implies} \quad d \Vdash \psi([f_1], \ldots, [f_k]).$$

By the induction hypothesis, this happens iff for all but finitely many $n \in d$

$$\phi(f_1(n), \ldots, f_k(n)), \tag{3.7}$$

implies for all but finitely many $n \in d$

$$\psi(f_1(n), \ldots, f_k(n)). \tag{3.8}$$

We need to show that for all but finitely many $n \in c$

$$\phi(f_1(n), \ldots, f_k(n)) \to \psi(f_1(n), \ldots, f_k(n)). \tag{3.9}$$

Let:

$$d = \{n \in c \mid \phi(f_1(n), \ldots, f_k(n))\}.$$

If d is finite, then the implication (3.9) is trivially true for all but finitely many $n \in c$. So suppose d is infinite. By definition of d: for all $n \in d$

$$\phi(f_1(n), \ldots, f_k(n)).$$

Also $d \leq c$, so by (3.7) and (3.8): for all but finitely many $n \in d$

$$\psi(f_1(n), \ldots, f_k(n)).$$

Hence for all but finitely many $n \in d$

$$\phi(f_1(n), \ldots, f_k(n)) \to \psi(f_1(n), \ldots, f_k(n)).$$

And therefore also for all but finitely many $n \in c$, $\phi(f_1(n), \ldots, f_k(n)) \to \psi(f_1(n), \ldots, f_k(n))$. (This implication is trivial for all $n \in c$ that are not in d.)

($\Leftarrow$) : Suppose for all but finitely many $n \in c$

$$\phi(f_1(n), \ldots, f_k(n)) \to \psi(f_1(n), \ldots, f_k(n)).$$

Let $d \leq c$ be chosen arbitrarily and suppose $d \Vdash \phi([f_1], \ldots, [f_k])$. Then by the induction hypothesis, for all but finitely many $n \in d$, $\phi(f_1(n), \ldots, f_k(n))$ is true. For all (but possibly finitely many of) these n, we also have:

$$\phi(f_1(n), \ldots, f_k(n)) \to \psi(f_1(n), \ldots, f_k(n)).$$

Hence for all but finitely many $n \in d$ we conclude $\psi(f_1(n), \ldots, f_k(n))$. By induction hypothesis again: $d \Vdash \psi([f_1], \ldots, [f_k])$. Hence for all $d \leq c$

$$d \Vdash \phi([f_1], \ldots, [f_k]) \quad \text{implies} \quad d \Vdash \psi([f_1], \ldots, [f_k]).$$

Thus by sheaf semantics: $c \Vdash \phi([f_1], \ldots, [f_k]) \to \psi([f_1], \ldots, [f_k])$.

- Negation: follows trivially.
- Existential quantification:

($\Rightarrow$) : Suppose $c \Vdash \exists x[\phi(x, [f_1], \ldots, [f_k])]$. Then according to sheaf semantics there exists a cover $S = \{d_i \mid i \in I\}$ and for each d_i, an element $[g_i] \in \mathbb{N}^{\mathbb{N}}/_{\sim}(d_i)$ such that

$$d_i \Vdash \phi([g_i], [f_1], \ldots, [f_k]).$$

By the induction hypothesis we have that for all but finitely many $n \in d_i$

$$\phi(g_i(n), f_1(n), \ldots, f_k(n)).$$

Now define the following function, $g : \mathbb{N} \to \mathbb{N}$, by:

$$g(n) = \begin{cases} g_i(n) & \text{if } n \in d_i \\ 0 & \text{if } n \notin \bigcup_{d_i \in S} d_i \end{cases}$$

Then, for all but finitely $n \in c$

$$\phi(g(n), f_1(n), \ldots, f_k(n)).$$

Hence for all but finitely $n \in c$: $\exists x[\phi(x, f_1(n), \ldots, f_k(n))]$
(namely: $x = g(n)$).

($\Leftarrow$) : Suppose that for all but finitely many $n \in c$: $\exists x[\phi(x, f_1(n), \dots, f_k(n))]$. Let:

$$d = \{n \in c \mid \exists x[\phi(x, f_1(n), \dots, f_k(n))]\}.$$

Then $\{d\}$ is a cover for c. For each $n \in d$, pick an m_n such that

$$\phi(m_n, f_1(n), \dots, f_k(n))$$

(we need countable choice here). Then define $[g]$:

$$g(n) = \begin{cases} m_n & \text{if } n \in d \\ 0 & \text{else} \end{cases}$$

Then, for all $n \in d$

$$\phi(g(n), f_1(n), \dots, f_k(n)).$$

Hence by the induction hypothesis

$$d \Vdash \phi([g], [f_1], \dots, [f_k]).$$

Therefore: there is a cover of c (namely $\{d\}$) and there is an element of $\mathbb{N}^{\mathbb{N}}/_{\sim}(d)$ (namely $[g]$) such that

$$d \Vdash \phi([g], [f_1], \dots, [f_k]).$$

Hence by sheaf semantics: $c \Vdash \exists x[\phi(x, [f_1], \dots, [f_k])]$

- Universal quantification:

($\Rightarrow$) : Suppose $c \Vdash \forall x[\phi(x, [f_1], \dots, [f_k])]$. Then (resulting from sheaf semantics applied to the identity arrow) for all $[g] \in \mathbb{N}^{\mathbb{N}}/_{\sim}(c)$

$$c \Vdash \phi([g], [f_1], \dots, [f_k]).$$

By the induction hypothesis, we have for all g, for all but finitely many $n \in c$

$$\phi(g(n), f_1(n), \dots, f_k(n)).$$

We need to show that for all but finitely many $n \in c$

$$\forall x[\phi(x, f_1(n), \dots, f_k(n))].$$

Suppose this is not the case. Suppose there are infinitely many $n \in c$ such that $\neg\forall x[\phi(x, f_1(n), \dots, f_k(n))]$. Define:

$$d = \{n \in c \mid \neg\forall x[\phi(x, f_1(n), \dots, f_k(n))]\}.$$

Then for each $n \in d$, we can pick (using countable choice) an m_n such that

$$\neg\phi(m_n, f_1(n), \dots, f_k(n)).$$

Then define $g : c \to \mathbb{N}$ as follows:

$$g(n) = \begin{cases} m_n & \text{if } n \in d \\ 0 & \text{else} \end{cases}$$

By definition of g, we have that for all $n \in d$, and hence for infinitely many $n \in c$

$$\neg\phi(g(n), f_1(n), \dots, f_k(n)).$$

But this contradicts the fact that for all g, for all but finitely many $n \in c$

$$\phi(g(n), f_1(n), \dots, f_k(n)).$$

Hence we must have that for all but finitely many $n \in c$

$$\forall x[\phi(x, f_1(n), \dots, f_k(n))].$$

$(\Leftarrow)$: Suppose that for all but finitely many $n \in c$, $\forall x[\phi(x, f_1(n), \dots, f_k(n))]$. Then we also have that for all $[g] \in \mathbb{N}^{\mathbb{N}}/_{\sim}(c)$, for all but finitely many $n \in c$

$$\phi(g(n), f_1(n), \dots, f_k(n)).$$

Hence by induction hypothesis, we have that for all $[g]$,

$$c \Vdash \phi([g], [f_1], \dots, [f_k]).$$

Hence $c \Vdash \forall x[\phi([x], [f_1], \dots, [f_k])]$.

□

We use this lemma to prove that the Peano axioms are valid in our model.

Proposition 3.2.5. *The following versions of the Peano axioms hold:*

1. 0 *is a number*
2. *The successor of a number is again a number*
3. $c \Vdash \forall x[s_c(x) \neq 0]$
4. $c \Vdash \forall x \forall y[s_c(x) = s_c(y) \to x = y]$
5. *External induction: For all formulas* Φ
   $c \Vdash (\Phi(0) \wedge \forall^{\mathrm{St}} x[\Phi(x) \to \Phi(s_c(x))]) \to \forall^{\mathrm{St}} x[\Phi(x)]$
6. *Internal induction: for all internal formulas* ϕ
   $c \Vdash (\phi(0) \wedge \forall x[\phi(x) \to \phi(s_c(x))]) \to \forall x[\phi(x)])$

Proof. The constant function $const_0$ is the zero in our model. The successor of a number was defined above and is hence also a number. Also 3 and 4 follow trivially from the definition of the successor function, external induction is immediate by the isomorphism between the natural numbers object and $\mathrm{St}_{\mathbb{N}}$. For internal induction, we need lemma 3.2.4:

Let ϕ be any internal formula. Define:

$$\phi' := (\phi(0) \wedge \forall x[\phi(x) \rightarrow \phi(s_c(x))]) \rightarrow \forall x[\phi(x)].$$

By lemma 3.2.4 $c \Vdash \phi'$ iff for all but finitely many $n \in c$, ϕ' is true (interpreted in the normal natural numbers). This is the normal induction axiom for the natural numbers, hence ϕ' is true, from which we conclude $c \Vdash \phi'$, proving internal induction.

□

The Peano axioms are valid, we have indeed found a model of natural arithmetic. Is it, however, a non-standard model? That is, are there any non-standard elements in our model? And if so, how do they behave with respect to the standard elements?

Proposition 3.2.6. *For all $c \in \mathbb{P}$:*

1. $c \Vdash \exists x[\mathrm{Inf}(x)]$
2. $c \Vdash \forall x[\mathrm{Inf}(x) \leftrightarrow \neg \mathrm{St}(x)]$
3. $c \Vdash \forall x[\mathrm{St}(x) \leftrightarrow \neg \mathrm{Inf}(x)]$
4. $c \Vdash \forall x \forall y[x \leq y \wedge \mathrm{St}(y) \rightarrow \mathrm{St}(x)]$
5. $c \Vdash \forall x \forall y[x \leq y \wedge \mathrm{Inf}(x) \rightarrow \mathrm{Inf}(y)]$
6. $c \Vdash \forall y[\mathrm{Inf}(y) \rightarrow \exists x(\mathrm{Inf}(x) \wedge x < y)]$

Proof.

1. The equivalence class of the identity function $\mathrm{Id} : n \mapsto n$ is larger than all standard natural numbers: Take any standard natural number $[f]$, then the image of f is finite. Hence there is an N such that $f(n) \leq N$ for all $n \in \mathbb{N}$. As there are only finitely many n for which $\mathrm{Id}(n) \leq N$, we have that $f(n) \leq \mathrm{Id}(n)$ for all but finitely many n, hence $[f] \leq [\mathrm{Id}]$ and $[\mathrm{Id}]$ is infinite.

2. The direction from left to right is trivial. The other direction is not immediately clear. Suppose $c \Vdash \neg \mathrm{St}([f])$. We prove by external induction that $c \Vdash \mathrm{Inf}([f])$, that is
$$c \Vdash \forall x[\mathrm{St}(x) \rightarrow x \leq [f]].$$

The basis $0 \leq [f]$ is immediately clear. For the induction step, suppose $[g]$ is a standard natural number and $c \Vdash [g] \leq [f]$. We need to show that

$$c \Vdash s_c([g]) \leq [f].$$

Consider:

$$\{n \in c \mid f(n) < s_c([g])(n)\}.$$

We need to show that this set is finite. Notice that:

$$\{n \in c \mid f(n) < s_c([g])(n)\} = \{n \in c \mid f(n) < g(n)\} \cup \{n \in c \mid f(n) = g(n)\}$$

The former set is finite, as we know $c \Vdash [g] \leq [f]$. Suppose that the latter set is infinite and call it d. Then $d \Vdash [g] = [f]$, and hence $d \Vdash \mathrm{St}([f])$. But we know by monotonicity of sheaf semantics that for all $d' \leq c$ we must have $d' \Vdash \neg\,\mathrm{St}([f])$. Therefore, we must have that d is finite, proving $s_c([g])(n) \leq f(n)$ for all but finitely many $n \in c$, which means

$$c \Vdash s_c([g]) \leq [f].$$

By external induction, we then have $c \Vdash \mathrm{Inf}([f])$.

3. Again, the direction from left to right is trivial. For the other direction, pick any $[f] \in \mathbb{N}^{\mathbb{N}}(c)$. Suppose $c \Vdash \neg\,\mathrm{Inf}([f])$. Then, for all $d \leq c$

$$d \Vdash \neg \forall^{\mathrm{St}} x [x \leq [f]].$$

We define, for each $N \in \mathbb{N}$, the following subsets of $\mathbb{N}$:

$$d_N = \{n \in c \mid f(n) \leq N\}$$

Claim: $S = \{d_N \mid N \in \mathbb{N}\}$ is a cover of c. That is: there are finitely many $d_N \in S$ such that their union is almost c. Proof of this claim: suppose not. Then for each d_N, there are infinitely many $n \in c$ which are not in d_N (otherwise, d_N would itself be a cover of c). For each N, pick an n_N such that

$$n_N \notin d_N$$
$$n_N \notin \{n_M \mid M < N\}.$$

Then, let:

$$d = \{n_N \mid N \in N\}.$$

Then d is infinite, and $d \Vdash \mathrm{Inf}([f])$: take any standard natural number $[g]$, then there is an N such that $g(n) < N$ for all $n \in d$. By construction of d, there are only finitely many $n \in d$ for which $f(n)$ might be smaller than $g(n)$: n_M for which $M < N$. for all other $n \in d$ we have per definition that $f(n) > N > g(n)$. Hence

$$d \Vdash [f] > [g].$$

But this is in contradiction with the fact that for all $d' \leq c$

$$d' \Vdash \neg\forall^{\mathrm{St}} x[x \leq [f]].$$

Therefore, S must be a cover of c. This means that there are finitely many d_N such that their union is almost c. That, in turn, implies that the image of f is finite. Hence $c \Vdash \mathrm{St}([f])$.

4. Almost trivial, proof left to the reader.

5. By transitivity of $\leq$.

6. Let $[f]$ be any infinite number. Now define f_{-1} as follows:

$$f_{-1}(n) = \begin{cases} f(n) - 1 & \text{if } f(n) \neq 0 \\ 0 & \text{else} \end{cases}$$

As $[f]$ is infinite, $f(n) = 0$ for at most finitely many (n), so we immediately have $[f_{-1}] < [f]$. It remains to show that $[f_{-1}]$ is infinite. Let $[g]$ be any standard natural number. Then $[s_c(g)]$ is also a standard natural number, hence $[s_c(g)] \leq [f]$ as $[f]$ is infinite. But then also $[g] \leq [f_{-1}]$ by definition of f_{-1}, proving $[f_{-1}]$ is infinite as well. □

Non-standard principles (overspill, underspill, and more)

As mentioned in the introduction, the overspill and underspill principles follow from the fact that at the level of first order logic, it is impossible to discriminate the standard numbers from the non-standard numbers of the model. Here, we show that these principles are indeed valid in our model. In addition, we show that transfer, idealisation and realisation hold.

Proposition 3.2.7. *Overspill*
The following overspill principle is true for any $c \in \mathbb{P}$

$$c \Vdash \forall^{\mathrm{St}} x[\phi(x, [f_1], \ldots, [f_k])] \rightarrow \exists y[\neg \mathrm{St}(y) \wedge \phi(y, [f_1], \ldots, [f_k])]$$

Proof. Suppose that $c \Vdash \forall^{\mathrm{St}} x[\phi(x, [f_1], \ldots, [f_k])]$. Then in particular, for all $const_m$

$$c \Vdash \phi([const_m], [f_1], \ldots, [f_k]).$$

By lemma 3.2.4, we hence have that for all $m \in \mathbb{N}$,

$$\phi(m, f_1(n), \ldots, f_k(n))$$

is true for all but finitely many $n \in c$. That is, for each $m \in \mathbb{N}$, the set

$$A_m = \{n \in c \mid \phi(m, f_1(n), \ldots, f_k(n))\}$$

is cofinite.

We need to construct a function $f : c \to \mathbb{N}$ such that $c \Vdash \neg\,\mathrm{St}([f])$ and

$$\{n \in c \mid \phi(f(n), f_1(n), \dots f_k(n))\}$$

is cofinite. We define $f : c \to \mathbb{N}$ inductively. Let $c_o < c_1 < \dots$ be the order-preserving enumeration of c and suppose f has been defined for all $c_i \leq c_n$. Consider the following two sets:

$$B_{c_{n+1}} = \{m \in \mathbb{N} \mid c_{n+1} \in A_m \text{ and } m > f(c_n)\}$$
$$C_{c_{n+1}} = \{m \in \mathbb{N} \mid c_{n+1} \in A_m\}$$

Then we define $f(c_{n+1})$ as:

$$f(c_{n+1}) = \begin{cases} m & \text{if } B_{c_{n+1}} \neq \emptyset \text{ and } m = \inf(B_{c_{n+1}}) \\ m' & \text{if } B_{c_{n+1}} = \emptyset, C_{c_{n+1}} \neq \emptyset \text{ and } m' = \sup(C_{c_{n+1}}) \\ 0 & \text{else} \end{cases}$$

Notice that for each $m \in \mathbb{N}$, we can have $f(n) = m$ for only finitely many $n \in c$. We prove this fact by induction:

$f(n) = 0$ only if C_n is empty or the singleton set. If either of those happens infinitely often, then there are infinitely many n such that $n \notin A_m$ for each $m \neq 0$. This contradicts the fact that each A_m is cofinite. Now suppose that for all $m' < m$, we know that $f(n) = m'$ for only finitely many $n \in c$. Then $f(n) = m$ can happen for two reasons:

1. $m \in B_n$. As we need $f(n-1)$ to be smaller than m for this to happen, this case occurs only finitely many times.

2. $m \in C_n$ and not in B_n. If this is the case, then m is the largest number for which $n \in A_m$. If there are infinitely many n for which m is the largest number for which $n \in A_m$, then for each $m'' > m$, there would be infinitely many $n \notin A_{m''}$. This contradicts the fact that $A_{m''}$ is cofinite, so also this case occurs only finitely many times.

Hence for each $m \in N$, $f(n) = m$ for only finitely many $n \in c$. This proves that is $[f]$ is non-standard. What is left to show is that

$$\{n \in c \mid \phi(f(n), f_1(n), \dots f_k(n))\}$$

is cofinite. By construction, the only n for which $\phi(f(n), f_1(n), \dots f_k(n))$ might not hold, are the ones such that $f(n) = 0$. We have just argued that these are only finitely many n, which proves the cofiniteness of $\{n \in c \mid \phi(f(n), f_1(n), \dots f_k(n))\}$.

□

Proposition 3.2.8. *Underspill*
The following underspill principle is true for any $c \in \mathbb{P}$

$$c \Vdash \forall x[\neg\,\mathrm{St}(x) \to \phi(x, [f_1], \dots, [f_k])] \to \exists^{\mathrm{St}} y[\phi(y, [f_1], \dots, [f_k])]$$

Proof. Suppose that $c \Vdash \forall x[\neg \mathrm{St}(x) \rightarrow \phi(x, [f_1], \ldots, [f_k])]$. As the identity function is non-standard, we may conclude that:

$$c \Vdash \phi([\mathrm{Id}], [f_1], \ldots, [f_k]).$$

By lemma 3.2.4, this means that for all but finitely many $n \in c$

$$\phi(n, f_1(n), \ldots, f_k(n)).$$

Define:

$$A_n = \{m \in \mathbb{N} \mid \phi(m, f_1(n), \ldots, f_k(n))\}.$$

Then A_n is non-empty for almost all $n \in C$. Define $f : c \to \mathbb{N}$ as follows:

$$f(n) = \begin{cases} \min(A_n) & \text{if } A_n \neq \emptyset \\ 0 & \text{else} \end{cases}$$

Claim:

$$c \Vdash \mathrm{St}([f]) \quad \text{and} \quad c \Vdash \phi([f], [f_1], \ldots, [f_k]).$$

The second part of the claim is immediate by definition, so lets concentrate on the first part. For $[f]$ to be standard, it needs to have a finite image. So suppose $\mathrm{Im}(f)$ is infinite. Then for each $N \in \mathbb{N}$

$$B_N := \{n \in c \mid f(n) > N\}$$

is infinite. Now pick the following elements:

$$\begin{aligned} &n_0 \in B_0 \\ &n_1 \in B_1 / \{n_0\} \\ &n_2 \in B_2 / \{n_0, n_1\} \\ &\quad \vdots \end{aligned}$$

And let:

$$d = \{n_N \mid N \in \mathbb{N}\}.$$

Then $d \leq c$, hence by monotonicity of sheaf semantics $d \Vdash \forall x[\neg \mathrm{St}(x) \rightarrow \phi(x, [f_1], \ldots, [f_k])]$. Now define $g : d \to \mathbb{N}$:

$$g(n_N) = N$$

Then we have the following facts about g, which are both true by construction:

- $d \Vdash \neg \mathrm{St}([g])$
- $\phi(g(n), f_1(n), \ldots, f_k(n))$ is not true for any n.

To see the latter fact: suppose $g(n) = N$. Then we must have $n = n_N$. That is: $n \in B_N$. By definition of B_N, we have that

$$\phi(N, f_1(n'), \dots, f_k(n'))$$

is *not* true for any $n' \in B_N$, hence

$$\phi(g(n), f_1(n), \dots, f_k(n))$$

is not true. But then, by lemma 3.2.4, we also do not have that

$$d \Vdash \phi([g], [f_1], \dots, [f_k])],$$

while $[g]$ is nonstandard. This is contradiction with the fact that $d \Vdash \forall x[\neg \mathrm{St}(x) \to \phi(x, [f_1], \dots, [f_k])]$. Hence $\mathrm{Im}(f)$ must be finite: $c \Vdash \mathrm{St}(f)$. This proves the underspill principle. □

We now prove the transfer principle. This principle expresses that the embedding of the natural numbers object into our model is elementary[Berg12]: any formula ϕ is true in our model if and only if it is true in the natural numbers object. ϕ can have parameters, but only standard ones, otherwise it would make no sense to interpret ϕ in the natural numbers object.

Proposition 3.2.9. *Transfer*
There are two ways to formalise the transfer principle[3]*, and they both hold in our model:*

1. $c \Vdash \forall^{\mathrm{St}} y_1, \dots, y_k \; [\; \forall^{\mathrm{St}} x \; \phi(x, y_1, \dots, y_k) \to \forall x \; \phi(x, y_1, \dots, y_k) \;]$
2. $c \Vdash \forall^{\mathrm{St}} y_1, \dots, y_k \; [\; \exists x \; \phi(x, y_1, \dots, y_k) \to \exists^{\mathrm{St}} x \; \phi(x, y_1, \dots, y_k) \;]$

Before proving the transfer principle, we consider a lemma that shows that whenever we talk about *all* standard numbers, it is enough to only consider the constant functions $const_m$. This greatly simplifies the proof of proposition 3.2.9.

Lemma 3.2.10. $c \Vdash \forall^{st} x \; [\Phi(x, [f_1], \dots, [f_k])] \iff c \Vdash \Phi([const_m], [f_1], \dots, [f_k])$ *for all constant functions* $const_m$.

Proof. The direction from left to right is trivial, so for the other direction: suppose that $c \Vdash \Phi([const_m], [f_1], \dots, [f_k])$ for all constant functions $const_m$. Let $[g]$ be any element of $\mathbb{N}^{\mathbb{N}}/_{\sim}(c)$ such that $c \Vdash \mathrm{St}([g])$. As $[g]$ has a finite image, there is a cover $\{d_i \mid i \in I\}$ such that $[g \restriction d_i]$ is constant. By monotonicity of sheaf semantics and our assumption that $c \Vdash \Phi([const_m], [f_1], \dots, [f_k])$, we know $d_i \Vdash \Phi([g \restriction d_i], [f_1], \dots, [f_k])$ for each d_i in the cover. By locality of sheaf semantics, we may then conclude that also $c \Vdash \Phi([g], [f_1], \dots, [f_k])$. □

We return to the transfer principle. We use the previous lemma to formulate the two principles slightly differently:

[3]In [Nelson77] they are treated as equivalent, and classically, they are. But as [Berg12] mentions, intuitionistically, they are not. As the internal logic of sheaves is constructive, we choose to treat them separately.

Proposition 3.2.11. *Transfer principle 2.0*
Let $const_{m_1}, \ldots, const_{m_k}$ be any constant functions. Then:

1. $c \Vdash \forall^{\mathrm{St}} x\ \phi(x, [const_{m_1}], \ldots, [const_{m_k}]) \rightarrow \forall x\ \phi(x, [const_{m_1}], \ldots, [const_{m_k}])$
2. $c \Vdash \exists x\ \phi(x, [const_{m_1}], \ldots, [const_{m_k}]) \rightarrow \exists^{\mathrm{St}} x\ \phi(x, [const_{m_1}], \ldots, [const_{m_k}])$

Proof.

1. Suppose that for all standard natural numbers x

$$c \Vdash \phi(x, [const_{m_1}], \ldots, [const_{m_k}]).$$

Then in particular for any constant function $const_m$

$$c \Vdash \phi([const_m], [const_{m_1}], \ldots, [const_{m_k}]).$$

By lemma 3.2.4, this is only true if for all but finitely many $n \in c$

$$\phi(const_m(n), const_{m_1}(n), \ldots, const_{m_k}(n)).$$

That is, we have that for all $m \in \mathbb{N}$

$$\phi(m, m_1, \ldots, m_k).$$

Now let $[g]$ be any element of $\mathbb{N}^{\mathbb{N}}/_{\sim}(c)$. We need to show that

$$c \Vdash \phi([g], [const_{m_1}], \ldots, [const_{m_k}]),$$

or, by lemma 3.2.4, that

$$\phi(g(n), m_1, \ldots, m_k)$$

is true for all but finitely many $n \in c$. The latter follows directly from the fact that $\phi(m, m_1, \ldots, m_k)$ is true for all $m \in \mathbb{N}$, so certainly for each $g(n)$.

2. Suppose that $c \Vdash \exists x\ \phi(x, [const_{m_1}], \ldots, [const_{m_k}])$. Then there is a cover $\{d_1, \ldots, d_l\}$ of c, together with functions $[g_1], \ldots, [g_l]$ such that:

$$d_i \Vdash \phi([g_i], [const_{m_1}], \ldots, [const_{m_k}]).$$

For each of these d_i, we have by lemma 3.2.4 that for all but finitely $n \in d_i$

$$\phi(g_i(n), m_1, \ldots, m_k).$$

Pick any $n \in d_i$ such that $\phi(g_i(n), m_1, \ldots, m_k)$ and call this number n_i. Then define $g : \mathbb{N} \rightarrow \mathbb{N}$ as follows:

$$g(n) = \begin{cases} g_i(n_i) & \text{if } n \in d_i \\ 0 & \text{if } n \notin d_1 \cup \ldots \cup d_l \end{cases}$$

Then g has a finite image, so that $c \Vdash \mathrm{St}([g])$ and by construction:

$$d_i \Vdash \phi([g \restriction d_i], [const_{m_1}], \ldots, [const_{m_k}]).$$

Hence by the local character of sheaf semantics:

$$c \Vdash \phi([g], [const_{m_1}], \ldots, [const_{m_k}]).$$

□

The next and last two principles we consider are each others dual: Idealisation and realisation. Idealisation has a compactness theorem-like feel: if for every finite sequence of standard natural numbers $n_1, \ldots, n_k$ there is an x such that $\phi(x, n_i)$ holds for any n_i in the sequence, then there is an x such that for $\phi(x, n)$ holds for any standard natural number n. Realisation states that if for all x there is a standard natural number n such that $\phi(x, n)$, then there exists a finite sequence of natural numbers $n_1, \ldots, n_k$ such that for all x, there is an n_i in this sequence such that $\phi(x, n_i)$.

Proposition 3.2.12. *Idealisation*
The following version of the Idealisation principle holds for all $c \in \mathbb{P}$:

$$c \Vdash \forall^{\mathrm{St}} x\ \exists y\ \forall z \leq x\ [\ \phi(y, z)\ ] \rightarrow \exists y'\ \forall^{\mathrm{St}} x\ [\ \phi(y', x)\ ]$$

Proof. We invoke lemma 3.2.10 to get the equivalent formulation:

$$c \Vdash \forall const_m\ \exists x\ \forall const_{m'} \leq const_m\ [\ \phi(x, [const_{m'}])\ ] \rightarrow \exists y\ \forall const_m\ [\ \phi(y, [const_m])\ ]$$

So suppose that

$$c \Vdash \forall const_m\ \exists x\ \forall const_{m'} \leq const_m\ [\ \phi(x, [const_{m'}])\ ].$$

With lemma 3.2.4 in our mind, we define A_k for each $k \in \mathbb{N}$:

$$A_k = \{n \in \mathbb{N} \mid \forall m' \leq k\ [\phi(n, m')]\}$$

Then our premiss gives us that A_k is never empty. For each k, pick an element $n_k \in A_k$. Then define $f : \mathbb{N} \rightarrow \mathbb{N}$ as follows:

$$f(k) = n_k$$

Then for all $const_m$ we have by definition that for all $k > m$, $\phi(n_k, m)$, that is:

$$\phi(f(k), const_m(k))$$

is true for all but finitely many $k \in \mathbb{N}$. Hence according to lemma 3.2.4, we have

$$c \Vdash \forall const_m[\ \phi([f], [const_m])\ ],$$

and hence:

$$c \Vdash \exists y\ \forall const_m\ [\ \phi(y, [const_m])\ ],$$

which proves Idealisation. □

Proposition 3.2.13. *Realisation*
The following version of the Realisation principle holds for all $c \in \mathbb{P}$:

$$c \Vdash \forall x\ \exists^{\mathrm{St}} y\ [\phi(x,y)] \to \exists^{\mathrm{St}} y\ \forall x\ \exists y' \leq y\ [\phi(x,y')]$$

Proof. Suppose that $c \Vdash \forall x\ \exists^{\mathrm{St}} y\ [\phi(x,y)]$. Define the following sets A_m:

$$A_m = \{n \in \mathbb{N} \mid \exists m' \leq m \text{ such that } \phi(n,m')\}$$

Then $A_m \subseteq A_{m+1}$ for all m. Claim: there exists an m such that $A_m = \mathbb{N}$. We prove this claim by contradiction: suppose that for each m, we could find an $k_m \in \mathbb{N}$ such that $k_m \notin A_m$. Then define $g : \mathbb{N} \to \mathbb{N}$ as follows:

$$g(n) = k_n$$

By our premiss, there exists a cover $d_1, \ldots, d_l$ and $[f_1], \ldots, [f_l]$ such that

$$d_i \Vdash \mathrm{St}([f_i]) \quad \text{and} \quad d_i \Vdash \phi([g],[f_i]).$$

And hence by lemma 3.2.4: for all but finitely many $n \in d_i$: $\phi(g(n), f_i(n))$. Define $f : \mathbb{N} \to \mathbb{N}$ as follows:

$$f(n) = \begin{cases} f_i(n) & \text{if } n \in d_i \\ 0 & \text{else} \end{cases}$$

As the image of each f_i is finite when restricted to d_i, the image of f is finite. We also have by construction that for all but finitely many $n \in c$, $\phi(g(n), f(n))$. Choose:

$$M \in \mathbb{N} \quad \text{such that} \quad f(n) < M \text{ for all } n \in \mathbb{N}.$$

As we have per construction that for all but finitely many $n \in c$: $\phi(g(n), f(n))$ and all $f(n) < M$, we may conclude that for all but finitely many $n \in c$, $g(n) \in A_M$. Remember that $g(n) = k_n$ is chosen in such a way that $g(n) \notin A_n$ and hence $g(n) \notin A_m$ for all $m \leq n$. So for every $n > M$, $g(n) \notin A_M$, which yields our contradiction.

So there exists an m such that $A_m = \mathbb{N}$. We need to show that there exists an $[f]$ with $c \Vdash \mathrm{St}([f])$, such that for all $[g]$ there exists an $f' \leq f$ such that $c \Vdash \phi([g],[f'])$. We let f be $const_m$, where m is such that $A_m = \mathbb{N}$. This fulfills the conditions: let $[g]$ be any function from $\mathbb{N}$ to $\mathbb{N}$. Then for each $n \in \mathbb{N}$, compute $g(n)$ and find $m'_n \leq m$ such that $\phi(g(n), m'_n)$ (As $A_m = \mathbb{N}$, this m'_n always exists). Then define f' as:

$$f'(n) = m'_n$$

As all $m'_n \leq m$, $c \Vdash \mathrm{St}([f'])$ and $c \Vdash [f'] \leq [const_m]$. By its construction (and via lemma 3.2.4), we also have that $c \Vdash \phi([g],[f'])$. This proves realisation. □

3.3 Conclusion and discussion

We found a non-standard model for natural arithmetic. This model, consisting of equivalence classes of functions from $\mathbb{N}$ to $\mathbb{N}$, models the Peano axioms as well as the non-standard principles overspill, underspill, transfer, idealisation and realisation. The natural numbers object, which consists of the functions from infinite subsets of $\mathbb{N}$ to $\mathbb{N}$ with finite image, is an elementary and strict subsheaf of our non-standard model.

The realisation principle, which we prove in proposition 3.2.13, can be extended to the *non-classical* realisation principle (or NCR) by allowing external formulas:

$$\forall x \; \exists^{\mathrm{St}} y \; [\Phi(x, y)] \rightarrow \exists^{\mathrm{St}} y \; \forall x \; \exists y' \leq y \; [\Phi(x, y')]$$

While realisation is classically equivalent to idealisation, NCR is incompatible with classical logic, as it implies the undecidability of the standard predicate:

$$\neg \forall x \; [\mathrm{St}(x) \;\vee\; \neg\, \mathrm{St}(x)]$$

For a nice proof of this fact, see proposition 3.5 in [Berg12]. It is still an open question whether NCR is valid in our model.

In proposition 3.2.6, we mention several relations between standard and infinite numbers in our model. Comparing this proposition to proposition 2.5 in [Moerdijk95], we see that 1,2,3,4 and 5 correspond to (iii), (i), (i), (vi) and (iv) respectively. We do not have clarity about (ii):

$$\neg \forall x \; [\mathrm{St}(x) \;\vee\; \mathrm{Inf}(x)]$$

(see the discussion on NCR above), but we have the additional 6:

$$\forall y \; [\mathrm{Inf}(y) \rightarrow \exists x \; (\mathrm{Inf}(x) \;\wedge\; x < y)].$$

However, our proofs of 2 and 3 use classical logic in the meta-theory, while Moerdijk is able to show everything constructively.

Our use of classical logic as meta-theory is not restricted to proposition 3.2.6. For example, the existential and universal cases and the case of disjunction of lemma 3.2.4 depend on the (countable) axiom of choice as well as the law of the excluded middle. To our regret, we believe that constructive versions of these proofs are unlikely to be found in the current setting. As is shown by Moerdijk and Palmgren in [Moerdijk97, proposition 2.1], the transfer principle together with the seven axioms of *HAI* (all of which are valid in our model) imply the law of the excluded middle. Therefore, it should not be possible to prove of all our results using a constructive meta-theory.

We frequently mention that sets are 'infinite' or 'finite', assuming that it is clear what finite sets and infinite sets are. This is of course a very classical approach. Proofs using the fact a certain set is finite should be treated with much more care if they should be fully constructive.

Moerdijk heavily uses the structure of his site to keep his proofs constructive. In proving his lemma 2.1 (our lemma 3.2.4), as well as in proving the overspill principle, he uses the fact that he can define suitable covers by taking clever products and then using the projection arrows as covering family. This construction makes both the use of the axiom of choice (as in the existential case of lemma 3.2.4) and the use of case-distinction (such as in the disjunction case of lemma 3.2.4) unnecessary.

Our site only has inclusion functions as morphisms, therefore such an approach is not possible in our current setting. In the next chapter we try to find a site that is still less complicated than the category of filters used by Moerdijk, but with enough structure so that we can profit from Moerdijk's proof methods.

Chapter 4

An intermediate model

Several proofs in the previous chapter are non-constructive, and we have little hope of finding constructive versions of these proofs. This is in great contrast to the work of Moerdijk [Moerdijk95] and Palmgren [Palmgren01], who are able to prove everything in a constructive setting. Moerdijk uses the structure of his site in many of his proofs (as for example, in [Moerdijk95, lemma 2.1]). The site discussed in the previous chapter, the poset of infinite subsets of the natural numbers, is not rich enough to be used for such purposes. In this chapter, we add more structure to our site in the hope that we can mimic his proof methods, while staying as close to the model $\mathbb{N}^{\mathbb{N}}/_{\sim}$ as possible. In doing so, we constructively find the natural numbers object of the new category of sheaves and a candidate non-standard model that closely resembles the non-standard model of the previous chapter. We are able to prove the equivalent of Moerdijk's lemma 2.1 constructively for quantifier-free formulas. However, for formulas containing quantifiers we still need both the axiom of choice and a proof by contradiction. The other results of chapter 3 (proposition 3.2.6 and the non-standard principles overspill, underspill, transfer, idealisation and realisation) are still true for the new non-standard model, but we did not succeed in making their proofs (more) constructive.

We again consider the steps from section 2.3:

Step 1: We take the full powerset of the natural numbers as base category. The morphisms in this category are not only the inclusion functions anymore. We also allow as many functions as possible, to provide a richer structure. However, just like Moerdijk, we have to impose some continuity condition on our morphisms, because we like to keep keep our covering condition (which is reformulated in Step 2 below). The covering condition would not yield a Grothendieck topology if we would allow *all* functions as morphisms. The following example illustrates this:

Example 4.0.1. Recall definition 2.2.3, where a Grothendieck topology was defined in terms of covering families. Such a topology had to satisfy three conditions, including stability and transitivity. It is the stability axiom that fails if we would allow all functions as morphisms. To see this, consider:

$$\begin{aligned}
c &= \mathbb{N} \\
d &= \{n \in \mathbb{N} \mid n \text{ is even } \} \\
S &= \text{ any covering family } \{f_i : d_i \to c\} \text{ such that } 0 \notin \bigcup_{f_i \in S} \mathrm{Im}(f_i) \\
f &: d \to c \\
&: n \mapsto 0
\end{aligned}$$

According to the stability axiom, the family of pullbacks

$$\{\pi_2 : d_i \times_c d \to d \mid d_i \text{ is the domain of } f_i \in S\}$$

should be a cover of d. However, $d_i \times_c d = \emptyset$ (as $0 \notin \mathrm{Im}(f_i)$ for all $f_i \in S$), so this set only contains the empty function, which is not a cover of d at all: the empty function includes none of the elements of d in its image, which means the image misses more than finitely many elements of d.

So the stability axiom is violated by functions that map infinitely many elements to only a finite set. A cover missing this finite set (which is allowed) then causes trouble. To prevent such situations, we say that the morphisms in our site are only those functions whose inverse images preserve finite sets. This is our continuity requirement.

The base category of our new site will hence consist of the powerset of $\mathbb{N}$ as objects, and equivalence classes of 'continuous' functions (inverse images of finite sets need to be finite) as morphisms. The equivalence relation is again $f \sim g$ iff $\{n \mid f(n) \neq g(n)\}$ is finite.

Step 2: Due to the extra functions in our base category the covering condition would now be formulated as: a family of morphisms $\{f_i : d_i \to c\}$ covers c if all but finitely many elements of c are in the union of the images of the f_i. However, this time, the transitivity axiom forces us to reconsider. When we leave the covering condition as it is currently formulated, we are not able to prove the transitivity axiom constructively.

Recall that the original covering condition for the site $(\mathbb{P}, J)$ was based on the observation that it resembled the covering condition of Moerdijk when his category of filters was restricted to only the Frechet filters. This resemblance is not an exact equality. Moerdijk's condition is stronger. Where we only require the union of the images of the covering family to be cofinite in c, Moerdijk requires that for every family of cofinite subsets $\{e_i \subseteq d_i\}$, the set

$$\bigcup_{i=1}^{k} f_i(e_i)$$

is still cofinite in c. Notice that by taking only $e_i = d_i$, we get the original formulation of the covering condition back.

We will take this translation of Moerdijk's covering condition as the new covering condition.

As we now intent to present everything in a fully intuistionistic setting, we must be very precise when we speak about 'finite' and 'infinite' sets. The usual intuitionistic interpretation of a finite set is that there is an initial segment of the natural numbers such that the finite set is in bijection with this initial segment. This, however, is too strong for our purposes. We therefore speak of *bounded* sets, which we define as:

Definition 4.0.1. *Bounded* and *unbounded* sets
A subset of the natural numbers is *bounded*, if there exists a natural number N such that every element of that subset is smaller than or equal to N.

A subset of the natural numbers is *unbounded* if for every natural number N, there is a natural number larger than N that is an element of the set.

Formalising the ideas mentioned above, we define our new site as follows: NB: From now on, we print the morphisms of our site in boldface, to avoid confusion later on with functions that live in different categories.

Definition 4.0.2. *The site* $(\mathcal{P}(\mathbb{N}), J')$

- **Objects:** All subsets of the natural numbers.
- **Morphisms:** A morphism between two objects $d, c \in \mathcal{P}(\mathbb{N})$ is an equivalence class of functions, $[\mathbf{f}] : d \to c$, where:

  $$\mathbf{f} \sim \mathbf{g} \iff \exists N \; \forall n \geq N \; [n \in d \to \mathbf{f}(n) = \mathbf{g}(n)]$$

  (**f** and **g** differ only on a bounded set)
  Such that:
  $$\forall N \in c \; \exists M \in d \; \forall m \in d \; [m > M \to \mathbf{f}(m) \geq N]$$

  (inverse images of bounded sets are bounded)
- **Covering families:** A family of morphisms S is covering c iff there are $d_1, \ldots, d_k \in \mathcal{P}(\mathbb{N})$ and $[\mathbf{f_i}] : d_i \to c \in S$ such that for all $M_1, \ldots, M_k$:

  $$\exists N \; \forall n \geq N \; \exists i \leq k \; \exists m \in d_i \; [n \in c \to (m \geq M_i \; \wedge \; \mathbf{f_i}(m) = n)]$$

  (the elements of c that are not included in the union of the images form a bounded set, and the described cofinite requirement holds).

We give the Grothendieck topology in terms of covering families rather than covering sieves. We may only do this if all pullbacks exist.

Proposition 4.0.1. *All pullbacks exist in $\mathcal{P}(\mathbb{N})$.*

Proof. Let $[\mathbf{f}] : d_1 \to c$ and $[\mathbf{g}] : d_2 \to c$ be two morphisms in the category.

$$\begin{tikzcd} & d_2 \arrow[d, "{[\mathbf{g}]}"] \\ d_1 \arrow[r, "{[\mathbf{f}]}"'] & c \end{tikzcd}$$

Then the following set, together with the following projection functions $[\boldsymbol{\pi}_1]$ and $[\boldsymbol{\pi}_2]$ form the pullback of the diagram above:

$$P = \{p \mid p \text{ is a code for } (n, m) \in d_1 \times d_2 \text{ such that } \mathbf{f}(n) = \mathbf{g}(m)\}$$
$$\boldsymbol{\pi_1}(p) = n$$
$$\boldsymbol{\pi_2}(p) = m$$

$$\begin{tikzcd} P \arrow[r, "{[\pi_2]}"] \arrow[d, "{[\pi_1]}"'] & d_2 \arrow[d, "{[\mathbf{g}]}"] \\ d_1 \arrow[r, "{[\mathbf{f}]}"'] & c \end{tikzcd}$$

We need to code the pair (n, m) as a single natural number, because sets of pairs of natural numbers are not objects in the category $\mathcal{P}(\mathbb{N})$. Take as coding for instance $p = 2^n 3^m$.

Apart from ensuring that the defined set is an object in our site, we also need $[\boldsymbol{\pi}_1]$ and $[\boldsymbol{\pi_2}]$ to be valid morphisms. That is, their inverse images should preserve bounded sets. We check that they do: Let $e_1 \subseteq d_1$ be a bounded subset of d_1. Then the pre-image of e_1 under $\boldsymbol{\pi_1}$ should be bounded as well. Let n_0 be any element of e_1, and consider the pre-image of n_0 under $\boldsymbol{\pi_1}$:

$$\boldsymbol{\pi_1}^{-1}(n_0) = \{p \mid p \text{ is a code for } (n_0, m) \text{ such that } \mathbf{f}(n_0) = \mathbf{g}(m)\}.$$

$[\mathbf{g}]$ is a morphism in the category, hence $\mathbf{g}$ has the property that the pre-image of a bounded set is bounded. Therefore, there are bounded many m such that $g(m) = f(n_0)$. Hence, $\boldsymbol{\pi_1}^{-1}(n_0)$ is also bounded.

For each $n_0 \in e_1$, the pre-image under $\boldsymbol{\pi_1}$ is bounded and e_1 is itself a bounded set, we may therefore conclude that the pre-image of e_1 under $\boldsymbol{\pi_1}$ is bounded. Therefore, $[\boldsymbol{\pi_1}]$ is a morphism in the site. The same argument proves that also $[\boldsymbol{\pi_2}]$ is a morphism in the site.

$[\boldsymbol{\pi_1}]$ and $[\boldsymbol{\pi_2}]$ clearly make the diagram commute, so the last thing to check is the UMP. Suppose there is some object P' and pair of morphisms $[\mathbf{h_1}] : P' \to d_1$ and $[\mathbf{h_2}] : P' \to d_2$ that makes the diagram commute, then there should be a unique morphism $[\mathbf{u}] : P' \to P$ such that $[\boldsymbol{\pi_1} \circ \mathbf{u}] = [\mathbf{h_1}]$ and $[\boldsymbol{\pi_2} \circ \mathbf{u}] = [\mathbf{h_2}]$.

$$\begin{tikzcd}
P' \arrow[rrd, bend left, "{[\mathbf{h_2}]}"] \arrow[rdd, bend right, "{[\mathbf{h_1}]}"'] \arrow[rd, dotted, "{[\mathbf{u}]}"] & & \\
& P \arrow[r, "{[\pi_2]}"] \arrow[d, "{[\pi_1]}"'] & d_2 \arrow[d, "{[\mathbf{g}]}"] \\
& d_1 \arrow[r, "{[\mathbf{f}]}"'] & c
\end{tikzcd}$$

This unique morphism is given by $[\langle \mathbf{h_1}, \mathbf{h_2} \rangle]$ sending $p' \in P'$ to the code p of the pair $(\mathbf{h_1}(p'), \mathbf{h_2}(p'))$ This is a well defined morphism, as by the commuting properties we always have that $\mathbf{f}(\mathbf{h_1}(p')) = \mathbf{g}(\mathbf{h_2}(p'))$, so the code p of the pair $(\mathbf{h_1}(p'), \mathbf{h_2}(p'))$ is an element of P. Uniqueness of $[\langle \mathbf{h_1}, \mathbf{h_2} \rangle]$ follows immediately from all the commuting properties.

Hence P and the projection morphisms are indeed the pullback. $\square$

We check that the given covering condition does indeed define a Grothendieck topology. In other words, that $(\mathcal{P}(\mathbb{N}), J')$ is a site.

Proposition 4.0.2. *$(\mathcal{P}(\mathbb{N}), J')$ is a site.*

Proof. We check the three defining axioms (see definition 2.2.3).

- Let $\mathbf{f} : d \to c$ be an isomorphism. Then $c \subseteq \mathrm{Im}(\mathbf{f})$, hence $\{[\mathbf{f}]\}$ is covering.
- Stability: Let $\{[\mathbf{f_i}] : d_i \to c \mid i \in I\}$ be a covering family, and let $[\mathbf{f}] : d \to c$ be any morphism. Consider the family of pullback projections

  $$\{[\boldsymbol{\pi}_{2i}] : d_i \times_c d \to d \mid i \in I\}$$

  coming from the following pullback diagrams:

  $$\begin{tikzcd}
  d_i \times_c d \arrow[r, "{[\pi_{2i}]}"] \arrow[d, "{[\pi_{1i}]}"'] & d \arrow[d, "{[\mathbf{f}]}"] \\
  d_i \arrow[r, "{[\mathbf{f_i}]}"'] & c
  \end{tikzcd}$$

  We need to show that this family covers d. That is, we need to show that there exist $d_1, \ldots, d_k$ such that for all $M_1, \ldots, M_k$:

  $$\exists N \ \forall m \geq N \ \exists i \leq k \ \exists p \in d_i \times_c d \ [m \in d \to (p \geq M_i \ \wedge \ \boldsymbol{\pi}_{2i}(p) = m)]$$

  As $\{[\mathbf{f_i}] : d_i \to c \mid i \in I\}$ is a covering family of c, there are $d_1, \ldots, d_k$ such that for all $M_1, \ldots, M_k$:

  $$\exists N' \ \forall n \geq N' \ \exists i \leq k \ \exists n' \in d_i \ [n \in c \to (n' \geq M_i \ \wedge \ \mathbf{f_i}(n') = n)]$$

Take the same $d_1, \ldots, d_k$ and consider $\boldsymbol{\pi_{21}}, \ldots, \boldsymbol{\pi_{2k}}$. Let $M_1, \ldots, M_k$ be given. We need to show that

$$\exists N\ \forall m \geq N\ \exists i \leq k\ \exists p \in d_i \times_c d\ [m \in d \to (p \geq M_i\ \wedge\ \boldsymbol{\pi_{2i}}(p) = m)]$$

From the fact that $d_1\ \ldots, d_k$ covers c, get N' such that for all $n \geq N'$ in c there is an i and $n' \in d_i$ such that $n' \geq M_i$ and $\mathbf{f_i}(n') = n$.

By definition (def 4.0.2, second bullet), $\mathbf{f}$ is such that for this N', there exists a N such that for all $m \in d$, whenever $m \geq N$, we have $\mathbf{f}(m) \geq N'$. So for all $m \in d$ with $m \geq N$, we know that there is an i and $n' \in d_i$ such that $\mathbf{f_i}(n') = \mathbf{f}(m)$. Pick $m \geq N'$ arbitrarily and find the corresponding i and n'.

Recall from proposition 4.0.1 that:

$$d_i \times_c d = \{p \mid p \text{ is a code for } (n'', m') : \mathbf{f_i}(n'') = \mathbf{f}(m')\}$$

As we know that $\mathbf{f_i}(n') = \mathbf{f}(m)$, there is a $p \in d_i \times_c d$ that is a code for (n', m), where the coding $p = 2^{n'}3^m$, so that we always have $p \geq n'$, and hence $p \geq M_i$. That is, we know there exists an N such that for all $m \geq N$ in d, there exists and i and a $p \in d_i \times_c d$ such that $p \geq M_i$ and $\boldsymbol{\pi_{2i}}(p) = m$ So the family of pullbacks does indeed cover d.

- Transitivity: Let

$$\{[\mathbf{f_i}] : d_i \to c\ \mid i \in I\}$$

be a covering family for c and for each d_i let

$$\{[\mathbf{g_{ij}}] : d_{ij} \to d_i \mid j \in I_i\}$$

be a covering family of d_i. We need to show that

$$\{[\mathbf{f_i} \circ \mathbf{g_{ij}}] : d_{ij} \to c \mid i \in I, j \in I_i\}$$

is again a covering family for c. As $\{[\mathbf{f_i}] : d_i \to c\ \mid i \in I\}$ is covering for c, we know that there are $[\mathbf{f_1}], \ldots, [\mathbf{f_k}]$ in there such that for all $M_1, \ldots, M_k$:

$$\exists N\ \forall n \geq N\ \exists i \leq k\ \exists m \in d_i\ [n \in c \to (m \geq M_i\ \wedge\ \mathbf{f_i}(m) = n)] \tag{4.1}$$

Also, for each $i \in I$, there are $[\mathbf{g_{i1}}], \ldots, [\mathbf{g_{il_i}}]$ such that for all $M_{i1}, \ldots, M_{il_i}$

$$\exists N_i\ \forall m \geq N_i\ \exists j \leq l_i\ \exists m' \in d_{ij}\ [m \in d_i \to (m' \geq M_{ij}\ \wedge\ \mathbf{g_{ij}}(m') = m)] \tag{4.2}$$

Claim: $[\mathbf{f_1} \circ \mathbf{g_{11}}], \ldots, [\mathbf{f_k} \circ \mathbf{g_{kl_k}}]$ are such that for all $\{M_{ij} \mid i \leq k, j \leq l_i\}$:

$$\exists N_0\ \forall n \geq N_0\ \exists i \leq k, j \leq l_i\ \exists m' \in d_{ij}\ [n \in c \to (m' \geq M_{ij}\ \wedge\ \mathbf{f_i} \circ \mathbf{g_j}(m') = n)]$$

Let $\{M_{ij} \mid i \leq k, j \leq l_i\}$ be given. From equation 4.2, find, for each $d_i, i \leq k$, an N_i such that for each $m \in d_i$, if $m \geq N_i$, there is a j and an $m' \in d_{ij}$ such that $m' \geq M_{ij}$ and $\mathbf{g_{ij}}(\mathbf{m'}) = m$.

Now use equation 4.1 for $N_1, \ldots, N_k$: this yields an N such that for all $n \geq N$, there is an i and an $m \in d_i$ such that $m \geq N_i$ and $\mathbf{f_i}(m) = n$. As $m \geq N_i$, there is a j an $m' \in d_{ij}$ such that $m' \geq M_{ij}$ and $\mathbf{g_{ij}}(\mathbf{m'}) = \mathbf{m}$.

In other words, for each $n \geq N$, there is an i, a j and an $m' \in d_{ij}$ such that $m' \geq M_{ij}$ and $\mathbf{f_i} \circ \mathbf{g_{ij}}(m') = n$. This proves transitivity.

□

We let $\mathcal{F}$ be the category of sheaves on the site $(\mathcal{P}(\mathbb{N}), J')$. In this category, we look for objects that are very similar to the objects we have treated before; the natural numbers object and a new non-standard model that looks like 'functions from $\mathbb{N}$ to $\mathbb{N}$'.

4.1 The natural numbers and a new non-standard model

Our approach is different from that in the previous chapter. In the previous chapter, we first looked for the natural numbers object, which we found by computing $\mathbf{a}\mathbb{N}$ and then proving an isomorphism between $\mathbf{a}\mathbb{N}$ and FinIm. Then we defined a candidate non-standard model and showed the natural numbers object was isomorphic to a subsheaf of this candidate non-standard model. Here, we will first define a candidate non-standard model of the natural numbers and prove that it is a sheaf. We then find a subsheaf of this sheaf which we prove to be the natural numbers object by showing that it has the required properties mentioned in definition 2.2.12.

The new candidate non-standard model sends an object c in $\mathcal{P}(\mathbb{N})$ to the set of all equivalence classes $[f]$ of functions form c to $\mathbb{N}$. Occasionally, we shall denote these functions by $[h]$ rather than $[f]$, to avoid confusion. However, to keep the similarity between chapter 3 and this chapter, we will use $[f]$ as much as possible.

Definition 4.1.1. *The sheaf* $\mathbb{N}^*$
Let $\mathbb{N}^*$ be the sheaf sending an object c to the set of equivalence classes of all functions from c to the natural numbers, under the now well known equivalence relation that identifies two functions h_1 and h_2 if the set $\{n \in c \mid h_1(n) \neq h_2(n)\}$ is bounded:

$$\mathbb{N}^*(c) = \{[h] : c \to \mathbb{N}\}$$
$$\mathbb{N}^*([\mathbf{f}] : d \to c)([h] : c \to \mathbb{N}) = [h \circ f] : d \to \mathbb{N},$$

where f is the same function as $\mathbf{f}$, but in the category Set rather than $\mathcal{P}(\mathbb{N})$. We will use this notation consistently: whenever a function could be a morphism of both $\mathcal{P}(\mathbb{N})$ and Set, then the boldface type indicates that we consider it as a morphism of $\mathcal{P}(\mathbb{N})$, while the usual italic print indicates that we consider it as a morphism in Set.

Proposition 4.1.1. $\mathbb{N}^*$ *is indeed a sheaf.*

Proof. We need to show that every matching family for $\mathbb{N}^*$ has a unique amalgamation. Take any $c \in \mathcal{P}(\mathbb{N})$ and let $\{[\mathbf{f_i}] : d_i \to c \mid i \in I\}$ be a cover of c. To get a matching family, we must assign to each $[\mathbf{f_i}]$, an element of $\mathbb{N}^*(d_i)$, that is, an equivalence class of functions $[h_i] : d_i \to \mathbb{N}$. So our matching family is:

$$\{([\mathbf{f_i}] : d_i \to c, [h_i] : d_i \to \mathbb{N}) \mid i \in I\}$$

Such that, when considering the pullback $d_i \times_c d_j$

$$[h_i] \cdot [\boldsymbol{\pi}^{\mathbf{1}}_{\boldsymbol{ij}}] = [h_j] \cdot [\boldsymbol{\pi}^{\mathbf{2}}_{\boldsymbol{ij}}]$$

Now $[h_i] \cdot [\boldsymbol{\pi}^{\mathbf{1}}_{\boldsymbol{ij}}]$ is just $[h_i \circ \pi^1_{ij}]$. So the matching condition just ensures that the outer diagram below commutes (notice that this diagram is entirely in Set):

$$\begin{tikzcd}
& d_i \arrow[dr, "{[f_i]}"'] \arrow[drr, "{[h_i]}"] & & \\
d_i \times_c d_j \arrow[ur, "{[\pi^1_{ij}]}"] \arrow[dr, "{[\pi^2_{ij}]}"'] & & c & \mathbb{N} \\
& d_j \arrow[ur, "{[f_j]}"] \arrow[urr, "{[h_j]}"'] & &
\end{tikzcd}$$

We need to find an amalgamation for this matching family. That is, we need to find an equivalence class of functions $[h] : c \to \mathbb{N}$ such that all these triangles commute:

$$\begin{tikzcd}
d_i \arrow[dr, "{[f_i]}"'] \arrow[drr, "{[h_i]}"] & & \\
& c \arrow[r, "{[h]}"'] & \mathbb{N}
\end{tikzcd}$$

The amalgamation is the equivalence class of the following function:

$$h(n) = \begin{cases} h_i(f_i^{-1}(n)), & \text{if } n \in \mathrm{Im}(f_i) \\ 0, & \text{otherwise} \end{cases}$$

It is not trivially clear that this is a well defined morphism: we do not require $[f_i]$ to be injective, so $h_i(f_i^{-1}(n))$ might not be a uniquely determined. Neither do we specify which of the $[f_i]$ we should pick, in the case that n is in the image of more than one covering morphism. The properties of the matching family ensure that these degrees of freedom do not cause $[h]$ to be ill-defined:

Suppose that $n \in \mathrm{Im}(f_i) \cup \mathrm{Im}(f_j)$, that is: there is an $m_i \in d_i$ and an $m_j \in d_j$ such that $f_i(m_i) = n = f_j(m_j)$. Consider the pullback of $[\mathbf{f_i}]$ and $[\mathbf{f_j}]$ in $\mathcal{P}(\mathbb{N})$:

$$\begin{tikzcd}
d_i \times_c d_j \arrow[r, "{[\pi^2_{ij}]}"] \arrow[d, "{[\pi^1_{ij}]}"'] & d_j \arrow[d, "{[\mathbf{f_j}]}"] \\
d_i \arrow[r, "{[\mathbf{f_i}]}"'] & c
\end{tikzcd}$$

Then by the properties of matching families, we must have: $h_i \circ \pi^1 = h_j \circ \pi^2$ (the commuting outer diagram of before). Now as $\mathbf{f_i}(m_i) = \mathbf{f_j}(m_j)$, there is a $p \in d_i \times_c d_j$ such that $\boldsymbol{\pi^1_{ij}}(p) = m_i$ and $\boldsymbol{\pi^2_{ij}}(p) = m_j$. We then must have that:

$$\begin{aligned}
h_i(m_i) =& (h_i \circ \pi^1_{ij})(p) \\
=& (h_j \circ \pi^2_{ij})(p) \\
=& h_j(m_j)
\end{aligned}$$

So it does not matter which morphism we choose to take the pre-image of n to compute $h(n)$. The same argument shows that possible non-injectivity of some $[f_i]$ causes no trouble: consider m_1 and m_2 in d_i such that $f_i(m_1) = f_i(m_2)$. Take the pullback of $[\mathbf{f_i}]$ along itself, and then the previous argument shows that we must have $h_i(m_1) = h_i(m_2)$.

The fact that $[h]$ is an amalgamation now follows trivially from its definition, which leaves us with one thing to check: its uniqueness.

Suppose $[g] : c \to \mathbb{N}$ is any amalgamation of

$$\{([\mathbf{f_i}] : d_i \to c, [h_i] : d_i \to \mathbb{N}) \mid i \in I\}.$$

Then we must have: $[g \circ f_i] = [h_i]$, so

$$g(n) = h(n) \quad \text{for all} \quad n \in \bigcup_{i \in I} \mathrm{Im}(\mathbf{f_i}).$$

As $\{[\mathbf{f_i}] \mid i \in I\}$ is covering, there is an N such that for all $n \geq N : n \in \bigcup_{i\in I} \mathrm{Im}(\mathbf{f_i})$ (a corollary from the covering condition with all $M_i = 0$). So there is a N such that for all $n \geq N : g(n) = h(n)$. So $g \sim h$, hence $[h]$ is unique.

So $\mathbb{N}^*$ is indeed a sheaf. □

We define some structure on $\mathbb{N}^*$, so that we can look for the subsheaf of standard natural numbers and verify the Peano axioms:

Definition 4.1.2. *Zero, Successor function, Equality, Order, Standard and Infinite numbers* For any $[f]$ and $[g]$ in $\mathbb{N}^*(c)$, we define:

- We define $0 : 1 \to \mathbb{N}^*$ as $0_c = [const_0 : c \to \mathbb{N}]$.
- The successor function, $s : \mathbb{N}^* \to \mathbb{N}^*$, does just what you would expect:

$$s_c([f])] = [f_{+1}]$$

- Equality:

$$c \Vdash [f] = [g] \iff \exists N\ \forall n \geq N\ [n \in c \to f(n) = g(n)]$$

- Order:

$$c \Vdash [f] \leq [g] \iff \exists N\ \forall n \geq N\ [n \in c \to f(n) \leq g(n)]$$

- The predicate St($\cdot$): We say that $[f]$ is a *standard natural number* in the eyes of c, if the image of f is bounded.

$$c \Vdash \mathrm{St}([f]) \iff \exists N\ \forall n\ [n \in \mathrm{Im}(f) \to n < N] \tag{4.3}$$

- The predicate Inf($\cdot$): We say that $[f]$ is an *infinite number* in the eyes of c, if it is larger than all standard natural numbers:

$$c \Vdash \mathrm{Inf}([f]) \iff c \Vdash \forall x[\mathrm{St}(x) \to x \leq [f]] \tag{4.4}$$

Of course, we hope that the predicate St captures exactly the standard natural numbers. That is, we hope that the subfunctor of $\mathbb{N}^*$ sending c to the set of all bounded functions from c to $\mathbb{N}$ is the natural numbers object of $\mathcal{F}$. We prove that this is the case, and call this subfunctor as $\mathrm{FinIm}^*/_\sim$, the new $\mathrm{FinIm}/_\sim$.

Proposition 4.1.2. *The subfunctor* $\mathrm{FinIm}^*/_\sim$ *is the natural numbers object in the sheaf category* $\mathcal{F}$.

Proof. First of all, we need to check wether $\mathrm{FinIm}^*/_\sim$ is actually a sheaf. By looking at the proof of proposition 4.1.1, we see that if all the $[h_i]$'s in the matching family are bounded functions, their amalgamation is again bounded. The uniqueness proof carries through, so $\mathrm{FinIm}^*/_\sim$ is a sheaf.

For $\mathrm{FinIm}^*/_\sim$ to be the natural numbers object, it should have the following universal mapping property: For any object A, together with $0' : 1 \to A$ and successor function $s' : A \to A$, there should exists a unique natural transformation $u : \mathrm{FinIm}^*/_\sim \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "0'"'] & \mathrm{FinIm}^*/_\sim \arrow[r, "s"] \arrow[d, dotted, "u"] & \mathrm{FinIm}^*/_\sim \arrow[d, dotted, "u"] \\
 & A \arrow[r, "s'"'] & A
\end{tikzcd}$$

Let A, $0'$ and s' be given. We use the monotonicity and the local character of sheaves: two elements a and b in $A(c)$ are equal if and only if there exists a cover $\{[\mathbf{f_i}] : d_i \to c\}$ of c such that $a \cdot [\mathbf{f_i}] = b \cdot [\mathbf{f_i}]$ for each $[\mathbf{f_i}]$ in the cover. So in order to define what the components $u_c : \mathrm{FinIm}^*/_\sim(c) \to A(c)$ do to $[f] \in \mathrm{FinIm}^*/_\sim(c)$, it is enough to define $u_{d_i}([f] \cdot [\mathbf{f_i}])$ for a cover $[\mathbf{f_i}] : d_i \to c$ of c.

Fixing c and $[f] \in \mathrm{FinIm}^* /_{\sim}(c)$, we choose a cover of c in such a way that $[f] \cdot [\mathbf{f_i}]$ is a constant function. That way, we can easily define $u_{d_i}([f] \cdot [\mathbf{f_i}])$ in terms of the successor functions. Such a cover exists because $[f]$ is bounded. That is:

$$\exists N_f \; \forall n \in c \; [f(n) \leq N_f]$$

Define, for each $i \leq N_f$, d_i as:

$$d_i = \{n \in c \mid f(n) = i\}$$

These $d_0, \ldots, d_{N_f}$, together with the inclusion functions, form a cover of c. In the eyes of d_i, $[f \upharpoonright d_i]$ is the i^{th} successor of 0_{d_i} (where the 0^{th} successor of 0_{d_i} is $0_{d_i} = const_0$) :

$$d_i \Vdash [f] = s_{d_i}^{(i)}(0_{d_i})$$

We hence define u on this cover as:

$$u_{d_i}([f \upharpoonright d_i]) = s_{d_i}'^{(i)}(0'_{d_i})$$

That is, we make $u_{d_i}([f \upharpoonright d_i])$ the i^{th} successor of $0'_{d_i}$.

We check the required properties: $u_c(0_c) = 0'_c$ by definition. For the square in the diagram, we need to have that $u \circ s = s' \circ u$, or componentwise we need for each $[f] \in \mathrm{FinIm}^* /_{\sim}(c)$ that

$$u_c(s_c([f])) = s'_c(u_c([f])).$$

Let $d_0, \ldots, d_{N_f}$ be the cover for $[f]$. Then this cover is the same as the cover for $s_c([f])$: if $[f \upharpoonright d_i]$ is constant, then $s_c([f \upharpoonright d_i])$ is also constant. However, in the definition of the cover for $s_c([f])$, these sets are not labeled as $d_0, \ldots, d_{N_f}$, but as $d'_0, \ldots, d'_{N_f+1}$. It is easy to see that d'_0 is empty and $d_i = d'_{i+1}$:

$$\begin{aligned} d'_{i+1} &= \{n \in c \mid s_c(f) = i + 1\} \\ &= \{n \in c \mid f(n) = i\} \\ &= d_i \end{aligned}$$

We then have (the component labels of the successor functions s and s' have been left out for clarity):

$$\begin{aligned} u_{d_i}(s([f \upharpoonright d_i])) &= u_{d'_{i+1}}(s([f \upharpoonright d'_{i+1}])) && \text{because } d_i = d'_{i+1} \\ &= s'^{(i+1)}(0'_{d'_{i+1}}) && \text{by definition} \\ &= s'(s'^{(i)}(0'_{d_{i+1}})) && \\ &= s'(s'^{(i)}(0'_{d_i})) && \text{because } d_i = d'_{i+1} \\ &= s'(u_{d_i}([f \upharpoonright d_i])) && \text{by definition} \end{aligned}$$

As this is true for all d_i in the cover, we must also have that $u_c(s([f])) = s'(u_c([f]))$.

Last thing to check is uniqueness of u. Suppose that u' is any other natural transformation such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "0'"'] & \mathrm{FinIm}^* /_\sim \arrow[r, "s"] \arrow[d, dotted, "u'"] & \mathrm{FinIm}^* /_\sim \arrow[d, dotted, "u'"] \\
 & A \arrow[r, "s'"'] & A
\end{tikzcd}$$

We have immediately that $u'_c(0_c) = 0'_c = u_c(0_c)$. We need to show for any $[f] \in \mathrm{FinIm}^* /_\sim(c) : u'_c([f]) = u_c([f])$. Let $d_0, \ldots, d_{N_f}$ be the cover of c defined above. Then $[f]$ is constant on this cover, so that for each d_i, $d_i \Vdash [f \restriction d_i] = s^{(i)}_{d_i}(0_{d_i})$. We therefore must have that

$$\begin{aligned} u'_{d_i}([f \restriction d_i]) &= s'^{(i)}_{d_i}(u'_{d_i}(0_{d_i})) \\ &= s'^{(i)}_{d_i}(u_{d_i}(0_{d_i})) \\ &= u_{d_i}([f \restriction d_i]) \end{aligned}$$

It now follows that also $u'_c([f]) = u_c([f])$, hence u is unique.

This proves that $\mathrm{FinIm}^* /_\sim$ is indeed the natural numbers object. □

As in the previous chapter, we shall denote the standard natural numbers object as $\mathbf{a}\mathbb{N}$, the sheafification of the constant presheaf $\mathbb{N}$. By proving that $\mathrm{FinIm}^* /_\sim$ is the natural numbers object, we have shown that $\mathrm{FinIm}^* /_\sim$ and $\mathbf{a}\mathbb{N}$ are isomorphic, justifying the use of either of the names '$\mathrm{FinIm}^* /_\sim$' and '$\mathbf{a}\mathbb{N}$' to refer to the standard natural numbers.

4.2 $\mathbb{N}^*$ as a non-standard model for natural arithmetic

As before, it is more convenient to prove statements about natural numbers than about functions from c to $\mathbb{N}$. The following lemma allows us to do so for quantifier-free formulas.

Lemma 4.2.1. *For every internal, quantifier-free formula $\phi([h_1], \ldots, [h_k])$ and every $c \in \mathcal{P}(\mathbb{N})$, we have:*
$c \Vdash \phi([h_1], \ldots, [h_k])$ iff $\exists N\ \forall n \geq N\ [n \in c \rightarrow \phi(h_1(n), \ldots, h_k(n))]$.

Proof. We prove this by induction on the complexity of formulas.

- For atomic formulas, this is the very definition.
- Conjunction: follows immediately.
- Disjunction:

($\Rightarrow$) : Suppose $c \Vdash \phi([h_1], \ldots, [h_k]) \vee \psi([h_1], \ldots, [h_k])$. We need to show that there exists an N such that for all $n \geq N$

$$n \in c \rightarrow \phi(h_1(n), \ldots, h_k(n)) \vee \psi(h_1(n), \ldots, h_k(n)).$$

As $c \Vdash \phi([h_1], \ldots, [h_k]) \vee \psi([h_1], \ldots, [h_k])$, there exists a cover $\{[\mathbf{f_i}] : d_i \to c \mid i \in I\}$ of c such that for every $i \in I$

$$d_i \Vdash \phi([h_1 \circ f_i], \ldots, [h_k \circ f_i]) \quad \text{or} \quad d_i \Vdash \psi([h_1 \circ f_i], \ldots, [h_k \circ f_i]).$$

Since $\{[\mathbf{f_i}] : d_i \to c \mid i \in I\}$ is a cover, there are $[\mathbf{f_1}], \ldots, [\mathbf{f_l}]$ among them such that for every $M_1, \ldots, M_l$:

$$\exists N_0 \; \forall n \geq N_0 \; \exists i \leq l \; \exists m \in d_i \; [n \in c \to (m \geq M_i \;\wedge\; \mathbf{f_i}(m) = n)] \tag{4.5}$$

By the induction hypothesis we have for each $[\mathbf{f_i}]$ in the cover that there exists an M'_i such that for all $m \geq M'_i$

$$m \in d_i \to \phi((h_1 \circ f_i)(m), \ldots, (h_k \circ f_i)(m))$$

or there exists an M'_i such that for all $m \geq M'_i$

$$m \in d_i \to \psi((h_1 \circ f_i)(m), \ldots, (h_k \circ f_i)(m)).$$

So certainly there exists an M'_i such that for all $m \in d_i$ such that $m \geq M'_i$

$$\phi((h_1 \circ f_i)(m), \ldots, (h_k \circ f_i)(m)) \vee \psi((h_1 \circ f_i)(m), \ldots, (h_k \circ f_i)(m)). \tag{4.6}$$

For each $[\mathbf{f_i}]$, find M'_i and consider $M'_1, \ldots, M'_l$. Use equation 4.5 to find N_0 such that for all $n \in c$, if $n \geq N_0$ there is an i and an $m \in d_i$ such that $m \geq M'_i$ and $\mathbf{f_i}(m) = n$.
Then as $m \geq M'_i$, by equation 4.6

$$\phi((h_1 \circ f_i)(m), \ldots, (h_k \circ f_i)(m)) \vee \psi((h_1 \circ f_i)(m), \ldots, (h_k \circ f_i)(m))]$$

And as we know $\mathbf{f_i}(m) = n$,

$$\phi(h_1(n), \ldots, h_k(n)) \vee \psi(h_1(n), \ldots, h_k(n))].$$

This is true for all $n \in c$ such that $n \geq N_0$, which was what we needed to show.

$(\Leftarrow)$: Suppose that there exists a N such that for all $n \geq N$: $n \in c \to \phi(h_1(n), \ldots, (h_k(n)) \vee \psi(h_1(n), \ldots, (h_k(n))$. Define d_1 and d_2 as follows:

$$d_1 = \{n \in c \mid \phi(h_1(n), \ldots, h_k(n))\}$$
$$d_2 = \{n \in c \mid \psi(h_1(n), \ldots, h_k(n))\}$$

Then the inclusion functions $[\mathbf{f_1}] : d_1 \hookrightarrow c$ and $[\mathbf{f_2}] : d_2 \hookrightarrow c$ form a cover of c. By the induction hypothesis, we have:

$$d_1 \Vdash \phi([h_1 \circ f_1], \ldots, [h_k \circ f_1])$$
$$d_2 \Vdash \psi([h_1 \circ f_2], \ldots, [h_k \circ f_2])$$

And hence:

$$d_1 \Vdash \phi([h_1 \circ f_1], \ldots, [h_k \circ f_1]) \vee \psi([h_1 \circ f_1], \ldots, [h_k \circ f_1])$$
$$d_2 \Vdash \phi([h_1 \circ f_2], \ldots, [h_k \circ f_2]) \vee \psi([h_1 \circ f_2], \ldots, [h_k \circ f_2])$$

By the local character of sheaf semantics, we may conclude that then also:

$$c \Vdash \phi([h_1], \ldots, [h_k]) \vee \psi([h_1], \ldots, [h_k])$$

Which was what we needed to show.

- Implication:

(⇒:) Suppose that $c \Vdash \phi([h_1], \ldots, [h_k]) \to \psi([h_1], \ldots, [h_k])$. We need to show that there exists an N such that for all $n \geq N$

$$n \in c \to (\phi(h_1(n), \ldots, h_k(n)) \to \psi(h_1(n), \ldots, h_k(n))).$$

Consider:

$$d = \{n \in c \mid \phi(h_1(n), \ldots, h_k(n))\}$$

Notice that for all $n \in c$ that are not in d, we already have the desired implication.
As $c \Vdash \phi([h_1], \ldots, [h_k]) \to \psi([h_1], \ldots, [h_k])$, we have for all $[\mathbf{f'}] : d' \to c$:

$$d' \Vdash \phi([h_1 \circ f], \ldots, [h_k \circ f']) \quad \text{implies} \quad d' \Vdash \psi([h_1 \circ f'], \ldots, [h_k \circ f']).$$

So in particular, this holds for d defined above together with the inclusion function $[\mathbf{f}] : d \hookrightarrow c$, where per induction hypothesis we have:

$$d \Vdash \phi([h_1 \circ f], \ldots, [h_k \circ f]).$$

Hence:

$$d \Vdash \psi([h_1 \circ f], \ldots, [h_k \circ f])$$

We again apply the induction hypothesis to get:

$$\exists N \; \forall n \geq N \; [n \in d \to \psi((h_1 \circ f)(n), \ldots, (h_k \circ f)(n))]$$

For this same N, we then also have:

$$\forall n \geq N \; [n \in c \to (\phi(h_1(n), \ldots, h_k(n)) \to \psi(h_1(n), \ldots, h_k(n)))]$$

(⇐) : Suppose there exists an N such that:

$$\forall n \geq N \; [n \in c \to (\phi(h_1(n), \ldots, h_k(n)) \to \psi(h_1(n), \ldots, h_k(n)))].$$

We need to show that for all $[\mathbf{f}] : d \to c$, if:

$$d \Vdash \phi([h_1 \circ f], \ldots, [h_k \circ f])$$

then also:

$$d \Vdash \psi([h_1 \circ f], \ldots, [h_k \circ f])$$

So let $[\mathbf{f}] : d \to c$ be any morphism and suppose that

$$d \Vdash \phi([h_1 \circ f], \ldots, [h_k \circ f]).$$

Then by the induction hypothesis, there exists an N_1 such that for all $m \geq N_1$:

$$m \in d \to \phi((h_1 \circ f)(m), \ldots, (h_k \circ f)(m))$$

We assumed that there exists an N such that:

$$\forall n \geq N\ [n \in c \to (\phi(h_1(n), \ldots, h_k(n)) \to \psi(h_1(n), \ldots, h_k(n)))].$$

Since $[\mathbf{f}]$ is a morphism of $\mathcal{P}(\mathbb{N})$, given this N there exists an N_2 such that for all $m \in d$:

$$m \geq N_2 \to \mathbf{f}(m) \geq N$$

And hence for all $m \in d$

$$m \geq N_2 \to (\phi((h_1 \circ f)(n), \ldots, (h_k \circ f)(n)) \to \psi((h_1 \circ f)(n), \ldots, (h_k \circ f)(n)))/$$

Define $N_3 = \max(N_1, N_2)$, then we have, for all $m \geq N_3$:

$$m \in d \to \phi((h_1 \circ f)(m), \ldots, (h_k \circ f)(m))$$

because $m \geq N_3$ implies that $m \geq N_1$ ánd, as $m \geq N_3$ implies that $m \geq N_2$:

$$m \in d \to (\phi((h_1 \circ f)(m), \ldots, (h_k \circ f)(m)) \to \psi((h_1 \circ f)(m), \ldots, (h_k \circ f)(m))$$

Hence: for all $m \geq N_3$, we have:

$$\psi((h_1 \circ f)(m), \ldots, (h_k \circ f)(m))$$

By the induction hypothesis, we may conclude that

$$d \Vdash \psi([h_1 \circ f], \ldots, [h_k \circ f]).$$

Therefore: for all $[\mathbf{f}] : d \to c$:

$$d \Vdash \phi([h_1 \circ f], \ldots, [h_k \circ f]) \quad \text{implies} \quad d \Vdash \psi([h_1 \circ f], \ldots, [h_k \circ f]).$$

Hence by sheaf semantics: $c \Vdash \phi([f_1], \ldots, [f_k]) \to \psi([f_1], \ldots, [f_k])$.

- Negation: follows trivially.

$\square$

To our regret, we still need a classical meta-theory to prove the same equivalence for formulas containing quantifiers:

Lemma 4.2.2. *For every formula* $\phi([h_1], \ldots, [h_k])$ *and every* $c \in \mathcal{P}(\mathbb{N})$*, we have:* $c \Vdash \phi([h_1], \ldots, [h_k])$ *iff* $\exists N\ \forall n \geq N\ [n \in c \to \phi(h_1(n), \ldots, h_k(n))]$.

Proof. We extend the induction proof of lemma 4.2.1 by adding the existential en universal quantifier cases.

- Existential quantification:

($\Rightarrow$) : Suppose that $c \Vdash \exists x\ [\phi(x, [h_1], \ldots, [h_k])]$. We need to show that there exists an N such that for all $n \geq N$, $n \in c \to \exists x\ [\phi(x, h_1(n), \ldots, h_k(n))]$. As $c \Vdash \exists x\ [\phi(x, [h_1], \ldots, [h_k])]$, there exists a cover $\{[\mathbf{f_i}] : d_i \to c \mid i \in I\}$ of c and elements $[g_i] : d_i \to \mathbb{N}$ of $\mathbb{N}^*(d_i)$ such that

$$d_i \Vdash \phi([g_i], [h_1 \circ f_i], \ldots, [h_k \circ f_i]).$$

By the induction hypothesis, there exists, for each $[\mathbf{f_i}]$, an N_i such that for all $m \geq N_i$

$$m \in d_i \to \phi(g_i(m), (h_1 \circ f_i)(m), \ldots, (h_k \circ f_i)(m)). \tag{4.7}$$

As $\{[\mathbf{f_i}] \mid i \in I\}$ is a cover, there exist $[\mathbf{f_1}], \ldots, [\mathbf{f_l}]$ such that for all $M_1, \ldots, M_l$

$$\exists N_0\ \forall n \geq N_0\ \exists i \leq l \exists m \in d_i\ [n \in c \to (m \geq M_i\ \wedge\ \mathbf{f_i}(m) = n)].$$

Take $M_i = N_i$ and find N_0. Then for all $n \in c$ such that $n \geq N_0$ we have by equation 4.7 an $m \in d_i$ such that

$$\phi(g_i(m), (h_1 \circ f_i)(m), \ldots, (h_k \circ f_i)(m)),$$

and $\mathbf{f}(m) = n$, so that:

$$\phi(g_i(m), (h_1(n), \ldots, (h_k(n)).$$

That is:

$$\exists x\ [\phi(x, h_1(n), \ldots, h_k(n))]$$

for all $n \in c$, $n \geq N_0$. (namely: $x = g_i(m)$.)

($\Leftarrow$) : Suppose that There exists an N such that for all $n \geq N$,

$$n \in c \to \exists x\ [\phi(x, h_1(n), \ldots, h_k(n))].$$

We need to show that

$$c \Vdash \exists x\ [\phi(x, [h_1], \ldots, [h_k])].$$

Let

$$d = \{n \in c \mid n \geq N\}.$$

Then for all $n \in d$, there is an x_n such that

$$\phi(x_n, h_1(n), \ldots, h_k(n)).$$

Define $g : d \to \mathbb{N}$ as

$$g(n) = x_n.$$

(Notice that we use the countable axiom of choice here). Then

$$d \Vdash \phi([g], [h_1], \ldots, [h_k]).$$

The inclusion function $d \to c$ is a cover of c, and hence by sheaf semantics

$$c \Vdash \exists x\, [\phi(x, [h_1], \ldots, [h_k])].$$

- Universal quantification: just as in the case of existential quantification, this case uses the same arguments as presented in the universal case of proposition 3.2.4, reformulated to fit the current setting. As nothing new is presented here, we leave the details to the reader.

□

The Peano axioms

The Peano axioms as formulated in proposition 3.2.5 are valid in our model. The proof of proposition 3.2.5 translates directly to our current setting. The only non-constructive part is the use of lemma 4.2.2, which means we have all results constructively if we restrict internal induction to quantifier-free formulas.

Standard and infinite numbers

We consider the proposition 3.2.6 in the new setting:

Proposition 4.2.3. *For all $c \in \mathbb{P}$:*

1. $c \Vdash \exists x[\mathrm{Inf}(x)]$
2. $c \Vdash \forall x[\mathrm{Inf}(x) \leftrightarrow \neg \mathrm{St}(x)]$
3. $c \Vdash \forall x[\mathrm{St}(x) \leftrightarrow \neg \mathrm{Inf}(x)]$
4. $c \Vdash \forall x \forall y[x \leq y \wedge \mathrm{St}(y) \to \mathrm{St}(x)]$
5. $c \Vdash \forall x \forall y[x \leq y \wedge \mathrm{Inf}(x) \to \mathrm{Inf}(y)]$
6. $c \Vdash \forall y[\mathrm{Inf}(y) \to \exists x(\mathrm{Inf}(x) \wedge x < y)]$

As we discussed in chapter 3, we used a classical meta-theory to prove item 2 and 3 of this proposition. In the new setting, 2 can be proved fully constructively:

Proof.

2 The direction from left to right is still trivial. So for other direction, suppose $c \Vdash \neg\,\mathrm{St}([f])$. We prove by external induction that $c \Vdash \mathrm{Inf}([f])$, that is

$$c \Vdash \forall x[\mathrm{St}(x) \to x \leq [f]].$$

The basis $0 \leq [f]$ is immediately clear. For the induction step, suppose $[g]$ is a standard natural number and $c \Vdash [g] \leq [f]$. We need to show that

$$c \Vdash s_c([g]) \leq [f].$$

Consider:

$$\{n \in c \mid f(n) < s_c([g])(n)\}.$$

We need to show that this set is bounded. Notice that:

$$\{n \in c \mid f(n) < s_c([g])(n)\} = \{n \in c \mid f(n) < g(n)\} \cup \{n \in c \mid f(n) = g(n)\}$$

The former set is bounded, as we know $c \Vdash [g] \leq [f]$. Call the latter set d:

$$d = \{n \in c \mid f(n) = g(n)\}$$

Then $d \Vdash [g] = [f]$, and hence $d \Vdash \mathrm{St}([f])$. By sheaf semantics, we know from the fact that $c \Vdash \neg\,\mathrm{St}([f])$, that for all $[\mathbf{f}'] : d' \to c$, if $d' \Vdash \mathrm{St}([f])$, then the empty family is a cover of d'. The inclusion function is a morphism from d to c, hence we know that the empty family must be a cover of d. This in turn implies that d is bounded. Which was what we needed to show. By external induction, we then have $c \Vdash \mathrm{Inf}([f])$.

□

The proof of 3 translates to the new setting, but still needs proof by contradiction. 1,4,5 and 6 have already been proved constructively in chapter 3, their proofs easily translate to prove proposition 4.2.3.

Non-standard principles

The principles overspill, underspill, transfer (formulation 1) and idealisation are all valid in our model and the proofs directly translate from the proofs in chapter 3, as does the equivalent of lemma 3.2.10.

For the proof of transfer (formulation 2) and realisation, we should take more care. We rewrite the proof for transfer below. Realisation needs a similar kind of adaptation.

Proposition 4.2.4. *Transfer principle (formulation 2)*
Let $const_{m_1}, \ldots, const_{m_k}$ be any constant functions. Then:

$$c \Vdash \exists x\ \phi(x, [const_{m_1}], \ldots, [const_{m_k}]) \to \exists^{\mathrm{St}} x\ \phi(x, [const_{m_1}], \ldots, [const_{m_k}])$$

Proof. Suppose that $c \Vdash \exists x\ \phi(x, [const_{m_1}], \ldots, [const_{m_k}])$. Then there is a cover $\{[\mathbf{f_1}], \ldots, [\mathbf{f_l}]\}$, $[\mathbf{f_i}] : d_i \to c$ of c, together with functions $[g_1], \ldots, [g_l]$ such that:

$$d_i \Vdash \phi([g_i], [const_{m_1} \circ f_1], \ldots, [const_{m_k} \circ f_k]).$$

For each of these $[\mathbf{f_i}]$, we have by lemma 4.2.2 an N_i such that

$$\forall n \geq N_i\ [n \in d_i \to \phi(g_i(n), m_1, \ldots, m_k)].$$

As $\{[\mathbf{f_1}], \ldots, [\mathbf{f_l}]\}$ is a cover, we know that for $N_1, \ldots, N_k$ there exists an N_0 such that for each $n \geq N_0$, if $n \in c$, then there exists an i and and $m_n \in d_i$ such that $m_n \geq N_i$ and $\mathbf{f_i}(m_n) = n$. We define two functions, $h : c \to \bigcup_{i=1}^{k} d_i$ and $g : c \to \mathbb{N}$. To define h, find, for each $n \in c$, $n \geq N_0$, an i_n and $m_n \in d_{i_n}$ such that $m_n \geq N_{i_n}$ and $\mathbf{f_{i_n}}(m_n) = n$. We define h inductively for each $n \in c$ as follows:

$$h(n) = 0 \quad \text{if} \quad n \leq N_0$$

Suppose that $h(n')$ has been defined for all $n' \in c : n' \leq n$, we then define $h(n)$ as:

$$h(n) = \begin{cases} h(n') & \text{if } n' \leq n \text{ and } i_{n'} = i_n \\ m_n & \text{otherwise.} \end{cases}$$

That is, h picks for each $n \in c$ that is greater than N_0 an $m' \in d_{i_n}$ such that $m' \geq N_{i_n}$ but not necessarily $\mathbf{f_{i_n}}(m') = n$. Notice that for each d_i, at most one $m' \in d_i$ is in the image of h. Now define g for each $n \in c$ as follows:

$$g(n) = \begin{cases} 0 & \text{if } n \leq N_0 \\ g_{i_n} \circ h & \text{otherwise.} \end{cases}$$

Then g has a bounded image by the construction of h, so that $c \Vdash \mathrm{St}([g])$ and by construction we have for all $n \in c$:

$$n \geq N_0 \to \phi(g(n), m_1, \ldots, m_k)$$

Hence by lemma 4.2.2:

$$c \Vdash \phi([g], [const_{m_1}], \ldots, [const_{m_k}]).$$

□

4.3 Conclusion and discussion

It was not trivial to find a site with more structure than $(\mathbb{P}, J)$ and less structure than Moerdijk's site. Trivial extensions of $(\mathbb{P}, J)$ (such as allowing all functions as morphisms instead of only inclusion functions) clashed with either the stability or the transitivity axiom of a site. Once we found a suitable site, finding a non-standard model went rather smoothly.

The intermediate model N^* is 'in between' our first model and Moerdijk's model: We have all the results from chapter 3 and we were able to prove some of these results fully constructively. However, for the crucial lemma 4.2.2 and quite a few other results we still need a classical meta-theory.

For quantifier-free formulas, we were able to improve: compare the proof of proposition 4.2.1 to the proof of proposition 3.2.4. In the case of disjunction, the direction ($\Rightarrow$) is now formulated much more precisely. The direction ($\Leftarrow$) was not constructive in 3.2.4, because we could not guarantee that the sets d_1 and d_2 were objects in the site. Now that our site consists of all subsets of the natural numbers, we do not need the case distinction anymore, making this part of the proof constructive. For the implication the direction ($\Rightarrow$) does not need case distinction anymore, again because we now have finite sets in our site.

We were also able to prove part 2 of proposition 4.2.3 constructively, also because the finite sets are no longer banned from our site.

We did not manage to mimic the clever proof method of Moerdijk for the quantifier cases of lemma 4.2.2 and the non-standard principles. We illustrate the reason for this failure with the proof of the ($\Leftarrow$) direction of the existential quantifier case in proposition 4.2.2:

Example 4.3.1. In the ($\Leftarrow$) direction of the existential quantifier case in proposition 4.2.2, we construct a function $g : d \to \mathbb{N}$ to prove that there exists a cover of c (consisting only of $[\mathbf{f}] : d \to c$) with for each $[\mathbf{f}]$ in that cover a function $g : d \to \mathbb{N}$ such that

$$d \Vdash \phi([g], [h_1 \circ f], \ldots, [h_k \circ f])$$

We use the axiom of choice to construct g.

Moerdijk avoids the axiom of choice by taking a subset of the product of $c \times \mathbb{N}$:

$$d = \{(n, m) \in c \times \mathbb{N} \mid \phi(m, h_1(n), \ldots, h_k(n)\}$$

He then shows that the projection arrow $\boldsymbol{\pi_1}$ is covering. π_2 (seen as function in $\mathbb{N}^*(c)$ instead of in the site) is our g. Where we need the axiom of choice to pick an m to pair with n, Moerdijk's projection functions make the choice unnecessary.

The reason that this approach fails in our case, is that we cannot guarantee that the projection function $[\boldsymbol{\pi_1}]$ is a morphism in our site: it could happen that for some n, there are unbounded many m such that $(n, m) \in d$. Then the projection function $[\boldsymbol{\pi_1}]$ would map unbounded many elements to a single element, which is not permitted in our site.

From the example, we conclude that although our site has all pullbacks, it does not have all products.

Chapter 5

Conclusions and suggestions for future research

In this thesis, we presented two sheaf models for non-standard arithmetic and one candidate non-standard sheaf model. We shortly summarise the descriptions of these models and list the principles that we found are valid in these models. We also comment on the remaining open questions, which we suggest for future research.

The model $\mathbb{N}^{\mathbb{N}}/_{\sim}$:

- is a model in the category $\mathrm{Sh}(\mathbb{P}, J)$, based on the site consisting of the poset of infinite subsets of the natural numbers with covering condition that a cover c should contain a finitely many $d_i \leq c$ such that the union of these d_i misses only finitely many elements of c.
- maps each $c \in \mathbb{P}$ to the set of equivalence classes of functions from $\mathbb{N}$ to $\mathbb{N}$:
$$\mathbb{N}^{\mathbb{N}}/_{\sim}(c) = \{[f : \mathbb{N} \to \mathbb{N}]\},$$
where $f \sim g$ iff $\{n \in c \mid f(n) \neq g(n)\}$ is finite.
- has the natural numbers object as a strict subsheaf, describing standard natural numbers as equivalence classes of functions with finite image.
- has both standard and non-standard, infinite numbers. The standard and infinite predicates are $\neg\neg$-stable.
- models the Peano axioms, as well as the non-standard principles overspill, underspill, transfer, idealisation and realisation.

To prove these results, we use a classical meta-theory. The remaining open questions are whether non-classical realisation is valid in this model. Also the question of whether the standard predicate is decidable remains unanswered. We hope future research will give an answer.

The model $\mathbb{N}^*$:

- is a model in the category $\mathrm{Sh}\,(\mathcal{P}(\mathbb{N}), J')$, based on the site consisting of the powerset of the natural numbers and equivalence classes of continuous functions. The covering condition states that a family of morphisms is covering if there are finitely many $[\mathbf{f_i}] : d_i \to c, i \leq k$ in this family such that for every $M_i, i \leq k$ there exists an N such that for every $n \in c$, if $n \geq N$ then there is an i and an $m \in d_i$ such that $m \geq M_i$ and $\mathbf{f_i}(m) = n$. This is a stricter version of the generalisation of the covering condition of the site $(\mathbb{P}, J)$, and equivalent to the Frechet filter restriction of the site described in [Moerdijk95].
- maps each $c \in \mathbb{P}$ to the set of equivalence classes of functions from c to $\mathbb{N}$:

  $$\mathbb{N}^{\mathbb{N}}/_{\sim}(c) = \{[c : \mathbb{N} \to \mathbb{N}]\},$$

  where $f \sim g$ iff there exists an N such that for all $n \in c$ with $n \geq N \to f(n) = g(n)$.
- has the natural numbers object as a strict subsheaf, describing standard natural numbers as equivalence classes of functions with bounded image.
- has both standard and non-standard, infinite numbers. The standard and infinite predicates are $\neg\neg$-stable.
- models the Peano axioms, as well as the non-standard principles overspill, underspill, transfer, idealisation and realisation.

For some of the results, we no longer needed a classical meta-theory. However, we still could not prove everything entirely constructively. A promising improvement would be to find a site that is close to the one we use here, but has all finite products.

The candidate model $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}$:

- is also a model in the category $\mathrm{Sh}\,(\mathbb{P}, J)$.
- maps each $c \in \mathbb{P}$ to the set of equivalence classes of functions from $\mathbb{N}$ to finite functions from $\mathbb{N}$ to $\mathbb{N}$:

  $$\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}(c) = \{[g] \mid g : \mathbb{N} \to \{[h : c \to \mathbb{N}] \mid \operatorname{Im}(h) \text{ is finite}\}\}$$

  Where:

  $$g \sim_U g' \iff \{d \leq c \mid \forall n \in d : g(n) \cdot d = g'(n) \cdot d\} \text{ covers } c$$

  and:

  $$h \sim h' \iff \{n \mid h(n) \neq h'(n)\} \text{ is finite}$$

- has the natural numbers object as a subsheaf, describing standard natural numbers as equivalence classes of constant functions.
- has the sheaf $\mathbb{N}^{\mathbb{N}}/_{\sim}$ as subsheaf.

The construction of this model was based on the well-known ultrafilter construction for non-standard models of natural arithmetic. We use a filter, and are curious if the ultrafilter property is needed in the categorical version of this construction.

We believe that the subsheaves $\mathbf{a}\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}/_{\sim}$ are strict subsheaves, but it could turn out that $\mathbb{N}^{\mathbb{N}}/_{\sim}$ is isomorphic to $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_u}$. Future research should point out whether $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_u}$ is also a non-standard model for natural arithmetic.

Appendix A

A second non-standard model in $\mathcal{E}$?

During our study of the site $(\mathbb{P}, J)$, we briefly considered another sheaf as possible non-standard model for natural arithmetic. In this appendix, we will describe the construction of this sheaf, and its relation to both the standard natural numbers and $\mathbb{N}^{\mathbb{N}}/_{\sim}$.

The construction of our sheaf is based on the ultrafilter construction, which is often used to obtain non-standard models for the natural numbers without using category theory: A non-principal ultrafilter U on the natural numbers induces an equivalence relation on $\mathbb{N}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathbb{N}\}$:

$$f \sim_U g \iff \{n \in \mathbb{N} \mid f(n) = g(n)\} \in U$$

The quotient $\mathbb{N}^{\mathbb{N}}/_{\sim_u}$ then yields a non-standard model.

We translate this to our categorical setting. Instead of $\mathbb{N}$, we take the natural numbers object $\mathbf{a}\mathbb{N}$. A filter on $\mathbf{a}\mathbb{N}$ is a subobject of the powerobject of $\mathbf{a}\mathbb{N}$ (compare to a filter in Set: a subset of the powerset of $\mathbb{N}$). The powerobject of $\mathbf{a}\mathbb{N}$ is $\Omega^{\mathbf{a}\mathbb{N}}$, where Ω is the *subobject classifier* of the topos $\mathcal{E}$. We did not treat the subobject classifier in the preliminaries. For an introduction to subobject classifiers and their relevance in sheaf theory, we refer to [MacLane&Mo92, chapter 1, section 3 and 4 and chapter 4] or the entry in NLab: `http://ncatlab.org/nlab/show/subobject+classifier`

To find out to which sets $\mathbf{a}\mathbb{N}^{\mathbf{a}\mathbb{N}}$ maps the c in $\mathbb{P}$, we use the following lemma.

Lemma A.0.1. *Let (C, J) be a site, $A \in \mathrm{Sh}(C, J)$ and $B \in \mathrm{PSh}(C)$. Then:*

1. *A^B is a sheaf*
2. *$A^B \cong A^{\mathbf{a}B}$*

Proof. For a proof of 1, we refer to [MacLane&Mo92, par III.6, proposition 1]. So for the proof of 2, we assume we already know A^B is a sheaf.

By the Yoneda lemma, we have $A^B \cong A^{\mathbf{a}B}$ iff $\mathrm{Hom}\left(-, A^B\right) \cong \mathrm{Hom}\left(-, A^{\mathbf{a}B}\right)$. Suppose $C \in \mathrm{Sh}(\mathcal{C}, J)$, then we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \to A^{\mathbf{a}B}}{C \times \mathbf{a}B \to A}}{\mathbf{a}(\imath C \times B) \to A}\,(1,2)}{\imath C \times B \to \imath A}\,(3)}{\imath C \to (\imath A)^B}}{\imath C \to \imath(A^B)}\,(1)}{C \to A^B}$$

Where we have used that:

(1) both $\mathbf{a}$ and $\imath$ preserve finite limits

(2) $\mathbf{a} \circ \imath$ is naturally isomorphic to the identity functor

(3) $\mathbf{a} \dashv \imath$

Which is enough to prove statement 2. □

The object $\mathbf{a}\mathbb{N}^{\mathbf{a}\mathbb{N}}$

Applying lemma A.0.1 to $\mathbf{a}\mathbb{N}^{\mathbf{a}\mathbb{N}}$ yields:

$$\mathbf{a}\mathbb{N}^{\mathbf{a}\mathbb{N}} \cong \mathbf{a}\mathbb{N}^{\mathbb{N}}$$

We use the same reasoning as in finding equation 3.4 and the description of $\mathbf{a}\mathbb{N}$ found in equation 3.1 to get:

$$\begin{aligned} \mathbf{a}\mathbb{N}^{\mathbb{N}}(c) &= \mathrm{Hom}\,(yc \times \mathbb{N}, \mathbf{a}\mathbb{N}) \\ &\cong \{g : \mathbb{N} \to \mathbf{a}\mathbb{N}(c)\} \\ &= \{g : \mathbb{N} \to \{[h : c \to \mathbb{N}] \mid \mathrm{Im}(h) \text{ is finite}\} \end{aligned}$$

To complete our description of the candidate non-standard model, we define an equivalence relation on $\mathbf{a}\mathbb{N}^{\mathbb{N}}$. This equivalence relation is induced by a filter on $\mathbf{a}\mathbb{N}$. As mentioned above, a filter is a subobject of the powerobject $\Omega^{\mathbf{a}\mathbb{N}}$ of $\mathbf{a}\mathbb{N}$. So to define the filter, we first need the subobject classifier Ω.

The subobject classifier Ω

Ω is the functor that sends $c \in \mathbb{P}$ to the set of all closed sieves on c:

$$\begin{aligned} \Omega(c) = \{S \mid\, & S \text{ is a sieve and for any } d \leq c : \\ & \text{if } d \cap S \text{ covers } d, \text{ then } d \in S\} \end{aligned}$$

For a proof of this statement, we refer the reader to [MacLane&Mo92, par III.7, proposition 3].

The powerobject $\Omega^{\mathbf{a}\mathbb{N}}$

From the subobject classifier, we build the powerobject $\Omega^{\mathbf{a}\mathbb{N}}$. By lemma A.0.1, it is isomorphic to $\Omega^{\mathbb{N}}$:

$$\Omega^{\mathbb{N}}(c) = \mathrm{Hom}\,(yc \times \mathbb{N}, \Omega)$$

Again, using the same reasoning that led to equation 3.4:

$$\Omega^{\mathbb{N}}(c) = \mathrm{Hom}\,(yc \times \mathbb{N}, \Omega) \cong \{X : \mathbb{N} \to \Omega(c)\}$$

The filter

Now that we have the powerobject of the natural numbers, we can define a filter U on the natural numbers, which should be a subobject of the powerobject $\Omega^{\mathbb{N}}$, so its components $U(c)$ are subsets of $\Omega^{\mathbb{N}}(c)$.

Definition A.0.1. We put $X : \mathbb{N} \to \Omega(c) \in U(c)$ iff the following set covers c:

$$\{d \leq c \mid \forall n \in d : d \in X(n)\}$$

Proposition A.0.2. *U has the following properties:*

1. *U is a subobject of $\Omega^{\mathbb{N}}$. That is:*
   - *(a) U is well defined: $U : \mathbb{P}^{\mathrm{op}} \to$ Sets*
   - *(b) Set inclusion: for all $c \in \mathbb{P}$, $U(c) \subseteq \Omega^{\mathbb{N}}(c)$*
   - *(c) For any morphism $d \leq c$, the restriction $U(c) \to U(d)$ agrees with $\Omega^{\mathbb{N}}(c) \to \Omega^{\mathbb{N}}(d)$*
   - *(d) For all $c \in \mathbb{P}$, all covers S of c and each $X \in \Omega^{\mathbb{N}}(c)$ we have: if for all $d \leq c$ in S the restriction of X is in $U(d)$, then X is in $U(c)$.*
2. *U is a filter. That is, for all $c \in \mathbb{P}$ we have:*
   - *(a) $U(c) \neq \emptyset$, $U(c) \neq \Omega^{\mathbb{N}}(c)$.*
   - *(b) if $X, Y \in U(c)$, then $X \wedge Y \in U(c)$.*
   - *(c) if $X \in U(c)$ and $X \leq Y$, then $Y \in U(c)$.*

Proof. 1. Parts (a), (b) and (c) are clear from the definition. For part (d), pick an arbitrary $c \in \mathbb{P}$, a cover S of c and an $X \in \Omega^{\mathbb{N}}(c)$. Suppose that for all $d \leq c$ in S the restriction of X is in $U(d)$. That is, for all $d \leq c$, the following set covers d:

$$\{e \leq d \mid \forall n \in e : e \in (X \cdot f)(n)\}$$

(Where $X \cdot f$ denotes the restriction of X along the arrow $f : c \to d$ in $\mathbb{P}^{\mathrm{op}}$.)

Recall that S covers c means that there are finitely many elements in S whose union is equal to c up to a finite set. So there are $d_1, \ldots, d_k$ in S and a finite set i_c such that

$$d_1 \cup \ldots \cup d_k \cup i_c = c.$$

For each of these d_j we have from the fact that the restriction of X is in $U(d_j)$, the covering set

$$S_{d_j} = \{e \leq d_j \mid \forall n \in e : e \in (X \cdot f)(n)\}.$$

So for each d_j, there are $e^j_1, \ldots e^j_{l_j}$ in S_{d_j} and a finite set i_j such that

$$e^j_1 \cup \ldots \cup e^j_{l_j} \cup i_j = d_j.$$

Now as

$$S_{d_j} \subseteq Q := \{e \leq c \mid \forall n \in e : e \in X(n)\},$$

we have that Q covers c:

$$\begin{aligned}
c &= d_1 \cup \ldots \cup d_k \cup i_c \\
&= (e^1_1 \cup \ldots \cup e^1_{l_1} \cup i_1) \cup \ldots \cup (e^k_1 \cup \ldots \cup e^k_{l_k} \cup i_k) \cup i_c \\
&= \bigcup_{j=1}^{k} (e^j_1 \cup \ldots \cup e^j_{l_j} \cup i_j) \cup i_c \\
&= \bigcup_{j=1}^{k} (e^j_1 \cup \ldots \cup e^j_{l_j}) \cup I
\end{aligned}$$

Where $I = \bigcup_j i_j \cup i_c$ is a finite union of finite sets, so again a finite set. So we have finitely many elements of Q whose union equals c up to some finite set. Hence Q covers c. Then by definition of U we have: $X \in U(c)$, which we needed to show.

2. (a) is immediately clear, as the function mapping n to the maximal sieve is always in $U(c)$, while the function mapping n to the empty sieve is never in $U(c)$. So $U(c)$ is neither empty nor equal to $\Omega^{\mathbb{N}}(c)$

   For (b), note that $X \wedge Y : \mathbb{N} \to \Omega(c)$ is well defined: $X \wedge Y(n) = X(n) \cap Y(n)$, which is again a closed sieve. Now we have by definition that $X \wedge Y \in U(c)$ iff the following set covers c:

$$\{d \leq c \mid \forall n \in d : d \in X(n) \cap Y(n)\}$$

   Now:

$$\begin{aligned}
\{d \leq c \mid \forall n \in d : d \in X(n) \cap Y(n)\} = &\{d \leq c \mid \forall n \in d : d \in X(n)\} \cap \\
&\{d \leq c \mid \forall n \in d : d \in Y(n)\}.
\end{aligned}$$

   As both X and Y are in $U(c)$, the latter two sets are known to cover c. The intersection of two covers is again a cover, so $X \wedge Y \in U(c)$.

   The proof of (c) is analogous.

□

The candidate non-standard model: the quotient $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}$

We use the filter U to define an equivalence relation $\sim_U$ on $\mathbf{a}\mathbb{N}^{\mathbb{N}}(c)$:

$$\begin{aligned} g \sim_U g' &\iff X : n \mapsto \{d \leq c \mid g(n) \cdot d = g'(n) \cdot d\} \in U \\ &\iff \{d \leq c \mid \forall n \in d : g(n) \cdot d = g'(n) \cdot d\} \text{ covers } c \end{aligned}$$

Then the second candidate non-standard model for natural arithmetic is:

$$\begin{aligned} \mathbf{a}\mathbb{N}^{\mathbb{N}}/U &= \mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U} \\ &= \{[g] \mid g : \mathbb{N} \to \{[h : c \to \mathbb{N}] \mid \operatorname{Im}(h) \text{ is finite}\}\} \end{aligned}$$

Where:

$$g \sim_U g' \iff \{d \leq c \mid \forall n \in d : g(n) \cdot d = g'(n) \cdot d\} \text{ covers } c$$

and:

$$h \sim h' \iff \{n \mid h(n) \neq h'(n)\} \text{ is finite}$$

Which is quite a mouthful of equivalence classes. Compare this to our first model: instead of functions from $\mathbb{N}$ to $\mathbb{N}$, we now have functions from $\mathbb{N}$ to functions from $\mathbb{N}$ to $\mathbb{N}$.

We elaborate on the relation between the natural numbers object and the two sheaves $\mathbb{N}^{\mathbb{N}}/_{\sim}$ and $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}$.

A.1 The relation between $\mathbf{a}\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}/_{\sim}$ and $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}$

We will prove that we have the following commuting diagram:

$$\begin{tikzcd} \mathbf{a}\mathbb{N} \arrow[r, hook] \arrow[d, hook] & \mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U} \\ \mathbb{N}^{\mathbb{N}}/_{\sim} \arrow[ur, hook] & \end{tikzcd}$$

We have already seen (proposition 3.2.2) that the natural numbers object is a subsheaf of $\mathbb{N}^{\mathbb{N}}/_{\sim}$. We now find that it is also a subsheaf of $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}$. The approach for finding this subsheaf is the same as in proposition 3.2.2; we first define a subsheaf $\mathrm{St}^U_{\mathbb{N}}$ of $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}$, and then we prove that this subsheaf is isomorphic to the natural numbers object.

Definition A.1.1. *Standard numbers in* $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}$
We define $\mathrm{St}^U_{\mathbb{N}}$ as:

$$\mathrm{St}^U_{\mathbb{N}}(c) = \{[g] \in \mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}(c) \mid \exists h : c \to \mathbb{N}\ \forall n\ [[g(n)] = [h]]\}$$

Proposition A.1.1.

1. *$\mathrm{St}^{\mathcal{U}}_{\mathbb{N}}$ forms a subsheaf of $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_{\mathcal{U}}}(c)$.*

2. *$\mathrm{St}^{\mathcal{U}}_{\mathbb{N}}$ is isomorphic to the natural numbers object $\mathbf{a}\mathbb{N}$.*

Proof.

1. $\mathrm{St}^{\mathcal{U}}_{\mathbb{N}}$ clearly forms a subpresheaf of $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_{\mathcal{U}}}$, so all we need to check is that every matching family has a unique amalgamation. Let $S = \{d_i \mid i \in I\}$ be a cover of some $c \in \mathbb{P}$, and $\{(d_i, [g_i])\}$ be a matching family, where all g_i are standard. We need to find an amalgamation of this matching family. That is, we need to find a $[g] \in \mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_{\mathcal{U}}}(c)$, such that $[g]$ is a standard natural number and $[g] \cdot d_i = [g_i]$. As $[g_i]$ are all standard, there are $h_i : d_i \to \mathbb{N}$ such that $[g_i(n)] = [h_i]$ for each $n \in d_i$. Now construct h as follows:

$$h(n) = \begin{cases} h_i(n) & \text{if } n \in d_i \\ 0 & \text{else} \end{cases}$$

   Then, let g be defined as $g(n) = h$ for all $n \in c$. Then per definition, $[g]$ is a standard natural number. Also, $[g]$ is an amalgamation of $[g_i]$, as $[g] \cdot d_i = [g_i]$. Uniqueness follows from the equivalence relations. So $\mathrm{St}^{\mathcal{U}}_{\mathbb{N}}$ is indeed a subsheaf of $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_{\mathcal{U}}}$.

2. Sending $[g]$ with constant image $g(n) = h$ to the equivalence class $[h]$ works as an isomorphism between $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_{\mathcal{U}}}(c)$ and $\mathbf{a}\mathbb{N}(c)$.

□

Which gives us:

$$\mathbf{a}\mathbb{N} \cong \mathrm{St}^{\mathcal{U}}_{\mathbb{N}} \hookrightarrow \mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_{\mathcal{U}}}$$

We complete the triangle by finding an embedding of $\mathbb{N}^{\mathbb{N}}/_{\sim}$ into $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_{\mathcal{U}}}$.

Proposition A.1.2.
The following natural transformation is an embedding of $\mathbb{N}^{\mathbb{N}}/_{\sim}$ into $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_{\mathcal{U}}}$:

$$\begin{aligned} \eta_c : \mathbb{N}^{\mathbb{N}}/_{\sim}(c) &\to \mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_{\mathcal{U}}}(c) \\ [f : \mathbb{N} \to \mathbb{N}] &\mapsto [g : \mathbb{N} \to \{[h : c \to \mathbb{N}]\}] \\ f &\mapsto g \textit{ with } g(n) = \mathit{const}_{f(n)} \end{aligned}$$

Proof. First of all, notice that η is well defined. That is, it is indeed a natural transformation and if $f \sim f'$, then $\eta_c(f) \sim_{\mathcal{U}} \eta_c(f')$, which follows by writing out the definitions of the two equivalence relations. Injectivity follows just as easily. □

This completes the triangle:

$$
\begin{tikzcd}
\mathbf{a}\mathbb{N} \arrow[r, hook] \arrow[d, hook] & \mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U} \\
\mathbb{N}^{\mathbb{N}}/_{\sim} \arrow[ur, hook] &
\end{tikzcd}
$$

The following questions remain open:

- Are $\mathbf{a}\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}/_{\sim}$ and $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}$ all non-isomorphic? We think a cardinality argument should suffice to prove this.

- Is $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}$ a non-standard model of natural arithmetic? And if so, how does it compare to $\mathbb{N}^{\mathbb{N}}/_{\sim}$ and Moerdijk's model

- Is U an ultrafilter? Although we based our ideas on the ultrafilter construction, it is interesting to know if for $\mathbf{a}\mathbb{N}^{\mathbb{N}}/_{\sim_U}$ to be a non-standard model we need that U is an ultrafilter or that having just a filter is enough.

Bibliography

[Avigad02] J. Avigad and J. Helzner. Transfer principles in nonstandard intuitionistic arithmetic. Arch. Math. Logic, volume 41, issue 6 (2002), p. 581 - 602.

[Awodey06] S. Awodey, Category Theory, Claredon Press, Oxford (2006).

[Berg12] B. van den Berg, E. Briseid, P. Safarik: A functional interpretation for nonstandard arithmetic. Annals of Pure and Applied Logic, volume 163, issue 12 (2012), p. 1962 - 1994.

[Boyer11] C. B. Boyer, U. C. Merzbach, A History of Mathematics, John Wiley & Sons, Jan 25 (2011).

[Dedekind61] R. Dedekind, Was sinds und was sollen die Zahlen?, neunte unveränderte Auflage, Fiedr. Vieweg & Sohn Braunschweig (1961).

[Frege1883] G. Frege, Grundgesetze der Arithmetik, band I, Verlag Hermann Pohle, (1893).
(pdf: `http://www.marilia.unesp.br/Home/Instituicao/Docentes/RicardoTassinari/DGA.pdf`)

[Gray08] J. Gray, Plato's Ghost, the modernist transformation of mathematics, Princeton University Press (2008).

[Heyting56] A. Heyting, Intuitionism: An Introduction, North-Holland Publishing Co. Amsterdam (1956).

[Johnstone02] P.T. Johnstone, Sketches of an Elephant, a topos theory compendicum, Volume 1, Claredon Press, Oxford (2002).

[MacLane&Mo92] S. Mac Lane and I. Moerdijk, Sheaves in Geometry and Logic, a First Introduction to Topos Theory, Springer-Verlag New York (1992).

[Moerdijk95] I. Moerdijk, A model for inuitionistic non-standard arithmetic, Annals of Pure and Applied Logic 73 (1995), p. 37-51.

[Moerdijk97] I. Moerdijk and E. Palmgren, Minimal models of Heyting arithmetic. Journal of Symbolic Logic, volume 62, issue 4 (1997), p. 1448 - 1460.

[Nelson77] E. Nelson, Internal set theory: a new approach to nonstandard analysis. Bull. Amer. Math. Soc., volume 83, issue 6 (1977), p. 1165-1198.

[Palmgren01] E. Palmgren, Real numbers in the topos of sheaves over the category of filters, Journal of Pure and Applied Algebra 160 (2001), p. 275 - 284.

[Peano1889] G. Peano, Arithmetices Principia Novo Methodo Exposita, Bocca(Augustae Taurinorum) (1889).

[Skolem34] Th. Skolem, Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen. Fundam. Math. volume 23 (1934), p. 150 - 161.

[Stanford-A] `http://plato.stanford.edu/entries/aristotle-mathematics/#10`

[Standford-P] `http://plato.stanford.edu/entries/pythagoras/`