Theories of size for infinite collections

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Abstract

There have been recent developments in mathematics that indicate that the theory of infinite collections exemplified by Cantor's cardinals and ordinals does not have to be the only alternative to be considered.

Here we are interested in investigating different ways of comparing the size of infinite collections — we will not enter the debate over their existence.

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Chapter 1

Introduction

Suppose you want to teach your child about counting the elements of infinite collections. They have already learnt how to count elements of finite collections, so for example they know that if they have a bag of oranges and a bag of apples, they can compare how many apples and oranges there are by putting all their oranges on a line and then juxtaposing to each one apple, and then if they have (at least) one orange with no apple by the side, there are more oranges, and vice versa if there is at least one apple with no orange by the side, it means there are more apples. Your child though also knows that if they count all of their apples, and then eat one and give another to a friend, they will have fewer apples than before, because what they have is only a part of what they used to have. Suppose now they want to know how many even numbers there are with respect to all natural numbers. They know that the even numbers are what remains of the natural numbers if you remove all the odds, so there should be fewer even numbers than all natural numbers. At the same time, though, you show them how to associate exactly one even number for each natural number, so that the apples-and-oranges scenario is recreated. What should your child conclude? that there are just as many evens as the natural numbers, or that the evens are strictly fewer?

The standard response[1] would be to say that the reasoning behind the 'fewer than' conclusion is ultimately flawed, because one finds oneself very quickly in situations where that reasoning does not help in determining the relative size of two infinite collections — Cantor has shown the one-to-one correspondence approach to be the most proficuous one, so we should forget about the other.

1. by 'standard' I mean the response that would mirror the default mathematical treatment of the size of infinite quantities, i.e. Cantorian cardinals.

I suspect that child wouldn't be particularly impressed by such an explanation. Offering Cantor's definition of cardinals as an answer to the problem of how many evens there are does not explain the pull exerted by the other answer.

1.1 Two intuitions, two principles

In the recent literature on the topic[2] the dilemma faced by the child above is presented via the classical paradox of the squares of the natural numbers in the form first verbalised by Galilei.[3] Briefly, the paradox is that the collection of the natural numbers on one hand, and the collection of all squared numbers on the other hand, seem to have both the same number of elements and to be one smaller than the other (the second collection smaller than the first). The contradiction arises as follows

(P1) If there is a one-to-one correspondence between two collections, then they have the same number of elements.

(P2) If one collection is a proper part of another, then it is strictly smaller than the other.

(P3) There is a one-to-one correspondence between the natural numbers on one hand, and the squares on the other.

(P4) The collection of all squares is properly included in the collection of all natural numbers.

(C1) (from (P1) and (P3)) The collection of all natural numbers and that of all squares have the same number of elements.

(C2) (from (P2) and (P4)) The collection of all squares is strictly smaller than the collection of all natural numbers.

(C) (from (C1) and (C2)) The two collections have the same number of elements, and at the same time one is smaller than the other.

Galilei (and Mancosu) take the conclusion to be contradictory, because the notion of 'smaller than' is translated into 'has fewer elements than', so that

2. Paolo Mancosu, "Measuring the size of infinite collections of natural numbers: was Cantor's theory of infinite number inevitable?," *The Review of Symbolic Logic* 2, no. 4 (December 2009); Matthew W. Parker, "Set-size and the Part-Whole Principle," *The Review of Symbolic Logic* 6, no. 4 (December 2013); Paolo Mancosu, "In Good Company? On Hume's Principle and the Assignment of Numbers to Infinite Concepts," *Review of Symbolic Logic* 8, no. 2 (2015): 370–410.

3. Mancosu, "Measuring the size of infinite collections of natural numbers: was Cantor's theory of infinite number inevitable?"

(C2) becomes a negation of (C1). Mancosu then argues that, in the course of history, this contradiction has led to four positions regarding the possibility of completed infinite collections:

1. The paradox illustrates that there is something fundamentally wrong with the very notion of different infinities, thus the resolution of the paradox should be to reject the existence of different infinities altogether. This is the position that concedes the least to the exploration of a mathematical treatment of the infinite.
2. There are infinite collections but it is misguided to try and apply concepts of size comparison to them.
3. Infinite collections exist and can be compared, but their part-whole relations are inherently different from those occurring within the class of finite collections, hence they cannot be treated in the same way as finite collections.
4. Infinite collections can and should be treated in a way that extends and generalises the arithmetical treatment of finite quantities.[4]

Of these four positions, the only one that clearly entails investigating extensions of the theory of finite arithmetic to infinite "numbers" is the fourth one. Whereas positions 1. to 3. seem to be mainly documented in ancient, medieval and modern texts [5], the fourth position is the one that is nowadays considered the default one, and the one advocated for by both Cantor and Bolzano.[6] Bolzano and Cantor[7] did nevertheless disagree on what it meant to try and develop an arithmetic of the infinite that would be similar to that of finite quantities. In order to phrase this disagreement, we need to reconsider the paradox of Galilei's outlined above. The premises that we labelled as (P1) and (P2) can be rephrased in set-theoretic terms as follows:

(CP) Given two sets A and B, $size(A) = size(B)$ if and only if there is a one-to-one correspondence between A and B.

(PW) If a set A is a proper subset of set B (written $A \subset B$) then $size(A) < size(B)$.

4. Mancosu, "Measuring the size of infinite collections of natural numbers: was Cantor's theory of infinite number inevitable?," p. 616.

5. the reader should consult Mancosu *ibid.* for a list of philosophers endorsing each of the views listed here

6. Bernard Bolzano, *Theory of Science* (Reidel, 1973); Bernard Bolzano, *Paradoxes of the Infinite* (Routledge / Kegan Paul, 1950).

7. Georg Cantor, "Ein Beitrag zur Mannigfaltigkeitslehre," in *Abhandlungen mathematischen und philosophischen Inhalts*, ed. Adolf Fraenkel (Georg Olms Verlagsbuchhandlung, 1966), 119–138.

The labels stand for Cantor's principle and Part-whole principle, respectively, because this is how the two principles are often designated in the literature. Thus the difference between Cantor and Bolzano is that Cantor would want to extend (CP) to be valid for all infinite collections, as well as the finite ones, whereas Bolzano believed that one should keep (PW) when developing a rigorous treatment of infinities.

We should be careful however in that (CP) is not quite just the translation of (P1) in set theoretic terms. (CP) is in fact a biconditional, whereas (P1) is only a conditional statement. (PW) is also sometimes called Euclid's principle,[8] and it is perceived to be a version of the fifth common notion of Euclid's *Elements*:

(CN5) The whole is larger than its parts.

Once Cantor introduced and developed his system of powers (what we nowadays call cardinals), the success of his approach eventually meant that defending the part-whole approach was regarded as a mistake, and Cantor's way of treating the infinite was to be seen as the only correct way.

In recent years, however, scholars have been proposing treatments of the size of infinite collections that would preserve (PW) for infinite collections, thus dropping (CP) as guiding principle for their notion of size. The proposal that we are going to discuss the most in this thesis is the so-called theory of numerosities, that is at once the most recent, and the most mathematically developed, of the non-Cantorian theories of size we are aware of. This seems to give reason to reconsider the 'solution' of the Galilean paradox, because now there seem to be three options for anyone who wants to develop an arithmetic for infinite quantities:

(O1) There is only one correct solution to the paradox, as there is only one correct notion of size for infinite collections. This is the solution that preserves (CP) and drops (PW). The correct mathematical treatment of size is the one offered by the theory of cardinals.

(O2) There is only one correct solution to the paradox, as there is only one correct notion of size for infinite collections. This is the solution that preserves (PW) and drops (CP). The one correct mathematical treatment of size is the one offered by the theory of numerosities.

(O3) The paradox admits for two incompatible solutions, that is either dropping (CP) and stipulating to use the theory of numerosities as the theory of size for infinite collections, or dropping (PW) and using the theory of

8. Parker, "Set-size and the Part-Whole Principle."

cardinals as the theory of size for infinite collections. The two principles (CP) and (PW) would be epistemically on equal standing

(O4) The paradox can be explained away by giving a different interpretation of one of the two principles, thus removing the incompatibility. This is the strategy followed in the present thesis, where we offer a new interpretation of the part-whole principle which does not weaken it, thus showing that (CP) and (PW) do not exemplify two competing notions of size for infinite collections.

Having said this, we give the full structure of the thesis in a nutshell.

1.2 Thesis plan

In the introduction, we offer a brief presentation of the problem of the treatment of the size of infinite collections in mathematics, problem this thesis proposes to address.

In Chapter 2 of the thesis, we introduce the standard solution to the problem, namely Cantor's theory of cardinals, and we compare it with the theory of ordinals; we also introduce the main other player, the theory of numerosities.

In Chapter 3 we consider the only known philosophical argument for the inevitability of Cantor's cardinals as the extension of the natural numbers into infinite quantities, namely Gödel's argument, and consider one set of responses. We conclude by introducing a way of reelaborating Gödel's defence of Cantor's cardinals offered recently by Parker.[9]

In Chapters 4 and 5 we articulate our reply to Parker's challenge and illustrate its significance within the context of giving a conceptual interpretation to the theory of numerosities.

In the conclusion we present a list of open problems and ways in which our proposal could be improved upon.

9. Matthew W. Parker, "Philosophical Method and Galileo's Paradox of Infinity," in *New Perspectives on Mathematical Practices: Essays in Philosophy and History of Mathematics: Brussels, Belgium, 26-28 March 2007*, ed. Bart Van Kerkhove (World Scientfic, 2009).

Chapter 2

Preliminaries

In what follows we are going to recall the standard definitions of ordinals and cardinals given in modern-day treatment of ZFC set theory, recall some fundamental results concerning cardinals and ordinals, and then summarise Cantor's own work on ordinal and cardinal numbers. We are aware that some readers might be afraid that, considered Cantor developed his own work outside of an axiomatic framework, a comparison between what is done with ZFC and what he achieved in his set-theoretic results is wrong-headed. We do not want to enter the merits of the debate, but we are indeed assuming that ZFC is, to some extent, an attempt at giving an axiomatic form to Cantor's work, and that is all we ask the reader to accept. Our exposition has only two objectives: to recall some fundamental definitions and concepts that are going to be central to this work, and to clarify why we are dealing with cardinals and not ordinals when discussing different options for theories of size.

2.1 Ordinals

In order to define the concept of ordinal number, it is best to give the following preliminary definitions

Definition 2.1.1 (Transitive set)**.** A set a is transitive if and only if for any $b \in a,\ b \subseteq a$.

Definition 2.1.2 (Well-ordered set, well-ordering)**.** A set a is well-ordered by a relation r iff r is a total order over a and for any non-empty subset b of a, there is $a_1 \in b$ that is the r-least element of b. A set a that is well-ordered by a relation r is also called a *well-ordering* and it is written as $\langle a, r\rangle$.

Definition 2.1.3 (Order isomorphic)**.** Given two well-orderings $\langle a, r\rangle$ and $\langle b, s\rangle$, we say that $\langle a, r\rangle$ and $\langle b, s\rangle$ are order isomorphic if and only if there is a bijection f between a and b such that for all $a_1, a_2 \in a, a_1 r a_2 \leftrightarrow f(a_1) s f(a_2)$.

Definition 2.1.4 (Order type)**.** Given a well-ordering $\langle a, r\rangle$, the order type (a, r) is the unique ordinal α such that $\langle a, r\rangle \cong \alpha$.

Finally, the main definition:

Definition 2.1.5 (Ordinal)**.** A set a is an *ordinal* iff a is transitive and well-ordered by $\in$.

Although we tend to refer to ordinals just as the respective underlying sets, without considering the order $\in$, an ordinal is not completely determined just by a list of its elements; the well-order needs to be specified, too. Each ordinal is in fact the canonical representative of a class of order-isomorphic sets. [1]

2.1.1 Arithmetical operations that are definable on the ordinals

There are a few arithmetical operations that are characteristic of the natural numbers but that nonetheless can be defined for all ordinals as well. Here we give in particular the definitions of successor ordinal, sum and multiplication.

Definition 2.1.6 (Successor)**.** If α is an ordinal, then $S(\alpha) = \alpha \cup \{\alpha\}$. S is the successor operation.

Definition 2.1.7 (Sum and multiplication)**.** Let α, β be ordinals. Then we can define

$\alpha + \beta = \mathit{type}(\alpha \times \{0\} \cup \beta \times \{1\}, r)$, where $r = \{\langle\langle \xi, 0\rangle, \langle \eta, 0\rangle\rangle : \xi < \eta < \alpha\} \cup \{\langle\langle \xi, 1\rangle, \langle \eta, 1\rangle\rangle : \xi < \eta < \beta\} \cup [(\alpha \times \{0\}) \cup (\beta \times \{0\})]$

$\alpha \cdot \beta = \mathit{type}(\beta \times \alpha, r)$, where $\langle \eta, \xi\rangle r \langle \xi', \eta'\rangle \leftrightarrow (\xi < \xi' \vee (\xi = \xi' \wedge \eta < \eta'))$

Informally, the definition for addition means that the sum of two ordinals α and β corresponds to the disjoint union of the underlying sets, paired with an order that will put the elements of the first addend first, and those from the second addend afterwards.

1. This is the definition first proposed by von Neumann in "On the introduction of transfinite numbers", although there the author defines each ordinal as the set of all preceding ordinals.

The definition for multiplication can similarly be understood in informal terms – multiplication works, just like for the natural numbers, as repeated addition after a fashion. Multiplying α and β means taking β-many copies of α, preserving their α-order.

Just like addition and multiplication of the natural numbers, addition and multiplication of the ordinals obey associativity, but distributivity of multiplication on the right fails, that is $\alpha\cdot(\beta+\gamma) = \alpha\cdot\beta+\alpha\cdot\gamma$, but $(\alpha+\beta)\cdot\gamma \neq \alpha\cdot\gamma+\alpha\cdot\beta$. On the other hand, there are these significant properties that are not valid for ordinals in general, but are valid for the natural numbers, such as commutativity of addition and multiplication, respectively. For multiplication, for example, 2ω is a series of ω-many copies of the finite ordinal 2, whereas $\omega \cdot 2$ are two copies of ω, and therefore in the first one, 2ω, each element has a predecessor, but that is not the case in $\omega \cdot 2$, so the two ordinals are not order-isomorphic, hence they are different.

More interesting features: the ordinal numbers satisfy the Peano axioms in the following form:

1. $0 \in \omega$.
2. $\forall n \in \omega(S(n) \in \omega)$
3. $\forall n, m \in \omega(n \neq m \rightarrow S(n) \neq S(m))$
4. $\forall x \subset \omega((0 \in x \wedge \forall n \in x(S(n) \in x)) \rightarrow x = \omega)$

The last one is the induction axiom for the natural numbers, and it is crucial as induction is seen as one of, if not *the*, defining characteristic of the structure of the natural numbers. Ordinals themselves satisfy a form of induction, that is transfinite induction. Kunen[2] formulates it as a least number principle: Given any class $C \subset On$ (where On is the class of ordinal numbers), if $C \neq 0$ then C has a least element.

2.2 Cardinals and cardinality

As for defining cardinals, there are different (equivalent) formulations of the definition once one assumes the axiom of Choice; here we are going to consider the one offered by Kunen[3] as the standard one. Before defining the cardinals, we need to give the following definition:

2. Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs* (North Holland, 1980).
3. Ibid.

Definition 2.2.1 (Cardinality)**.** Let a be a well-orderable set[4]. Then the cardinality of a, written as $|a|$, is the least ordinal α such that there is a one-to-one correspondence between a and α.

Thus we can define cardinals in terms of cardinality.

Definition 2.2.2 (Cardinal)**.** An ordinal α is a cardinal if and only if $|\alpha| = \alpha$.

We can think of the distinction between cardinalities and cardinals as a distinction between equivalence classes and their canonical representative. The relation from which the equivalence classes arise is that of being in a one-to-one correspondence, and it is usually perceived as partitioning the whole class of sets on the basis of size.

So for example, while it is correct to say that $|\omega \cdot 2| = \aleph_0 = |\omega|$, it is not the case that $\omega \cdot 2 = \omega$ (as ordinals, because they are not order isomorphic).

Definition 2.2.3 (Sum and multiplication)**.** Let κ, λ be cardinals. Then
$\kappa + \lambda = |\kappa \times \{0\}| \cup |\lambda \times \{1\}|$
$\kappa \cdot \lambda = |\kappa \times \lambda|$.

2.2.1 Remarks

From the definition 2.2.3 of cardinal addition and multiplication it becomes apparent that, when computing the cardinality of a set, one forgets about any order-relation among the members, and this allows for addition and multiplication to be commutative, unlike in the case of ordinals; moreover, multiplication distributes over addition, and both operations are associative. All these algebraic properties are desirable, in the sense that they give substance to the claim that cardinal numbers allow one to carry on arithmetic into the infinite in a way that is analogous to finite arithmetic (arithmetic on finite numbers). In this respect, however, there is an important rupture of the analogy: whenever either κ or λ are transfinite cardinals, $\kappa \cdot \lambda = \kappa + \lambda = \max\{\kappa, \lambda\}$. This point is important to mention because the so-called Euclidean theories of size we are going to introduce later on in the thesis are partly motivated by dissatisfaction with this 'flat' arithmetic of infinite cardinals.

Secondly, a point on terminology: some authors (Mancosu included) use 'cardinality' to mean 'size' of a set – cardinality is then the underlying feature of sets that is measured by cardinals or whatever may serve the purpose. In

4. If one assumes the axiom of choice, any set can be well-ordered.

this thesis, however, we will refer to the feature of sets as their ‘size’, and will keep using ‘cardinality’ and ‘cardinal’ strictly in their technical sense. When reporting Cantor’s work and views, we will use ‘power’ instead of ‘cardinal’, as it is Cantor’s own terminology.

Before introducing our representative of choice of Euclidean theories, namely the theory of numerosities, we want to consider the development of ordinals and cardinals as effected by Cantor, because we think it is important to investigate what was the role Cantor himself wanted these two classes of numbers to play.

2.3 Cantor's cardinals and ordinals

Ordinals and cardinals were first introduced by Cantor. The full development of the respective theories started around 1878[5] and continued through decades, at least until 1897, year of publication of Cantor's Beiträge.[6] Here we summarise Cantor's work, with the intent of uncovering the role that ordinals and cardinals were to cover, according to him.

The first mention of the power of a set is in Cantor (1878).[7] Here he introduces such notion by giving a criterion for when two sets have the same power, namely when they are in a one-to-one correspondence. They are also said then to be equivalent – equipollent, in modern terms. (Cantor (1966), p.119). Over the rest of the initial paragraph of this paper Cantor introduces the notion that for any two sets a and b, if a and b themselves are not equipollent, then either a is equipollent to some subset of b or viceversa. Notice that this is tantamount to imposing trichotomy[8] over the notion of cardinality of a set, although the context of the paper and of the letters Cantor was exchanging at the time with Dedekind[9] suggest that Cantor was not taking himself as stipulating trichotomy, rather he seems to take it as an obvious feature implicit in the concept of cardinality itself. The way in which it is phrased also suggests that Cantor already viewed powers as a way of measuring relative size of sets, for he writes: "In the first case [when set M is in one-one correspondence with a subset of set N] we call the power of M smaller; in the second case [when set N is in one-one correspondence with a subset of M] we call the power of M bigger than the power of N". (ibid.)

After this initial mention of powers, Cantor continued developing a theory of

5. Cantor, "Ein Beitrag zur Mannigfaltigkeitslehre."

6. Georg Cantor, "Beiträge zur Begründung der transfiniten Mengenlehre," in *Abhandlungen mathematischen und philosophischen Inhalts*, ed. Adolf Fraenkel (Georg Olms Verlagsbuchhandlung, 1966), 282–356.

7. Cantor, "Ein Beitrag zur Mannigfaltigkeitslehre."

8. Given an order relation $<$ over a class R, the law of trichotmoy is the proposition: $\forall x, y \in R, x < y \lor x = y \lor y < x$. To impose trichotomy for cardinality would mean to impose that, for R the class of cardinalities of all sets, given any two cardinalities x, y, either $x = y$ or $x < y$ or $y > x$. Informally, trichotomy can be considered as total comparability: all elements of a given domain are comparable with one another with respect to a certain ordering relation.

9. José Ferreirós, *Labyrinth of thought: a history of set theory and its role in modern mathematics* (Springer Science & Business Media, 2008).

powers that was consigned to its finished form in his 1897 Beiträge. According to Hallett's and Dauben's reconstructions, Cantor developed the ordinals while trying to expose his theory of cardinal numbers.

The paper considered to be programmatic regarding the theory of cardinals is "Über unendliche, lineare Punktmannigfaltigkeiten, 5";[10] in a letter to Dedekind of November 1882 (Ewald From Kant to Hilbert, p. 875 and ff.), Cantor claims to have found a way of proving that, given sets a, b and c such that a and c can be put into a one-one correspondence and $a \subseteq b \subseteq c$, then b also has the same power as a and c. What Cantor is referring to is the work that is then published in this article, in which he introduces the transfinite ordinals and argues for the conceptual coherence of these, thus concluding in favour of their existence. The steps of his argument are as follows. Firstly, he considers both transfinite ordinals and natural numbers as ordinal numbers, and these in turn are explained as numerals (*Anzahlen*) of well-ordered sets – Cantor's definition of well-ordered sets is different from the current one, although extensionally equivalent.

What remains to explain then is what these numerals are; it emerges from Cantor's 1883 that these numerals 'stand for', represent, well-ordered sets (Hallett, p. 52). Recall (definition 2.1.2) that a well-ordering is not completely determined just by a list of its elements; one needs to specify also the order relation among them to obtain a full determination of the set. Numerals represent different ways of ordering sets, and so they can be seen as a way of enumerating sets (Dauben, pp. 101-102).

Although he does not fully explain the notion of numeral in this paper, Cantor still defines the operations of addition and multiplication of ordinals in the usual way (see 2.1.7), thus suggesting that whatever the relationship of 'representation' between a well-ordered set and its ordinal number is, it is such that the arithmetical operations on the ordinals can be carried out as set-theoretic operations. So the ordinals are not (yet) said to be sets, but one can manipulate them as such.

Having sketched a theory of ordinals, Cantor is in a position to elaborate on his (1878) remarks regarding powers, by developing the "number-classes". He defines the first number-class, (I), as the set of all finite sets; the second, (II), as the set of all ordinals standing for countable well-ordered sets, *and so on*(Cantor (1966), p. 167). Unfortunately, even though he envisages the existence of limit steps in this recursive procedure, he does not offer an explicit way of extending the recursion to limit ordinals as well. This though suffices to understand what

10. Georg Cantor, "Über unendliche lineare Punktmannigfaltigkeiten, 5., §1-3," in *Abhandlungen mathematischen und philosophischen Inhalts*, ed. Adolf Fraenkel (Georg Olms Verlagsbuchhandlung, 1966), 165–169.

Cantor's project is: he wants to use the ordinals to define a scale of powers, via number-classes.[11] These number-classes have to behave 'like natural numbers' (cfr. Cantor (1966), p.119), when it comes to fulfilling the role of cardinals, i.e. instruments to measuring the size of sets via counting how many members a set has.

In the finite case, the ordinary nonnegative integers – or equivalently, the natural numbers – do double duty: the number 5, for example, can be used both to designate the size of a group of five objects, and the fifth position on the scale of the natural numbers, not counting zero as the starting position. Hallett (p.62) expresses the situation in the following way: in the finite case, counting the elements suffices as a way of measuring size. In the infinite case, this may not be so.

2.4 What this tells us about ordinals and cardinals as extensions of the natural numbers

Let us briefly review what has been said in the previous section. We have seen that Cantor first mentioned powers in his 1878, and he already considered them as a tool to measure the size of sets via counting the elements. In order to be able to solve the problem of defining what these powers are, he developed a theory of *ordinals* – (representatives of) equivalence classes of sets that track the 'length' of a set once it is well-ordered (Hallett, p. 62). Cardinals (number-classes) are obtained from ordinals by identifying those that can be put in a one-to-one correspondence.

From this summary of Cantor's discovery and development of cardinals and ordinals one can conclude that Cantor perceived the need for having both cardinals and ordinals – as different extensions of the natural numbers. The cardinals, first thought of as powers of sets, were needed to measure the size of sets. For Cantor, this meant they had to provide a way of counting the elements in a set, even a transfinite set, because the size of a set just is the number of elements it has. Ordinals were needed to give a sufficiently clear definition of how one obtains cardinals, and also they stand for equivalence classes – to use modern terms – of order type, that is they track something more fine-grained than just size.

Both ordinal and cardinal numbers can be seen as expansions of the natural numbers. They are expansions of the natural numbers in the following sense:

11. At this stage, it is not said whether the transfinite cardinals are just to be identified with the number classes or if they are special representatives, or primitive entities that are somehow related to these classes.

if we imagine the class of ordinals (cardinals) as ordered in the standard way, then the 'initial segment' of this order is exhausted by the natural numbers. This occurs because Cantor defined ordinals and cardinals in such a way that the set-theoretic representation of the natural numbers coincides with the initial segment of the class of ordinals, and of cardinals, respectively. The definitions of ordinals and cardinals in ZFC also preserve this characteristic. So the concept of ordinal number, as well as that of cardinal number, expands that of natural number in the sense that it encompasses the same entities *and some more*.

Cardinals and ordinals are also generalisations of the natural numbers in the following sense: the ordinals are defined by recursion, just like the natural numbers, only they also allow a third step in the recursion, different from the successor step: once all successors have been formed, one can consider the union of all the ordinals formed at the previous steps and that constitutes a new element of the class of ordinals. To express the idea more clearly, consider the following as a recursive definition of natural numbers:

- 0 is a natural number
- If n is a natural number then $n \cup \{n\}$ is a natural number

And compare with the recursive definition of ordinals:

- $\emptyset$ is an ordinal
- if α is an ordinal, then $\alpha \cup \{\alpha\}$ is a successor ordinal
- given a series of ordinals α_i, then their union $\bigcup_i \alpha_i$ is also an ordinal, called a limit ordinal.

Then it is easier to see where the ordinals generalise the natural numbers, and that is in extending the domain on which one can apply induction, thanks to the recursive definition: while with the natural numbers induction can only be performed on 0 and successor elements, with the ordinals this can be extended to a broader class of sets.

2.4.1 Why the ordinals cannot measure the size of sets

Even if by assuming choice one can always find a well-ordering for a set, and hence an isomorphism with some ordinal, so that each set has an ordinal-representative, ordinals have never been perceived as entities measuring the size of sets. One plausible reason is that ordinals track order-type, that is the different ways in which a collection of elements can be ordered. The same collection of elements – the same set – can have a representative belonging to different order-types, and then if order-types, hence ordinals, were used to determine size, size could not be univocally determined. One would need to choose which order-type somehow mirrors the size of the set, among some equally good candidates, for they would all have in common the underlying set of elements, that is what one is trying to measure. This sort of difficulty does not arise with cardinals, for in the case of cardinals, the elements of sets are enough to determine their size and size-relations with other sets. As we already noted in 2.2, $\aleph_0 = |\omega| = |2\omega| = |\omega \cdot 2|$, but then $\omega \cdot 2 \neq 2\omega$, and yet, order aside, these sets are virtually the same.

2.5 Numerosities

From the section concerning cardinal numbers there are two takeaway points: one, equality of cardinalities is sanctioned by the existence of a bijection between two sets; two, Cantor himself takes cardinal numbers as measuring the size of sets – and if one believes in set theoretic reductionism, then cardinal numbers can measure the size of any mathematical object.

From the first point it follows that cardinalities do not respect the so-called part-whole principle, that is, if $a \subset b$, then it is not the case that $card(a) < card(b)$, because we have many proper subsets of the natural numbers that can be put into a one-one correspondence with the whole set of natural numbers. If one also accepts to interpret cardinality as a faithful size measurement, the theory of cardinals is to be considered as the proper theory of size, therefore the proper theory of size will not preserve the part-whole principle (PW): if two sets a and b are such that a is a proper subset of b then the size of a is strictly smaller than the size of b. The theory of cardinals, however, does satisfy the following: given two sets a and b such that $a \subset b$, then size$(a)\leq$ size (b), where size(–) stands for card(–). Call this the weak part-whole principle (WPW).

The failure of Cantorian cardinals to obey the full part-whole principle has recently given spur to the formulation of alternative systems that would measure size by means other than cardinals, in an effort to present a coherent theory of size that does not share this – in the eyes of the proponents – weakness. Failure of satisfying the part-whole principle, though, is not the only aspect that has made people wary of endorsing the theory of size exemplified by cardinals. Some alternative axiomatisations of set theory [12] also seem motivated by dissatisfaction with the Cantorian treatment of infinite quantities, so the concern that cardinals may not capture size after all goes beyond the observance of the part-whole principle in particular. Nevertheless, since there is a tradition of considering the two principles, Cantor's principle and the part-whole principle, that is, as the two premises that lead to contradiction in the traditional paradoxes of the infinite, we are interested in the alternative theories that try to preserve the part-whole principle. We focus specifically on numerosities because they are Parker's prime target in his criticism of Euclidean theories (theories that choose to preserve the part-whole principle over Cantor's), and because we think it is the easiest to compare with the Cantorian theory, since they can both be expressed in the language of ZFC.

12. e.g. the 'pocket set theory' presented in the SEP article by Holmes, M. Randall, "Alternative Axiomatic Set Theories", The Stanford Encyclopedia of Philosophy (Fall 2014 Edition), Edward N. Zalta (ed.), `<http://plato.stanford.edu/archives/fall2014/entries/settheory-alternative/>`. I am indebted to Prof. Löwe for bringing this to my attention

Context

The theory of numerosities was first presented in a paper published in Italian in 1995, then translated into English and published in 2003 as "Numerosities of labelled sets: A new way of counting". At that early stage, the system was developed only to the point of measuring the size of countable subsets of ℕ. In the papers that have come after that, the authors have tried to make the theory strong enough to enable measurement of a broader class of sets, and they have also discussed some fundamental issues that arise in the required set-ups. As the last paper on the subject was published only last year, it is likely that numerosities are still work in progress. The circle of people working on numerosities, however, does not seem to have widened significantly.

In philosophy, numerosities were first mentioned in an article by Parker,[13] only to then be extensively discussed by Mancosu in (2009), and at later dates they were once again considered by Parker in what is a reply to Mancosu of sorts; subsequent citations occur always in the context of a discussion of one of the papers authored by Mancosu. This situation then seems to call for a justification for discussing numerosities beyond their presence in these papers.

The only people who are working on the theory of numerosities are the same who first articulated the theory. So we cannot really say (perhaps it is too early?) that their work has proved influential in any of their fields of research – applied maths, foundations for analysis and so forth. It is also not the case that numerosities are the only 'alternative' to cardinalities when it comes to measuring set sizes. It is not even the only alternative that preserves the principle that we are interested in, the part-whole principle. The reason why we picked numerosities as the token alternative is because it is nevertheless the most thoroughly motivated theory in its details when compared to other Euclidean theories[14],[15] and because it is the theory on which there is still ongoing research – even though as already said, always by and large by the people who came up with the idea in the first place – and because, at least in some papers, the authors take themselves as trying to forge an alternative theory of size that is a 'better' one than Cantor's.[16]

13. Parker, "Philosophical Method and Galileo's Paradox of Infinity."

14. By Euclidean in this context we mean a theory of size for sets that obeys the part-whole principle

15. Mancosu, "Measuring the size of infinite collections of natural numbers: was Cantor's theory of infinite number inevitable?"

16. Vieri Benci and Mauro Di Nasso, "Numerosities of labelled sets: a new way of counting," *Advances in Mathematics* 173 (2003): pp. 50-1.

2.5.1 Motivation for a theory of numerosities

The authors of most of the papers on numerosities are Benci, Di Nasso and Forti. The authors usually work in the fields of analysis and its applications (Benci), and mathematical logic and foundations (Di Nasso, Forti), which seems to explain why they would be interested in making 'an attempt to extending the notion of finite cardinality' (quoted from Benci, Di Nasso (2003)). As this quote shows, the theory of numerosities was initially presented as an alternative to Cantor's cardinals, first and foremost. In a number of subsequent publications, however, numerosities are also presented as offering a way of unifying the construction of 'infinitely large' quantities and 'infinitely small' ones, as numerosities can be interpreted as the set of hypernatural numbers, and hence be used to construct infinitesimals. We can then say that numerosities seem to be motivated by, on the one hand, developing a theory of size that preserves the fundamental algebraic properties of finite cardinals, and on the other by offering a unified foundation for the infinitely small and the infinitely large. For present purposes, we will not discuss numerosities as a unified treatment for the infinitely large and the infinitely small, but only as an attempt to provide a theory of size that, unlike Cantor's, preserves (PW) for (countably) infinite sets, as well as finite ones.

After the first few papers in fact, where the authors only mention the part-whole principle (PW) as the feature missing from the usual transfinite cardinals, which they yet want a theory of size to satisfy, they subsequently frame the discussion around Euclid's five 'common notions' from the first book of the *Elements*. They claim that Euclid's five common notions 'traditionally embody the properties of magnitudes' (Benci, Di Nasso, Forti (2007), p.43), and this seems to be their reason to argue that these notions are to be incorporated in any theory of size. They are the following:

(CN1) Things equal to the same thing are also equal to one another

(CN2) If equals are added to equals, the wholes are equals

(CN3) If equals are subtracted from equals, the remainders are equal

(CN4) "Things *applying onto* one another are equal to one another" [17]

(CN5) The whole is bigger than the part.

The requirement that any theory of size satisfy all five common notions allows the proponents of numerosities to make a stronger case in favour of their work,

17. This is the exact way in which Benci et al. translate the text from Euclid because they feel *applying onto* to be the most precise rendition of ἐφαρμόζοντα.

because they can now say that cardinals are defective not just with respect to some principle, but to a more comprehensive conception of what size should be. They do in fact highlight that (CN3) as well as (CN5) is violated by the theory of transfinite cardinals, and so somehow this also serves as further motivation to prefer numerosities over cardinals. We have already discussed how (CN5) is not respected by the theory of cardinals if we interpret it as (PW); the way (CN3) fails is even subtler. There is no way of defining subtraction of cardinals in a meaningful way.

The weak aspect in motivating their work in terms of the five common notions is that they do not offer a justification as per why one should consider these principles as prescriptive any more than just the axioms of Peano arithmetic, for example. Since the authorship of the principles is also debated by philologists[18], it seems one cannot even appeal to the authority of Euclid to justify them. In Chapters 4 and 5 of the thesis, we will propose a different way of motivating the theory that does not incur in this sort of issue.

2.5.2 The theory of numerosities for countable subsets of $\mathbb{N}$

Intuitively, the underlying idea for numerosities is that one procedure for counting infinite sets is to partition them into finite subsets, that then can be counted in the usual way and the sequence of the finite cardinalities of the subsets forms the approximate size of the original set. While this procedure does not yield any different result from cardinals when the set one is trying to measure is finite, this 'counting via partitions' makes the results for infinite sets nontrivial. Different numerosities arise from different ways of partitioning the given set.

The Benci-Di Nasso original paper starts off by giving three desiderata any counting system should satisfy. Letting ν denote the size-measuring function:

1. if there exists a bijection between A and B, then $\nu(A) = \nu(B)$.
2. if A is a proper subset of B then $\nu(A) < \nu(B)$.
3. if $\nu(A) = \nu(A')$ and $\nu(B) = \nu(B')$, then $\nu(A \uplus B) = \nu(A' \uplus B')$ and $\nu(A \times B) = \nu(A' \times B')$ (where $\uplus$ denotes disjoint union).

The second item in the list is of course a version of the part-whole principle in set-theoretic terms, and clearly Cantor's ordinals and cardinals do respect

18. T.L. Heath, for example; compare his Introduction to Euclid's Elements

(CN1) and (CN3) but they do not satisfy (CN2). This is what motivates the authors' investigation in what they dub numerosities – which are a special kind of function.

In order to present the definition of numerosities, we need to first define labelled sets, labelled subset-hood, and sum and product of labelled sets.

Definition 2.5.1. A *labelled set* **A** is a pair $\langle A, l_A \rangle$ where A is a set, and the labelling $l_A : A \to \mathbb{N}$ is a finite-one mapping.

The set A can be viewed as a union of finite sets $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \ldots$. $A_i := \{a : l_A(a) \leq i\}$. Each A_i has therefore a cardinality (in the standard sense of the term), and the sequence formed by the cardinalities of the A_i's is called the *approximating sequence* of **A**.

Definition 2.5.2. Given $\mathbf{A} = \langle A, l_A \rangle$ and $\mathbf{B} = \langle B, l_B \rangle$ we say that $\mathbf{A} \subseteq \mathbf{B}$ if and only if $A \subseteq B$ and for all $a \in A, l_A(a) = l_B(a)$.

Definition 2.5.3 (Sum and Product)**.** The sum of two labelled sets $\langle \mathbf{A}+\mathbf{B},\ l_A \oplus l_B \rangle$, where $\mathbf{A} = \langle A, l_A \rangle, \mathbf{B} = \langle B, l_B \rangle$ is defined as $\langle A \uplus B, l_A \uplus l_B \rangle$ where for any $x \in A \cup B$, $l_{A+B}(x) = l_A(x)$ if $x \in A$, $l_{A+B}(x) = l_B(x)$ if $x \in B$.

The product of two labelled sets $\langle \mathbf{A}\times\mathbf{B},\ l_A \odot l_B \rangle$ is given by the set $A \times B = \{(a, b) : a \in A, b \in B\}$ and the labelling for each $(a, b) \in A \times B$ is $(l_A \odot l_B)(a, b) = max\{l_A(a); l_B(b)\}$

The definition of sum is not quite complete, because the domain of the sum $\langle \mathbf{A} + \mathbf{B}, \oplus \rangle$ is the disjoint union $A \uplus B$ of A and B; hence if an element x is such that $x \in A$ and $x \in B$, the definition does not determine what should be the label of x. It is clear though that it can be defined as any binary function of the labels $l_A(x)$ and $l_B(x)$, such as their sum or product.

Definition 2.5.4 (Numerosity function)**.** A *numerosity function* is thus a map $\mu : \mathcal{L} \to \mathcal{N}$, where $\mathcal{L}$ is the class of labelled sets and $\mathcal{N}$ is a linearly ordered set according to an order $\leq$, such that μ satisfies the desiderata 1-3 expressed above.

Even when extended beyond countable subsets of the natural numbers, the theory of numerosities easily satisfies common notions (CN1)-(CN3) and (CN5), suitably formulated. Whether numerosities do indeed respect (CN4) or not depends on what kind of mapping one takes to capture the relation "applying onto one another". Benci, Forti and Di Nasso argue for a narrow interpretation, so to speak, that should encompass only permutations and regroupings. Since this feature of numerosities is related to one of Parker's criticisms against Euclidean theories, we will briefly discuss this in the relevant chapter.

So defined, numerosities enjoy some nice features, notably that the operations of addition and multiplication, defined as $num(\mathbf{A}+\mathbf{B}) = num(\mathbf{A}) + num(\mathbf{B})$ and $num(\mathbf{A}) \cdot num(\mathbf{B}) = num(\mathbf{A} \cdot \mathbf{B})$, behave like the usual addition and multiplication of finite cardinalities whenever both A and B are finite. More than that, $\mathcal{N}$ is a positive semi-ring.

An important aspect that will be discussed later is in which sort of models of ZFC numerosities do exist. Benci and Di Nasso[19] have proved that the existence of numerosities is equivalent to the existence of a selective ultrafilter.

An *ultrafilter* $\mathcal{U}$ over some set I is a nonempty family of sets such that, if $D \in \mathcal{U}$ and $D \subseteq D'$ then $D' \in \mathcal{U}$, and if $D_0, \ldots, D_n \in \mathcal{U}$, for any finite n $\bigcap_n D_n \in \mathcal{U}$.
An ultrafilter $\mathcal{U}$ is called *nonprincipal* if it does not contain any finite subset of I.
A nonprincipal ultrafilter $\mathcal{U}$ on I is called *selective* if for any function $\varphi : I \to I$ there is a $D \in U$ such that φ restricted to D is nondecreasing.[20]

Once there is a selective ultrafilter $\mathcal{U}$ over $\mathbb{N}$, we consider its ultrapower[21] over $\mathbb{N}$, namely the set $\mathcal{N} := \{[\varphi]_{\mathcal{U}} | \ \varphi : \mathbb{N} \to \mathbb{N}\}$, where $[\varphi]_{\mathcal{U}}$ is the equivalence class of the map $\varphi : \mathbb{N} \to \mathbb{N}$ determined by the equivalence relation:

$$\varphi \sim_{\mathcal{U}} \psi \Leftrightarrow \{n | \ \varphi(n) = \psi(n)\} \in \mathcal{U}.^{22}$$

The claim is that this set $\mathcal{N}$ satisfies the axioms of numerosities. The proof is too elaborated to try and relay here, but the interested reader can find it in the original paper.

Having defined the set $\mathcal{N}$, we can define an order $[\varphi]_{\mathcal{U}} \leq [\psi]_{\mathcal{U}} \Leftrightarrow \{n : \varphi(n) \leq \psi(n)\} \in \mathcal{U}$, and this order turns out to be a linear order whenever φ, ψ are approximating sequences – this means that the order thus defined over $\mathcal{N}$ satisfies trichotomy. If we set that for each labelled set $\mathbf{A}$, $num(\mathbf{A}) = [\gamma_A]_{\mathcal{U}}$ then we obtain that this yields numerosities.

Numerosities then appear to be equivalence classes of non-decreasing functions having as both domain and codomain the set of natural numbers.

19. Benci and Di Nasso, "Numerosities of labelled sets: a new way of counting."
20. The interest in nondecreasing functions from $\mathbb{N}$ to $\mathbb{N}$ is that they simulate the process of counting the elements of subsets of $\mathbb{N}$ itself.
21. An *ultrapower* is a specific case of the *ultraproduct* construction. For a definition of ultraproduct, see Insall, Matt. "Ultraproduct" From MathWorld–A Wolfram Web Resource, created by Eric W. Weisstein. `http://mathworld.wolfram.com/Ultraproduct.html`
22. Clearly, if $\varphi \sim_{\mathcal{U}} \psi$ then $[\varphi]_{\mathcal{U}} = [\psi]_{\mathcal{U}}$.

2.5.3 Remarks

As the reader may have realised, building numerosities from a selective ultrafilter leaves many issues unsolved. Since the construction only assumes the existence of an ultrafilter, we do not know which sets are members and which are not – all we know is that we cannot have both a set A and its complement $\mathbb{N} - A$ as an element of an ultrafilter, and also that the finite subsets of $\mathbb{N}$ cannot be in a selective ultrafilter (for if a finite subset $A \subseteq \mathbb{N}$ were an element of the ultrafilter, it would fail to be a nonprincipal ultrafilter).

For any set A that is neither finite nor cofinite, then, it is undetermined whether it is a member of the selective ultrafilter or not. As an example, the set $3\mathbb{N}$ (the set of multiples of 3) might be in the ultrafilter, or alternatively its complement might – and similarly, the set of the odd numbers could be in the ultrafilter, or the set of the evens – they are equally good alternatives. There is no constraint that suggests which of these sets should be in the ultrafilter, and this has immediate consequences for size relations.

Consider the following example, given by Benci and Di Nasso (2003, p. 63) and Mancosu (2009, pp. 634-35): suppose the selective ultrafilter $\mathcal{U}$ is in the model of ZFC you are considering, and $2\mathbb{N} \in \mathcal{U}$ (and $0 \in 2\mathbb{N}$). We know that the numerosity of the labelled set of even numbers, **Even**, is defined as the (equivalence class of) the approximating sequence of $2\mathbb{N}$. This is $\gamma_{even} = \langle 1, 1, 2, 2, 3, 3, \dots \rangle$. Similarly, for the labelled set **Odd**, the approximating sequence is $\gamma_{odd} = \langle 0, 1, 1, 2, 2, 3, 3, \dots \rangle$. Since, given any two labelled sets **A, B**, $num(\mathbf{A}) + num(\mathbf{B}) = num(\mathbf{A}+\mathbf{B})$, and **Even**+**Odd**= **N** (where **N** is the set $\mathbb{N}$ together with the canonical labelling, i.e. identity), we have that $num(\mathbf{N}) = \gamma_{\mathbf{N}} = \langle 1, 2, 3, \dots \rangle = num(\mathbf{Even}) + num(\mathbf{Odd})$. At the same time, though, since $2\mathbb{N} \in \mathcal{U}$, $num(\mathbf{Even}) = num(\mathbf{Odd}) + \mathbf{1}$: by definition, $num(\mathbf{Odd}) + \mathbf{1} = num(\mathbf{Odd} + \mathbf{1})$, and $num(\mathbf{Odd} + \mathbf{1}) = \gamma_{\mathbf{Odd}} + \gamma_{\mathbf{1}} = \langle 0, 1, 1, 2, 2, 3, 3, 4 \dots \rangle + \langle 1, 1, 1, 1, 1, \dots \rangle$, and the latter equals $\gamma_{\mathbf{Even+1}} = \langle 1, 2, 2, 3, 3, 4, 4, \dots \rangle$. It is easy to check that for every nth position where n is an even number, $\gamma_{\mathbf{Odd+1}}(n) = \gamma_{\mathbf{Even}}(n)$. Thus the set $\{n | \gamma_{\mathbf{Odd+1}}(n) = \gamma_{\mathbf{Even}}(n)\} = 2\mathbb{N} \in \mathcal{U}$, and this, by definition 2.5.2 of $\varphi \sim_{\mathcal{U}} \psi$, means that $[\gamma_{\mathbf{Odd+1}}] = [\gamma_{\mathbf{Even}}]$, hence $num(\mathbf{Odd+1}) = num(\mathbf{Even})$, so $num(\mathbf{Odd}) + \mathbf{1} = num(\mathbf{Even})$, as anticipated.

The problem is that then one can substitute this last result in the equation $num(\mathbf{N}) = num(\mathbf{Odd}) + num(\mathbf{Even})$ and obtain $num(\mathbf{N}) = \mathbf{2}num(\mathbf{Even}) + \mathbf{1}$, that can be interpreted as showing that the numerosity of the natural numbers is odd (because it can be written as $2k + 1$ for some k). On the other hand, if $\mathcal{U}$ actually contained the set of the odd numbers instead, we would have $num(\mathbf{Odd}) = num(\mathbf{Even})$ and so $num(\mathbf{N})$ would be even. So if on the one hand numerosities, unlike cardinals, allow to make sense of the notion that an infinite number may be even or odd, on the other they cannot determine for any infinite set (such as $\mathbb{N}$) whether their numerosity is even or odd – this may change depending on the ultrafilter one chooses.

Chapter 3

Gödel's defence of Cantor's proposal

3.1 An argument in favour of cardinals

Cardinals were famously defended as the sole genuine extension of the concept of number (into the infinite) by Gödel in his (1947) paper. Since we are interested in discussing the interpretation of this argument, it seems necessary to give the reader an opportunity to judge the faithfulness of the competing interpretations by themselves. Thus, we report the whole passage in which Gödel's argument is found.

> [...] Closer examination, however, shows that Cantor's definition of infinite numbers really has this character of uniqueness, and that in a very striking manner. For whatever "number" as applied to infinite sets may mean, we certainly want it to have the property that the number of objects belonging to some class does not change if, leaving the objects the same, one changes in any way whatsoever their properties or mutual relations (*e.g.*, their colors or their distribution in space). From this however, it follows at once that two sets (at least two sets of changeable objects of the space-time world) will have the same cardinal number if their elements can be brought into a one-to-one correspondence, which is Cantor's definition of equality between numbers. For if there exists such a correspondence between two sets A and B it is possible (at least theoretically) to change the properties and relations of each element of A into those

> of the corresponding element of B, whereby A is transformed into a set completely indistinguishable from B, hence of the same cardinal number.[1]

We are going to restructure the argument as follows:

thesis (T) Cantor's definition of cardinal number is the unique extension of the concept of number that can account for infinite quantities. ('Cantor's definition of infinite numbers really has this character of uniqueness')

premise (P1) The number of objects of a set is invariant under change of qualities or relations. ('For whatever "number" as applied to infinite sets may mean, we certainly want it to have the property that the number of objects belonging to some class does not change if, leaving the objects the same, one changes in any way whatsoever their properties or mutual relations')

premise (P2) If a feature of a set is invariant under change of qualities or relations among its members, then it is invariant under bijections. ('For if there exists such a correspondence between two sets A and B it is possible (at least theoretically) to change the properties and relations of each element of A into those of the corresponding element of B[...] hence of the same cardinal number.').

Conclusion(C) Any mathematically viable way of extending the concept of number coincides with Cantor's way.

So the argument made of (P1), (P2) and (C) supports thesis (T). Mancosu refutes the argument because he believes that the existence of numerosities is enough to show (P1) as unwarranted for infinite quantities: according to him, (P1) is uncontroversial for finite sets, but it needs more justification than an argument from analogy to be extended to infinite sets as well. To declare Cantor's cardinals as the only sensible way of extending the concept of number by appeal to (P1) is almost a *petitio principii*, or so Mancosu seems to suggest. The reasoning goes as follows: when Gödel asserts that Cantor's way of extending the concept of number has a claim to uniqueness as an extension to infinite sets, he likely means that Cantor's is the only "mathematically viable" way of extending the concept. Mancosu thus considers the theory of numerosities to be well-developed enough to count as a "mathematically viable" way of extending the concept of number to the infinite, and yet it does not coincide with Cantor's – moreover, this way of extending the concept is such that the number

1. Kurt Godel, "What is Cantor's Continuum Problem?," *The American Mathematical Monthly* 54, no. 9 (1947): 515–525, ISSN: 00029890, 19300972, `http://www.jstor.org/stable/2304666`.

of elements of an infinite set does change under certain alterations of relations among the elements of the set, so premise (P1) must be unsound, or at the very least is not easy to defend as sound unless one is already committed to Cantor's cardinals.

We should recognise that the attribute of "mathematical viability" plays a key role in our presentation of Gödel's argument; Gödel claims uniqueness for Cantor's extension of the natural numbers, but this claim of uniqueness means that cardinals are the only (unique) extension of the natural numbers into the infinite *of a certain kind*, that meets certain requirements. We express these requirements as "mathematical viability" to convey the fact that more than consistency with known characteristics of finite arithmetic, say, is needed. It is however difficult to be more specific than this – and this is what allows for different readings of Gödel's argument. Crucially, the meaning one may attach to this mathematical viability determines what sort of inevitability Gödel is attaching to Cantor's definition of the cardinals.

When arguing against Gödel's conclusion, Mancosu leaves unexplained what is the 'inevitability' he interprets Gödel as advocating. This leaves his rebuttal of the argument open to criticism, but at the same time he offers tools to make his attack on Gödel's argument more precise and forceful. Mancosu[2] quotes the work of Meir Buzaglo[3] on concept expansion in mathematics as an additional sceptical voice towards this 'inevitability' of Cantor's theory of cardinals. Mancosu himself though is not interested in using Buzaglo's formalisation of concept expansion to explain exactly what it means for Cantor's theory of cardinals to (fail to) be inevitable.

This is the work we carry out in the next section.

3.2 Buzaglo's concept expansion and expanding the concept of 'number of'

In his book *The Logic of Concept Expansion*, Meir Buzaglo offers a logical treatment of the phenomenon of concept expansion. To that end, he first presents an enriched first-order language – still countable though – and then interprets concept expansions as embeddings of models of such language. In chapter 3 he offers formal definitions of different sorts of such embeddings, basing his account on Tarski's definition of truth in a model. The 'concept' is formalised as a (partial) function, and the expansion is interpreted as an enlargement of the

2. Mancosu, "Measuring the size of infinite collections of natural numbers: was Cantor's theory of infinite number inevitable?," p. 638.
3. Meir Buzaglo, *The logic of concept expansion* (Cambridge University Press, 2002).

domain over which the function is defined. Formally, this means that he considers models in a signature that contains function symbols and a distinguished constant X, and interprets a function f as *defined* on an element a in a model M whenever $f(a)_M \neq X$.

Of the various definitions of expansion offered in the book, we are only going to consider the following:

Definition 3.2.1 (Internal expansion)**.** Let M, N be models of the first order language of concept expansion. N is an internal expansion of M if and only if $Dom(M) = Dom(N)$, $X_M = X_N$, and if a function f is in the signature of M and N, then for all $a, b \in Dom(M) \setminus X_M$, if $f(a)_M = b$, then $f(a)_N = b$. Notation: $M << N$ or $N >> M$.

Informally, an internal expansion of a concept f occurs when there is a context of interpretation (a model, to follow the definition) of the concept in which f still applies to the same objects as in the restricted context (M) – and possibly some more. Note that this kind of expansion only has to preserve the values of the concept f on the original domain, but it does not have to meet any other constraints.

Definition 3.2.2 (Forced Internal Expansion)**.** Let $N >> M$ as defined above. Let Φ be a set of sentences from the language of concept expansion. Then N is a forced (internal) expansion of M with respect to Φ (equivalently, $N >> M$ is forced by Φ) if $M \vDash \Phi$, $N \vDash \Phi$ and for every $K >> M$ such that $K \vDash \Phi$, the functions in K and N agree on their common part and therefore compatible, i.e., for any function symbol f in the signature, and any $a \in Dom(M) \setminus X_M$, if $f(a)_N \neq X_N$ and $f(a)_K \neq X_K$, then $f(a)_N = f(a)_K$.

This second kind of expansion is more demanding. Informally, A forced internal expansion N occurs whenever, given a certain context M for interpreting f, there may be several ways of expanding M, but once one assumes that a certain set of conditions Φ is true, then all these ways of expanding M are consistent with N.

What this means is that, if M has a forced internal expansion, even though there may be different ways of expanding M, given a certain set of assumptions, there is a core of "facts" about the concepts interpreted in M that are fixed across all expansions.

Definition 3.2.3 (Strongly Forced Expansion)**.** Let N, M, Φ as above. We say N is strongly forced by Φ if and only if

(i) $N >> M$ is forced by Φ;

(ii) for any set Ψ of sentences in the language, if Ψ forces an extension $N' >> M$, N' is compatible with N.

This means that not only do all expansions of M which satisfy Φ agree on the ways in which N evaluates functions (i.e., which elements fall under which concepts in interpretation N), but any expansion of M that is forced by any other set of conditions Ψ is going to agree with N on the ways in which N evaluates functions.

The second type of expansion is the one under which the expansion of the concept of relation 'x has the same number of elements as y' from only finite to also infinite sets falls, according to Buzaglo. His formalization of the situation is the following. He considers a partial function $f(x, y)$ from V, the hierarchy of sets, into $\{0, 1\}$ and defines it as such:

$$f(x,y) = \begin{cases} 0 \text{ if } x \text{ and } y \text{ are both finite and do not have the same number of elements} \\ 1 \text{ if } x \text{ and } y \text{ are both finite and have the same number of elements} \\ \text{undefined otherwise} \end{cases}$$

As it stands, f is a binary function, or a function defined on pairs of finite sets; equivalently, its domain is $V_{fin} \times V_{fin}$, where V_{fin} stands for the finite sets in V. In the forced expansion N, f will be defined as $f(x, y) = 1$ iff there is a one-one correspondence from x to y; $f(x, y) = 0$ otherwise. Since f is now a function defined for all pairs of sets, with no restrictions, it is defined on $V \times V$. Clearly, the situation with respect to f in N corresponds to Cantor's expansion of the concept of sameness of number to the infinite sets.

As noted by Mancosu (2009) already, Buzaglo here comments that it may be possible to expand f differently, if one considers a set of sentences other than Φ, and goes on by producing an example thereof. For our purposes, we are interested in considering $\{(PW)\}$ as Φ'. Then we can indeed define $f(x, y) = 1$ iff $num(\mathbf{x}) = num(\mathbf{y})$, where num stands for the numerosity function [4].

Assuming for the sake of argument that Buzaglo's proposal is faithful enough as a formalisation of concept expansion in mathematics, comparing Cantor's definition on one hand and the one via numerosities on the other leads to the conclusion that, from the conceptual point of view, the two expansions are on a par. They can both be formalised as forced expansions of the standard model,

4. to see this, one needs to check that the whole definition of $num(x) = num(y)$ can indeed be expressed in first order logic, which it can. The problem is that we have checked this only for the definition of numerosities which encompasses countable sets, nothing more and by far not the whole of V, so this expansion does not go as far as that formalising the work of Cantor's.

and since they're clearly incompatible with one another, neither of them is a strongly forced expansion. There is also no way of pushing either of the expansions into the stronger category of strongly forced expansion, because that would require compatibility with other expansions which is ruled out automatically for both, given the existence of the other. This observation offers a specific sense in which claims of inevitability and uniqueness of Cantor's definition might be refused.[5]

3.3 Another interpretation of Gödel's argument

There is reason to doubt though that Mancosu's (and Buzaglo's) reading is the only one supported by the text of Gödel's article. A few lines after the long passage quoted above, in fact, Gödel writes: " [By accepting Cantor's definition of equality of numbers] it becomes possible to extend (again without any arbitrariness) the arithmetical operations to infinite number." (Gödel p. 515). When remarking that Cantor's definitions of sameness of number and less than, and greater than, allow for a reconstruction of ordinary arithmetical results for infinite quantities, he writes in parenthesis: '*again* without any arbitrariness' (emphasis mine). Yet the word 'arbitrariness' has not appeared before – so what is Gödel talking about? why *again*? I think the answer is that Gödel views Cantor's definition of cardinal number as the only one that is not arbitrary. That ought to be the meaning of the idea of iteration here. The choice of Cantor's definition is not arbitrary, the way the ensuing transfinite arithmetic develops is not arbitrary. Neither of them is arbitrary, in the sense that Cantor's definition was guided by a natural principle, the one Gödel discusses at the beginning of the first section of his paper, and the arithmetic of these new numbers does not betray the basic mechanisms of its preexisting finite cousin, and so they share some of the guiding principles.

Suppose Gödel is arguing here that Cantor's extension is the only logically possible way of extending the concept of infinite number. Although the part-whole principle by itself does not determine a theory of size, let alone a comprehensive mathematical theory of the infinite comparable to Cantor's, even in Gödel's times it was a conceptual possibility to define the size of infinite sets in such a way that it would respect the part-whole principle. Hence, we feel it is doubtful that Gödel might have believed that Cantor's conception of size was

5. Luca Incurvati has rightfully objected that the defender of Cantor's cardinals as the one right expansion of the concept of natural number could claim that the definition of equality of cardinality in terms of one-to-one correspondence (that is, (CP)) is already a constituent part of the concept of number, albeit implicit, hence it cannot just be treated as a requirement or constraint to be met by an expansion of the concept. My reply to such a concern is that whoever may want to pursue such defence would still have to produce an argument to assert that (CP), not restricted to just finite cardinalities, is part of the concept of number.

the only consistent one – he might have well thought though that it was the only extension amenable to a thorough mathematical treatment.

Both interpretations we propose therefore agree that Gödel must have thought that Cantor's was the only *adequate* extension of the concept. The problem is to decide what are the minimal requirements of adequacy. If adequacy just means that the concept extension is such that it can support a rigorous arithmetisation of the infinite, by defining operations of addition and multiplication among different magnitudes of the infinite, then Mancosu's reply is a satisfactory one that proves Gödel wrong. If, on the other hand, adequacy means more than that, for example, that Cantor's is the only way of expanding the concept and develop an arithmetic of the infinite non-arbitrarily, then further work is needed to refute his claim. More specifically, one would need to show that e.g. the numerosities considered by Mancosu are not arbitrary.

3.4 Parker's counterattack

In the first section of the current chapter, we have introduced Gödel's argument defending the uniqueness of Cantor's definition of number. We have also remarked that Gödel might be interpreted as claiming that Cantor's is the unique proposal satisfying some desiderata – yet it is unclear which ones these might be. We have therefore entertained the option that Gödel defends Cantor's as the unique extension of the concept of number (into the infinite) that can have a satisfactory mathematical treatment (that is mathematically viable). If that is the right interpretation, then Mancosu can indeed appeal to numerosities as a counterexample. If the assertion being defended is that the concept of number for finite quantities is such that it can only be extended in one way to infinite quantities, then, depending on what one takes to be constitutive of the concept of number, Cantor's may or may not be the unique expansion of the concept. In particular, if one agrees that the concept of number does not contain Cantor's principle (CP) as an implicit component, then it is possible to use Buzaglo's framework to show precisely how uniqueness fails for Cantor. That is what we have done in the second section.

From the work carried out so far we conclude that Gödel's argument falls short of establishing uniqueness of Cantor's definition of cardinals as extension of the concept of number – at least so long as one does not think that (CP) itself is part of the concept of number.

The Cantorian however is not left empty-handed when trying to defend cardinals as the one extension. Parker (2013)[6] offers an argument that can

6. Parker, "Set-size and the Part-Whole Principle."

be seen as a more contemporary take on Gödel's; instead of trying to defend cardinals as the only true extension in fact, Parker aims at establishing that theories of size which forfeit (CP) in favour of (PW) – which he calls *Euclidean* theories of size – are not good enough as theories of size, because they are of limited epistemic use (compared to the standard Cantorian theory of cardinals).

He explains that for a theory of size to count as epistemically useful, said theory needs to satisfy four criteria: it has to be 'strong, general, well motivated, and informative'[7], and then explains the four criteria as follows:

> By a *general* theory I mean one that applies to a broad domain or many domains, including, in particular, subsets of the whole numbers and countable point sets. By *strong* I mean logically strong – a theory that leaves little undecided [. . .] By *well-motivated* I mean that the details of the theory – all of the particular sizes and size relations it assigns – are so assigned for some reason; they are not chosen arbitrarily. And finally, *informative* here means that the consequences of the theory (or concept) indicate something of interest that holds independently of the theory itself.[8]

Before we move on to consider how Parker embeds this framework in his general argument against Euclidean theories, we should comment on the criteria he singles out as criteria for an epistemically useful theory of size. First of all, these are not, in and of themselves, specific to a theory of size – they seem more like general criteria of epistemic adequacy for any scientific theory. Secondly, in spite of Parker's explanations, it remains quite hard to grasp the precise signification of those four criteria, or the relations they might bear to one another. We will come back later on to this point. For the moment however, I would like to focus on the last two criteria. Given Parker's explanations, it seems that being well motivated and being informative are related, because if a theory is well motivated, then it is informative: if each size relation has a reason – is not arbitrarily imposed – then whatever does the motivating is what the theory is informative about. So in the case of the Cantorian theory, the single size-relations are motivated by the existence of bijections, and the theory is informative about properties of bijections, in a way. The converse however seems to fail. Imagine a theory of size such that it assigns one same size to all sets, except precisely when one is a proper subset of the other, in which case the proper subset gets assigned a stricly smaller size. This size assignment could be considered as informative, because whenever $size(A) < size(B)$ is true in the theory, for sets A and B, that means that A is a proper subset of B, and conversely if $size(A) = size(B)$ then A and B are certainly disjoint, so size would be tracking subsethood – it would be informative about subsethood.

7. Parker (2013), op. cit., p. 590.
8. Parker, *ibid.*

This theory however could still count as arbitrary, in Parke's sense of the term, because this size assignment would fail to discriminate between substantially different sets, and would identify, for example, the size of the set of negative integers to that of positive real numbers for seemingly no good reason.

Let us now turn to the main goal of the paper, though. Parker's claim regarding Euclidean theories of size is that they are either too weak and not general, or arbitrary (non well motivated) and uninformative, to count as good theories of size. The strategy is to argue that any Euclidean theories cannot satisfy all four criteria at the same time, although the heart of the paper is to show that any Euclidean theory that meets his criteria of strength and generality is then either arbitrary or not informative. In the specific case of numerosities they seem to be judged as arbitrary. Either way, he makes the immediate inference that a theory that is either weak and with a narrow domain of application or arbitrary and uninformative is going to be limitedly useful from an epistemic point of view to conclude that Euclidean theories are not mathematically useful as theories of size.

In the next section, we consider Parker's argument in somewhat more detail.

3.4.1 Structure of the argument

Parker builds his argument by presenting the two only principles he is going to assume any Euclidean theory of size subscribes to. The first one is (PWS), or the part-whole principle in its *set* form:

$$\text{(PWS) If } A \subset B \text{ then } size(A) < size(B).$$

He refers to this principle as Euclid's principle, as noted in the introduction **??**.

The second principle is discreteness:

$$\text{If } size(A) < size(B) \text{ then } size(A \cup \{x\}) \leq size(B).$$

Despite presenting his main thesis as a disjunction ("Any Euclidean theory of size is either too weak or arbitrary to be useful"), Parker devotes most of the discussion in his paper to establishing that, under the assumption of discreteness, Euclidean theories of size are arbitrary. His presupposition is that rigid transformations should preserve size, but some of them do not, if the theory of size adopted is Euclidean (and discrete). Recall the definition of rigid transformation:

Definition 3.4.1 (Rigid transformation)**.** Let $\langle S, s\rangle, \langle T, t\rangle$ be two metric spaces[9], and let $f : S \to T$ be a bijection. Then f is a rigid transformation (or isometry) if and only if, for any $a, b \in S$, $s(a, b) = t(f(a), f(b))$.

Informally, then, rigid transformations are transformations that preserve distances – it seems reasonable then to expect them to preserve size, but this is not the line followed by Parker. Rather, he asserts that the guiding principle behind the requirement that rigid transformations preserve size is: the more properties a map can preserve, the more it has a claim to preserving size, too. Parker is also adamant though that his argument for arbitrariness itself does not rest on this principle. These are the invariance principles he considers:

(ATI) If T is a translation on a metric space $\langle S, d\rangle$, then for any $A \subseteq S$ we want that the size of TA equals the size of A.

(RTI) If $T : S \to S$ is a translation over a metric space $\langle S, d\rangle$ and $A, B \subseteq S$ then the size of A is no larger than the size of B if and only if the size of TA is no larger than the size of TB.

(ARI) For any rotation R on a Euclidean metric space $\langle S, d\rangle$ and any $A \subseteq S$, $size(A) = size(RA)$.

(RRI) For any rotation R on a Euclidean metric space $\langle S, d\rangle$ and any $A, B \subseteq S$, $size(A) \leq size(B)$ if and only if $size(RA) \leq size(RB)$.

The first two principles are the ones discussed more generously in the paper, hence we will consider these in particular. An easy example to visualise the meaning of the ATI principle is that of $\mathbb{N}$ and $\mathbb{N} - \{0\} = \mathbb{Z}^+$. $\mathbb{Z}^+$ can be considered as the result of translating $\mathbb{N}$ by the translation $T(n) = n+1\ \forall n \in \mathbb{N}$. Then according to the principle, $\mathbb{N}$ and $\mathbb{Z}^+$ should have the same size. But if we want size to agree with the part-whole principle, since $\mathbb{Z}^+$ is a proper subset of $\mathbb{N}$, it should be strictly smaller than $\mathbb{N}$, too. The problem of course is not just that the adoption of the part-whole principle generates different theorems

9. A *metric space* $\langle S, d\rangle$ is a set S together with a distance map d satisfying the following:

(i) $d(x, y) = 0$ if and only $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

A *Euclidean* metric space is a metric space $\langle S, d\rangle$ such that, for some $n \in \mathbb{N}, \langle S, d\rangle$ is isometric to $\mathbb{R}^n$ with respect to d and the metric $d'(a, b) = \sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2}$.

when it comes to sizes, but that it seems to violate results which it seems unmotivated to drop. For example, if we accept that **Odd** and $\mathbf{Odd}^2$ (the set of all odd numbers and the set of the squares of all odd numbers, respectively) have different sizes [10], because $\mathbf{Odd}^2 \subset \mathbf{Odd}$, then we are letting two sets with the exact same algebraic structure differ in size. This, says Parker, is misleading, because the two sets have exactly the same algebraic structure (the map $f(m) = m^2$ from **Odd** to $\mathbf{Odd^2}$ preserves multiplication, and also addition on both sets, vacuously, as neither is closed under addition) and yet they do not have the same size.

This is an instance of how the criterion of informativeness is not specific enough to serve its purpose. Informativeness of a theory of size, if interpreted as the requirement that two objects A and B are assigned different sizes only if there are significant differences between their algebraic structures, is not satisfied even by Cantor's theory of cardinals, because we can have homomorphic algebraic objects, for example, two groups$(G, +), (H, \oplus)$, that are homomorphic – namely there is a map $f : G \to H$ such that for any $g_1, g_2 \in G$, $f(g_1 + g_2) = f(g_1) \oplus f(g_2)$, but they may not be in a bijection (i.e. they may not be isomorphic), hence, they would not have the same cardinality. I am aware that the reply here could be that there is a significant structural difference between being just homomorphic and being isomorphic groups, hence cardinality still is informative; but it seems that Parker has left the notion of informativeness too underspecified to both be able to use it against Euclidean theories and not harm the status of cardinalities.

3.5 Taking stock

Let us summarise once more Parker's overall argument.

A Euclidean theory of size is a theory of size that obeys the part-whole principle for sets, and the discreteness principle.

From those two principles alone, it is provable that principles (ATI),(RTI), (ARI) and (RRI) fail. Since these principles are necessary and sufficient to guarantee that size is invariant under common rigid transformations, it means that Euclidean theories of size are incapable of warranting invariance under these maps.

This is particularly damaging because (even though Parker does not frame his attack this way) Euclid's fourth common notion (see 2.5.1) is interpretable as

10. This is an example of Parker's, *op.cit.* pp.600-601

requiring invariance of size under a suitable class of transformations. The most common interpretation of what these transformations might be is congruences – however Benci, Di Nasso and Forti have argued for a narrower interpretation of what the transformations should be. Parker's attack makes their position rather delicate, because it pushes them to defend the exclusion of rigid transformations from the class of transformations that should preserve size – without presenting it as an ad hoc move.

Parker's main line of attack against numerosities and Euclidean theories of size and/or size assignments is that of pointing out that, if we adopt Euclidean theories across the board, their applications will incur a number of limitations that would limit and deform ordinary mathematical activity in a way that is downplayed by e.g. the authors of the papers on numerosities themselves, even when they do know that these limitations occur (see for example the failure of ATI).

The general strategy of Parker is therefore to point at situations in which we would expect (it would be useful from a mathematical point of view, he claims) that size is preserved, only to be forced, by the assumption of (PWS), to conclude that this translation or rotation does not preserve size, or it does not preserve size relations. However, there seems to be essentially three ways in which his claim to that the maps he considers *should* preserve size may be convincing.

The first possibility is that the rhetorical strength of his argument relies on some sort of geometric intuition about size. In particular the failure of the principles mentioned above in Euclidean theories is a problem because, in plain Euclidean geometry, *the area* of any geometric figure that undergoes rigid transformations remains unchanged. What is being discussed here, however, is whether the number of points should be left unchanged by such transformations, which is not the same thing as preservation of the area. So if Parker's idea is to argue that maps such as translations and rotations should preserve the number of points, appealing to geometric intuitions does not seem like a cogent argument.

The second possibility is that, perhaps more that he would admit it, Parker is relying on the principle that the more mathematical properties a map between two sets preserves, the more likely it is to also preserve size. In particular, the maps that Parker considers as likely to preserve size are all bijections with certain additional properties, and so, by this principle, they are more likely to preserve size than bijections themselves. On this respect however, it seems difficult to see what kind of justification of this idea could be given. For if one wants to argue against Euclidean theories without presupposing the background of the Cantorian position, then one has to discard the fact that the maps Parker considers are bijections as a justification for them to preserve size.

On the other hand, it is not clear either why the fact that a given map preserves some other properties of certain mathematical structures should be any indication that it is more likely to preserve also size. We discussed earlier the reasons why preservation of algebraic structures is not a sufficient reason for preservation of size even if one understands size as cardinality. Moreover, this argument can be extended to a much wider class of mathematical structures. Indeed, the well-known Löwenheim-Skolem theorem for First-Order Logic shows that two mathematical structures that can be so similar that they are both models of a complete first-order theory can nevertheless have different cardinalities. So it seems that, even in the case of Cantor's cardinals, a wide number of mathematical properties can be shared by two different structures without that being a guarantee that they also have the same size.

The last possibility is that the maps Parker consider should preserve size because otherwise our theory of size for infinite collections would not be as mathematically useful as one such that translations and rotations would preserve size. This is most likely the point Parker is trying to make. In order to assess if his argument goes through under *this* specific reading, we need however a clearer picture of what exactly the role of a theory of size for infinite collections should be. As noted earlier, Parker's own four criteria for a theory of size seem too general to provide us with such a picture. In the next chapter, I will try to expand this part of Parker's work, and come up with a more precise picture of what a theory of measurement of size should be like, in the hope that this will allow us to determine whether or not he is right in dismissing Euclidean theories of size.

Chapter 4

Measuring size of infinite collections

This thesis has seen us entangled in at least two problems: on the one hand, one wonders what is the right way of extending the concept of number to encompass infinite quantities; on the other, there is the problem of how to measure the size of infinite objects – and more specifically, infinite collections. In this chapter, we are going to focus on this second problem, trying to keep it separate from the former.

4.1 Principles for a theory of size of (infinite) collections

The main lesson from Parker is that there may well be no notion of size justifying the details of size assignments that respect (PWS). Parker presents a few principles that are not validated by Euclidean theories of size and argues that failure of these principles makes sets which share all sorts of properties fundamentally different. In other words, Euclidean size assignments are not guided by any significant property of sets.

In the previous chapter we have introduced the reader to Parker's attempts to revisit Gödel's argument. Parker defends the primacy of the theory of cardinals indirectly, that is, by questioning the epistemic adequacy of other theories of set size – Euclidean theories. His strategy is to consider four criteria (strength, generality, being well motivated, being informative) that are necessary conditions for any theory of size to be epistemically useful, and then show that Euclidean theories cannot satisfy all four criteria at once. In particular, if Euclidean theories are strong and general, then they violate certain principles of size invariance under rigid transformations, and these violations imply that the theories are arbitrary, i.e. not well motivated.

Although Parker himself is not interested in investigating the adequacy of Euclidean theories of size from a conceptual point of view (i.e. he is not interested in establishing that Euclidean theories do not meet the epistemic standards any theory of set size should, *as* a theory of the size concept), his four criteria provide a starting point for an analysis of theories of size based on the criteria they need to meet as theories of size. Better still, they provide a starting point for a discussion for what a theory of measuring size should be like, for collections.

It might be the case that difference in size as sanctioned by Euclidean theories is no reliable indicator of variations in one (family of) mathematical properties of sets, as Parker stresses; yet this does not rule out the possibility that measuring size by preserving the part-whole principle might be useful, or interesting.

Let us start with a seemingly uncontroversial statement, that has nonetheless important consequences. The size of a collection is a quantitative magnitude, where a quantitative magnitude is defined as any magnitude that can be measured on a numerical scale.[1] As for any theory of magnitudes, a theory of size should provide one with a way of measuring a certain property that is shared by a wide class of objects – collections are the objects, in our case. We can list a certain number of desiderata for such a theory, some depending on general principles of a theory of measurement, some depending on the property that one seeks to measure, i.e. size, and finally some depending on the specific measuring procedure we are interested in – namely, counting.

4.2 General principles of a theory of measurement of size

Any theory of measurement of property X for a class of objects $\mathcal{C}$ should provide the following:

1. An assignment A of values to objects in $\mathcal{C}$.
2. An effective rule or method M to determine the value of a given object, and mathematical relations between values.

1. Eran Tal, "Measurement in Science," in *The Stanford Encyclopedia of Philosophy*, Summer 2015, ed. Edward N. Zalta (2015).

3. A mathematical structure S associated to the set of values, that allows for mathematical investigations via algebraic operations to be performed.

4. A conception J of property X, containing justifications of why the theory provides a meaningful and valuable way of tracking property X.

Note that we can reformulate Parker's four criteria for epistemic usefulness as conditions to impose on A, M, S, and J respectively. Thus a theory T of size measurement for collections (sets) is *general* if the assignment A is total; it is *strong*, if M can determine size for as many objects of the domain of A as possible. On the other hand, T is *well motivated* if the algebraic relations established by S are interpreted in a natural way as relations among the objects of the domain of A; lastly, T is *informative* if J is nonempty.

One needs to be slightly careful when making comparisons with Parker, though, because we use the same word, 'theory', to mean somewhat different things: Parker uses 'theory' in its logical sense of deductively closed set of sentences, whereas in the rest of the thesis theory is used in the more generic sense of body of propositions about one concept – size and size measurement, in this particular instance.

We now turn to the discussion of some principles that any theory of measurement (A, M, S, J) should satisfy. We provide here a list of plausible ones.

(a) The assignment A should be functional: for every object P in $\mathcal{C}$, at most one value should be assigned to P.

(b) The method M should be consistent, sound and useful: it should allow to determine unambiguously and whenever possible which value has been assigned to an object C – this is what we mean by *useful*. In particular, this means that the method should not lead one to determine different values for the same object, nor to determine a value different from that assigned by A – this is what we mean by *consistent*. Finally, it should also guarantee a certain independence of the result from the choices made when one tries to determine the size of such an object. This determines *usefulness*.

(c) The mathematical structure S should offer a faithful representation of the behaviour of the property it models, and operations in S, should have a natural interpretation on the property X. In other words, every syntactic manipulation in S should have a semantic counterpart on X (note the similarity with our reinterpretation of well-motivatedness).

(d) The set of justifications J should define a conception of the property X that should give reasons as per why all and only those operations defined in S make sense (for example, why addition of Celsius temperatures does not make sense, but ordering of intervals does).

(e) Moreover, the conception should present a justification for why the whole theory of measurement effectively tracks property X, i.e. it should at the very least provide justifications for equality or inequality between the values assigned to two different objects. That is, if two given objects are assigned the same value by A, then the conception should explain why they both have property X in exactly the same respect, or with negligible differences, and, similarly if they are assigned different values.

The principles of generality, strength, avoidance of ad-hocness and conceptual justification are often mentioned in general philosophy of science as some of the requirements any scientific theory should aspire towards. We have simply formulated them in a way that is more specific to our specific interests over theories of measurement. The following list is a list of characteristics that seem to be essential to the concept of size, and at least some of the items on the list have already been discussed in the existing literature. To my knowledge, though, none of them has been presented in the context of a programmatic attempt of giving an account of what a theory of size measurement for collections should look like.

(f) Size is a property exhibited by any collection of objects: whatever the size of a collection is, and regardless of whether or not we can always determine the size of a collection, we want to argue that the property we are after is exhibited by any collection. As we already mentioned in Chapter 1, this is not a position that has always been universally accepted by philosophers. This requirement implies that the assignment A be total: even if it may not always be possible to determine the size of some collection, it should be the case that every collection has a size.

(g) Size is a quantitative property. This implies that the mathematical structure S should contain that of a partial order, since, if both things have a quantity, it makes sense to ask whether one has more or less than the other. Whether this question must always have an answer is the point of the next requirement.

(h) Any two sizes should be comparable. This, like (f), is a supposition that can already be found in Cantor's (1878),[2] where the mathematician simply assumes trichotomy, as already noted in Chapter 2. Katz,[3] author of another system for a Euclidean theory of size – different from numerosities - also retains trichotomy among the axioms of his theory, because he believes it is essential to the very concept of size of collections. In terms

2. Cantor, "Beiträge zur Begründung der transfiniten Mengenlehre," p. 119.
3. Fred M. Katz, "Sets and Their Sizes" (PhD diss., M.I.T., 1981).

of characteristics of the structure S, should be total. Note that a stronger version of this requirement could be adopted, namely that any two sizes should be expressible as a multiple of one another, or as a ratio.

(i) For any collection of objects P, the size of P should be such that method M yields an effective way of determining the size of A, and for any two collections of objects P and Q, M should allow one to determine the order relation between the size of P and that of Q. This requirement is strength, formulated in a way that is specific to theories of size: not only it assigns a size to everything, but also in such a way that the full mathematical structure can be known. This is very hard to achieve, and in particular, the problem of (CH) is a witness to that.

(j) Addition and multiplication should be definable on S, and satisfy certain equations (commutativity, distributivity, associativity): here the requirement that addition be definable seems to be justified by the fact that by the homogeneity of sizes of collections, and the fact that every two collections can be unified into a bigger one, we would like to have a function or relation in the structure S such that it mirrors the fact that the size of the bigger collection is a function of the size of the smaller ones – this would be the role played by addition. Multiplication would be justified because it can be easily seen as shorthand for iterated addition – especially if we want to interpret multiplication as a relation among collections. For example, in the finite case we want a way of measuring size of collections that enables us to compute how many apples we need to feed a certain group of people, if we know that each person needs two apples.

(k) The size of a collection is compositional, i.e. it is a product of the sizes of its elements. This is a requirement that has two justifications: first, the idea that extensive magnitudes are magnitudes compositionally (see Kant[4]). Second, the requirement that M be both effective and as strong as possible forces us towards a recursive, or bottom-up way of determining the size of a collection, and the compositionality principle fits that requirement.

As the last step in our construction of a theory of measurement for infinite collections, we introduce two requirements that are justified by the specific purposes of this thesis:

(l) The method M for determining the size of a collection (i.e. the value assigned by A) should be via counting. Now by counting one means a successive and discrete procedure, and this seems to require the existence of a unit, or a 'quantum' of size, i.e. a value that is both *significant*(i.e. such

4. for Kant, extensive magnitudes are those "in which the representation of the parts makes possible the representation of the whole"(1787: A162/B203). This is what we mean here by 'compositionality'

that if added to a collection A, it leads to an increase of size) and *irreducible* (i.e. there is no smaller quantity that can be added to a collection and that will increase the size of the collection).

(m) The whole theory should be formalised in standard set theory, and collections are to be identified with sets.

Now we see how the standard conception fares with respect to these requirements.

4.3 Framing the standard conception

Cantor's principle is the main component of the Cantorian theory of size: it allows to define the assignment A, provides an effective way of determining relations of size, and is the basis for the definition of the algebraic properties of S (the cardinal number scale). Cantor's principle plays the role of an abstraction principle, which means it can be put in the shape

$$\forall A \forall B(\spadesuit x(Ax) = \spadesuit x(Bx) \leftrightarrow A \equiv B)$$

where A and B are sets, $\spadesuit$ is the ' abstraction operator' to be interpreted as 'number of', and $A \equiv B$ means (always just in the specific case of (CP)) 'A and B are in a one-to-one correspondence'. So (CP) is an abstraction principle, because it can be interpreted as defining a second-order abstraction operator (number of) by using an equivalence relation between classes ((CP) is about not proper classes, i.e. sets). Since we are singling out an equivalence relation, we can partition the domain of discourse into equivalence classes, determined by (CP).

Hence (CP) provides everything we need for an assignment A all at once: a set of values (equivalence classes), and the assignment itself (sending every set to its equivalence class). It also provides in a natural way the order that has to be on the set of values, although the proof that every set has a cardinality relies on the well-ordering theorem (but the totality of the order, on the other hand, is a consequence of the linearity of ordinals). Finally, the definitions for addition and multiplication make use of (CP) as well as very simple concepts like disjoint union and product. The weaknesses of the theory on the other hand may lie in the conception of size that CP is supposed to promote, and the sense in which bijections truly are a way of counting sets. Gödel's argument seems to be meant to motivate a certain conception of the size of sets.

Let us reconsider then the argument under this new interpretation (for the full extract, see p. 28 of the thesis). It can be summarised as follows:

The number of objects is invariant under permutation of qualities and relations of said objects. But then every two sets that are in one-to-one correspondence should have the same size. For if there is such a correspondence between A and B, then replacing every element in A by its image in B amounts to changing properties of elements, which preserves cardinality. Then by the axiom of extensionality the new A and B are one and the same set, and so their cardinality should be the same.

Here we have explicitly mentioned the role played by the axiom of extensionality in the argument, which we had not highlighted in the reconstruction of the argument in Chapter 2. There is also another implicit premise, that as we will soon realise, is less trivial than at first sight: The size of an infinite set just is the number of its elements.

4.3.1 Making the implicit explicit in Gödel's argument

There are a few points that can be extracted from Gödel's argument thus interpolated. First of all, size is a property of collections qua sets, therefore determining the size of a set is enough to determine that of the corresponding collection. Second, to determine the size of a set it is enough to determine the number of its objects. This implies that one only has to count the number of objects in order to determine the size of a set (this is closely related to the idea that the unit in counting is the element). Such assumption seems to dovetail with the extensionality of sets that George Boolos[5][6], for one, considered as analytic of the concept of set. Under the assumption of extensionality, then, it is a forced choice that the size of a set be completely determined by its elements.

4.4 Framing the Euclidean conception

Difficulties aside, (CP) provides a clear way of defining A, T and S in a unique way. It is far from clear that the same can be said about (PW). As noted above, (CP) defines implicitly equivalence classes of sets, and then plays the role of an *abstraction principle*. It is not clear that this is what is happening with (PW). Prima facie, (PW) just says something about relations between

5. George Boolos, "The Iterative Conception of Set," *The Journal of Philosophy* 68, no. 8 (1971): 215–231, ISSN: 0022362X, http://www.jstor.org/stable/2025204.
6. The full quote can be found on p. 119 of the article.

some sets, namely, that under certain hypotheses (a part-whole relation) some are smaller than others. As it stands, it does not tell us what should be the case when those conditions do not hold, i.e. when something is not a part of something else, nor does it prove that those conditions always hold. Moreover, (PW) does not immediately suggest an effective method for determining the size of some collection. One could try some kind of "sandwich" method: if we know the size of A and C, and also that A is a part of B and B a part of C, then we can conclude that the size of B is somewhere in between that of A and that of C. So if one is to make use of (PW) for a theory of measurement of size for infinite collections, it seems that PW would be unable to play the same determining role as (CP) does for the standard theory. Instead, it seems like (PW) itself needs to be derived from some conception of size, thus earning its justification.

There might be two ways of giving a conception of size that would entail (PW). One is to say that the size of infinite collections should obey the same rules as that of finite collections, i.e. that the axioms for finite arithmetic should also hold for the infinite one. The other is to appeal to some geometric intuitions about sizes, in particular via Euclid's common notions.

4.4.1 The argument from finite arithmetic

We already stipulated that we should assume an homogeneity in kind between finite and infinite collections. If so, then any theory of size for infinite collections should be as continuous with the theory of size for finite collections as possible – which is finite arithmetic.

Formally, if one concedes that Peano's axioms capture the essence of finite arithmetic, this means that the structure S given by our theory should be a model of PA, which means it should be a discrete linear order with successor; associative, commutative and distributive additions and multiplication; and cancellation for addition. In particular, 0 (the 'null size') should be the only case in which adding it to some non-zero element is not strictly increasing. So adding something that is not null in size should cause the size to change. It is well known that this does not hold for cardinals, though. If there is a smallest non-zero size, then why is it acceptable that adding such a size suddenly does not change anything anymore? We know that the reason why not all of the properties of the finite numbers is that then one can derive a contradiction in the system. The question is though, what in the notion of size for collections, and size measurement, justifies the choice of preserving commutativity of addition and multiplication, but not the cancellation property. One intuitive motivation for commutativity and associativity of addition is that it should capture the summation of collections into a bigger collection. But now if the size of a collection is somehow non-relative to other collections (i.e., it is an intrinsic feature of a collection), then the result of adding two collections should not depend on the way we add them (provided it is a disjoint addition). In particular, it should not depend on whether we are adding the first one to the second one, or the second one to the first one, hence commutativity should hold. Similarly, if we actually want to add the size of three collections, then the result should not depend on whether we add the first two to the third, or the first one to the sum of the two others, hence associativity should hold. It is hard to explain why one could not motivate the cancellation property in a very similar way. Suppose we have two collections A and B, and we decide to add them, and them remove one of them, say A. If their sizes are intrinsic properties of collections, then combining collections or separating them should not make a difference on the collections themselves. But of course, in the Cantorian view, if one decides to add an infinite collection to a finite one, and then subtract this infinite collection, the theory predicts that the result should be zero.

Problems with the argument

The argument 'from analogy' with the finite case is too quick. It seems reasonable that Cantor would have wanted to preserve as much as possible of the features of finite arithmetic, but this was not possible because of the paradox of the infinite we have mentioned in the introduction.

Once faced with a contradiction between preserving the law of cancellation on one hand, and choosing to define size of sets in terms of cardinals (instead of ordinals, say), it seems sensible to argue that the best choice is to resort to some guiding principle that is not descended from the concept of size or size measurement, if those do not seem to be able to give a priority to either the cancellation law, or commutativity of the sum of sizes, say.

One could interpret the Cantorian 'choice'[7] as guided my a cost/benefit analysis of what was there to lose in forgoing the cancellation law in order to keep cardinals.

The payoff for dropping some of the algebraic features of addition and multiplication – the never-ending scale of cardinals – must have outweighed the cost of interrupting the continuity between the finite and the infinite at the level of arithmetical operations. This at least would allow for the very existence of different infinities.

7. We cannot talk about choice *simpliciter*, because we do not know whether Cantor himself actually did consider a Euclidean option to define "cardinals".

4.4.2 The argument from geometric intuition

Euclid's five common notions include the part-whole principle as the fifth common notion, but they also provide an example of what a theory of size based on geometric intuitions could look like, in particular because notions CN1 to CN4 are properties that can be interpreted as setting requirements for equivalence classes of magnitudes of the same size, the first common notion being exactly that having the same size is transitive (or euclidean, depending on how it is read; cfr. p.22). Common notions CN2 and CN3 put as a condition that this equivalence relation of size also be a congruence with respect to the operations of adding and subtracting quantities: $A \sim B$ and $C \sim D$ imply that $A + C \sim B + D$, and $A - C = B - D$.

Problems with the geometric conception

Unfortunately, CN4 is the only one that seems to determine when two magnitudes belong to the same equivalence class – but it is also the most obscure: it states that magnitudes are equal when they "apply onto one another". It is not clear what would be the application of CN4, and it is in any case not nearly as helpful as CP.

Moreover, there is a more general problem with the appeal to the geometric discourse in the case at hand here. Even if one is charitable and grants that the five common notions indeed offer a conceptualisation of size measurement, we need to make an argument as per why this is the conceptualisation that one would want to extend to the infinite.

4.5 A difficulty for any theory of size?

From the analysis of the two competing approaches we are mainly preoccupied with, it seems that the standard (i.e. Cantorian) conception fares better than Euclidean approaches, and that (CP) provides a solid basis for a theory of measurement of sizes of infinite collections, while the same cannot be said of (PW). However, I would like to raise a concern that stems from the requirements that we have imposed on any theory of measurement of size.

In the case of geometry, the measure is done by counting, but a unit has to be specified, and, in general, this unit is not the point. Classical definitions of the point describe it as an entity that has no dimension, no size, and no quantity. But geometric figures that are being measured, like lines or figures, exist in at least one dimension, and, in the case of a line, it seems like an odd claim to make that its size ought to be the "number of points" it contains. The parallel with elements in a collection here is quite tempting: in infinite collections, according to the Cantorian view, the size of elements is negligible, just like the size of a point is negligible in a line: adding a point to a line will not affect its size, for only the addition of a unit will make such a difference. The important distinction that has to be drawn between points and lines on the one hand, and units and (finite) collections on the other, however, is that points never make a difference in size, while elements do make a difference in finite sizes. How can we explain the difference? When it comes to continuous magnitudes, the size of a magnitude is always relative to a unit that has been fixed. The null quantity is always the same, but the interval that counts as a unit is a matter of convenience and significance. When considering collections, there is a canonical unit that can be defined in the finite setting, namely the element of a collection. However, when we move to infinite collections, the element becomes negligible, and virtually of size 0. All of this poses a challenge to any theory of measurement of infinite quantities that operates via counting, and in particular it seems to threaten the standard conception. In fact, it seems that we can argue for the following

Claim 4.5.1. No theory can consistently satisfy all the desiderata for a theory of measurement of size for infinite collections.

For, assume you have a theory of measurement of size (A, M, S, J) that is both via counting and compositional (we have defined these notions earlier in the chapter). This means that given any collection P, we have an effective way of measuring the value A(P) by counting how many units one can find in P and that A(P) is a direct product[8] of the size of the elements in P. Now if collections are sets, then they are completely determined by their elements, and there is nothing more to them apart their elements. But then the only thing that can be picked as a unit are the elements themselves, and we must conclude that every element in a set has size "unit". Hence counting the units means counting the elements. But then, by Gödel's argument, counting the elements implies that Cantor's principle should be satisfied. If Cantor's principle is implied by our theory, though, then it is not true anymore that units always make a difference (in the strict sense we considered before), since adding an element to an infinite set does not change its cardinality. So we reach the following paradoxical situation:

1. adding units makes a difference (they are *significant* magnitudes: for if this is not the case, then how can one be certain that counting units is a sound method for determining size?)

8. not in the technical sense.

2. In any set, an element has always size "unit" (for this is the only way to fix a unit if collections are sets)
3. adding elements to a set does not always make a difference in size (consequence of Cantor's principle).

It seems hard to imagine how any conception of size could justify such a feature of the theory, namely that objects that are assigned a significant size do not make a difference in size when added to certain collections.

4.6 Conclusions of this chapter

If one grants that the argument above poses a problem for any theory of size in the sense given in this chapter, then it seems that a diagnosis of the paradoxical situation above is necessary.

In the argument offered for Claim 4.6.1, it seems that the very assumption that the unit in the counting for determining the size of infinite collections should be the element is what triggers the paradoxical conclusion: on one hand, elements have a size that is significant enough to stand as the unit for counting collections, but on the other hand, they are in some cases not significant enough for an increase in elements to be an increase in size. It is not clear whatGödel would reply to this sort of objection; to be fair, it seems that Gödel was more interested in arguing that transfinite numbers are the only viable and non-arbitrary extension of the finite numbers, than that Cantor's way is the right way of measuring the size of infinite collections. More importantly, it seems that Gödel took for granted that counting the elements of a set was the right way of determining the size of a collection. So it seems that if the assumption that is to blame for the paradox is that the unit in the counting should be the element, this could have important consequences, and ultimately force us to abandon the assumption that the notion of a set captures enough about collections for us to be able to determine their size by regarding them as sets alone. This is the hypothesis that I would like to explore in the next chapter, and I will focus especially on the possibility hat it may open for a reassessment of Euclidean theories, and of numerosities in particular.

Chapter 5

Reframing numerosities

5.1 Chapter plan

Goal of this final chapter is to provide an argument for considering numerosities not as an alternative theory to that of Cantorian cardinalities, but as a complement of it.

The key idea is that the theory of numerosities captures the size relations between "wholes and parts", whereas cardinals treat size relations of sets and subsets, and sets and subsets do not necessarily coincide with parts and wholes.

There are two main reasons to believe that subsets and parts are not identical: first, intuitive reasons – we do not seem to individuate, say, the even numbers, merely as a list of objects. We conceive of them as certain spaces on the line of the natural numbers. The second reason to distinguish parts from subsets comes from the discussion in chapter 4.

If we do not equate subsets and parts any more, then Parker's arguments against Euclidean theories suddenly lose their grip: they are based on the set-theoretic reading of the part-whole principle and they do not seem amenable to an obvious reworking with another formulation of the principle. Thus here we have a possibly strong strategy against Parker's arguments, at last.

We are left only to answer the question of whether we still can formulate (PW) in the language of set theory, and the answer is yes, through a 'toy theory' of parts. We conclude by suggesting that numerosities may be seen as a generalisation of this toy theory.

5.2 Why subsets are not necessarily parts

It seems that there is a substantial difference between considering the set $\{1, 2, \dots\}$ as $\mathbb{N} - \{0\}$ on one hand, and as $\mathbb{Z}^+$ on the other hand. In the first instance the most correct visualisation is to draw a line starting from 0 and then highlighting it only from 1 onwards; in the second case, it is enough to just draw the line starting with 1. Parker recognises that there is a case to be made that in the context of sets of integers, failure of preserving translation invariance is not per se a reason to consider the theory of size one is evaluating as arbitrary[1]. He does not go as far as to distinguish between $\mathbb{N} - \{0\}$ and $\mathbb{Z}^+$, though, and that prevents him from giving a good reason for why, in this particular case, translation invariance is not the right criterion.

It is our opinion that the reason why translation invariance is not the right criterion is because there is a substantial difference between $\mathbb{N} - \{0\}$ and $\mathbb{Z}^+$, namely the first is a proper part of $\mathbb{N}$ while the latter is an independent mathematical entity, as it were. Hence, the part-whole principle really should be reformulated as

(PWP) If A is a proper part of B then $[A] < [B]$.

One reason to consider the part-whole principle as not specifically about sets and subsets is that the part-whole principle for sets (PWS) is one among several possible specifications of 1.1 (CN5). Another reason is that it seems that we do make a distinction whenever we think of (to take our favourite example) the even numbers as a set, independently of whether they are a subset of $\mathbb{N}$ or $\mathbb{Q}$ and so forth, or as a structure embedded into the natural numbers, or the rationals, et cetera. Once again, considering how we would represent the set graphically helps. In the case we want to represent the evens without worrying what set they are a subset of, we may draw them on a line just like the one we would use for all nonnegative integers, only instead of writing the sequence $\{0, 1, 2, 3, \dots\}$ underneath, we would write $\{0, 2, 4, 6, \dots\}$ in exactly the same positions. If instead we wanted to stress the fact that the evens are a subset of

1.

> there is a qualitative difference between $\mathbb{N}$ and $\mathbb{Z}^+$, and likewise between any two sets of whole numbers. Hence, the failure of ATI for number sets does not directly imply that their sizes are arbitrary. (...)

Parker, *op.cit.*, p.600.

the set ℕ, we would most likely draw them by leaving exactly one 'empty spot' between each element of the set.

The main feature that is missing from subsets and that we think is instead essential of parts is something that can witness that they are extracted from some other entity – something that can keep track of the 'empty spots'. This deficiency (not meant in a negative way, we only want to stress that it is a missing feature) is somehow mirrored by how subsets are formed according to the iterative conception of sets. Once we have a set m at a certain stage λ of the cumulative hierarchy, its subsets are formed, i.e.'come into existence', at the following stage $\lambda+1$, although the members of all of these new sets were already present at stage λ,[2] so it feels like forming subsets out of m is more than just partitioning m into overlapping parts. This is our first justification for the claim that subsets and parts are not quite the same thing, at least intensionally.

A second justification comes from the outcome of the analysis we carried out in Chapter 4: the contradiction we derived from trying to apply the desiderata to the Cantorian theory of cardinals shows that there is something amiss with cardinals as the scale for measuring the size of infinite sets.

Before moving on with an exposition of our 'theory of parts', we should briefly mention what is its relation with traditional mereology, since mereology is defined[3] as the theory of parthood.

As Burgess[4] notes, one of the central differences between mereology and set theory is their treatment of the singleton. While the singleton set is considered in set theory as the atomic part of any set (meaning there can be no smaller part than that), this is not so for the mereologist, who does not see the positing of the existence of singletons as metaphysically innocent. In particular, the mereologist seems to have issues with the idea that several things can be collected into a set in such a way that they remain distinct *elements*. As Burgess puts it:

> Nominalists have traditionally objected [to the] implication that set-formation is less a process of merger, like that by which Italy was formed from various minor states, than a process of federation, by which thirteen colonies became the United States. The implication, to be more explicit, is that even after the many have been collected together into a one, it is still discernible which many they were: that just as the set is determined by its elements, so also the elements are determined by the set. Mereological fusion, by contrast, obliterates

2. Boolos, "The Iterative Conception of Set," pp. 220-221.
3. Achille Varzi, "Mereology," in *The Stanford Encyclopedia of Philosophy*, Spring 2016, ed. Edward N. Zalta (2016).
4. John P. Burgess, "Lewis on set theory" (https://www.princeton.edu/ jburgess/Lewis.pdf).

> the separate identities of the fused: A single whole can be taken apart in many ways, and there is no one way of taking it apart of which it can be said that the genuine parts of which the set is composed are just those pieces into which it is disassembled when taken apart in that way and no other.[5]

Mereology and our theory of parts have in common that they reject the following equivalence:

(E) X is a part of set A if and only if $X \subseteq A$.

However, the mereologist rejects the left-to-right direction of the biconditional (not every part of some whole is a subset of that whole), while our theory rejects the right-to-left direction (part of wholes are a certain kind of subsets). While mereology still has an extensional understanding of parts, here we are inviting the reader to consider parts as not just a 'collection' of elements all coming from the same 'supercollection' – we are proposing an intensional conception of parts. Another difference is that we are not interested in giving an account of the real metaphysics of parts, but just an alternative understanding of what are parts of sets when they cannot be fully identified with subsets.

The next issue we need to address is whether we can talk about parts in set theoretic terms without reducing them to subsets, and that is technically challenging, but not impossible. Here we are going to present a toy idea to solve the problem, at least in part.

5.3 A theory of parts

Suppose we restrict our attention to developing a framework in which we can talk about the subsets of $\mathbb{N}$, as well as its parts. The first thing we do is to observe that $\mathbb{N}$ is a (nonproper) part of itself, and we are going to express that by introducing the set $\mathbb{N}_p$, where the subscript stands for 'part', and $\mathbb{N}_p = \{(0,1),(1,1),(2,1),\dots\}$. In other words, $\mathbb{N}_p$ is the graph of the characteristic function of $\mathbb{N}$ as a subset of itself. Next, we are going to consider the graphs of the characteristic functions of all subsets of $\mathbb{N}$ *as subsets of* $\mathbb{N}$. We will call them the *parts* of $\mathbb{N}$.

In order to present things more precisely, we need to introduce a couple of definitions.

5. ibidem

Definition 5.3.1 (Significant map)**.** A significant map between two graphs $\mathbf{a}$ and $\mathbf{b}$ is a map $s : \mathbf{a} \to \mathbf{b}$ such that for all $c \in C$, if $\mathbf{a}(c) = 1$ then $s(\mathbf{a}(c)) = 1$ and if $s(\mathbf{a}(c)) = 0$ then $\mathbf{a}(c) = 0$.

Definition 5.3.2 (Part)**.** Given two sets a, b, we say that $\mathbf{a} \subseteq_p \mathbf{b}$ (read: $\mathbf{a}$ is a part of $\mathbf{b}$) if and only if the second projection of $\mathbf{b}$ is the constant function 1 and the second projection of $\mathbf{a}$ is the characteristic function of a subset of $\mathbf{b}$.

Definition 5.3.3 (Proper part)**.** Let a be the set that is to be considered as the whole. Then $\mathbf{b}$ is a proper part of a iff $\mathbf{b} \subseteq_p \mathbf{a}$ and there is no significant map between $\mathbf{a}$ and $\mathbf{b}$.

Next, we want to define a size-order between the different parts of a set. One could attempt the following:

Definition 5.3.4 (Less than ($\leq_p$))**.** Given $\mathbf{a}$ and $\mathbf{b}$ both parts of the same set C, we say $\mathbf{a} \leq_p \mathbf{b}$ if and only if there is a significant map $s : \mathbf{a} \to \mathbf{b}$ such that for all $c \in C$, if $\mathbf{a}(c) = 1$ then $s(\mathbf{a}(c)) = 1$ and if $s(\mathbf{a}(c)) = 0$ then $\mathbf{a}(c) = 0$.

This defines an equivalence relation in the usual way: given parts $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \leq_p \mathbf{b}$ and $\mathbf{b} \leq_p \mathbf{a}$, then $\mathbf{a} \sim \mathbf{b}$. [6] From now on, we will consider $\mathbb{N}$ to be the set whose parts we are treating[7].

We can also render the $\leq_p$ order a strict order in the usual way: $\mathbf{a} <_p \mathbf{b}$ iff $\mathbf{a} \leq_p \mathbf{b}$ but $\mathbf{b} \not\leq_p \mathbf{a}$.

Claim 5.3.1. Given $\mathbf{a}, \mathbf{b} \subset_p \mathbb{N}$, either $\mathbf{a} <_p \mathbf{b}$ or $\mathbf{b} <_p \mathbf{a}$ or $\mathbf{a} \sim \mathbf{b}$.

(for the proof, see appendix A.1).

This theory is no doubt riddled with limitations. Within this framework, the part-whole principle for parts (PWP) holds, but not in an interesting way, because a consequence of the given definitions is that only the set that is considered the "whole" has parts. For a concrete example, if $\mathbb{N}$ is the set we are working with, given any two parts of $\mathbb{N}$, $\mathbf{a}$ and $\mathbf{b}$, by our definitions $\mathbf{a}$ is not a part of $\mathbf{b}$, and vice versa, and it is relatively easy to see that there cannot be a significant map between $\mathbb{N}_p$ and any of its proper parts, so (PWP) is safe, although because we got rid of the most problematic situations quite artificially. It is possible to give a definition of nested parts such that a part of a part of ... a part of $\mathbb{N}$ is still a part of $\mathbb{N}$, by keeping track of the characteristic functions, so to speak, but it would be extremely cumbersome. Moreover, the advantage would be limited

6. The proof is straightforward hence left to the reader.
7. Up to this point, there was nothing of the definitions that depended on features peculiar to $\mathbb{N}$; however, we were presupposing the well-ordering theorem in the background.

because it would still not solve the other heavy limitation our toy theory has to face, namely the absence of any sort of arithmetical manipulations.

When it comes to try to define addition, for example, the situation is problematic because somehow we have created a closed system with a top element, that is the set whose parts we are studying (in our specific case, that is $\mathbb{N}$). It seems then like a reasonable request that the structure composed by our whole set and its parts be closed under addition, that is, given two parts **a** and **b**, the sum of their sizes should not exceed the size of $\mathbb{N}$. The obvious candidate then becomes union,but this is incompatible with how we have defined sizes for parts and wholes, because we used equivalence classes – and union is dependent on the choice of representatives for each class. At the time of writing, I do not see a way of avoiding the problem.

This obstacle motivates a reconsideration of numerosities. At first sight they might not seem helpful, because they are presented as an alternative way of counting for sets, and more crucially the version of the part-whole principle they consider is a variation of the part-whole principle for sets. In their (2003) paper in fact they write the part-whole principle for sets as one of the desiderata a counting system should satisfy. The conceptual similarities with our approach are however striking: the construction of numerosities starts by converting each set into what they call a labelled set. Recall the formal definition of labelled set: 2.5.1. If the reader compares it with the definition of part, they will see that a part can also be defined as a labelled set where the domain of the labelled set is e.g. C and the label l_C is the characteristic function of C as a subset of $\mathbb{N}$. In the (2003) paper, Benci and Di Nasso mention the idea of a canonical labelling. What they define as such is the identity labelling on all subsets of $\mathbb{N}$, but we can also view the labels of parts as another special sort of labelling, that we can call the part-labelling.

Indeed, this definition allows them to then define the approximating sequence of a set in the following way: the labelled sets are used to partition A into finite subsets $A_0, A_1, A_2, \ldots$ such that they are ordered by inclusion and each of them has finite cardinality. So A can be represented by the sequence made of the cardinalities of its partitions. Now the similarity with our suggestions are made clear: the approximating sequence is akin to the characteristic function of a set, in the sense that it does not progress as long as one does not add elements to the set, and each repetition keeps track of a number that has been left out, just like the 0s do with the characteristic function approach. More formally, one can systematically retrieve the approximating sequence $\#(A_0), \#(A_1), \ldots$ by defining it inductively on the characteristic function, that we will denote by f.[8]

8. To make this more precise, from an approximating sequence $s_0, s_1, s_2 \ldots$, one can define a characteristic function $r_0, r_1, r_2 \ldots$ as follows:

(i) $r_0 = s_0$

5.4 A reassessment of the contribution of numerosities

I would like to conclude with some remarks about which role numerosities can play under this reinterpretation of the part-whole principle. My idea is that, perhaps contrary to what their creators believe, the philosophical value of numerosities is not that they provide an alternative way of measuring the size of subsets, but as a new way of determining the relative size of parts with respect to their wholes. Indeed the authors themselves seem to interpret numerosities as giving the size of subsets, and fail to appreciate that their work could well be interpreted as being about the size of parts of sets instead. They seem to be framing their proposal in the following way: right now there is one favourite scale, *Card* (the class of all cardinal numbers) that is chosen to determine the size of sets in the set theoretic universe. This scale has many nice properties, but it does not satisfy the (set theoretic) part-whole principle: $2\mathbb{N} \subset \mathbb{N}$ and yet $|\mathbb{N}| = |2\mathbb{N}| = \aleph_0$. If one were to insist that this principle is essential to the very concept of size, then the cardinals are not the right scale. Because of their nice features though it is advantageous to develop an alternative that is consistent with the existence of cardinals; the project is of constructing another scale that can coexist in the set theoretic universe with the cardinals, and that would take over from them the role of size-scale for sets.

I think this is what makes numerosities vulnerable to Parker's criticisms: the ambitious goal of developing a system that is *alternative* to Cantor's inevitably calls for the reaction of looking for some results that have now become commonplace and then showing them to fail under the new theory that is meant to supplant or provide an alternative to the old one. Once a certain paradigm is entrenched in the practice of a discipline there must be very strong reasons to eradicate it and replace it with something new. This is another reason – strategic almost – to wanting to redefine the theory of numerosities and clarify that it is not meant to be an attempt at supplanting Cantorian cardinals [9].

Moreover, I would like to argue that under our interpretation of the significance of numerosities, the sense in which they are "a new way of counting" that preserves part-wholes relations becomes clearer. When introducing numerosities to the reader, Mancosu uses an analogy from how to stack pegs numbered from 1 to 90 on a table of squares numbered from 1 to 90. He claims that the way numerosities work is similar to carrying out the counting of the pegs by placing

(ii) $r_{n+1} = s_{n+1} - s_n$

. Conversely, one can retrieve an approximating sequence $s_0, s_1, \ldots$ from a characteristic function $r_0, r_1, \ldots$ by defining it recursively as follows:

(a) $r_0 = s_0$

(b) $r_{n+1} = r_n + s_{n+1}$.

9. This is something touched upon briefly in Mancosu (2009) as well.

the pegs numbered from 1 to 10 all on the square labelled as '10', then the pegs from 11 to 20 are placed on the square '20'; and so on, until we have exhausted all pegs. Count up the number of pegs in each stack, and the sequence you obtain in the process is the approximating sequence of the collection of pegs for the game.

On our understanding of numerosities as a way of measuring the size of 'parts', the following picture seems more fitting. Suppose you have three bags, one with (plastic specimens, say, of) all the natural numbers from 0 to infinity in their natural ordering, and two empty bags, one labelled as the '1'-bag, the other as the '0'-bag. You want to have a faithful representation of how your collection comes from the bigger one. And so you take each number and as you go, you add it to the '0'-bag if you are discarding it, and to the '1'-bag if you're adding it to the collection you want to form. It is clear that what you are doing is not simply keeping track of how many elements your collection has, but also which slot, so to say, they occupy in the collection they come from. This is what numerosities as a theory of parts and their sizes can allow you to do.

And so at the very least we can view numerosities as a tool to give an articulate answer to the question: can two sets with the same number of elements still have different sizes? they allow us to answer that in a sense, yes: to pick up again one of our running examples, if we view the even numbers just as a subset, in the standard ZFC sense, of the naturals, they can be put into one-to-one correspondence with the set $\mathbb{N}$, hence they have the same cardinality, that is to say, they share the same size, as sets. If we conceive of the even numbers really as a part of the natural numbers which occupies specific portions of the "line" on which we can represent them, they do not have the same numerosity, hence they do not have the same size when considered as objects in a part-whole relation.

Chapter 6

Conclusion

6.1 Galileo's paradox

We started this thesis by considering one of the traditional paradoxes of infinity, we offered four possible resolutions of the paradox, and promised to argue for one of them, namely, that it is possible to explain away the paradox.

In the second chapter of the thesis, we analysed the framework that is typically used to define the size of sets, namely Cantorian cardinal numbers, and what we chose as the paradigmatic theory for the competing conception of size, the theory of numerosities.

In the third chapter we moved on to examine one important argument that has been offered to defend cardinals as the right tool to define set size (Gödel's), together with a more recent version of the argument (Parker's), trying to see if they would settle the question of solving the paradox by arguing for unquestionably for Cantorian cardinals. Whereas Gödel's argument did not seem particularly compelling, Parker's indirect argument for cardinals (I say indirect because it is more of an attack on Euclidean theories than a direct defence for cardinal numbers) was harder to tackle, and that required us to first propose a tentative theory of measurement of size for (infinite) collections, and then to argue for an intensional understanding of the notion of 'part' in the part-whole principle, thus in effect shifting the domain of application of (PW) from subsets to 'parts'.

This last move allows us to regard (CP) and (PW) as not in direct conflict any more, but as two principles governing two different kinds of size relations: (CP) is the principle regulating absolute size of sets, while (PW) is the principle underlying the assessment of size relations between sets and their parts, where parts are not subsets *simpliciter*. Hence, the solution to Galilei's paradox is as follows: if considered as an independently generated set, $\mathbb{N}^2$ (the set of all squared natural numbers) has exactly as many elements as $\mathbb{N}$, for we count the elements by using cardinal numbers. If on the other hand we want to consider it as a *part* of $\mathbb{N}$, then we should compute its size via numerosities and conclude that $\mathbb{N}^2$ is strictly smaller than $\mathbb{N}$.

6.2 Further work

There are several interesting issues that we would have liked to explore in the context of this thesis, but couldn't because of lack of time. Here I am going to mention just a few.

First of all, in building our defence of numerosities from Parker's criticisms of arbitrariness, we had to make the theory less general than what it currently is: we only discussed (and incorporated in our framework of Chapter 5) numerosities for countable sets; Benci, Forti and Di Nasso, however, have been able to define numerosities for larger classes of sets, so it would be interesting to attempt to extend our defence of countable numerosities to the uncountable ones, too. The problem is that the theory of numerosities becomes very sophisticated very quickly, and it was not possibile in the limited amount of time we had to try and build a comprehensive defence of the theory.

The second element that needs further work is the tentative theory of measurement of size we propose in Chapter 4: as it is a kind of work that is absent in the literature, it would be beneficial to investigate further its links with more traditional general philosophy of science, and with the problem of applicability of mathematics specifically. Considerations regarding how we teach measurement of size for sets would also need to be incorporated for the theory to be really well developed.

Finally, a deeper investigation into the origins of the so-called Euclidean theory of size as exemplified by the five common notions could help us give more historical perspective to the whole question of what is size, and to what extent we can have a unified conception of size for geometrical and arithmetical/algebraic objects.

Appendix A

Appendix

A.1 Proofs

Proof of the claim that the order of parts is indeed a total order over parts.

Proof. This is an easy proof by cases.

Case 1 The second projection of $\mathbf{a}$ contains finitely many 1s and infinitely many 0s, and so does the second projection of $\mathbf{b}$. Then our p-order collapses to usual comparison of finite cardinalities.

Case 2 The second projection of $\mathbf{a}$ contains finitely many 1s and infinitely many 0s, whereas $\mathbf{b}$ has infinitely many 1s and finitely many 0s. Then $\mathbf{a} <_p \mathbf{b}$.

Case 3 The second projection of $\mathbf{a}$ contains finitely many 1s and infinitely many 0s, whereas $\mathbf{b}$ has infinitely many 1s and infinitely many 0s. Then $\mathbf{a} <_p \mathbf{b}$.

Case 4 The second projection of $\mathbf{a}$ contains infinitely many 1s and finitely many 0s, whereas $\mathbf{b}$ has finitely many 1s and infinitely many 0s. Then, since both $\mathbf{a}$ and $\mathbf{b}$ have finitely many 0s, $\mathbf{a} <_p \mathbf{b}$ if and only if the number of 0s in the projection of $\mathbf{b}$ is strictly smaller than the number of 0s in the projection of $\mathbf{a}$.

Case 5 The second projection of $\mathbf{a}$ contains infinitely many 1s and finitely many 0s, and so does the second projection of $\mathbf{b}$. Then $\mathbf{a} \sim_p \mathbf{b}$.

Case 6 The second projection of **a** contains infinitely many 1s and finitely many 0s, whereas **b** has infinitely many 1s and infinitely many 0s. Then $\mathbf{b}{<_p}\mathbf{a}$.

Case 7 The second projection of **a** contains infinitely many 1s and infinitely many 0s, whereas **b** has finitely many 1s and infinitely many 0s. Then $\mathbf{b}{<_p}\mathbf{a}$.

Case 8 The second projection of **a** contains infinitely many 1s and infinitely many 0s, whereas **b** has infinitely many 1s and finitely many 0s. Then $\mathbf{a} <_p \mathbf{b}$.

Case 9 The second projection of **a** contains infinitely many 1s and infinitely many 0s,and so does that of **b**. Then $\mathbf{a} \sim_p \mathbf{b}$.

□

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