Generic Structures *

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Abstract

In this paper, the theory of generic structures is put in a general, unified framework. The framework is applied to Hrushovski's [8] *universal domains* and Poizat's [11] *bicoloured fields*. Of particular interest will be the conditions under which generic structures are saturated.

Introduction

Generic structures have found wide application in contemporary research in model theory. Most of it stems from two seminal papers by Hrushovski [6, 7] in which he adapts Fraïssé's [4] generic construction to refute conjectures by Lachlan and Zil'ber. More recently, Hrushovski's techniques have been adapted by Poizat [11, 12], Baudisch [3], Baldwin [1], and Baldwin and Holland [2] among others to construct a variety of ω-stable generic structures.

Fraïssé's original notion of a generic structure in [4] is that of a countable universal-homogeneous structure with respect to a class $\mathbf{K}$ of finite relational structures. Hrushovski and his followers replace the substructure relation by stronger relations (usually denoted by $\leqslant$) and adapt the definitions of universality and homogeneity accordingly.

Keuker and Laskowski [9] investigate the theory of generic structures in this context. Wagner [14] looks at the relation between generic structures and notions of closure, dimension and independence which are related to the relation $\leqslant$. In both cases (and in all cases I've come across) attention is restricted to classes of finite structures.

In this paper, the theory of generics is put in a general, unified framework. The class $\mathbf{K}$ is no longer required to contain only finite structures and generic structures may be uncountable. As the definitions will make clear, genericity will be regarded as a weaker form of saturation. Unsurprisingly, the conditions under which generics are actually saturated will be of particular interest.

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A general notion of *morphism* is taken as primitive rather than the more restrictive relation $\leqslant$ which is treated as derived. In part I, an axiomatisation for morphisms is provided and many of the standard results regarding generics are generalised to higher cardinalities. In part II, the *nice* case of morphisms as Δ-elementary maps for a certain class of formulas Δ is considered. It is shown that Hrushovski's [8] *universal domains* are examples of generic structures and conditions for their saturation are investigated.

Part III applies the framework of this paper to a sample of recent research in the construction of ω-stable structures, *viz.* Poizat's [11] *bicoloured fields* (or - negligibly more general - bicoloured *strongly minimal structures*). Constructions of this type yield very interesting and important results and it is therefore important that our framework is applicable to them. Aside from this fact, Poizat's construction is of interest also purely in the context of this paper, for it provides an example of morphisms which are not Δ-elementary for some Δ.[1] As will become evident, proving the saturation of generic structures in this context can be far from trivial.

I General theory of generics

After defining the central notions of this paper - that of a *morphism* and that of a *generic structure* - we prove some essential results and give some examples.

Throughout, L will denote a fixed countable language. We also fix a class of L-structures $\mathbf{K}$ closed under isomorphism and elementary substructure. Henceforth, by 'structure' - or 'model' if $\mathbf{K}$ is elementary - we shall mean an element of $\mathbf{K}$.

§1 Morphisms

For each pair $\mathcal{M}, \mathcal{N} \in \mathbf{K}$ we associate a set of *morphisms* $\mathrm{Mor}_{\mathbf{K}}(\mathcal{M}, \mathcal{N})$ (we shall usually omit the subscript). The sets $\mathrm{Mor}(\mathcal{M}, \mathcal{N})$ satisfy the following axioms.

M1 $\mathrm{Mor}(\mathcal{M}, \mathcal{N}) \neq \emptyset$.

M2 Every $f \in \mathrm{Mor}(\mathcal{M}, \mathcal{N})$ is a partial isomorphism $f : M \to N$.

M3 If $f : M \to N$ is a partial elementary map, then $f \in \mathrm{Mor}(\mathcal{M}, \mathcal{N})$.

M4 If $f \in \mathrm{Mor}(\mathcal{M}_0, \mathcal{M}_1)$ and $g \in \mathrm{Mor}(\mathcal{M}_1, \mathcal{M}_2)$, then $gf \in \mathrm{Mor}(\mathcal{M}_0, \mathcal{M}_2)$.

M5 If $f \in \mathrm{Mor}(\mathcal{M}, \mathcal{N})$, then $f^{-1} \in \mathrm{Mor}(\mathcal{N}, \mathcal{M})$.

M6 If $f \in \mathrm{Mor}(\mathcal{M}, \mathcal{N})$, then for any set A, the restriction map $f \restriction_A \in \mathrm{Mor}(\mathcal{M}, \mathcal{N})$.

We write $\mathcal{M} \leqslant \mathcal{N}$ if $\mathcal{M} \subseteq \mathcal{N}$ and $\mathrm{id}_M \in \mathrm{Mor}(\mathcal{M}, \mathcal{N})$.

[1] Or are they? See the concluding remarks.

Remarks

1. M3 and M4 ensure that $\leqslant$ is reflexive and transitive.
2. The addition of M5 and M6 gives us the following useful properties.
   - If $f \in \mathrm{Mor}(\mathcal{M},\mathcal{N})$, $\mathcal{M} \leqslant \mathcal{M}'$ and $\mathcal{N} \leqslant \mathcal{N}'$, then $f \in \mathrm{Mor}(\mathcal{M}',\mathcal{N}')$.
   - If $f \in \mathrm{Mor}(\mathcal{M},\mathcal{N})$, $\mathrm{dom}\, f \subseteq \mathcal{M}' \leqslant \mathcal{M}$ and $\mathrm{im}\, f \subseteq \mathcal{N}' \leqslant \mathcal{N}$, then $f \in \mathrm{Mor}(\mathcal{M}',\mathcal{N}')$.
   - If $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{N}'$ and $\mathcal{M} \leqslant \mathcal{N}'$, then $\mathcal{M} \leqslant \mathcal{N}$ if and only if $\mathcal{N} \leqslant \mathcal{N}'$.
3. M1 and M6 ensure that $\mathrm{id}_\emptyset \in \mathrm{Mor}(\mathcal{M},\mathcal{N})$.

We say $f \in \mathrm{Mor}(\mathcal{M},\mathcal{N})$ is an *embedding* if $\mathrm{dom}\, f = M$. That is, an embedding is a total morphism. We shall write $f : \mathcal{M} \to \mathcal{N}$ for a morphism and $f : \mathcal{M} \hookrightarrow \mathcal{N}$ for an embedding. Note that $f : \mathcal{M} \hookrightarrow \mathcal{N}$ if and only if $f : \mathcal{M} \cong \mathcal{M}' \leqslant \mathcal{N}$ for $\mathcal{M}' = \mathrm{im}\, f$.

§2 Genericity

An infinite structure $\mathcal{G}$ is said to be *generic* if it satisfies the following conditions.

G1 (Homogeneity) Every morphism $f : \mathcal{G} \to \mathcal{G}$ with $|\mathrm{dom}\, f| < |G|$ extends to an automorphism of $\mathcal{G}$.

G2 (Universality) If $\mathcal{M}$ is a structure with $|M| \leq |G|$, then there is an embedding $f : \mathcal{M} \hookrightarrow \mathcal{G}$.

We isolate the following useful property which will turn out to be equivalent to genericity.

(1.1) If $\mathcal{M}$ is a structure with $|M| \leq |G|$ and $f : \mathcal{M} \to \mathcal{G}$ is a morphism with $|\mathrm{dom}\, f| < |G|$, then f extends to an embedding $F : \mathcal{M} \hookrightarrow \mathcal{G}$.

§3 Uniqueness

We will show that two generics of the same cardinality are isomorphic.

Lemma 1.1 *If $\mathcal{M}$ and $\mathcal{N}$ are two structures of the same cardinality satisfying (1.1) and $f : \mathcal{M} \to \mathcal{N}$ is a morphism with $|\mathrm{dom}\, f| < |M|$, then f extends to an isomorphism $F : \mathcal{M} \cong \mathcal{N}$.*

Proof. First assume that $\mathcal{M}$ and $\mathcal{N}$ are countable. Let $\{a_i\}_{i<\omega}$ and $\{b_i\}_{i<\omega}$ be enumerations of $M \setminus \mathrm{dom}\, f$ and $N \setminus \mathrm{im}\, f$ respectively. We will construct the isomorphism F as the limit of a chain of morphisms $(f_i : i < \omega)$ where $|\mathrm{dom}\, f_i|$ is finite for each i. We put $f_0 = f$. Assume that a chain $(f_i : i \leq n)$ as above has already been constructed.

Suppose n is even. By (1.1), we can extend $f_n : \mathcal{M} \to \mathcal{N}$ to an embedding $F_n : \mathcal{M} \hookrightarrow \mathcal{N}$. Let a be the element with least index in $M \setminus \mathrm{dom}\, f_n$. We define $f_{n+1} : \mathcal{M} \to \mathcal{N}$ as $F_n \restriction_{\mathrm{dom}\, f_n \cup \{a\}}$. By M6, f_{n+1} is a morphism. Clearly $|\mathrm{dom}\, f_{n+1}|$ is finite.

Now suppose n is odd. Again, by (1.1), we can extend $f_n^{-1} : \mathcal{N} \to \mathcal{M}$ to an embedding $F'_n : \mathcal{N} \hookrightarrow \mathcal{M}$. Let b be the element with least index in $N \setminus \operatorname{im} f_n$. Define $f_{n+1} = (F'_n \restriction_{\operatorname{im} f_n \cup \{b\}})^{-1}$.

Define $F = \bigcup_{i<\omega} f_i$ which is clearly an isomorphism between $\mathcal{M}$ and $\mathcal{N}$.

Now assume that $|M| = |N| = \kappa$ where κ is an uncountable cardinal. Let $\{a_i\}_{i<\kappa}$ and $\{b_i\}_{i<\kappa}$ be enumerations of $M \setminus \operatorname{dom} f$ and $N \setminus \operatorname{im} f$ respectively. We will need to construct a chain of morphisms $(f_i : i < \kappa)$ and a family of morphisms $\{F_i : i < \kappa\}$ in parallel.

We define

$$f_0 = F_0 = f$$

Let $\alpha < \kappa$ be an ordinal and assume that the chain $(f_i : i \leq \alpha)$ and the family $\{F_i : i \leq \alpha\}$ of morphisms have already been constructed satisfying

- $F_i \subseteq F_{i+1}$ for each i.
- $|\operatorname{dom} f_{i+1} \setminus \operatorname{dom} f_i| = 1$ and $|\operatorname{dom} F_i| < \kappa$ for each i.
- $f_i \subseteq F_i$ for each i and $F_\delta = f_\delta = \bigcup_{i<\delta} f_i$ at limits δ.
- $\operatorname{dom} F_\beta$ is the universe of an elementary substructure of $\mathcal{M}$ when β is an odd successor.
- $\operatorname{im} F_\beta$ is the universe of an elementary substructure of $\mathcal{N}$ when β is an even successor.

Suppose that α is even. Extend F_α to an embedding $F'_\alpha : \mathcal{M} \hookrightarrow \mathcal{N}$ by (1.1). Let a be the element with least index in $M \setminus \operatorname{dom} f_\alpha$ and let $A \supseteq \operatorname{dom} F_\alpha \cup \{a\}$ be the universe of an elementary substructure of $\mathcal{M}$ of cardinality $< \kappa$. Define

$$F_{\alpha+1} = F'_\alpha \restriction_A \quad \text{and} \quad f_{\alpha+1} = F'_\alpha \restriction_{\operatorname{dom} f_\alpha \cup \{a\}}$$

Now suppose α is odd. Extend F_α^{-1} to an embedding $F''_\alpha : \mathcal{N} \hookrightarrow \mathcal{M}$. Let b be the element with least index from $N \setminus \operatorname{im} f_\alpha$ and let $B \supseteq \operatorname{im} F_\alpha \cup \{b\}$ be the universe of an elementary substructure of $\mathcal{N}$ of cardinality $< \kappa$. Put

$$F_{\alpha+1} = (F''_\alpha \restriction_B)^{-1} \quad \text{and} \quad f_{\alpha+1} = (F''_\alpha \restriction_{\operatorname{im} f_\alpha \cup \{b\}})^{-1}$$

Finally, let $\delta < \kappa$ be a limit ordinal and assume that the chain $(f_i : i < \delta)$ and family $\{F_i : i < \delta\}$ satisfy all the conditions above. We define $F_* = \bigcup_{i<\delta} F_i$. Clearly F_* is a partial elementary map which is a morphism by M3. Put

$$F_\delta = f_\delta = \bigcup_{i<\delta} f_i$$

Then F_δ, being the restriction of F_* is also a partial elementary map, hence a morphism. Moreover, $|\operatorname{dom} F_\delta| < \kappa$.

By putting $F = \bigcup_{i<\kappa} f_i$ we get an isomorphism between $\mathcal{M}$ and $\mathcal{N}$ as required. □

Remark 4. The above proof would have been much simpler if we had assumed that $|M|$ was regular or that the union of a chain of morphisms was a morphism. However, as our proof shows, these assumptions are in fact redundant.

Lemma 1.2 *A structure $\mathcal{G}$ is generic if and only if it satisfies (1.1).*

Proof. ($\Rightarrow$) Assume $\mathcal{G}$ is generic. Let $\mathcal{M}$ be a structure with cardinality $\leq |G|$ and let $f : \mathcal{M} \to \mathcal{G}$ be a morphism with $|\operatorname{dom} f| < |G|$. By universality, there is an embedding $g : \mathcal{M} \hookrightarrow \mathcal{G}$. So $g^{-1} : \mathcal{G} \to \mathcal{M}$ is a morphism with domain $\operatorname{im} g$. Consider the composition morphism $fg^{-1} : \mathcal{G} \to \mathcal{G}$. $|\operatorname{dom} fg^{-1}| = |\operatorname{dom} f| < |G|$, so by homogeneity fg^{-1} extends to an automorphism $\sigma : \mathcal{G} \cong \mathcal{G}$. Clearly $F = \sigma g$ is the desired embedding.

($\Leftarrow$) Assume $\mathcal{G}$ satisfies (1.1). By putting $\mathcal{G} = \mathcal{M} = \mathcal{N}$ in Lemma 1.1, we see that $\mathcal{G}$ is homogeneous. To see that it's universal, let $\mathcal{M}$ be a structure with $|M| \leq |G|$ and extend $\operatorname{id}_\emptyset : \mathcal{M} \to \mathcal{G}$ to an embedding. □

Corollary 1.3 *Two generics of the same cardinality are isomorphic.*

Proof. Let $\mathcal{G}$ and $\mathcal{G}'$ be generic structures of the same cardinality. By Lemma 1.2, they both satisfy (1.1) and by Lemma 1.1, we can extend $\operatorname{id}_\emptyset : \mathcal{G} \to \mathcal{G}'$ to an isomorphism. □

§4 Elementary equivalence

We will show that any two generics are elementarily equivalent. Here we use the assumption that $\mathbf{K}$ is closed under elementary substructure.

Proposition 1.4 *Let $\mathcal{M}$ and $\mathcal{N}$ be generic structures with $|M| \leq |N|$. If $f : \mathcal{M} \to \mathcal{N}$ is a morphism with $|\operatorname{dom} f| < |M|$, then f extends to an elementary embedding $F : \mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N}$. In particular, any morphism between generic structures is a partial elementary map.*

Proof. First assume M is countable. Let $\{a_i\}_{i<\omega}$ be an enumeration of $M \setminus \operatorname{dom} f$. Let $\operatorname{odd}(\omega)$ be the set of odd numbers in ω and let $\pi : \operatorname{odd}(\omega) \times \omega \to \operatorname{odd}(\omega)$ be a bijection such that $\pi(i,j) \geq i$.

We will construct an elementary embedding as the union of a chain of morphisms $(f_i : \mathcal{M} \to \mathcal{N})_{i<\omega}$ with $|\operatorname{dom} f_i|$ finite. We put $f_0 = f$. Assume that a chain $(f_i : i \leq n)$ as above has already been constructed.

Suppose n is even. Extend f_n to an embedding $F_n : \mathcal{M} \hookrightarrow \mathcal{N}$. Let a be the element with least index from $M \setminus \operatorname{dom} f_n$ and define

$$f_{n+1} = F_n \restriction_{\operatorname{dom} f_n \cup \{a\}}$$

Now suppose n is odd. Let $(\psi_{ni} : i < \omega)$ be an enumeration of all formulas of the form $\exists x \varphi(x, \bar{b})$ true in $\mathcal{N}$ where $\bar{b} \subseteq \operatorname{im} f_n$. We consider the formula ψ_{jk} where

$n = \pi(j,k)$. If this formula has a witness in $\operatorname{im} f_n$ then we simply proceed as in the even case. However, if ψ_{jk} doesn't already have a witness in $\operatorname{im} f_n$, let b be any witness for ψ_{jk} in $N \setminus \operatorname{im} f_n$. By Löwenheim-Skolem downward, there is an $\mathcal{N}'$ with $|N'| \leq |M|$ s.t. $\operatorname{im} f_n \cup \{b\} \subseteq N' \preccurlyeq N$. Since $\mathbf{K}$ is closed under elementary substructure, $\mathcal{N}' \in \mathbf{K}$. So by genericity, we can extend f_n^{-1} to an embedding $F_n' : \mathcal{N}' \hookrightarrow \mathcal{M}$ which can be written as a morphism $F_n' : \mathcal{N} \to \mathcal{M}$. Define

$$f_{n+1} = (F_n' \restriction_{\operatorname{im} f_n \cup \{b\}})^{-1}$$

Define $F = \bigcup_{i<\omega} f_i$. F is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$ since the even stages ensure totality and the odd stages ensure elementariness.

Now assume M has uncountable cardinality κ and fix an enumeration $\{a_i\}_{i<\kappa}$ of $M \setminus \operatorname{dom} f$. The argument is now similar to that in the proof of Lemma 1.1. Assume the chain $(f_i : i \leq \alpha)$ and associated family $\{F_i : i \leq \alpha\}$ as in Lemma 1.1 have already been constructed.

If α is even, let a be the element with least index in $M \setminus \operatorname{dom} f_\alpha$ and let $\mathcal{M}' \in \mathbf{K}$ be an elementary substructure of $\mathcal{M}$ containing $\operatorname{dom} F_\alpha \cup \{a\}$ with $|M'| < \kappa$. Let $F_{\alpha+1}$ be an embedding of $\mathcal{M}'$ into $\mathcal{N}$ extending F_α and put

$$f_{\alpha+1} = F_{\alpha+1} \restriction_{\operatorname{dom} f_\alpha \cup \{a\}}$$

If α is odd and $\operatorname{im} F_\alpha \preccurlyeq \mathcal{N}$, simply define $F_{\alpha+1} = F_\alpha$ and $f_{\alpha+1} = f_\alpha$. If $\operatorname{im} F_\alpha$ is not elementary in $\mathcal{N}$, let $\mathcal{N}' \in \mathbf{K}$ be an elementary substructure of $\mathcal{N}$ containing $\operatorname{im} F_\alpha$ with $|N'| < \kappa$ and let $F'_{\alpha+1}$ be an extension of F_α^{-1} to an embedding of $\mathcal{N}'$ into $\mathcal{M}$. Put $F_{\alpha+1} = (F'_{\alpha+1})^{-1}$ and

$$f_{\alpha+1} = F_{\alpha+1} \restriction_{\operatorname{dom} f_\alpha \cup \{b\}}$$

where b is any element of $\operatorname{dom} F_{\alpha+1} \setminus \operatorname{dom} f_\alpha$.

At the limit stage $\delta < \kappa$ we define

$$F_\delta = f_\delta = \bigcup_{i<\delta} f_i$$

Then we get a partial elementary map $F_\delta : \mathcal{M} \to \mathcal{N}$ with $|\operatorname{dom} F_\delta| < \kappa$. At stage κ, we get the required elementary embedding.

For the 'in particular' part, let $f : \mathcal{M} \to \mathcal{N}$ be any morphism between generic structures $\mathcal{M}$ and $\mathcal{N}$. Then, by above, for each finite subset A, $f \restriction_A : \mathcal{M} \to \mathcal{N}$ is partial elementary. Hence f itself must be partial elementary. □

Corollary 1.5 *If $\mathcal{G}$ and $\mathcal{G}'$ are generic structures, then one is elementarily embeddable in the other. In particular, they are elementarily equivalent.*

Proof. By universality, we can always embed one generic into another. By Proposition 1.4 this embedding must be elementary. □

§5 Saturation

We say that a structure $\mathcal{M}$ is *weakly saturated* if it realises all types $p(\tilde{x})$ over $\emptyset$ where $\tilde{x}$ is an infinite tuple of length $< |M|$. The following two results are true of homogeneous structures and hence also of generic structures.

Lemma 1.6 *Let $\mathcal{M}$ be a weakly saturated, homogeneous structure. Then $\mathcal{M}$ is saturated.*

Proof. Let $p(x)$ be a complete type over $\tilde{a} \subseteq M$ where $\ell(\tilde{a}) < |M|$. Define

$$q(x, \tilde{y}) = \{\varphi(x, \tilde{y}) : \varphi(x, \tilde{a}) \in p(x)\}$$

Clearly $q(x, \tilde{y})$ is a type of $\mathcal{M}$ over $\emptyset$ with $\ell(x\hat{\ }\tilde{y}) < |M|$. By the weak saturation of $\mathcal{M}$, $q(x, \tilde{y})$ is realised in $\mathcal{M}$ by a tuple $(c, \tilde{b})$. The map $f : \tilde{b} \mapsto \tilde{a}$ defines a partial elementary map $f : \mathcal{M} \to \mathcal{M}$. By the homogeneity of $\mathcal{M}$, f extends to an automorphism $F : \mathcal{M} \cong \mathcal{M}$. Hence $F(c)$ realises $p(x)$ in $\mathcal{M}$. □

Corollary 1.7 *Let $\mathcal{M}$ be a homogeneous structure. If $\mathrm{Th}\,\mathcal{M}$ is $|M|$-categorical, then $\mathcal{M}$ is saturated.*

Proof. Suppose $\mathrm{Th}\,\mathcal{M}$ is $|M|$-categorical. Let $p(\tilde{x})$ be a type over $\emptyset$ with $\ell(\tilde{x}) < |M|$. Then $p(\tilde{x})$ is realised in a structure $\mathcal{N} \equiv \mathcal{M}$ with $|N| = |M|$. Hence, by $|M|$-categoricity, $\mathcal{N} \cong \mathcal{M}$ and $p(\tilde{x})$ is also realised in $\mathcal{M}$. So $\mathcal{M}$ is weakly saturated and by Lemma 1.6, $\mathcal{M}$ must be saturated. □

The following lemmas show that if one generic structure is κ-saturated, then they all are.

Lemma 1.8 *Let $\mathcal{G}$ be a κ-saturated generic. If $\mathcal{G}'$ is generic with $|G'| \leq \kappa$, then $\mathcal{G}'$ is saturated.*

Proof. Let $p(x)$ be a type of $\mathcal{G}'$ over a set $A \subseteq G'$ with $|A| < |G'|$. By Corollary 1.5, we may assume that $\mathcal{G}' \preccurlyeq \mathcal{G}$. By the κ-saturation of $\mathcal{G}$, $p(x)$ is realised in $\mathcal{G}$ by an element b. By Löwenheim-Skolem downward, there is an $\mathcal{M}$ with $|M| \leq |G'|$ satisfying $A \cup \{b\} \subseteq \mathcal{M} \preccurlyeq \mathcal{G}$. By the genericity of $\mathcal{G}'$, there is an embedding $f : \mathcal{M} \hookrightarrow \mathcal{G}'$ fixing A. We can write f as a morphism $f : \mathcal{G} \to \mathcal{G}'$. By Proposition 1.4, f is partial elementary. Hence $f(b)$ realises $p(x)$ in $\mathcal{G}'$. I.e., $\mathcal{G}'$ is saturated. □

Lemma 1.9 *Let $\mathcal{G}$ be a κ-saturated generic. If $\mathcal{G}'$ is generic with $|G'| > \kappa$, then $\mathcal{G}'$ is also κ-saturated.*

Proof. Let $p(x)$ be a type of $\mathcal{G}'$ over a set $A \subseteq G'$ with $|A| < \kappa$. By Löwenheim-Skolem downward, there is a structure $\mathcal{M}$ with $|M| \leq \kappa$ s.t. $A \subseteq \mathcal{M} \preccurlyeq \mathcal{G}'$. By the universality of $\mathcal{G}$, there is an embedding $f : \mathcal{M} \hookrightarrow \mathcal{G}$ which we can write as $f : \mathcal{G}' \to \mathcal{G}$. By Proposition 1.4, f is partial elementary. By the κ-saturation of $\mathcal{G}$, the conjugate type $f(p)$ is realised in $\mathcal{G}$ by an element b. By Löwenheim-Skolem downward, there is an $\mathcal{N}$ with $|N| \leq \kappa$ satisfying $f[A] \cup \{b\} \subseteq \mathcal{N} \preccurlyeq \mathcal{G}$. By the genericity of $\mathcal{G}'$, we can extend $f^{-1} \restriction_{f[A]}$ to an embedding $g : \mathcal{N} \hookrightarrow \mathcal{G}'$ which we can write as $g : \mathcal{G} \to \mathcal{G}'$. By Proposition 1.4, g is partial elementary, hence $g(b)$ realises $p(x)$ in $\mathcal{G}'$. □

§6 Amalgamation & existence

A subclass $\mathbf{K}_0 \subseteq \mathbf{K}$ is said to be an *amalgamation class* if the following *amalgamation property* holds.

AP If $\mathcal{M}, \mathcal{N} \in \mathbf{K}_0$ and $f : \mathcal{M} \to \mathcal{N}$ is a morphism, then there is an $\mathcal{N}' \in \mathbf{K}_0$ with $\mathcal{N} \leqslant \mathcal{N}'$ and there is an embedding $F : \mathcal{M} \hookrightarrow \mathcal{N}'$ extending f.

$\mathbf{K}_0$ is *inductive* (in $\mathbf{K}$) if it satisfies the following condition.

IND If $\mathcal{M}_i \in \mathbf{K}_0$ and $\mathcal{M}_i \leqslant \mathcal{M}_j$ for each $i \leq j < \kappa$, then $\bigcup_{i<\kappa} \mathcal{M}_i \in \mathbf{K}$ and $\mathcal{M}_j \leqslant \bigcup_{i<\kappa} \mathcal{M}_i$ for each $j < \kappa$.

$\mathbf{K}_0$ is said to be *initial* if whenever $\mathcal{M} \in \mathbf{K}_0$ and $\mathcal{N} \in \mathbf{K}$ s.t. $|N| < |M|$, also $\mathcal{N} \in \mathbf{K}_0$. $\mathbf{K}_0$ is *bounded* if there is a cardinal κ (an *upper bound*) s.t. for every $\mathcal{M} \in \mathbf{K}_0$, $|M| \leq \kappa$. An upper bound κ is a *strict upper bound* if it is an upper bound but no structure $\mathcal{M} \in \mathbf{K}_0$ satisfies $|M| = \kappa$. $\mathbf{K}_0$ is *strictly bounded* if it is bounded and its least upper bound is strict. $\mathbf{K}_0$ is *inductively bounded* if it is bounded and whenever $\mathcal{M} \in \mathbf{K}$ has cardinality equal to the least upper bound of $\mathbf{K}_0$, $\mathcal{M}$ is embedded in the union of a $\leqslant$-chain in $\mathbf{K}_0$. Finally, we write $I(\mathbf{K}_0)$ for the number of isomorphism types in $\mathbf{K}_0$.

The following theorem is a generalisation of a construction due to Fraïssé [4].

Theorem 1.10 *Let $\mathbf{K}_0 \subseteq \mathbf{K}$ be a bounded, initial, inductive amalgamation class. Assume the least upper bound of $\mathbf{K}_0$ is a regular cardinal κ which is a strict bound if κ is uncountable and an inductive bound if κ is countable. Assume further that for all $\mathcal{M}, \mathcal{N} \in \mathbf{K}_0$, $|\mathrm{Mor}(\mathcal{M}, \mathcal{N})| \leq \kappa$ and $I(\mathbf{K}_0) \leq \kappa$. Then there is a generic structure $\mathcal{G} \in \mathbf{K}$ of cardinality κ.*

Proof. We will construct a $\leqslant$-chain $(\mathcal{M}_i : i < \kappa)$ in $\mathbf{K}_0$ satisfying

(1.2) If $\mathcal{A} \in \mathbf{K}_0$ and $f : \mathcal{A} \to \mathcal{M}_i$, then there is a $j \geq i$ and an embedding $F : \mathcal{A} \hookrightarrow \mathcal{M}_j$ extending f.

Let $\pi : \kappa \times \kappa \to \kappa$ be a bijection such that $\pi(i,j) \geq i$ and let $\mathbf{P}$ be a set containing exactly one representative for each isomorphism type in $\mathbf{K}_0$. We construct the chain by induction as follows. Let $\mathcal{M}_0$ be an arbitrary structure from $\mathbf{K}_0$. Assume the continuous $\leqslant$-chain $(\mathcal{M}_i : i \leq \alpha)$ in $\mathbf{K}_0$ has already been constructed. Enumerate as $((\mathcal{A}_{\alpha i}, f_{\alpha i}) : i < \kappa)$ all pairs $(\mathcal{A}, f)$ s.t. $\mathcal{A} \in \mathbf{P}$ and $f : \mathcal{A} \to \mathcal{M}_\alpha$. If $\alpha = \pi(\beta, \gamma)$, we choose $\mathcal{M}_{\alpha+1} \geqslant \mathcal{M}_\alpha$ by AP using the morphism $f_{\beta\gamma} : \mathcal{A}_{\beta\gamma} \to \mathcal{M}_\alpha$.

At limit ordinals $\delta < \kappa$ we simply put $\mathcal{M}_\delta = \bigcup_{i<\delta} \mathcal{M}_i$. By inductiveness, $\mathcal{M}_\delta \in \mathbf{K}$ and for each $i < \delta$, $\mathcal{M}_i \leqslant \mathcal{M}_\delta$. As κ is regular and is a strict bound, $|M_\delta| < \kappa$. As κ is the least upper bound of $\mathbf{K}_0$ and $\mathbf{K}_0$ is initial, we must have $\mathcal{M}_\delta \in \mathbf{K}_0$.

By induction we get a $\leqslant$-chain $(\mathcal{M}_i : i < \kappa)$ with $\mathcal{M}_i \in \mathbf{K}_0$ for each i satisfying (1.2). So if we put $\mathcal{G} = \bigcup_{i<\kappa} \mathcal{M}_i$, inductiveness yields $\mathcal{G} \in \mathbf{K}$ and $\mathcal{M}_i \leqslant \mathcal{G}$ for each $i < \kappa$. Obviously $|G| = \kappa$.

We claim $\mathcal{G}$ is generic. To see this, let $\mathcal{M}$ be a structure with $|M| \leq \kappa$ and let $f : \mathcal{M} \to \mathcal{G}$ be a morphism with $|\mathrm{dom}\, f| < \kappa$. If $\mathcal{M} \in \mathbf{K}_0$, then, by (1.2), we have taken care to extend f to an embedding of $\mathcal{M}$ into $\mathcal{G}$ at some stage of the construction. So we may suppose that $\mathcal{M} \notin \mathbf{K}_0$. This implies $|M| = \kappa$ since $\mathbf{K}_0$ is initial.

First assume $\kappa = \aleph_0$. Since $\aleph_0$ is an inductive bound, $\mathcal{M} \leqslant \mathcal{A} = \bigcup_{i<\omega} \mathcal{A}_i$ where each $\mathcal{A}_i \in \mathbf{K}_0$ and $\mathcal{A}_i \leqslant \mathcal{A}_j \leqslant \mathcal{A}$ for $i \leq j < \omega$. We'll construct an embedding $F : \mathcal{A} \hookrightarrow \mathcal{G}$ as the limit of a chain of morphisms. $F \restriction_M$ will then be the required embedding of $\mathcal{M}$ into $\mathcal{G}$. Note that $\mathrm{dom}\, f \subseteq \mathcal{A}_k$ for some k. So let f_k be the extension of f to an embedding of $\mathcal{A}_k$ into $\mathcal{G}$. Assume that the chain $(f_i : k \leq i \leq n)$ has already been constructed where $\mathrm{dom}\, f_i = \mathcal{A}_i$ for each i. By (1.2), we have already extended f_n to an embedding $F_n : \mathcal{A}_{n+1} \hookrightarrow \mathcal{G}$. Put $f_{n+1} = F_n : \mathcal{M} \to \mathcal{G}$. By putting $F = \bigcup_{i<\omega} f_i$ we get an embedding of $\mathcal{A}$ into $\mathcal{G}$ as required.

Now assume $\kappa > \aleph_0$. Then we can write $\mathcal{M} = \bigcup_{i<\kappa} \mathcal{A}_i$ where each $\mathcal{A}_i \in \mathbf{K}_0$, $\mathcal{A}_i \preccurlyeq \mathcal{A}_j$ for $i \leq j$, and $\mathrm{dom}\, f \subseteq A_0$. We can now simply proceed as above, taking unions at limit stages. $\square$

Corollary 1.11 *If* $\mathbf{K}$ *is an elementary, inductive amalgamation class and* κ *is a strongly inaccessible cardinal, then there exists a generic structure of cardinality* κ.

Proof. Put $\mathbf{K}_0 = \{\mathcal{M} \in \mathbf{K} : |M| < \kappa\}$. Then clearly $\mathbf{K}_0$ is an initial, inductive amalgamation class strictly bounded by κ. So we need only establish that $I(\mathbf{K}_0) \leq \kappa$ and $|\mathrm{Mor}(\mathcal{M}, \mathcal{N})| \leq \kappa$.

To see that $I(\mathbf{K}_0) \leq \kappa$, note that on any set M of cardinality $\alpha < \kappa$, there are at most 2^α ways of choosing an interpretation for any nonlogical symbol from L. As L is countable, there can be at most countably many nonlogical symbols. So there are at most $2^{\alpha \times \aleph_0} = 2^\alpha$ isomorphism types of structure of cardinality α. As κ is strongly inaccessible, this number is smaller than κ. So there are at most $\kappa \times \kappa = \kappa$ isomorphism types in $\mathbf{K}_0$.

Now for $|\mathrm{Mor}(\mathcal{M}, \mathcal{N})| \leq \kappa$. Let $|M| = \alpha$ and $|N| = \beta$. Then $|\mathrm{Mor}(\mathcal{M}, \mathcal{N})| \leq 2^\alpha \times 2^\beta = 2^{\max\{\alpha,\beta\}} < \kappa$. This proves the corollary. $\square$

§7 Examples

– An unsaturated generic

Let L be the language with a unary function symbol f. Let $\mathbf{K}$ be the class of L-structures $\mathcal{M}$ s.t.

$$\text{For every } a \in M, \text{ there exists an } n < \omega \text{ s.t. } f^n(a) = a$$

So every element of M generates a finite cycle

$$a \to f(a) \to f^2(a) \cdots \to f^{n-1}(a) \to f^n(a) = a$$

Define $\text{Mor}_{\mathbf{K}}(\mathcal{M},\mathcal{N})$ as the set of partial isomorphisms $f : M \to N$. Then $\text{id}_\emptyset \in \text{Mor}(\mathcal{M},\mathcal{N})$ since there are no quantifier-free sentences. Define

$$\mathbf{K}_0 = \{\mathcal{M} \in \mathbf{K} : |M| < \aleph_0\}$$

$\mathbf{K}_0$ is easily seen to satisfy the hypothesis of Theorem 1.10. We therefore get a countable generic structure $\mathcal{G} \in \mathbf{K}$. However, $\mathcal{G}$ is not saturated. To see this, put

$$\Phi(x) = \{f^n(x) \neq x : n < \omega\}$$

$\Phi(x)$ is easily seen to be a type of $\mathcal{G}$ which is omitted.

If we replace $\aleph_0$ by a strongly inaccessible cardinal κ in the definition of $\mathbf{K}_0$ we see that we get a generic structure $\mathcal{U} \in \mathbf{K}$ of cardinality κ. By Lemma 1.8, $\mathcal{U}$ is not even $\aleph_0$-saturated. In fact, $\mathcal{U}$ also omits $\Phi(x)$.

– A generic countable discrete linear ordering

Let T be the theory of linear orderings and define

$$\mathbf{K}_0 = \{\text{finite models of } T\}$$

Let Δ be the boolean closure of the set of quantifier free formulas union the set of formulas

$$\{\exists_n z(x < z \wedge z < y) : n < \omega\}$$

Note that this includes the case $n = 0$ which says that there are no points between x and y. Let $\mathbf{K}$ be the closure of $\mathbf{K}_0$ under unions of $\preccurlyeq_\Delta$-chains and define

$$\text{Mor}_{\mathbf{K}}(\mathcal{M},\mathcal{N}) = \{\text{partial } \Delta\text{-elementary maps } f : M \to N\}$$

It is relatively straightforward to verify that $\mathbf{K}_0$ is as in the hypothesis of Theorem 1.10. Hence, there is a countable generic linear ordering $\mathcal{U} \in \mathbf{K}$.

We claim that $\mathcal{U}$ is discrete and has no end points. Let $a \in U$ and let $\mathcal{M}$ be the linear ordering with universe $\{a, b\}$ s.t. $a < b$. Then $\text{id}_{\{a\}} : \mathcal{M} \to \mathcal{U}$ is a morphism and hence, by the genericity of $\mathcal{U}$, extends to an embedding $F : \mathcal{M} \hookrightarrow \mathcal{U}$. Clearly $a < F(b)$ and there is no $z \in U$ s.t. $a < z < F(b)$. Similarly, there is a $c \in U$ s.t. $c < a$ and for no z do we have $c < z < a$. So $\mathcal{U}$ is indeed discrete and has no end points.

We now show that there are only finitely many points between any two points in $\mathcal{U}$. Let $a, b \in U$ s.t. $a < b$. Since $\mathcal{U} \in \mathbf{K}$, we can write $\mathcal{U} = \bigcup_{i<\omega} \mathcal{M}_i$ where the $\mathcal{M}_i$ form a $\preccurlyeq_\Delta$-chain in $\mathbf{K}_0$. Let k be the minimum index such that $a, b \in M_k$. Since $\mathcal{M}_k$ is a finite linear ordering, there must be an n s.t. $\mathcal{M}_k \models \exists_n z(a < z \wedge z < b)$. Hence, $\mathcal{M}_k \preccurlyeq_\Delta \mathcal{U}$ implies $\mathcal{U} \models \exists_n z(a < z \wedge z < b)$. That is, there are only finitely many points between a and b.

All this means that $\mathcal{U} \cong \mathbb{Z}$. So obviously $\mathcal{U}$ is not saturated. Explicitly: $\mathcal{U}$ omits the type

$$\Phi(x, y) = \{\exists_{\geq n} z(x < z \wedge z < y) : n < \omega\}$$

So this is yet another example of an unsaturated generic. We will see examples of saturated generic structures in II and III.

II Universal domains

In the last example of part I, we considered a class of morphisms that were Δ-elementary for some Δ. The generic structure that was obtained - the linear ordering of the integers - was not saturated. In fact, it omitted a Δ-type. In this part, we examine the case of morphisms as Δ-elementary maps in more detail, restricting our class $\mathbf{K}$ to the model class of a universal theory. In this context, generic structures realise all Δ-types and we'll call them *universal domains* after Hrushovski [8].

It will turn out that there is a natural property - called *quantifier separation* - which determines a necessary and sufficient condition on $\mathbf{K}$ for the existence of a universal domain. The conditions for a universal domain to be saturated will also turn out to be quite natural. We conclude this section by giving some simple examples.

§1 Basic formulas

Throughout this section we let L be a countable language. We fix a set $\Delta \subseteq L$ of *basic formulas* closed under subformulas, substitution of variables and boolean combinations. We assume further that Δ contains all quantifier-free formulas. We shall write $\Delta(A)$ for the set of basic formulas with parameters from a set A. Π will denote the set of *universal formulas*. I.e., formulas of the form $\forall \bar{x} \varphi$ where $\varphi \in \Delta$. Similarly, Σ will denote the set of *existential formulas*. All standard notation (such as $T_\forall$, $\mathrm{tp}_\exists(\bar{a})$, $\equiv_\exists$, $\preccurlyeq_\exists$ etc.) is interpreted in this context. That is, $\forall$ always refers to the set Π and $\exists$ to the set Σ.

A set defined in some L-structure $\mathcal{M}$ by a formula from a set Θ over $A \subseteq M$ is a Θ-*set over* A. When $\Theta = \Delta$ the corresponding sets will be called *basic sets*. A subset A of M is called *small* if $|A| < |M|$. A collection Φ of formulas from Θ with parameter set A will be called *small* if A is small. A collection of Θ-sets defined by a small set of formulas will also be called *small*.

We will write x for a single variable, $\bar{x}$ for a finite tuple and $\tilde{x}$ for an infinite tuple. We will write x_i for element i of the tuple $\bar{x}$ or $\tilde{x}$. Let $\tilde{a}$ be a tuple in an L-structure $\mathcal{M}$. By the (open) diagram of $\tilde{a}$, written $\mathrm{diag}_\Delta \tilde{a}$, we shall mean the set $\{\varphi(\tilde{x}) \in \Delta : \mathcal{M} \models \varphi(\tilde{a})\}$. If $\tilde{a}$ enumerates M we will also sometimes write simply $\mathrm{diag}_\Delta \mathcal{M}$. If a structure $\mathcal{N}$ satisfies $\mathrm{diag}_\Delta \tilde{a}$ at a tuple $\tilde{b}$, then the partial function $f : M \to N$ defined by $f(a_i) = b_i$ is clearly a partial Δ-elementary map from $\mathcal{M}$ to $\mathcal{N}$.

There will be times when we need to consider the *full* diagram of a tuple $\tilde{a}$. This is simply the set of $L(\tilde{a})$ sentences $\{\varphi(\tilde{a}) \in \Delta(\tilde{a}) : \mathcal{M} \models \varphi(\tilde{a})\}$. We will denote this set by $\mathrm{Diag}_\Delta \tilde{a}$ or $\mathrm{Diag}_\Delta \mathcal{M}$ when $\tilde{a}$ enumerates M.

§2 Morphisms

We fix a theory T deciding all basic sentences. We define

$$\mathbf{K} = \mathrm{Mod}\, T_\forall$$

$$\mathrm{Mor}_{\mathbf{K}}(\mathcal{M},\mathcal{N}) = \{\text{partial } \Delta\text{−elementary maps from } \mathcal{M} \text{ to } \mathcal{N}\}$$

Axioms M1-M6 are easily seen to hold noting that nonemptiness is guaranteed by the fact that $\mathrm{id}_\emptyset : \mathcal{M} \to \mathcal{N}$ is partial Δ-elementary since T decides all basic sentences.

Trivially, for all $\mathcal{M}, \mathcal{N} \in \mathbf{K}$,

$$\mathcal{M} \leqslant \mathcal{N} \quad \Leftrightarrow \quad \mathcal{M} \preccurlyeq_\Delta \mathcal{N}$$

§3 Morleyisation

Let T be an L-theory deciding all basic sentences. Let Δ_1 be the set of formulas in Δ with at least one free variable (i.e., all formulas that are not sentences). If $\varphi \in \Delta_1$ has n free variables, we let R_φ be an n-ary relation symbol and define

$$L^\Delta = L \cup \{R_\varphi : \varphi \in \Delta_1\}$$

$$T^\Delta = T \cup \{\forall \bar{x}(R_\varphi \bar{x} \leftrightarrow \varphi(\bar{x})) : \varphi \in \Delta_1\}$$

Then it is clear that any basic formula is equivalent modulo T^Δ to a quantifier-free L^Δ-formula. Obviously, any L-model of T has a unique expansion to an L^Δ-model of T^Δ. The process of expanding the language and theory in this way is known as *Morleyisation*.

There is no loss of generality if the discussion of this section is assumed to take place in an expanded language and expanded theory of this type: I.e., $\mathbf{K} = \mathrm{Mod}\,(T_\forall)^\Delta$. We may therefore safely assume that Δ is the set of quantifier-free formulas, that morphisms are partial isomorphisms, and that $\leqslant$ is simply $\subseteq$. Essential uses of this assumption (e.g. in §8 when discussing existential closure and model completeness) will always receive explicit mention.

§4 Universal domains

Following Hrushovski [8], we define a *universal domain* $\mathcal{U}$ as an infinite structure satisfying the following properties.

U1 $\mathrm{Th}_\forall \mathcal{U} = T_\forall$.

U2 $\mathcal{U}$ is homogeneous.

U3 (Compactness) Every small collection of basic subsets of U^n (for finite n) with the finite intersection property has nonempty intersection.

Clearly compactness is equivalent to the following statement.

(2.1) $\mathcal{U}$ realises all basic types over a small set of parameters.

We make some useful remarks about homogeneity and prove a lemma on compactness.

Remarks

1. If an L-structure $\mathcal{M}$ is homogeneous, then for any tuples $\bar{b}, \bar{c} \subseteq M$ and any small set of parameters A, if $\mathrm{tp}_\Delta(\bar{b}/A) = \mathrm{tp}_\Delta(\bar{c}/A)$ then $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$.

2. Let $\mathcal{M}$ be a homogeneous structure. For any definable set X over $\bar{a}$, if $p(\bar{x})$ is a complete Δ-type over $\bar{a}$, then either $p(\mathcal{M}) \subseteq X$ or $p(\mathcal{M}) \subseteq \neg X$. This is an easy consequence of the previous remark: If $p(\mathcal{M}) \nsubseteq \neg X$ then there is a $\bar{b} \subseteq M$ s.t. $\bar{b} \in p(\mathcal{M}) \cap X$. Now take any $\bar{c} \in p(\mathcal{M})$. This implies $p(\bar{x}) = \mathrm{tp}_\Delta(\bar{b}/\bar{a}) = \mathrm{tp}_\Delta(\bar{c}/\bar{a})$. The homogeneity of $\mathcal{M}$ implies that $\mathrm{tp}(\bar{b}/\bar{a}) = \mathrm{tp}(\bar{c}/\bar{a})$. Hence $\bar{c} \in X$.

Lemma 2.1 *Let $\mathcal{U}$ be a compact structure. Σ-sets in $\mathcal{U}$ are compact. I.e., any small basic covering of an Σ-set has a finite subcover.*

Proof. Let X be the set defined by the existential formula $\exists \bar{y} \psi(\bar{x}, \bar{y})$. Let $\mathcal{C}$ be a small basic covering of X so that $X \subseteq \bigcup \mathcal{C}$. Put $\Phi(\bar{x}) = \{\neg\varphi(\bar{x}) : \varphi(\mathcal{U}) \in \mathcal{C}\}$. Suppose $\Phi(\bar{x}) \cup \{\psi(\bar{x}, \bar{y})\}$ is finitely satisfied in $\mathcal{U}$. Then by compactness, $\Phi(\bar{x}) \cup \{\psi(\bar{x}, \bar{y})\}$ is realised in $\mathcal{U}$. I.e. $X \cap \bigcap\{\neg\varphi(\mathcal{U}) : \varphi(\mathcal{U}) \in \mathcal{C}\} \neq \emptyset$ which is a contradiction. So there are finitely many formulas $\neg\varphi_0(\bar{x}), \ldots, \neg\varphi_{n-1}(\bar{x}) \in \Phi(\bar{x})$ such that $\{\bigwedge_{i<n} \neg\varphi_i(\bar{x}), \psi(\bar{x}, \bar{y})\}$ is not satisfied in $\mathcal{U}$. I.e., $\mathcal{U} \models \forall \bar{x}(\exists \bar{y} \psi(\bar{x}, \bar{y}) \to \bigvee_{i<n} \varphi_i(\bar{x}))$. Hence $X \subseteq \bigcup_{i<n} \varphi_i(\mathcal{U})$. $\square$

§5 Genericity

We'll show that the universal domains are precisely the generic structures. But first a lemma.

Lemma 2.2 *Let $\mathcal{U}$ be a universal domain. Let $\tilde{x}$ be an infinite tuple of length $\leq |U|$ and $\Phi(\tilde{x})$ a small set of existential formulas finitely satisfied in $\mathcal{U}$. Then $\Phi(\tilde{x})$ is realised in $\mathcal{U}$.*

Proof. It is enough to prove the lemma for a small set of *basic* formulas $\Phi(\tilde{x})$. For suppose that $\Phi(\tilde{x})$ is a small set of existential formulas as in the hypothesis. Then we can replace all existentially quantified variables with new free variables, resulting in a set $\Theta(\tilde{x}, \tilde{y})$ finitely satisfied in $\mathcal{U}$ with $|\tilde{y}| \leq |U|$. Let $\tilde{z}$ be a tuple well-ordering $\tilde{x} \cup \tilde{y}$. Then $|\tilde{z}| \leq |U|$ and we have reduced the problem to a small set of basic formulas $\Theta(\tilde{z})$.

So let $\Phi(\tilde{x})$ be a small set of basic formulas with parameters from A. Put $\kappa = \ell(\tilde{x})$. Let $\Phi(\tilde{a}_\alpha)$ be the result of replacing the first α variables in $\Phi(\tilde{x})$ with a sequence $\tilde{a}_\alpha$ of length α. Suppose a sequence $\tilde{a}_\alpha = (a_i : i < \alpha)$ (where $\alpha < \kappa$) of parameters have already been chosen such that $\Phi(\tilde{a}_\alpha)$ is finitely satisfiable in $\mathcal{U}$. Let x_α be the next free variable in the tuple $\tilde{x}$. The set

$$\Psi(x_\alpha) = \{\varphi(x_\alpha) \in \Delta(A, \tilde{a}_\alpha) : \mathrm{Th}_\forall(\mathcal{U}, A, \tilde{a}_\alpha) \cup \Phi(\tilde{a}_\alpha) \vdash \varphi(x_\alpha)\}$$

is small. Each $\varphi(x_\alpha) \in \Psi(x_\alpha)$ is a consequence of finitely many formulas in $\mathrm{Th}_\forall(\mathcal{U}, A, \tilde{a}_\alpha) \cup \Phi(\tilde{a}_\alpha)$. As $\Phi(\tilde{a}_\alpha)$ is finitely satisfied in $\mathcal{U}$ (by hypothesis) each $\varphi(x_\alpha)$ must be satisfied in $\mathcal{U}$. This implies $\Psi(x_\alpha)$ is finitely satisfied in $\mathcal{U}$ as it is closed under conjunctions. By compactness, $\Psi(x_\alpha)$ is realised in $\mathcal{U}$ by some $a_\alpha \in U$.

Let $\psi(x_\alpha, \bar{y}, \bar{a})$ be a basic formula with $\bar{a} \subseteq A \cup \tilde{a}_\alpha$ s.t. $\Phi(\tilde{a}_\alpha) \vdash \psi(x_\alpha, \bar{y}, \bar{a})$ and suppose that $\mathcal{U} \models \forall \bar{y} \neg\psi(a_\alpha, \bar{y}, \bar{a})$. By Remark 2 the complete Δ-type $p(x_\alpha)$ of a_α over $\bar{a}$ satisfies $p(\mathcal{U}) \subseteq \forall \bar{y} \neg\psi(\mathcal{U}, \bar{y}, \bar{a})$. So by Lemma 2.1 there is a basic formula $\theta(x_\alpha, \bar{a}) \in p(x_\alpha)$ such that $\exists \bar{y} \psi(\mathcal{U}, \bar{y}, \bar{a}) \subseteq \neg\theta(\mathcal{U}, \bar{a})$. I.e.,

$$\mathcal{U} \models \forall x_\alpha \bar{y} (\psi(x_\alpha, \bar{y}, \bar{a}) \rightarrow \neg\theta(x_\alpha, \bar{a}))$$

As $\Phi(\tilde{a}_\alpha) \vdash \psi(x_\alpha, \bar{y}, \bar{a})$ this means $\mathrm{Th}_\forall(\mathcal{U}, A, \tilde{a}_\alpha) \cup \Phi(\tilde{a}_\alpha) \vdash \neg\theta(x_\alpha, \bar{a})$. I.e., $\neg\theta(x_\alpha, \bar{a}) \in \Psi(x_\alpha)$ and so $\mathcal{U} \models \neg\theta(a_\alpha, \bar{a})$. But this contradicts that $\theta(x_\alpha, \bar{a}) \in p(x_\alpha) = \mathrm{tp}_\Delta(a_\alpha/\bar{a})$. Hence if $\Phi(\tilde{a}_\alpha) \vdash \psi(x_\alpha, \bar{y}, \bar{a})$ then $\psi(a_\alpha, \bar{y}, \bar{a})$ is satisfied in $\mathcal{U}$. But this means that $\Phi(\tilde{a}_\alpha)$ remains finitely satisfied in $\mathcal{U}$ if we replace x_α with a_α.

Let $\delta \leq \kappa$ be a limit ordinal and suppose that $\Phi(\tilde{a}_\alpha)$ is finitely satisfied in $\mathcal{U}$ for each $\alpha < \delta$. Then obviously $\Phi(\tilde{a}_\delta)$ is finitely satisfied in $\mathcal{U}$. By induction, we get a realisation $\tilde{a}_\kappa$ of $\Phi(\tilde{x})$. □

Proposition 2.3 *$\mathcal{U}$ is a universal domain if and only if $\mathcal{U}$ is generic.*

Proof. ($\Rightarrow$) Assume $\mathcal{U}$ is a universal domain. We need to establish universality. To this end, let $\mathcal{M} \in \mathbf{K}$ be s.t. $|M| \leq |U|$. Suppose there is a $\varphi(\bar{x}) \in \mathrm{diag}_\Delta \mathcal{M}$ such that $\mathcal{U} \models \forall \bar{x} \neg\varphi(\bar{x})$. Then also $\mathcal{M} \models \forall \bar{x} \neg\varphi(\bar{x})$ as $\mathcal{M} \models \mathrm{Th}_\forall \mathcal{U}$. But this is a contradiction. So $\mathrm{diag}_\Delta \mathcal{M}$ is finitely satisfied in $\mathcal{U}$. By Lemma 2.2 $\mathrm{diag}_\Delta \mathcal{M}$ is realised in $\mathcal{U}$. Hence $\mathcal{M} \hookrightarrow \mathcal{U}$.

($\Leftarrow$) Assume $\mathcal{U}$ is generic. We first prove U1. Obviously $T_\forall \subseteq \mathrm{Th}_\forall \mathcal{U}$ since $\mathcal{U} \in \mathbf{K}$. So we need only establish $\mathrm{Th}_\forall \mathcal{U} \subseteq T_\forall$. To this end, Let $\mathcal{M} \models T_\forall$. Then there is an elementary substructure $\mathcal{N} \preccurlyeq M$ with $|N| \leq |U|$ s.t. $\mathcal{N} \hookrightarrow \mathcal{U}$. So $\mathcal{U} \Rrightarrow_\forall \mathcal{M}$. Hence $\mathrm{Th}_\forall \mathcal{U} \subseteq T_\forall$ as required.

It now suffices to establish compactness. To this end, let $\Phi(\bar{x})$ be a small collection of basic formulas over A finitely satisfied in $\mathcal{U}$. There is an elementary extension $\mathcal{M} \succcurlyeq \mathcal{U}$ of cardinality $|U|$ realising $\Phi(\bar{x})$. I.e., $\mathcal{M} \models \Phi(\bar{a})$ for some $\bar{a} \subseteq \mathcal{M}$. By the universality of $\mathcal{U}$, there is an embedding $f : \mathcal{M} \hookrightarrow \mathcal{U}$. Hence $f(\bar{a}) \in U$ realises the conjugate type $f(\Phi)$ over $f[A]$. Since $\mathcal{U} \preccurlyeq \mathcal{M}$, $f \restriction_A : \mathcal{U} \rightarrow \mathcal{U}$ is a morphism. By the homogeneity of $\mathcal{U}$, there is a $\sigma \in \mathrm{Aut}\,\mathcal{U}$ extending $f \restriction_A$. So $\mathcal{U} \models \Phi(\sigma^{-1} f(\bar{a}))$. □

§6 Quantifier separation

Following Hrushovski [8], we say T admits *quantifier separation* if, whenever $\varphi(\bar{x})$ is existential and $\psi(\bar{x})$ universal such that

$$T \vdash \forall \bar{x} (\varphi(\bar{x}) \rightarrow \psi(\bar{x}))$$

there is a basic formula $\theta(\bar{x})$ s.t.

$$T \vdash \forall \bar{x} (\varphi(\bar{x}) \rightarrow \theta(\bar{x})) \quad \text{and} \quad T \vdash \forall \bar{x} (\theta(\bar{x}) \rightarrow \psi(\bar{x}))$$

Note that the formulas on the right hand side of the above turnstiles are all universal formulas. The following lemma is immediate from the definitions.

Lemma 2.4 *Let $\mathcal{U}$ be a universal domain. Then T admits quantifier separation if and only if whenever X and Y are disjoint Σ-sets of $\mathcal{U}$ over $\emptyset$, there is a basic set Z over $\emptyset$ such that $X \subseteq Z$ and $Y \cap Z = \emptyset$.* □

Proposition 2.5 *Let $\mathcal{U}$ be a universal domain. Then*
(a) *Every Σ-set of $\mathcal{U}$ over $\bar{a}$ is the intersection of a collection of basic sets over $\bar{a}$ (which is therefore small).*
(b) *T admits quantifier separation.*

Proof. (a) Let X be an Σ-set over $\bar{a}$. Let $\bar{b} \in \neg X$ and put $p_{\bar{b}}(\bar{x}) = \mathrm{tp}_\Delta(\bar{b}/\bar{a})$. By Remark 2 we must have $p_{\bar{b}}(\mathcal{U}) \subseteq \neg X$. So $X \subseteq \bigcup\{\neg\varphi(\mathcal{U}) : \varphi(\bar{x}) \in p_{\bar{b}}(\bar{x})\}$. By Lemma 2.1 there is a basic formula $\theta_{\bar{b}}(\bar{x}) \in p_{\bar{b}}(\bar{x})$ (as $p_{\bar{b}}(\bar{x})$ is complete) such that $X \subseteq \neg\theta_{\bar{b}}(\mathcal{U})$. So we have $X = \bigcap\{\neg\theta_{\bar{b}}(\mathcal{U}) : \bar{b} \in \neg X\}$.

(b) Let X and Y be disjoint Σ-sets of $\mathcal{U}$ over $\emptyset$. Then by (a) $X = \bigcap \mathcal{C}$ and $Y = \bigcap \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are collections of basic sets over $\emptyset$. As $\bigcap\{\mathcal{C}, \mathcal{D}\} = \emptyset$, compactness implies that there are basic sets $X' \supseteq \bigcap \mathcal{C}$ and $Y' \supseteq \bigcap \mathcal{D}$ such that $X' \cap Y' = \emptyset$. So $X \subseteq X'$ and $X' \cap Y = \emptyset$. By Lemma 2.4 this means T admits quantifier separation. □

Remark 3. Clearly, we also have that $\mathrm{Th}\,\mathcal{U}$ admits quantifier separation.

§7 Amalgamation & existence

The property IND from I, §6 is immediately verified on **K**. We now verify AP.

Lemma 2.6 *Assume T admits quantifier separation. Then* **K** *is an amalgamation class.*

Proof. Let $\mathcal{M}, \mathcal{N} \in \mathbf{K}$ and let $f : \mathcal{M} \to \mathcal{N}$ be a morphism. Let $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ be tuples enumerating $\mathrm{dom}\, f$, $M \setminus \mathrm{dom}\, f$, and $N \setminus \mathrm{im}\, f$ respectively. Bear in mind that $\tilde{a}$ could be empty. Let

$$p(\tilde{x}, \tilde{y}) = \{\varphi(\tilde{x}, \tilde{y}) \in \Delta : \mathcal{M} \models \varphi(\tilde{a}, \tilde{b})\}$$

and

$$q(\tilde{x}, \tilde{z}) = \{\varphi(\tilde{x}, \tilde{z}) \in \Delta : \mathcal{N} \models \varphi(f(\tilde{a}), \tilde{c})\}$$

Suppose that $T \cup p(\tilde{x}, \tilde{y}) \cup q(\tilde{x}, \tilde{z})$ is unsatisfiable. Then, by compactness, there are formulas $\varphi(\bar{x}, \bar{y}) \in p(\tilde{x}, \tilde{y})$ and $\psi(\bar{x}, \bar{z}) \in q(\tilde{x}, \tilde{z})$ such that $T \cup \{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{z})\}$ is unsatisfiable. I.e., $T \vdash \forall\bar{x}(\exists\bar{y}\varphi(\bar{x}, \bar{y}) \to \forall\bar{z}\neg\psi(\bar{x}, \bar{z})\})$. By quantifier separation this implies that

$$T \vdash \forall\bar{x}(\exists\bar{y}\varphi(\bar{x}, \bar{y}) \to \theta(\bar{x})) \quad \text{and} \quad T \vdash \forall\bar{x}(\theta(\bar{x}) \to \forall\bar{z}\neg\psi(\bar{x}, \bar{z}))$$

for some basic formula $\theta(\bar{x})$. Let $\bar{a}$ be the tuple from $\mathrm{dom}\, f$ such that $\mathcal{M} \models \exists\bar{y}\varphi(\bar{a}, \bar{y})$ and $\mathcal{N} \models \exists\bar{z}\psi(f(\bar{a}), \bar{z})$. Then we have $\mathcal{M} \models \theta(\bar{a})$ and $\mathcal{N} \models \neg\theta(f(\bar{a}))$. But this means that $\mathcal{M} \models \theta(\bar{a})$ and $\mathcal{M} \models \neg\theta(\bar{a})$ which is a contradiction. So there is a structure $\mathcal{N}'$ satisfying $T \cup p(\tilde{x}, \tilde{y}) \cup q(\tilde{x}, \tilde{z})$. So $\mathcal{N}' \in \mathbf{K}$ and $\mathcal{N}' \models p(\tilde{d}, \tilde{r}) \cup q(\tilde{d}, \tilde{s})$ for some $\tilde{d}, \tilde{r}, \tilde{s} \subseteq N'$. Now define the map $g : M \to N'$ by $g(a_i)) = d_i$ and $g(b_i) = r_i$. Define the map $h : N \to N'$ by $h(f(a_i)) = d_i$ and $h(c_i) = s_i$. Then $g : \mathcal{M} \hookrightarrow \mathcal{N}'$, $h : \mathcal{N} \hookrightarrow \mathcal{N}'$ and $g \restriction_{\mathrm{dom}\, f} = hf$. By isomorphic correction we can ensure that $\mathcal{N} \leqslant \mathcal{N}'$. □

Remark 4. Note that the proof goes through just as well for $\mathbf{K} = \mathrm{Mod}\, T$. By Remark 3, we have that $\mathrm{Th}\,\mathcal{U}$ (or rather, its model class) has the amalgamation property. This fact will be used in the proof of Proposition 2.10.

By Corollary 1.11 there exists a universal domain of strongly inaccessible cardinality provided T admits quantifier separation. By Proposition 2.5(b), we get the following equivalence.

Proposition 2.7 *T admits quantifier separation if and only if there exists a universal domain.* □

§8 Model completeness

In the following results we make explicit reference to the process of Morleyisation from §3.

Proposition 2.8 *A universal domain is an existentially closed L^{Δ}-model of $(T_{\forall})^{\Delta}$.*

Proof. Let $\mathcal{U}$ be a universal domain and $\mathcal{M}$ any other model of $T_{\forall}$ such that $\mathcal{U} \leqslant \mathcal{M}$. Then, by Löwenheim-Skolem downward, there is an $\mathcal{N} \preccurlyeq \mathcal{M}$ s.t. $|N| = |U|$ and $\mathcal{U} \leqslant \mathcal{N}$. By universality, there is an embedding $f : \mathcal{N} \hookrightarrow \mathcal{U}$.

If $\psi(\bar{x})$ is a basic formula, then $\mathcal{U} \models \psi(\bar{a})$ if and only if $\mathcal{N} \models \psi(\bar{a})$ if and only if $\mathcal{U} \models \psi(f(\bar{a}))$. So for any $\bar{a} \subseteq U$, $\mathrm{tp}_{\Delta}(\bar{a}) = \mathrm{tp}_{\Delta}(f(\bar{a}))$ and hence, by the homogeneity of $\mathcal{U}$ (and Remark 1), $\mathrm{tp}(\bar{a}) = \mathrm{tp}(f(\bar{a}))$.

Let $\bar{a} \subseteq U$ and $\varphi(\bar{x})$ be an existential formula s.t. $\mathcal{M} \models \varphi(\bar{a})$. Then $\mathcal{N} \models \varphi(\bar{a})$ and so $\mathcal{U} \models \varphi(f(\bar{a}))$. This implies, by above, that $\mathcal{U} \models \varphi(\bar{a})$.

So we've established that $\mathcal{U} \leqslant \mathcal{M}$ implies $\mathcal{U} \preccurlyeq_{\exists} \mathcal{M}$ as required. □

The following theorem is due to A. Robinson and will be useful in §9. Its proof can be found in any introductory text such as [13].

Theorem 2.9 *The following are equivalent for any L-theory T.*
(i) *T^{Δ} is a model complete L^{Δ}-theory.*
(ii) *If $\mathcal{M} \models T$ then $T \cup \mathrm{Diag}_{\Delta}\mathcal{M}$ is a complete $L(M)$-theory.*
(iii) *If $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M} \leqslant \mathcal{N}$, then $\mathcal{M} \preccurlyeq_{\exists} \mathcal{N}$.*
(iv) *If $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M} \leqslant \mathcal{N}$, then $\mathcal{M} \preccurlyeq \mathcal{N}$.*
(v) *If $\mathcal{M}, \mathcal{N} \models T$ and $f : \mathcal{M} \hookrightarrow \mathcal{N}$, then $f : \mathcal{M} \overset{\equiv}{\hookrightarrow} \mathcal{N}$.* □

§9 Elimination & saturation

The following results show how smooth the theory is when morphisms are Δ-elementary maps.

Proposition 2.10 *A universal domain is saturated if and only if* $\mathrm{Th}\mathcal{U}$ *admits* Δ-*elimination.*

Proof. ($\Rightarrow$) We first show that $(\mathrm{Th}\mathcal{U})^{\Delta}$ is a model complete L^{Δ}-theory. Let $\mathcal{M}, \mathcal{N} \models \mathrm{Th}\mathcal{U}$ s.t. $\mathcal{M} \leqslant \mathcal{N}$. Let $\varphi(\bar{x})$ be an existential formula. By Proposition 2.5(a), $\varphi(\bar{x})$ is $\mathcal{U}$-equivalent to a set of basic formulas $\Phi(\bar{x})$. So if $\bar{a} \subseteq M$ s.t. $\mathcal{N} \models \varphi(\bar{a})$, then $\mathcal{N} \models \Phi(\bar{a})$. But this means that $\mathcal{M} \models \Phi(\bar{a})$ since $\mathcal{M} \leqslant \mathcal{N}$. Let $\mathcal{M}' \preccurlyeq \mathcal{M}$ s.t. $\bar{a} \subseteq M'$ and $|M'| \leq |U|$. By the saturation of $\mathcal{U}$, there is an elementary embedding $f : \mathcal{M}' \overset{\equiv}{\hookrightarrow} \mathcal{U}$. So $\mathcal{U} \models \Phi(f(\bar{a}))$. But this means $\mathcal{U} \models \varphi(f(\bar{a}))$. Hence $\mathcal{M} \models \varphi(\bar{a})$. So we've established $\mathcal{M} \preccurlyeq_{\exists} \mathcal{N}$. By Theorem 2.9, $(\mathrm{Th}\mathcal{U})^{\Delta}$ is model complete.

We now show that $\mathrm{Th}\mathcal{U}$ admits Δ-elimination. We need to show that for any $\mathcal{M} \models \mathrm{Th}\mathcal{U}$ and any $\bar{a} \subseteq M$, $\mathrm{Th}\mathcal{U} \cup \mathrm{Diag}_{\Delta}\bar{a}$ is a complete $L(\bar{a})$-theory. Suppose it is incomplete. Then there is a formula $\varphi(\bar{a})$ such that both $\mathrm{Th}\mathcal{U} \cup \mathrm{Diag}_{\Delta}\bar{a} \cup \{\varphi(\bar{a})\}$ and $\mathrm{Th}\mathcal{U} \cup \mathrm{Diag}_{\Delta}\bar{a} \cup \{\neg\varphi(\bar{a})\}$ are satisfiable. This means there are $\mathcal{N}, \mathcal{N}' \models \mathrm{Th}\mathcal{U}$ and morphisms $e : \mathcal{M} \to \mathcal{N}$ and $f : \mathcal{M} \to \mathcal{N}'$ with $\operatorname{dom} e = \operatorname{dom} f = \bar{a}$ s.t. $\mathcal{N} \models \varphi(e(\bar{a}))$ and $\mathcal{N}' \models \neg\varphi(f(\bar{a}))$. Put $g = fe^{-1} : \mathcal{N} \to \mathcal{N}'$. As $\mathrm{Th}\mathcal{U}$ has the amalgamation property, there is an $\mathcal{M}' \models \mathrm{Th}\mathcal{U}$ s.t. $\mathcal{N}' \leqslant \mathcal{M}'$ and g extends to an embedding $F : \mathcal{N} \hookrightarrow \mathcal{M}'$. By Theorem 2.9, the model completeness of $(\mathrm{Th}\mathcal{U})^{\Delta}$ implies that $\mathcal{N}' \preccurlyeq \mathcal{M}'$ and $F : \mathcal{N} \overset{\equiv}{\hookrightarrow} \mathcal{M}'$. But this implies $\mathcal{M}' \models \varphi(f(\bar{a})) \wedge \neg\varphi(f(\bar{a}))$ which is a contradiction.

($\Leftarrow$) Assume $\mathrm{Th}\mathcal{U}$ admits Δ-elimination. Since $\mathcal{U}$ realises all Δ-types, it must realise all types. So $\mathcal{U}$ is saturated. □

Proposition 2.11 *Let* $\mathcal{U}$ *be a universal domain and assume* $\mathrm{Th}\mathcal{U}$ *has a saturated model of cardinality* $|U|$. *Then the following are equivalent.*
(i) *$\mathcal{U}$ is saturated.*
(ii) *All universal domains are saturated.*
(iii) *All saturated models of* $\mathrm{Th}\mathcal{U}$ *are universal domains.*

Proof. (i)$\Rightarrow$(ii) Suppose $\mathcal{U}$ is saturated. Then $\mathrm{Th}\mathcal{U}$ admits Δ-elimination. Let $\mathcal{V}$ be any other universal domain. Then $\mathcal{V} \equiv \mathcal{U}$. As $\mathcal{V}$ realises all Δ-types it therefore realises *all* types.

(ii)$\Rightarrow$(i) Trivial.

(i)$\Rightarrow$(iii) Let $\mathcal{M}$ be a saturated model of $\mathrm{Th}\mathcal{U}$. We need to establish homogeneity. To this end, let $f : \mathcal{M} \to \mathcal{M}$ be a morphism with small domain. Once again $\mathcal{U}$ saturated implies $\mathrm{Th}\mathcal{U}$ admits Δ-elimination. Hence $f : \mathcal{M} \to \mathcal{M}$ is partial elementary. By the saturation of $\mathcal{M}$, f extends to an automorphism of $\mathcal{M}$.

(iii)$\Rightarrow$(i) By hypothesis, $\mathrm{Th}\mathcal{U}$ has a saturated model $\mathcal{V}$ of cardinality $|U|$. By assumption $\mathcal{V}$ is a universal domain. Hence, by uniqueness $\mathcal{U} \cong \mathcal{V}$ and $\mathcal{U}$ must also be saturated. □

§10 Examples

– The countable random graph

Let L be a language with just one binary relation symbol R. Let T be the universal set of axioms stating that R is irreflexive and symmetric. Models of T are called *graphs*. Put $\mathbf{K} = \operatorname{Mod} T$ and $\operatorname{Mor}_{\mathbf{K}}(\mathcal{M}, \mathcal{N}) = \{\text{partial isomorphisms } f : M \to N\}$. Define

$$\mathbf{K}_0 = \{\text{finite models of } T\}$$

Then $\mathbf{K}_0$ is easily seen to satisfy the hypothesis of Theorem 1.10. This gives us a unique countable generic graph $\mathcal{G}$. $\mathcal{G}$ is called the *random graph*. By Proposition 2.5(b), T admits quantifier separation.

Seeing as our signature is finite, the open diagram of any finite model of T is logically equivalent to a single formula. So for each $\mathcal{M} \in \mathbf{K}_0$, we let $\varphi_{\mathcal{M}}(\bar{x})$ denote the open diagram of M. Let T_ω be the set of sentences

$$\forall \bar{x}(\varphi_{\mathcal{M}}(\bar{x}) \to \exists \bar{y} \varphi_{\mathcal{N}}(\bar{x}, \bar{y}))$$

for all $\mathcal{M} \subseteq \mathcal{N} \in \mathbf{K}_0$. It is not hard to see that $\mathcal{G} \models T_\omega$ and any countable $\mathcal{M} \models T_\omega$ is generic. Hence $\operatorname{Th} \mathcal{G}$ is $\aleph_0$-categorical. By Corollary 1.7 and Proposition 2.10, $\mathcal{G}$ is saturated and $\operatorname{Th} \mathcal{G}$ admits quantifier elimination.

The random graph $\mathcal{G}$ is characterisable by a nice property: A countable graph $\mathcal{M}$ is the random graph if and only if it satisfies

> For any disjoint finite sets of vertices $X, Y \subseteq M$, there is an element $a \notin X \cup Y$ adjacent to all vertices in X and no vertex in Y.

For a proof of this fact see [5].

– The countable generic partial ordering

Let L be the language with a single binary relation symbol $<$ and let T be the universal theory of partial orderings: I.e., T states that $<$ is irreflexive and transitive. Define $\mathbf{K}$, $\operatorname{Mor}_{\mathbf{K}}(\mathcal{M}, \mathcal{N})$ and $\mathbf{K}_0$ as above. Reasoning analogously to the above example, we get a countable generic partial ordering $\mathcal{U}$ s.t. $\operatorname{Th} \mathcal{U}$ is $\aleph_0$-categorical. Hence T admits quantifier separation, $\mathcal{U}$ is saturated and $\operatorname{Th} \mathcal{U}$ admits quantifier elimination.

For any subsets X, Y of a partial ordering $\mathcal{M}$, we define

$$\begin{aligned} X < Y \quad &\Leftrightarrow \quad \text{for every } x \in X \text{ and } y \in Y,\ x < y \\ X \not< Y \quad &\Leftrightarrow \quad \text{for every } x \in X \text{ and } y \in Y,\ x \not< y \\ X \mid Y \quad &\Leftrightarrow \quad X \not< Y \text{ and } Y \not< X \end{aligned}$$

When $X = \{x\}$ we omit the braces and write simply $x < Y$ (or $Y < x$). The same goes for $\not<$ and $|$.

We claim that a countable partial ordering $\mathcal{M}$ is the generic partial ordering if and only if it has the following property.

(2.2) For any disjoint finite sets $X, Y, Z \subseteq M$, satisfying $X < Y$, $Y \not< Z$ and $Z \not< X$, there is a point $a \notin X \cup Y \cup Z$ s.t. $X < a < Y$ and $a \mid Z$.

To see this, first assume $\mathcal{M}$ is the countable generic partial ordering $\mathcal{U}$. Let X, Y, Z be finite subsets of M as in (2.2). Let a be a point disjoint from $X \cup Y \cup Z$ and let $\mathcal{N}$ be the partial ordering with universe $X \cup Y \cup Z \cup \{a\}$ and s.t. $X < a < Y$ and $a \mid Z$. It is easily verified that $\mathcal{N}$, so defined, is indeed a partial ordering. By the genericity of $\mathcal{M}$, $\mathrm{id}_{X\cup Y\cup Z} : \mathcal{N} \to \mathcal{M}$ extends to an embedding $F : \mathcal{N} \hookrightarrow \mathcal{M}$. Hence $F(a)$ satisfies the conditions in (2.2).

Now for the other direction. Let $\mathcal{M}$ be a countable partial ordering satisfying (2.2). We need to show that $\mathcal{M}$ is the generic $\mathcal{U}$. To this end, let $\mathcal{N}$ be a finite partial ordering and $f : \mathcal{N} \to \mathcal{M}$ a partial isomorphism with $\mathrm{dom}\, f = A$. We need to extend f to an embedding $F : \mathcal{N} \hookrightarrow \mathcal{M}$. It is enough if we consider the case $|N \setminus A| = 1$; the rest is by induction.

So suppose $N \setminus A = \{a\}$. Define

$$X = \{x \in A : x < a\}$$
$$Y = \{x \in A : a < x\}$$
$$Z = \{x \in A : x \mid a\}$$

Then $X \cup Y \cup Z$ forms a partition of A. By transitivity, we must have $X < Y$,$Y \not< Z$ and $Z \not< X$. So also $f[X] < f[Y]$, $f[Y] \not< f[Z]$ and $f[Z] \not< f[X]$. By (2.2), there is a $b \in M \setminus f[A]$ s.t. $f[X] < b < f[Y]$ and $b \mid f[Z]$. Obviously $F = f \cup (a, b)$ is the desired embedding.

The property (2.2) implies some further nice properties. Here are a few examples.

- For any finite $X \subseteq M$, there are $a, b \in M \setminus X$ s.t. $a < X < b$.
- For any finite set $X \subseteq M$, there is an infinite set $Y \subseteq M$ which is totally disconnected from X. I.e., $X \mid Y$.

We leave the verifications to the reader.

– The countable dense linear ordering without end points

Let L be the language with a single binary relation symbol $<$ and let T be the (universal) theory of linear orderings. Define $\mathbf{K}$, $\mathrm{Mor}_{\mathbf{K}}(\mathcal{M}, \mathcal{N})$ and $\mathbf{K}_0$ as above. Again, it is easy to see that $\mathbf{K}_0$ satisfies the hypothesis of Theorem 1.10. Hence we get a countable generic linear ordering $\mathcal{U}$. So T admits quantifier separation.

We claim that $\mathcal{U}$ is a dense linear ordering without end points. We first establish density. Let $a_1, a_3 \in U$ s.t. $a_1 < a_3$. Let $\mathcal{M}$ be a model with universe $\{b_1, b_2, b_3\}$ s.t. $b_1 < b_2 < b_3$. The map $f : (b_1, b_3) \mapsto (a_1, a_3)$ defines a partial isomorphism $f : \mathcal{M} \to \mathcal{U}$. By genericity f extends to an embedding $F : \mathcal{M} \hookrightarrow \mathcal{U}$. Hence $a_1 < F(b_2) < a_3$.

We now show that $\mathcal{U}$ has no end points. Let $a_1 \in U$ and let $\mathcal{M}$ be a model with universe $\{b_1, b_2\}$ s.t. $b_1 < b_2$. The map $f : b_1 \mapsto a_1$ defines a partial isomorphism

$f : \mathcal{M} \to \mathcal{U}$ which extends to an embedding $F : \mathcal{M} \hookrightarrow \mathcal{U}$. Hence $a_1 < F(b_2)$. So $\mathcal{U}$ has no right end point. A similar argument shows that $\mathcal{U}$ has no left end point either.

By Cantor's theorem $\mathcal{U} \cong \mathbb{Q}$. Hence Th$\mathcal{U}$ is $\aleph_0$-categorical. So $\mathcal{U}$ is saturated and Th$\mathcal{U}$ admits quantifier elimination.

– The countable dense linear ordering with a left but no right end point

Let L, T, $\mathbf{K}$ and $\mathbf{K}_0$ be as above. Let φ be the formula

$$\forall y(x = y \vee x < y)$$

and define Δ to be the smallest set of formulas containing φ and all quantifier-free formulas which is closed under boolean combinations and substitution of variables. Put

$$\mathrm{Mor}_{\mathbf{K}}(\mathcal{M}, \mathcal{N}) = \{\text{partial } \Delta\text{-elementary maps } f : M \to N\}$$

It is easy to see that if a morphism $f : \mathcal{M} \to \mathcal{N}$ has the minimal element of $\mathcal{M}$ in its domain, then it maps this element to the minimal element of $\mathcal{N}$. Conversely, any partial isomorphism which maps minimal elements to minimal elements is a morphism. Given these facts, it is easy to verify the amalgamation property for $\mathbf{K}_0$. (The reader should also check that $\mathbf{K}_0$ is inductively bounded.) As usual, this yields a countable generic linear ordering $\mathcal{U}$.

We first show that $\mathcal{U}$ has a left end point. For let $\mathcal{M}$ be the linear ordering consisting of the single point a. Then by universality there is an embedding $f : \mathcal{M} \hookrightarrow \mathcal{U}$. Clearly $f(a)$ is the left end point of $\mathcal{U}$.

We claim that $\mathcal{U}$ is dense. Let $a_1, a_3 \in U$ s.t. $a_1 < a_3$. Let $\mathcal{M}$ be the linear ordering with universe $\{b_0, b_1, b_2, b_3\}$ s.t. $b_0 < b_1 < b_2 < b_3$. If a_1 is a minimal element of $\mathcal{U}$, then the map $f_0 : (b_0, b_3) \mapsto (a_1, a_3)$ defines a morphism $f_0 : \mathcal{M} \to \mathcal{U}$. If on the other hand a_1 is no minimal element of $\mathcal{U}$, then the map $f_1 : (b_1, b_3) \mapsto (a_1, a_3)$ defines a morphism $f : \mathcal{M} \to \mathcal{U}$. In each case we get an extension to an embedding $F_i : \mathcal{M} \hookrightarrow \mathcal{U}$. Hence $a_1 < F_i(b_2) < a_3$.

Similar reasoning shows that $\mathcal{U}$ has no right end point. Hence Th$\mathcal{U}$ is the complete theory of dense linear orderings with a left but no right end point. So Th$\mathcal{U}$ is $\aleph_0$-categorical. Hence $\mathcal{U}$ is saturated and Th$\mathcal{U}$ admits Δ-elimination.

III Bicoloured strongly minimal structures

We now turn to our main application: *bicoloured strongly minimal structures*. The morphisms that we shall need to work with will not have a simple definition in terms of Δ-elementary maps. Our definition of a morphism will require three other notions which we'll need to define and study first: *predimension*, *closure* and *self-sufficiency*. Once our definition of a morphism has been laid down, we will prove the existence of both a countable and a strongly inaccessible generic structure. We finish by proving that all generic models are saturated.

Throughout, we let T be a strongly minimal theory with quantifier elimination in a countable language L. We assume moreover that models of T have the following natural algebraic closure property.

ACP $\quad \operatorname{acl}(X \cup \{a\}) \setminus \operatorname{acl} X$ is either empty or infinite.

This property is satisfied by algebraically closed fields of any characteristic as well as any structure in which $\operatorname{acl} \emptyset$ is infinite. ACP of course implies that T has models of dimension n for each $n \geq 1$.

§1 Predimension

Define $L_0 = L \cup \{P\}$ where P is a new unary predicate. Let T_0 be the L_0-theory axiomatised by T and all sentences of the form

$$(3.1) \qquad \forall y_1 \dots y_m (\exists_n x \varphi(x, y_1, \dots, y_m) \to \exists_{\leq 2m} x (Px \wedge \varphi(x, y_1, \dots, y_m)))$$

where $0 < n, m < \omega$ and $\varphi \in L_{m+1}$. See Lemma 3.1 for the significance of these sentences. Following Poizat [11], a model $\mathcal{M}$ of T_0 will be called a *bicoloured* strongly minimal structure and the set $P(\mathcal{M})$ will be its set of *black points* and $\neg P(\mathcal{M})$ its set of *white points*.

Henceforth, $\mathcal{M}$, $\mathcal{N}$, etc. will refer to L_0-structures that model T. However, all talk of algebraic closure (acl) and dimension (dim) will be relative to the language L. So given a structure $\mathcal{M}$, we define a *predimension* function δ mapping finite dimensional subsets X of M to the integers by

$$\delta_{\mathcal{M}}(X) = 2 \dim_{\mathcal{M}} X - |P(\mathcal{M}) \cap X|$$

I'll omit the subscript if the ambient model is clear (this goes for all subsequent definitions). The following lemma links δ with T_0.

Lemma 3.1 *$\mathcal{M} \models T_0$ if and only if for all finite dimensional $X \subseteq M$, $\delta(X) \geq 0$.*

Proof. ($\Rightarrow$) Suppose $\mathcal{M} \models T_0$ and let $X \subseteq M$ s.t. $\dim X = m$. Let A be a basis for X. Suppose $\delta(X) < 0$. Then $|P(\mathcal{M}) \cap X| \geq 2m+1$. Let $B = \{b_1, \dots, b_{2m+1}\} \subseteq P(\mathcal{M}) \cap X$. $B \subseteq \operatorname{acl} A$ so for each $1 \leq i \leq 2m+1$ there is an algebraic formula $\varphi_i(x)$ over A s.t. $b_i \in \varphi_i(\mathcal{M})$. Let $\varphi(x) = \bigvee_{i=1}^{2m+1} \varphi_i(x)$. Then $\varphi(x)$ is algebraic over A but $|P(\mathcal{M}) \cap \varphi(\mathcal{M})| > 2m$ contradicting (3.1).

($\Leftarrow$) Let $\varphi(x)$ be algebraic over an m-tuple $\bar{a}$. Let $X = \varphi(\mathcal{M})$. Then $\delta(X) \geq 0$ implies $|P(\mathcal{M}) \cap X| \leq 2 \dim X \leq 2m$. □

If X and Y are finite dimensional subsets in $\mathcal{M}$, we put

$$\delta_{\mathcal{M}}(X/Y) = \delta_{\mathcal{M}}(X \cup Y) - \delta_{\mathcal{M}}(Y)$$

Remark 1. Note that $\delta(X/Y) = 2(\dim(X \cup Y) - \dim Y) - |P(\mathcal{M}) \cap X| = 2\dim(X/Y) - |P(\mathcal{M}) \cap X|$. So if $Z \subseteq Y$, then $\dim(X/Y) \leq \dim(X/Z)$ and hence $\delta(X/Y) \leq \delta(X/Z)$.

§2 Closure

Let $\mathcal{M} \models T_0$. For a finite dimensional $X \subseteq M$ we define

$$d_{\mathcal{M}}(X) = \min\{\delta_{\mathcal{M}}(Y) : X \subseteq Y \subseteq M,\ \dim_{\mathcal{M}} Y < \aleph_0\}$$

This is the *d-rank* of X. The following defines the *closure* of X.

$$\mathrm{cl}_{\mathcal{M}} X = \bigcap\{\mathrm{acl}_{\mathcal{M}} Y : X \subseteq Y \subseteq M,\ \dim_{\mathcal{M}} Y < \aleph_0,\ \delta(Y) = d_{\mathcal{M}}(X)\}$$

Remarks

2. Note that if $\delta(Y) = d(X)$ then also $\delta(\mathrm{acl}\, Y) = d(X)$. For $\dim Y = \dim(\mathrm{acl}\, Y)$ implies that $\delta(\mathrm{acl}\, Y) \leq \delta(Y)$. So by minimality $\delta(\mathrm{acl}\, Y) = \delta(Y) = d(X)$. In particular, $\mathrm{acl}\, Y \setminus Y \subseteq \neg P(\mathcal{M})$.

3. If $P(\mathcal{M}) = \emptyset$, then $\delta(X) = d(X)$ and so $\mathrm{cl}\, X = \mathrm{acl}\, X$.

4. If $X \subseteq Y$ then $d(X) \leq d(Y)$.

5. $\mathrm{acl}(\mathrm{cl}\, X) = \mathrm{cl}\, X$, $X \subseteq \mathrm{acl}\, X \subseteq \mathrm{cl}\, X$, and $\mathrm{cl}(X) = \mathrm{cl}(\mathrm{acl}\, X)$.

6. If X is an infinite but finite dimensional set, then there is a finite $A \subseteq X$ s.t. $\delta(X) = \delta(A)$. Just take A to be $(P(\mathcal{M}) \cap X) \cup X_0$ where X_0 is a basis for X. Similarly if $Y \subseteq X$ then there is a finite $A \subseteq X \setminus Y$ s.t. $\delta(Y \cup A) = \delta(X)$.

Lemma 3.2 *Let $\mathcal{M} \models T_0$ and X be a finite dimensional subset of M. Then $\delta(\mathrm{cl}\, X) = d(X)$.*

Proof. Let $\mathrm{cl}\, X = \bigcap_{i \in I} X_i$ where $\delta(X_i) = d(X)$ for each $i \in I$. Fix $j \in I$ and let B be a finite subset of $X_j \setminus \mathrm{cl}\, X$. Obviously there must be a $k \in I$ s.t. B is disjoint from X_k. By minimality, $\delta(B \cup X_k) \geq \delta(X_k)$. That is, $\delta(B/X_k) \geq 0$. By Remark 1, $\delta(B/\mathrm{cl}\, X) \geq \delta(B/X_k) \geq 0$. Hence $\delta(B \cup \mathrm{cl}\, X) \geq \delta(\mathrm{cl}\, X)$. As B is arbitrary, by Remark 6, $\delta(X_j) \geq \delta(\mathrm{cl}\, X)$. By minimality, $\delta(\mathrm{cl}\, X) = \delta(X_j) = d(X)$. □

We shall write $A \subseteq_0 X$ for 'A is a finite subset of X'.

Lemma 3.3 *Let $\mathcal{M} \models T_0$. For any finite dimensional $X, Y \subseteq M$, the following hold.*
(a) *(Reflexivity)* $X \subseteq \mathrm{cl}\, X$.
(b) *(Idempotence)* $\mathrm{cl}(\mathrm{cl}\, X) = \mathrm{cl}\, X$.
(c) *(Monotonicity) If* $X \subseteq Y$, *then* $\mathrm{cl}\, X \subseteq \mathrm{cl}\, Y$.
(d) *(Finite Nature)* $\mathrm{cl}\, X = \bigcup\{\mathrm{cl}\, A : A \subseteq_0 X\}$.

Proof. (a) Obvious.

(b) By Remark 5, $\mathrm{cl}\, X$ is algebraically closed. So we need only check that $\delta(\mathrm{cl}\, X) = d(\mathrm{cl}\, X)$. That is, that $d(X) = d(\mathrm{cl}\, X)$. But this is trivial.

(c) Let $X \subseteq Y$ and suppose $\mathrm{cl}\, X \not\subseteq \mathrm{cl}\, Y$. Then $\mathrm{cl}\, X \setminus \mathrm{cl}\, Y \neq \emptyset$. Put $X_0 = \mathrm{cl}\, X \setminus \mathrm{cl}\, Y$ and $Y_0 = \mathrm{cl}\, Y \setminus X$ (Y_0 could be empty). Then $\delta(X_0 \cup Y_0 \cup X) = \delta(\mathrm{cl}\, X) = d(X)$. Since

$X_0 \cap \operatorname{acl}(X \cup Y_0) = \emptyset$, we have $\delta(X_0 \cup X \cup Y_0) < \delta(X \cup Y_0)$. Hence $\delta(X_0/X \cup Y_0) < 0$ and so also $\delta(X_0/\operatorname{cl} Y) < 0$. But this is a contradiction.

(d) As X is finite dimensional, there is an $A \subseteq_0 X$ with $\operatorname{acl} A = \operatorname{acl} X$. So $\operatorname{cl} A = \operatorname{cl}(\operatorname{acl} A) = \operatorname{cl}(\operatorname{acl} X) = \operatorname{cl} X$. For the other direction, let $A \subseteq_0 X$. Then, by monotonicity $\operatorname{cl} A \subseteq \operatorname{cl} X$. This proves the statement. □

Let $\mathcal{M} \models T_0$. By Lemma 3.3(d) we can consistently make the following definition for *any* infinite set $X \subseteq M$.

$$\operatorname{cl}_{\mathcal{M}} X = \bigcup \{\operatorname{cl}_{\mathcal{M}} A : A \subseteq_0 X\}$$

Lemma 3.3 now generalises to infinite (particularly infinite dimensional) sets.

Lemma 3.4 *Let $\mathcal{M} \models T_0$ and suppose X, Y are infinite subsets of M. Then*
(a) *(Reflexivity) $X \subseteq \operatorname{cl} X$.*
(b) *(Idempotence) $\operatorname{cl}(\operatorname{cl} X) = \operatorname{cl} X$.*
(c) *(Monotonicity) If $X \subseteq Y$, then $\operatorname{cl} X \subseteq \operatorname{cl} Y$.*
(d) *(Algebraic Closure) $\operatorname{acl}(\operatorname{cl} X) = \operatorname{cl} X$.*

Proof. (a) $X = \bigcup_{a \in X} \{a\} \subseteq \bigcup_{a \in X} \operatorname{cl}\{a\} \subseteq \operatorname{cl} X$.

(b) Let $a \in \operatorname{cl}(\operatorname{cl} X)$. Then $a \in \operatorname{cl} A$ for some $A \subseteq_0 \operatorname{cl} X$. This implies $A \subseteq \operatorname{cl} B_1 \cup \cdots \cup \operatorname{cl} B_n$ for $B_i \subseteq_0 X$. By monotonicity and idempotence for finite dimensional sets, $A \subseteq \operatorname{cl}(B_1 \cup \cdots \cup B_n)$ and $\operatorname{cl} A \subseteq \operatorname{cl}(B_1 \cup \cdots \cup B_n) \subseteq \operatorname{cl} X$. So $a \in \operatorname{cl} X$ as required.

(c) Let $X \subseteq Y$. Then $\operatorname{cl} X = \bigcup\{\operatorname{cl} A : A \subseteq_0 X\} \subseteq \bigcup\{\operatorname{cl} A : A \subseteq_0 Y\} = \operatorname{cl} Y$.

(d) Let $a \in \operatorname{acl}(\operatorname{cl} X)$. Then $a \in \operatorname{acl} X_0$ for some $X_0 \subseteq_0 \operatorname{cl} X$. So $a \in \operatorname{cl} X_0 \subseteq \operatorname{cl}(\operatorname{cl} X) = \operatorname{cl} X$. □

Remark 7. Note that $\operatorname{cl} X$ is always a substructure since it's algebraically closed.

§3 Self-sufficiency

Let $\mathcal{N} \models T_0$. We write $\mathcal{A} \leqslant_{ss} \mathcal{N}$ ($\mathcal{A}$ is *self-sufficient* in $\mathcal{N}$) if $\mathcal{A} \subseteq \mathcal{N}$ and $\operatorname{cl}_{\mathcal{N}} A = A$.

Remarks

8. If $\mathcal{M}, \mathcal{N} \models T_0$ and $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{M} \preccurlyeq_L \mathcal{N}$. In general, if $\mathcal{M} \preccurlyeq_L \mathcal{N}$, then for any subset $X \subseteq M$, $\operatorname{acl}_{\mathcal{M}} X = \operatorname{acl}_{\mathcal{N}} X$ and $\dim_{\mathcal{M}} X = \dim_{\mathcal{N}} X$. So in such cases we will be able to omit subscripts when talking about algebraic closure or dimension. In particular, we can usually omit the subscript from δ without risk of confusion.

9. If $\mathcal{M}, \mathcal{N} \models T_0$ and $\operatorname{cl}_{\mathcal{N}} X \subseteq \mathcal{M} \subseteq \mathcal{N}$, then $\operatorname{cl}_{\mathcal{N}} X = \operatorname{cl}_{\mathcal{M}} X$. Hence, if $\mathcal{A} \subseteq \mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{A} \leqslant_{ss} \mathcal{N}$, then also $\mathcal{A} \leqslant_{ss} \mathcal{M}$.

10. If $\mathcal{A} \leqslant_{ss} \mathcal{N}$ and $\mathcal{B} \leqslant_{ss} \mathcal{N}$, then $\operatorname{cl}(A \cap B) \subseteq \operatorname{cl} A \cap \operatorname{cl} B = A \cap B$ and so $\mathcal{A} \cap \mathcal{B} \leqslant_{ss} \mathcal{N}$.

Lemma 3.5 *Let $\mathcal{M}, \mathcal{N} \models T_0$ and let $f : M \to N$ be a partial map with $\operatorname{dom} f$ finite. If f is elementary w.r.t. existential formulas, then f extends to an isomorphism $F : \operatorname{cl}_{\mathcal{M}}(\operatorname{dom} f) \cong \operatorname{cl}_{\mathcal{N}}(\operatorname{im} f)$.*

Proof. Put $X = \operatorname{dom} f$ and $Y = \operatorname{im} f$. Clearly $\delta(X) = \delta(Y)$. We claim that $d(X) = d(Y)$. To see this, suppose that $d(X) \neq d(Y)$. Without loss, we may assume that $d(X) < d(Y)$. Then $d(X) < d(Y) \leq \delta(Y) = \delta(X)$. So there is a tuple of distinct black points $\bar{a} = (\operatorname{cl} X \setminus X) \cap P(\mathcal{M})$ so that $\operatorname{cl} X = \operatorname{acl}(X \cup \bar{a})$. In particular, we must have $\delta(\bar{a}/X) < 0$. I.e., $\dim(\bar{a}/X) < |\bar{a}|/2$. Note that we also have that for any non-trivial partition of $\bar{a}$ into two disjoint subsets $\bar{a}^1$ and $\bar{a}^2$, $\delta(\bar{a}^2 \cup X) > d(X) = \delta(\bar{a} \cup X)$. That is, $\delta(\bar{a}^1/\bar{a}^2 \cup X) < 0$.

Put $n = |\bar{a}|$ and $m = \dim(\bar{a}/X)$. Without loss, we may assume that the first m elements of $\bar{a}$ form a basis for $\bar{a}$ over X. Let $\bar{c}$ be an enumeration of X and define $\varphi(\bar{x}, \bar{y})$ to be a formula s.t. $\mathcal{M} \models \varphi(\bar{a}, \bar{c})$ which expresses the following.

- $\bigwedge_{i=1}^{n} Px_i$
- The x_i's are distinct from each other and every y_j
- $\bigwedge_{i=m+1}^{n} (\exists_{k_i} x \psi_i(x, x_1, \ldots, x_{i-1}, \bar{y}) \wedge \psi_i(x_i, x_1, \ldots, x_{i-1}, \bar{y}))$ where ψ_i is an algebraic formula isolating the L-type of a_i over $\{a_1, \ldots, a_{i-1}\} \cup X$.
- For any non-trivial partition $\bar{x} = \bar{x}^1 \cup \bar{x}^2$, $\dim(\bar{x}^1/\bar{x}^2 \cup \bar{y}) < \ell(\bar{x}^1)/2$.

As f preserves existential formulas, $\mathcal{N} \models \exists \bar{x} \varphi(\bar{x}, f(\bar{c}))$. So there is a tuple of distinct black points $\bar{b} \subseteq N$ disjoint from Y s.t. $\dim(\bar{b}/Y) \leq \dim(\bar{a}/X) < |\bar{a}|/2 = |\bar{b}|/2$. This implies that $\delta(\bar{b} \cup Y) \leq \delta(\bar{a} \cup X) = d(X)$ since $\delta(Y) = \delta(X)$. I.e. $d(Y) \leq d(X)$ which is a contradiction. So we indeed have $d(X) = d(Y)$.

If $\delta(X) = d(X)$, then $\operatorname{cl} X = \operatorname{acl} X$ and $\operatorname{cl} X \setminus X$ is white. Since $d(X) = d(Y)$ and $\delta(X) = \delta(Y)$, we must have $\operatorname{cl} Y = \operatorname{acl} Y$ and $\operatorname{cl} Y \setminus Y$ is also white. Hence, obviously f extends to an isomorphism $F : \operatorname{cl} X \cong \operatorname{cl} Y$.

Assume now that $d(X) < \delta(X)$. Then as above, there is a tuple of distinct black points $\bar{a} \subseteq M \setminus X$ s.t. $\operatorname{cl} X = \operatorname{acl}\{X \cup \bar{a}\}$ and a tuple $\bar{b}$ of distinct black points in $N \setminus Y$ of the same length s.t. $\delta(\bar{b} \cup Y) \leq \delta(\bar{a} \cup X) = d(X)$. As $d(X) = d(Y)$, we must have $\delta(\bar{b} \cup Y) = \delta(\bar{a} \cup X)$ and so $\delta(\bar{b} \cup Y) = d(Y)$. This means that $\operatorname{acl}(\bar{b} \cup Y)$ can contain no other black point.

Since $\dim(\bar{a}/X) = \dim(\bar{b}/Y)$, $\bar{b}$ has a basis over Y of cardinality m. The third condition on φ says that the last $n - m$ elements of $\bar{b}$ are algebraic over the first m. So we can take the set $\{b_1, \ldots, b_m\}$ to be a basis. So by putting $f(a_i) = b_i$ for $1 \leq i \leq m$, we can extend f to a partial isomorphism $f : X \cup \{a_1, \ldots, a_m\} \to Y \cup \{b_1, \ldots, b_m\}$. Since we have taken the ψ_i to isolate the types of the a_i over $\{a_1 \ldots, a_{i-1}\}$, we can extend f even further to a partial isomorphism $f : X \cup \bar{a} \to Y \cup \bar{b}$. It follows that we can extend f to an isomorphism $F : \operatorname{cl} X \cong \operatorname{acl}(Y \cup \bar{b})$. We now only need to show that $\operatorname{acl}(Y \cup \bar{b}) = \operatorname{cl} Y$ and we're done.

This is where the remaining condition on φ comes into play. Whenever we split $\bar{b}$ into two disjoint subsets $\bar{b}^1$ and $\bar{b}^2$, we have $\delta(\bar{b}^1 \cup Y) > d(Y)$. Since $d(Y) < \delta(Y)$, no tuple of black points $\bar{d}$ outside $\operatorname{acl}(\bar{b} \cup Y)$ can satisfy $\delta(\bar{d} \cup Y) = d(Y)$. So we must have that $\operatorname{cl} Y = \operatorname{acl}(\bar{b} \cup Y)$. This establishes the result. □

Corollary 3.6 *If $\mathcal{M} \preccurlyeq_\exists \mathcal{N} \models T_0$, then also $\mathcal{M} \leqslant_{ss} \mathcal{N}$.*

Proof. We need to show that for every finite $A \subseteq_0 M$, $\operatorname{cl}_\mathcal{N} A \subseteq M$. So let A be any such finite subset. Since $\operatorname{id}_A : M \to N$ is partial $\exists$-elementary, Lemma 3.5 implies that there is an isomorphism $f : \operatorname{cl}_\mathcal{M} A \cong_A \operatorname{cl}_\mathcal{N} A$. Since $\dim_\mathcal{N}(\operatorname{cl}_\mathcal{N} A) = \dim_\mathcal{N}(\operatorname{cl}_\mathcal{M} A)$, we must have $d_\mathcal{N}(A) = \delta(\operatorname{cl}_\mathcal{N} A) = \delta(\operatorname{cl}_\mathcal{M} A)$. Hence $\operatorname{cl}_\mathcal{N} A = \operatorname{cl}_\mathcal{M} A \subseteq M$. □

§4 Morphisms

We define

$$\mathbf{K} = \operatorname{Mod} T_0$$

$$\operatorname{Mor}_\mathbf{K}(\mathcal{M}, \mathcal{N}) = \left\{ \begin{array}{l} \text{partial maps } f : M \to N \text{ which extend to} \\ \text{an isomorphism } F : \operatorname{cl}_\mathcal{M}(\operatorname{dom} f) \cong \operatorname{cl}_\mathcal{N}(\operatorname{im} f) \end{array} \right\}$$

We now verify axioms M1 -M6.

M1: Note that $\mathcal{M}, \mathcal{N} \in \mathbf{K}$ implies that $\mathcal{M} \equiv_L \mathcal{N}$. Since $\delta(\emptyset) = 0$, $\operatorname{cl}_\mathcal{M} \emptyset$ and $\operatorname{cl}_\mathcal{N} \emptyset$ must both be white and equal to their algebraic closures. Hence there is an isomorphism $f : \operatorname{cl}_\mathcal{M} \emptyset \cong \operatorname{cl}_N \emptyset$. That is, $\operatorname{id}_\emptyset \in \operatorname{Mor}(\mathcal{M}, \mathcal{N})$.

M2: This is immediate from the definition.

M3: Let $f : M \to N$ be a partial elementary map. We may assume that f is $\operatorname{id}_A : M \to N$ for some set A. By elementary amalgamation, $\mathcal{M}$ and $\mathcal{N}$ are both elementary substructures of a larger structure $\mathcal{M}^*$. By Corollary 3.6, $\mathcal{M} \leqslant_{ss} \mathcal{M}^*$ and $\mathcal{N} \leqslant_{ss} \mathcal{M}^*$. Hence $\operatorname{cl}_{\mathcal{M}^*} A = \operatorname{cl}_\mathcal{M} A = \operatorname{cl}_\mathcal{N} A$. So the required extension of f is simply the identity on $\operatorname{cl}_\mathcal{M} A$.

M4: Let $f : \mathcal{M}_0 \to \mathcal{M}_1$ and $g : \mathcal{M}_1 \to \mathcal{M}_2$ be morphisms and put $\operatorname{im} f = X$, $\operatorname{dom} g = Y$. Without loss, we may assume that $\operatorname{dom} f, X, Y, \operatorname{im} g$ are all self-sufficient. We have $\operatorname{dom} gf = f^{-1}[X \cap Y]$ and $\operatorname{im} gf = g[X \cap Y]$. Since $X \cap Y \leqslant_{ss} \mathcal{M}_1$, we must have $\operatorname{dom} gf \leqslant_{ss} \mathcal{M}_0$ and $\operatorname{im} gf \leqslant_{ss} \mathcal{M}_2$ as required.

M5: Immediate.

M6: This is also immediate.

Clearly, for all $\mathcal{M}, \mathcal{N} \in \mathbf{K}$,

$$\mathcal{M} \leqslant \mathcal{N} \quad \Leftrightarrow \quad \mathcal{M} \leqslant_{ss} \mathcal{N}$$

To simplify notation, we shall henceforth only ever write $\mathcal{M} \leqslant \mathcal{N}$, even when $\mathcal{M}$ is a self-sufficient substructure which isn't necessarily in $\mathbf{K}$.

The following lemma gives a useful characterisation of types in a generic model.

Lemma 3.7 *Let $\mathcal{G}$ be a generic model. For any set A of cardinality $< |G|$ and any tuples $\bar{a}, \bar{b} \in G^n$, $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$ if and only if there is an isomorphism $f : \mathrm{cl}(A \cup \bar{a}) \cong \mathrm{cl}(A \cup \bar{b})$ s.t. $f(\bar{a}) = \bar{b}$ and $f \restriction_A = \mathrm{id}$.*

Proof. ($\Rightarrow$) If $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$, then the map $f : \bar{a} \cup A \to \bar{b} \cup A$ fixing A and taking $\bar{a}$ to $\bar{b}$ is partial elementary. Hence f extends to an isomorphism $F : \mathrm{cl}(A \cup \bar{a}) \cong \mathrm{cl}(A \cup \bar{b})$.

($\Leftarrow$) Let f be as in the hypothesis. Then, by homogeneity, f extends to an automorphism of $\mathcal{G}$. Hence $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$. □

§5 Amalgamation & existence

We first verify the property IND.

Lemma 3.8 *If $\mathcal{M}_i \in \mathbf{K}$ and $\mathcal{M}_i \leqslant \mathcal{M}_j$ for $i \le j < \xi$, then for each $k < \xi$, $\mathcal{M}_k \leqslant \bigcup_{i<\xi} \mathcal{M}_i \models T_0$.*

Proof. Define $\mathcal{M} = \bigcup_{i<\xi} \mathcal{M}_i$. Clearly $\mathcal{M} \models T_0$ since $\mathcal{M} \models T$ and every finite subset $X \subseteq_0 M$ is contained in some M_i, yielding $\delta(X) \ge 0$.

Fix a $k < \xi$. Let $A \subseteq_0 M_k$. Since $\mathrm{cl}_{\mathcal{M}} A$ is finite dimensional, $\mathrm{cl}_{\mathcal{M}} A \subseteq M_j$ for some $j > k$. Hence $\mathrm{cl}_{\mathcal{M}} A = \mathrm{cl}_{\mathcal{M}_j} A$. But this means $\mathrm{cl}_{\mathcal{M}} A \subseteq M_k$ since $\mathcal{M}_k \leqslant \mathcal{M}_j$. □

We define

$$\mathbf{K}_0 = \{\mathcal{M} \in \mathbf{K} : \dim M < \aleph_0\}$$

For convenience, we define $\mathbf{K}_1 = \mathbf{K}$. We now verify AP for $\mathbf{K}_\sigma$ ($\sigma = 0, 1$).

Let $\mathcal{M}, \mathcal{N} \in \mathbf{K}_\sigma$ and let $\mathcal{A} = \mathcal{M} \cap \mathcal{N}$. Suppose further that $\mathcal{A} \leqslant \mathcal{M}$ and $\mathcal{A} \leqslant \mathcal{N}$. We construct the *free amalgam* of $\mathcal{M}$ and $\mathcal{N}$ over $\mathcal{A}$ in three steps.

Step 1: Let $\mathbb{M}$ be a large saturated (or, equivalently, large infinite dimensional) model of T containing $\mathcal{M}$. Let $C \subseteq N \setminus A$ be independent over A s.t. $N = \mathrm{acl}(C \cup A)$. Choose a set $C' \subseteq \mathbb{M} \setminus M$ independent over M with $|C'| = |C|$. Put $N' = \mathrm{acl}(C' \cup A)$.

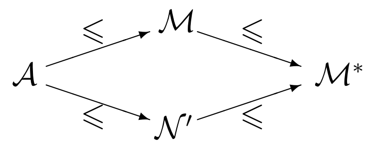
It is easy to see that $N' \cap M = A$. For suppose that $b \in (N' \cap M) \setminus A$. Let C_* be a minimal subset of C' s.t. $b \in \mathrm{acl}(A \cup C_*)$. Then if $c \in C_*$, $b \notin \mathrm{acl}(A \cup C_* \setminus \{c\})$. By exchange, $c \in \mathrm{acl}(A \cup \{b\} \cup C_* \setminus \{c\})$ which contradicts that C' was chosen independently of M.

It is clear that our choice of N' ensures that for any $X \subseteq N' \setminus A$ and $Y \subseteq M \setminus A$, $\dim(X/A) = \dim(X/A \cup Y)$ and (by exchange) $\dim(Y/A) = \dim(Y/A \cup X)$.

Step 2: As N' is an infinite algebraically closed subset of $\mathbb{M}$, we must have $\mathcal{N}' \preccurlyeq_L \mathbb{M}$. In particular, $\mathcal{N}' \models T$. As $\dim(N'/A) = \dim(N/A)$, it follows that there is an L-isomorphism $f : \mathcal{N} \cong \mathcal{N}'$ fixing A. By putting $P(\mathcal{N}') = f[P(\mathcal{N})]$ we make $\mathcal{N}'$ a bicolored strongly minimal structure isomorphic to $\mathcal{N}$. So in particular, $\mathcal{A} \leqslant \mathcal{N}'$.

Step 3: Let $M^* = \mathrm{acl}_{\mathbb{M}}(M \cup N')$. Then M^* is the universe of an L-elementary substructure $\mathcal{M}^*$ of $\mathbb{M}$. So $\mathcal{M}^* \models T$. By painting $M^* \setminus (M \cup N')$ white, we make $\mathcal{M}^*$ an L_0-structure. This is the free amalgam of $\mathcal{M}$ and $\mathcal{N}$ over $\mathcal{A}$. Clearly $\mathcal{M}^*$ is finite dimensional if $\mathcal{M}$ and $\mathcal{N}$ are.

Lemma 3.9 *Let $\mathcal{M}, \mathcal{N} \in \mathbf{K}_\sigma$ s.t. $\mathcal{M} \cap \mathcal{N} = \mathcal{A}$, $\mathcal{A} \leqslant \mathcal{M}$ and $\mathcal{A} \leqslant \mathcal{N}$. Then there is an $\mathcal{M}^* \in \mathbf{K}_\sigma$ and $\mathcal{N}' \cong_A \mathcal{N}$ s.t. $\mathcal{A} = \mathcal{M} \cap \mathcal{N}'$, $\mathcal{A} \leqslant \mathcal{N}'$, $\mathcal{M} \leqslant \mathcal{M}^*$ and $\mathcal{N}' \leqslant \mathcal{M}^*$. In a picture:*



Proof. Take $\mathcal{M}^*$ to be the free amalgam of $\mathcal{M}$ and $\mathcal{N}$ over $\mathcal{A}$ and take $\mathcal{N}'$ to be as in the above construction. We first need to verify that $\mathcal{M}^*$ is indeed a model of T_0. That is, we need to check that $\delta(X) \geq 0$ when $X \subseteq_0 M^*$.

As $M^* \setminus (M \cup N') \subseteq \neg P(\mathcal{M}^*)$, we have $\delta(X) \geq \delta(X \cap (M \cup N'))$ for all $X \subseteq_0 M^*$. So we need only consider finite subsets of $M \cup N'$. Let X be such a finite set. Then $X = X_a \cup X_m \cup X_n$ where $X_a \subseteq A$, $X_m \subseteq M \setminus A$ and $X_n \subseteq N' \setminus A$. $\mathcal{A} \leqslant \mathcal{M}$ implies that $\delta(\mathrm{cl}_{\mathcal{M}} X_a) \leq \delta(\mathrm{cl}_{\mathcal{M}} X_a \cup X_m)$. That is, $\delta(X_m / \mathrm{cl}_{\mathcal{M}} X_a) \geq 0$. By the choice of $\mathcal{N}'$, $\delta(X_m / \mathrm{cl}_{\mathcal{M}} X_a) = \delta(X_m / \mathrm{cl}_{\mathcal{M}} X_a \cup X_n)$. So $\delta(X_m / X_a \cup X_n) \geq \delta(X_m / \mathrm{cl}_{\mathcal{M}} X_a \cup X_n) \geq 0$. Hence $\delta(X_a \cup X_m \cup X_n) \geq \delta(X_a \cup X_n) \geq 0$ since $\mathcal{N}' \models T_0$.

We now need to show that $\mathcal{M} \leqslant \mathcal{M}^*$ (and $\mathcal{N}' \leqslant \mathcal{M}^*$, but this case is similar). That is, we need to verify

$$\text{For every } X \subseteq_0 M,\ \mathrm{cl}_{\mathcal{M}^*} X \subseteq M$$

So fix an $X \subseteq_0 M$ and put $X_a = X \cap A$. Let B be a finite set of black points in $N' \setminus B$. Then $\delta(\mathrm{cl}_{\mathcal{M}} X_a \cup B) \geq \delta(\mathrm{cl}_{\mathcal{M}} X_a)$ since $\mathcal{A} \leqslant \mathcal{N}'$. I.e., $\delta(B / \mathrm{cl}_{\mathcal{M}} X_a) \geq 0$ and so $\delta(B / \mathrm{cl}_{\mathcal{M}} X) = \delta(B / \mathrm{cl}_{\mathcal{M}} X_a) \geq 0$. But this means that $\mathrm{cl}_{\mathcal{M}^*} X = \mathrm{cl}_{\mathcal{M}} X \subseteq M$ as required. □

Lemma 3.10 $\mathbf{K}_\sigma$ *is an amalgamation class.*

Proof. Let $\mathcal{M}, \mathcal{N} \in \mathbf{K}_\sigma$ and let $f : \mathcal{M} \to \mathcal{N}$ be a morphism. We need to extend f to an embedding $F : \mathcal{M} \hookrightarrow \mathcal{N}' \geqslant \mathcal{N}$ where $\mathcal{N}' \in \mathbf{K}_\sigma$.

Put $X = \mathrm{dom}\, f$ and $Y = \mathrm{im}\, f$. Then, f extends to a partial isomorphism $f_0 : \mathrm{cl}_{\mathcal{M}} X \cong \mathrm{cl}_{\mathcal{N}} Y$. Put $\mathcal{A} = \mathrm{cl}_{\mathcal{N}} Y$. By isomorphic correction, we can find an $F_0 : \mathcal{M} \cong \mathcal{M}'$ extending f_0 s.t. $\mathcal{A} \leqslant \mathcal{M}'$ and $M' \cap N = A$. By Lemma 3.9, there is an $\mathcal{M}''$, an isomorphism $F_1 : \mathcal{M}' \cong_A \mathcal{M}''$, and an $\mathcal{N}' \in \mathbf{K}_\sigma$ s.t. $\mathcal{M}'' \leqslant \mathcal{N}'$ and $\mathcal{N} \leqslant \mathcal{N}'$. The morphism $F = F_1 F_0 : \mathcal{M} \to \mathcal{N}'$ is the required embedding. □

Proposition 3.11
(a) *There exists a strongly inaccessible generic model.*
(b) *There exists a countable generic model.*

Proof. (a) By Lemma 3.10, $\mathbf{K}$ is an inductive amalgamation class. By Corollary 1.11, there exists a generic model of strongly inaccessible cardinality.

(b) Again, by Lemma 3.10, $\mathbf{K}_0$ is an inductive amalgamation class. $\mathbf{K}_0$ is also obviously initial. So we need only establish that $\mathbf{K}_0$ is inductively bounded, that $I(\mathbf{K}_0) \leq \aleph_0$ and that for all $\mathcal{M}, \mathcal{N} \in \mathbf{K}_0$, $|\mathrm{Mor}(\mathcal{M}, \mathcal{N})| \leq \aleph_0$.

We first show $\mathbf{K}_0$ is inductively bounded. That is, we need to show that every countable $\mathcal{M} \in \mathbf{K}$ can be written as the union of a $\leqslant$-chain of finite dimensional models. To this end, fix an enumeration $\{a_i\}_{i<\omega}$ of M. Let A be any independent subset of M os size $\iota(T)$ and put $M_0 = \mathrm{cl}\, A$. Suppose the chain $\mathcal{M}_0 \leqslant \mathcal{M}_1 \leqslant \cdots \leqslant \mathcal{M}_n$ has already been chosen. If $M_n = M$ we can stop. If $M_n \neq M$, let a be the element with least index from $M \setminus M_n$ and put $M_{n+1} = \mathrm{cl}(M_n \cup \{a\})$.

Now for the number of isomorphism types. For each natural number $n \geq \iota(T)$, T has (up to isomorphism) exactly one model of dimension n which necessarily has cardinality $\aleph_0$. Each of these can be made into a model of T_0 in at most $\aleph_0$ ways since the interpretation of P must be finite. Hence T_0 has (up to isomorphism) at most $\aleph_0$ finite dimensional models.

Finally, we show that $|\mathrm{Mor}(\mathcal{M}, \mathcal{N}) \leq \aleph_0$. Let $\mathcal{M}, \mathcal{N}$ be finite dimensional models of T_0. Since every substructure of $\mathcal{M}$ is generated by a finite set, there are at most $\aleph_0$ substructures in $\mathcal{M}$. Similarly, there are at most $\aleph_0$ substructures in $\mathcal{N}$. Since morphisms are partial isomorphisms, $|\mathrm{Mor}(\mathcal{M}, \mathcal{N})|$ = the number of morphisms $f : \mathcal{M} \to \mathcal{N}$ with $\mathrm{dom}\, f$ and $\mathrm{im}\, f$ substructures. Hence $|\mathrm{Mor}(\mathcal{M}, \mathcal{N})| \leq \aleph_0$ as required. □

§6 Saturation

We will show that all generic models of T_0 are saturated. In order to prepare the way, we first need to analyse minimal extensions.

Define $\mathcal{M} < \mathcal{N}$ as $\mathcal{M} \leqslant \mathcal{N}$ but $\mathcal{M} \neq \mathcal{N}$. Further, define $\mathcal{M} <_{\min} \mathcal{N}$ if $\mathcal{M} < \mathcal{N}$ and there is no $\mathcal{M}_0$ s.t. $\mathcal{M} < \mathcal{M}_0 < \mathcal{N}$. If $\mathcal{M}$ and $\mathcal{N}$ are finite dimensional s.t. $\mathcal{M} < \mathcal{N}$, then there is a finite tower $\mathcal{M} = \mathcal{M}_0 <_{\min} \mathcal{M}_1 <_{\min} \cdots <_{\min} \mathcal{M}_n = \mathcal{N}$ (for if this tower was infinite, then $\mathcal{N}$ would have to have infinite dimension).

We distinguish three types of minimal extension $\mathcal{M} <_{\min} \mathcal{N}$.

Type 1: $N \setminus M \subseteq \neg P(\mathcal{N})$. Take any $a \in N \setminus M$ and put $M_0 = \mathrm{acl}(M \cup \{a\})$. As $N \setminus M$ is white, we have $\delta(M_0) = d_{\mathcal{N}}(M_0)$. I.e., $\mathcal{M}_0 \leqslant \mathcal{N}$ which implies that $\mathcal{M}_0 = \mathcal{N}$ since $\mathcal{M} <_{\min} \mathcal{N}$. So $\dim(N/M) = 1$ and $\delta(N/M) = 2$. This type of extension is called the *white generic extension.*

Type 2: $(N \setminus M) \cap P(\mathcal{N}) \neq \emptyset$ and $\delta(N/M) \neq 0$. This implies $\delta(N/M) > 0$ as $\mathcal{M} \leqslant \mathcal{N}$. Let $a \in (N \setminus M) \cap P(\mathcal{N})$ and put $M_0 = \mathrm{acl}(M \cup \{a\})$. Then $\delta(M) \leq \delta(M_0) \leq \delta(M) + 1$. $\delta(M_0) = \delta(M)$ implies that $\mathcal{M}_0 \leqslant \mathcal{N}$ and hence $\mathcal{M}_0 = \mathcal{N}$ by minimality. But this implies $\delta(N/M) = 0$ which is a contradiction. So we must have $\delta(M_0) = \delta(M) + 1$.

Suppose $M_0 \neq N$. Then there is an $X \subseteq N \setminus M_0$ s.t. $\delta(X \cup M_0) = \delta(M)$ since $\mathcal{M}_0 \not\leqslant \mathcal{N}$. So, by putting $M_1 = \mathrm{acl}(X \cup M_0)$ we obtain $\delta(M_1) = \delta(M)$ and $\mathcal{M}_1 \leqslant \mathcal{N}$. By minimality, $\mathcal{M}_1 = \mathcal{N}$. But this implies $\delta(N/M) = 0$ which is a contradiction.

So we have $\mathcal{M}_0 = \mathcal{N}$. Hence $\delta(N/M) = \dim(N/M) = 1$ and $N \setminus M$ contains exactly one black point. This extension is the *black generic extension.*

Type 3: $(N \setminus M) \cap P(\mathcal{N}) \neq \emptyset$ and $\delta(N/M) = 0$. Let $X = (N \setminus M) \cap P(\mathcal{N})$. We must have $N = \mathrm{acl}(M \cup X)$ since $\mathcal{M} <_{\min} \mathcal{N}$. Hence $\dim(X/M) = \dim(N/M) = |X|/2$ since $\delta(N/M) = 0$. So we also have $\delta(X/M) = 0$.

Partition X into two nonempty disjoint subsets X_0 and X_1. Suppose $\delta(X_0 \cup M) = \delta(M)$. Then, if we put $M_0 = \mathrm{acl}(X_0 \cup M)$, also $\delta(M_0) = \delta(M)$. But this means $\mathcal{M}_0 \leqslant \mathcal{N}$. As $X_1 \subseteq N \setminus M_0$ we have $\mathcal{M}_0 \neq \mathcal{N}$ which contradicts $\mathcal{M} <_{\min} \mathcal{N}$. So we must have $\delta(X_0 \cup M) > \delta(M)$. I.e., $\delta(X_0/M) > 0$. As $\delta(M) = \delta(X \cup M) = \delta(X_0 \cup X_1 \cup M)$, we also have $\delta(X_1/X_0 \cup M) < 0$.

Let $\mathcal{M} <_{\min} \mathcal{N}$ be a minimal extension of the third type. (We shall implicitly work in some monster model of T.) Let $\bar{a}$ be the set of black points in $N \setminus M$ with $|\bar{a}| = 2n$. Let $\bar{b}$ be a tuple containing a basis for $\mathcal{M}$ and all its black points. Define $\varphi(\bar{x}, \bar{y})$ as a formula s.t. $\varphi(\bar{a}, \bar{b})$ which expresses the following.

- The x_i's are distinct from each other and every y_j
- $\dim(\bar{x}/\bar{y}) \leq n$[1]
- For any non-trivial partition $\bar{x} = \bar{x}^1 \cup \bar{x}^2$, $\dim(\bar{x}^1/\bar{x}^2 \cup \bar{y}) < \ell(\bar{x}^1)/2$.

Now let $\psi(\bar{y})$ be a formula which defines the set

$$\{\bar{c} : \mathrm{RM}(\varphi(\bar{x}, \bar{c})) \geq n\}$$

where RM stands for *Rang de Morley.* That this set is indeed definable in a strongly minimal structure is explained in [10]. It is also explained there that in strongly minimal structures we have $\mathrm{RM}(\mathrm{tp}(\bar{c}/A)) = \dim(\bar{c}/A)$. Since $\dim(\bar{a}/\bar{b}) = n$, we have $\mathrm{RM}(\varphi(\bar{x}, \bar{b})) \geq \mathrm{RM}(\mathrm{tp}(\bar{a}/\bar{b})) = \dim(\bar{a}/\bar{b}) = n$. So we certainly have $\psi(\bar{b})$. Conversely, if $\bar{c} \subseteq C$ is a tuple s.t. $\psi(\bar{c})$, then there is a complete type $p(\bar{x})$ over C containing $\varphi(\bar{x}, \bar{c})$ s.t. $\mathrm{RM}(p) \geq n$. Hence there is a tuple $\bar{g}$ s.t. $\varphi(\bar{g}, \bar{c})$ and $\dim(\bar{g}/C) \geq n$. But $\dim(\bar{g}/C) \leq \dim(\bar{g}/\bar{c}) \leq n$. So $\dim(\bar{g}/C) = n$. We call $\bar{g}$ the *generic* realisation of $\varphi(\bar{x}, \bar{c})$ over C.

Let T_* be the set of sentences

$$\forall \bar{y}(\psi(\bar{y}) \rightarrow \exists \bar{x}(\varphi(\bar{x}, \bar{y}) \wedge \bigwedge P x_i))$$

for every pair of formulas (φ, ψ) as above.

The proof of the following lemma is essentially that of Poizat [11].

Lemma 3.12 *If $\mathcal{G}$ is a generic model, then $\mathcal{G} \models T_*$. In particular, T_* is consistent.*

[1]Specifically, we need

$$\bigwedge_{i=n+1}^{2n} (\exists_{k_i} x \psi_i(x, x_1, \ldots, x_{i-1}, \bar{y}) \wedge \psi_i(x_i, x_1, \ldots, x_{i-1}, \bar{y}))$$

where ψ_i is an algebraic formula isolating the L-type of a_i over $\{a_1, \ldots, a_{i-1}\} \cup M$ as in Lemma 3.5.

Proof. Let $\bar{c}$ be a tuple in G satisfying $\psi(\bar{c})$ and put $M = \mathrm{cl}_{\mathcal{G}}(\bar{c})$. Let $\bar{g}$ be the generic realisation of $\varphi(\bar{x}, \bar{c})$ over M. Assume without loss that $\bar{g}$ is outside G and let $\mathcal{N} = \mathrm{acl}(M \cup \bar{g})$. We add colour to $\mathcal{N}$ by painting $\bar{g}$ black and all points in $N \setminus (M \cup \bar{g})$ white. Clearly $\mathcal{M} <_{\min} \mathcal{N} \models T_0$ is a minimal extension of type 3 since $\dim(\bar{g}/M) = n$ and whenever we split $\bar{g}$ up into $\bar{g}^1 \cup \bar{g}^2$, $\delta(\bar{g}^1/M) > 0$. By the genericity of $\mathcal{G}$, there is an embedding $f : \mathcal{N} \hookrightarrow \mathcal{G}$ over M. Obviously $f(\bar{g})$ is black and $\mathcal{G} \models \varphi(f(\bar{g}), \bar{c})$. That is, $\mathcal{G} \models T_*$. □

Proposition 3.13 *Let $\mathcal{U}$ be a strongly inaccessible generic. Let $\mathcal{M} \equiv \mathcal{U}$ be an $\aleph_0$-saturated model satisfying $\mathcal{M} \leqslant \mathcal{U}$. Then $\mathcal{M} \preccurlyeq \mathcal{U}$.*

Proof. We need to establish the Tarski-Vaught test. To this end, let $\chi(x)$ be a formula over $A \subseteq_0 M$ s.t. $\mathcal{U} \models \chi(a)$ where $a \in U \setminus M$. Put $\mathcal{M}_0 = \mathrm{cl}_{\mathcal{U}} A$ and $\mathcal{N} = \mathrm{cl}_{\mathcal{U}}(M_0 \cup \{a\})$. Since $\mathcal{M} \leqslant \mathcal{U}$, we have $M_0 \subseteq M$. We will try to find an isomorphic copy $\mathcal{N}'$ of $\mathcal{N}$ over M_0 s.t. $\mathcal{N}' \subseteq \mathcal{M}$ and $\mathcal{N}' \leqslant \mathcal{U}$. By the homogeneity of $\mathcal{U}$, it will then follow that there is an automorphism $\sigma \in \mathrm{Aut}_{M_0}\mathcal{U}$ s.t. $\sigma(a) \in M$.

Note that we need only consider the case where $\mathcal{M}_0 <_{\min} \mathcal{N}$. For suppose that $\mathcal{M}_0 <_{\min} \mathcal{M}_1 <_{\min} \mathcal{N}$ and we have found an $\mathcal{M}_1' \subseteq \mathcal{M}$ and an isomorphism $f : \mathcal{M}_1 \cong_{M_0} \mathcal{M}_1' \leqslant \mathcal{U}$. Then there is a $\sigma \in \mathrm{Aut}_{M_0}\mathcal{U}$ extending f. Hence we get $\mathcal{M}_0 <_{\min} \mathcal{M}_1' <_{\min} \sigma[\mathcal{N}]$ and can continue our argument with the minimal extension $\mathcal{M}_1' <_{\min} \sigma[\mathcal{N}]$.

So let's assume first of all that $\mathcal{M}_0 <_{\min} \mathcal{N}$ is of type 3. Let $\bar{b}$ be a tuple containing a basis for M_0 and all its black points. Let $\bar{a}$ be the tuple of black points in $N \setminus M_0$. Let (φ, ψ) be the tuple from T_* corresponding to this situation. We clearly have $\mathcal{M} \models \psi(\bar{b})$. Since $\mathcal{M} \models T_*$, there must be a tuple of black points $\bar{d}$ in $M \setminus M_0$ s.t. $\varphi(\bar{d}, \bar{b})$. This implies that $\delta(\bar{d}/M_0) \leq 0$. However, since $\mathcal{M}_0 \leqslant \mathcal{U}$, we must have $\delta(\bar{d}/M_0) = 0$ and $\mathrm{acl}(\bar{d} \cup M_0)$ can contain no other black point. Hence, if we put $\mathcal{N}' = \mathrm{acl}(\bar{d} \cup M_0)$, there is an isomorphism $f : \mathcal{N} \cong_{M_0} \mathcal{N}' \subseteq \mathcal{M}$ s.t. $f(\bar{a}) = \bar{d}$. We also have $\mathcal{N}' \leqslant \mathcal{U}$ since $\mathcal{M}_0 \leqslant \mathcal{U}$ and $\delta(N'/M_0) = 0$.

Now let's consider the black generic extension. $\mathcal{M}$ has an isomorphic copy of the black generic extension over M_0 if and only if there is a black point $b \in M \setminus M_0$ s.t. $\mathrm{acl}(M_0 \cup \{b\}) \setminus (M_0 \cup \{b\})$ is white and no tuple of black points $\bar{d}$ outside $\mathrm{acl}(M_0 \cup \{b\})$ satisfies $\delta(\bar{d}/M_0 \cup \{b\}) < 0$.

Let $\bar{c}$ be a tuple containing a basis for M_0 and all its black points. We will define a set of sentences $\Phi(x)$ over $\bar{c}$ s.t. $\mathcal{M} \models \Phi(b)$ if and only if $\mathcal{M}_0 <_{\min} \mathcal{N}' = \mathrm{acl}(M_0 \cup \{b\})$ is of type 2.

Let $Q(x, \bar{y}, \bar{c})$ be the formula

$$\bigwedge_i P y_i \wedge \bigwedge_{i \neq j} y_i \neq y_j \wedge \bigwedge_{i,j} y_i \neq c_j \wedge \bigwedge_i y_i \neq x$$

If $\bar{y}$ is an n-tuple, we define

$$\bar{y}/2 = \begin{cases} () & \text{for } n = 1, 2 \\ (y_1, \ldots, y_{\lceil n/2 \rceil - 1}) & \text{for } n \geq 3 \end{cases}$$

Now define $\Gamma_{k,\varphi}(x)$ to be the formula

$$\forall \bar{y}(Q(x,\bar{y},\bar{c}) \wedge \exists_k z\varphi(z,x,\bar{y}/2,\bar{c}) \to \bigvee_{y\in\bar{y}\setminus\bar{y}/2} \neg\varphi(y,x,\bar{y}/2,\bar{c}))$$

The case $n = 1$ says that $\mathrm{acl}(M_0 \cup \{x\}) \setminus (M_0 \cup \{x\})$ is white. Clearly

$$\Phi(x) = \{Px \wedge \bigwedge x \neq c_i\} \cup \{\Gamma_{k,\varphi}(x) : \varphi \in L,\ k < \omega\}$$

is the desired set of sentences. We need to prove that $\Phi(x)$ is a type of $\mathcal{M}$. The $\aleph_0$-saturation of $\mathcal{M}$ will then imply that $\Phi(x)$ is realised in $\mathcal{M}$.

By ACP, we may assume that $N \setminus M_0$ is infinite. Fix a $\Psi(x) \subseteq_0 \Phi(x)$. Let b be the black point in $N \setminus M_0$ and paint another point b' in $N \setminus M_0$ black. We now get a minimal extension of the third type. So for any tuple $\bar{y}$ disjoint from N, $\Gamma_{k,\varphi}(b)$ is satisfied. That is, if $\bar{y}$ is a counter-example to one of the $\Gamma_{k,\varphi}(b)$'s in $\Psi(b)$, then $\bar{y}$ must include the point b'.

We'll show that the set of $b' \in N \setminus M_0$ which are included in a tuple $\bar{y}$ contradicting some $\Gamma_{k,\varphi}(b)$ in $\Psi(b)$ is finite.

Assume $\ell(\bar{y}) = 1$. Then $\bar{y} = b'$. Obviously the algebraic closure of $M \cup \{b\}$ restricted to the formulas in $\Psi(b)$ is finite. So this case is clear.

If $\ell(\bar{y}) = 2$, then we may assume without loss that $\bar{y} = (b', y)$ for some black point $y \notin N$. Hence y is not in the algebraic closure of $\bar{c} \cup \{b\}$ and $\bar{y}$ cannot be a counter-example to any $\Gamma_{k,\varphi}(b)$.

Now assume $\ell(\bar{y}) = n \geq 3$. Suppose that $b' \in \bar{y}/2$ and that $\varphi(z, b, \bar{y}/2, \bar{c})$ is the relevant algebraic counter-example. I.e., φ algebraises $\bar{y} \setminus \bar{y}/2$ over $\bar{y}/2 \cup \{b\} \cup \bar{c}$. Note that we can immediately rule out the case $b' = \bar{y}/2$. So we may assume that $n \geq 5$. Let $\bar{y}'$ be the result of removing b' from $\bar{y}$. Then it is easy to see that there must be a formula φ' algebraising $\bar{y}' \setminus \bar{y}'/2$ over $\bar{y}'/2 \cup \{b\} \cup \bar{c}$. But this is a contradiction.

Suppose now that $b' \in \bar{y} \setminus \bar{y}/2$ and that $\bar{y}$ is a counter-example to some $\Gamma_{k,\varphi}(b)$. If $\bar{y}/2$ is not independent over $\bar{c} \cup \{b\}$, then $\dim(\bar{y}/\bar{c} \cup \{b\}) < |\bar{y}|/2 - 1$. Again we let $\bar{y}'$ be the result of removing b' from $\bar{y}$. Then

$$\dim(\bar{y}'/\bar{c} \cup \{b\}) = \dim(\bar{y}/\bar{c} \cup \{b\}) < \frac{|\bar{y}|}{2} - 1 = \frac{|\bar{y}'| + 1}{2} - 1 = \frac{|\bar{y}'| - 1}{2}$$

But this implies that $\delta(\bar{y}'/N) < -1$ which is a contradiction. So we may assume that $\bar{y}/2$ is independent over $\bar{c} \cup \{b\}$. Note that any two such sets have the same L-type over $\mathrm{acl}(\bar{c} \cup \{b\})$. Hence, if $\bar{y}^1$ and $\bar{y}^2$ are two tuples of black points with $\bar{y}^1/2$ and $\bar{y}^2/2$ independent over $\bar{c} \cup \{b\}$, and $\varphi(z, b, \bar{y}^1/2, \bar{c})$ is algebraic, then so is $\varphi(z, b, \bar{y}^2/2, \bar{c})$ and for all $b' \in N \setminus M_0$, $\varphi(b', b, \bar{y}^1/2, \bar{c})$ iff $\varphi(b', b, \bar{y}^2/2, \bar{c})$. Hence there are only finitely many b''s s.t. $\varphi(b', b, \bar{y}/2, \bar{c})$. So this confirms that there can be only finitely many counter-examples in $N \setminus M_0$ to $\Psi(b)$.

Now, since we are assuming that $N \setminus M_0$ is infinite, we can paint a $b' \in N \setminus M_0$ black without contradicting any of the formulas in $\Psi(b)$. As our minimal extension is now of type 3, $\mathcal{M} \models T_*$ implies that there is an isomorphism $f : \mathcal{N} \cong_{M_0} \mathcal{N}' \leqslant \mathcal{M}$. Since

f extends to an automorphism of $\mathcal{U}$ we have $\mathcal{U} \models \Psi(f(b))$. But $\Psi(f(b))$ is universal, hence also $\mathcal{M} \models \Psi(f(b))$. So every finite subset of $\Phi(x)$ is satisfied in $\mathcal{M}$. By the $\aleph_0$-saturation of $\mathcal{M}$, $\Phi(x)$ is realised in $\mathcal{M}$ as required.

Finally, let us consider the white generic extension. Clearly we need to realise the set

$$\Phi'(x) = \{\neg Px\} \cup \{\exists_k z\varphi(z,\bar{c}) \rightarrow \neg\varphi(x,\bar{c}) : \varphi \in L,\ k < \omega\} \cup \{\Gamma_{k,\varphi}(x) : \varphi \in L,\ k < \omega\}$$

in $\mathcal{M}$. Fix a white point $a \in N \setminus M_0$ and fix a finite subset $\Psi(x)$ of $\Phi'(x)$. As with the black generic extension, we can paint a point $b' \in N \setminus M_0$ black without contradicting any of the formulas in $\Psi(a)$. As we now have a black generic extension, we get an isomorphism $f : \mathcal{N} \cong_{M_0} \mathcal{N}' \leqslant \mathcal{M}$. Hence $\mathcal{M} \models \Psi(f(a))$ and $\Phi'(x)$ is finitely satisfied in $\mathcal{M}$. □

Corollary 3.14 *All generic models are saturated.*

Proof. Let $\mathcal{U}$ be a strongly inaccessible generic. By Lemma 1.8, it suffices to show that $\mathcal{U}$ is saturated. To this end, let $p(x)$ be a type of $\mathcal{U}$ over a set of parameters A s.t. $|A| < |U|$. Then there is an $\aleph_0$-saturated model $\mathcal{M} \equiv \mathcal{U}$ with $|M| \leq |U|$ s.t. $A \subseteq M$ and $\mathcal{M} \models p(a)$ for some $a \in M$. By the genericity of $\mathcal{U}$, we may assume that $\mathcal{M} \leqslant \mathcal{U}$. Hence, by Proposition 3.13, $\mathcal{M} \preccurlyeq \mathcal{U}$. So $\mathcal{U} \models p(a)$. □

Concluding remarks

The existence of a saturated strongly inaccessible generic $\mathcal{U}$ provides a useful tool for establishing the stability of $\mathrm{Th}\mathcal{U}$: We need only count the types of $\mathcal{U}$.

Poizat [11] proves the following crucial lemma.

Lemma 4.1 *If $\bar{a} \subseteq U$ and $\mathcal{M} \leqslant \mathcal{U}$ with $|M| < |U|$, then* $\mathrm{cl}(M \cup \bar{a})$ *has finite dimension over M and contains just finitely many black points.* □

Let $\mathcal{M}$ be as in the hypothesis of the above lemma. In order to show that $\mathrm{Th}\mathcal{U}$ is ω-stable, we need to establish

$$|S_n(M)| = |M|$$

Given Lemma 3.7, $|S_n(M)| = |M|$ + the number of isomorphism types over M of substructures $\mathrm{cl}(M \cup \bar{a})$ where $\bar{a}$ is an n-tuple. So our task boils down to counting these isomorphism types.

For each $m = \dim(\mathrm{cl}(M \cup \bar{a})/M)$ there is one L-isomorphism type over M. So there are $\aleph_0$ L-isomorphism types over M in total. As each of these can contain only finitely many black points in $\mathrm{cl}(M \cup \bar{a}) \setminus M$, the number of *full* isomorphism types over M is $|\mathrm{cl}(M \cup \bar{a}) \setminus M| \cdot \aleph_0 = |M| \cdot \aleph_0 = |M|$. Hence $|S_n(M)| = |M| + |M| = |M|$ and $\mathrm{Th}\mathcal{U}$ is ω-stable.

Poizat [11] verifies that his generic structures have Morley rank $\omega \cdot 2$. Similar techniques can be applied to construct structures of Morley rank $\omega \cdot k$ for arbitrary k.

It is also possible to construct an ω-stable structure of Morley rank 2 (see [2]) but this is significantly trickier.

A study which is more in the spirit of this paper is that of identifying the general conditions under which a generic structure is ω-stable. This is an interesting and useful question. For more on this, I refer the reader to Wagner's [14].

There is one obvious question which emerges unanswered from part III.

> Is it possible to characterise the morphisms of part III as Δ-elementary maps for some set Δ? If so what does Δ look like?

A positive answer to this question would imply that the theory of the generics admits Δ-elimination since all generics are saturated. It would also mean that every saturated model of the generic theory is a generic model. I leave this question open.

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