Duals of subdirectly irreducible modal algebras

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Abstract

We give a characterization of the simple, and of the subdirectly irreducible boolean algebras with operators (including modal algebras), in terms of the dual descriptive frame. These characterizations involve a special binary *quasi-reachability* relation on the dual structure; we call a point u a quasi-root of the dual structure if every ultrafilter is quasi-reachable from u. We prove that a boolean algebra with operators is simple iff every point in the dual structure is a quasi-root; and that it is subdirectly irreducible iff the collection of quasi-roots has measure nonzero in the Stone topology on the dual structure.

1 Introduction

The duality theory between modal algebras, or more generally, boolean algebras with operators on the one hand, and (topological) frames or relational structures on the other, has been well developed, see for instance Sambin & Vaccaro [5] or Goldblatt [2]. Dualities allow the transfer, from one field to the other, of concepts, techniques and results alike. For instance, when it comes to the fundamental algebraic concept of *subdirect irreducibility*, there is a nice connection with the frame theoretical notion of *rootedness*. Without too much difficulty one can show that a Kripke frame is rooted iff its complex algebra is subdirectly irreducible (s.i.).

Unfortunately, this connection seems to be less nice when we look at arbitrary algebras: Sambin [4] gives examples of subdirectly irreducible modal algebras of which the dual ultrafilter frame is not rooted, and conversely, of non-s.i. modal algebras with a rooted dual frame. In the same paper, Sambin brings the Stone topology of the dual structure into the picture, showing that for any modal algebra $\mathbb{A}$:

$$\text{if } I_{\mathbb{A}_*} \text{ has measure nonzero, then } \mathbb{A} \text{ is s.i.} \tag{1}$$

($I_{\mathbb{A}_*}$ denotes the collection of roots, or initial points, of the dual frame $\mathbb{A}_*$; a set has measure nonzero iff it has a clopen subset.) He also proves that for **K4**-algebras the converse of (1) holds as well. In the closely related field of intuitionistic logic, similar characterizations of s.i. Heyting algebras in terms of their dual structures had been known for some time, cf. Esakia [1]. It is straightforward to verify that the the converse of (1) goes through for all *ω-transitive* logics, but the general picture, however, is not so nice: Kracht [3] provides an example of a subdirectly irreducible (in fact: simple) algebra $\mathbb{A}$ with $I_{\mathbb{A}_*} = \varnothing$. Kracht's result indicates that in the general case there is no simple characterization of s.i. algebras in terms of the *roots* of their dual frames.

In this paper we will show that a fairly transparent characterization of s.i. algebras is possible once we consider a new kind of relation in the dual structure. Given a modal algebra $\mathbb{A} = (A, \wedge, -, \bot, \Diamond)$, we define the following *quasi-reachability* relation $R^\star$ on ultrafilters:

$$R^\star uv \text{ iff for all } a \in v \text{ there is some } n \in \omega \text{ such that } \Diamond^n a \in u. \tag{2}$$

One may give various alternative characterizations of $R^\star$; for instance, it is not hard to prove that $R^\star uv$ iff v belongs to the *topological closure* of the subframe generated from u. This may explain why we call an ultrafilter a *quasi-root* of $\mathbb{A}_*$ if every ultrafilter is quasireachable from it.

Our characterization of subdirect irreducibility will be in terms of the collection $Q_{\mathbb{A}_*}$ of quasi-roots of the dual frame associated with an algebra $\mathbb{A}$. For modal algebras, we can formulate the main result of this paper, Theorem 2 below, as the following equivalence:

$$\mathbb{A} \text{ is s.i. iff } Q_{\mathbb{A}_*} \text{ is of measure nonzero,} \tag{3}$$

or, for readers that prefer a frame-theoretic formulation:

$$\mathbb{A} \text{ is s.i. iff there is a non-empty admissible set of quasi-roots in } \mathbb{A}_*.$$

In a similar way, we can characterize *simplicity*. For a modal algebra $\mathbb{A}$, Theorem 1 below states that $\mathbb{A}$ is simple iff each of its ultrafilters is a quasi-root of the dual structure:

$$\mathbb{A} \text{ is simple iff } Q_{\mathbb{A}_*} = A_*. \tag{4}$$

Our results will be formulated in the more general setting of boolean algebras with operators; for this purpose we will give a definition of the relation $R^\star$ which generalizes (2). Before going into the details, let us mention that a number of well known dual characterizations of subdirect irreducibility for special algebras, such as for ω-transitive ones, can easily be seen as special cases of our results.

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2 Preliminaries

Notation and terminology

We assume that the reader is familiar with basic concepts from universal algebra, with *boolean algebras with operators*, or BAOs for short, with *frames* and *Kripke frames*, and in particular, with the duality between the categories of BAOs and algebraic homomorphisms on the one hand, and descriptive frames with bounded morphisms on the other hand. Nevertheless, we review some of the terminology and notation.

First, let R be a binary relation on a set S. Inductively we define $R^0 = \{(s,s) \mid s \in S\}$ and $R^{n+1} = R \circ R^n$. The reflexive-transitive closure of R is denoted as R^ω. For a point s, define $R[s] = \{t \in S \mid Rst\}$. For a subset T of S, define $[R]T = \{s \in S \mid R[s] \subseteq T\}$ and $\langle R\rangle T = \{s \in S \mid Rst \text{ for some } t \in T\}$.

A boolean algebras with operators is denoted as $\mathbb{A} = (\mathbb{B}_\mathbb{A}, (f_i)_{i\in I})$ where $\mathbb{B}_\mathbb{A}$ is the boolean reduct of the algebra and the f_i are the operators; unary operators are called *diamonds*. We speak of *modal* algebras in the case that there is exactly one operator, and this operator is a diamond; such algebras are denoted $\mathbb{A} = (\mathbb{B}_\mathbb{A}, \Diamond)$. Inductively we define the notion of a *compound diamond*; first, given an n-ary operator ∇, define (for $1 \leq k \leq n$) its k-th *induced diamond* as the operation $\lambda x.\nabla(\top, \ldots, \top, x, \top, \ldots, \top) : A \to A$; (that is, all arguments are $\bot$ except for the i-th). The collection $CD(\mathbb{A})$ of compound diamonds is then defined as the smallest collection of operations containing the identity map, such that if $\blacklozenge_1$ and $\blacklozenge_2$ are in $CD(\mathbb{A})$, and $\Diamond$ is an induced diamond, then $\lambda x.\Diamond\blacklozenge_1 x$ and $\lambda x.\blacklozenge_1 x \vee \blacklozenge_2 x$ are compound diamonds. (For instance, in the case of a modal algebra $(\mathbb{B}, \Diamond)$, the compound diamonds are the maps of the form $\lambda x. \bigvee_{k\in K} \Diamond^k x$ for some finite $K \subseteq \omega$.) It is easy to see that compound diamonds are indeed diamonds, i.e., unary operations preserving all finite joins. The boolean dual map of a compound diamond $\blacklozenge$ is denoted by $\blacksquare$; that is, $\blacksquare a = -\blacklozenge - a$.

Given a BAO $\mathbb{A} = (\mathbb{B}_\mathbb{A}, (f_i)_{i\in I})$, the dual structure of $\mathbb{A}$ is denoted as $\mathbb{A}_* = (A_*, (R_i)_{i\in I}, \widehat{A})$. Here A_* is the set of ultrafilters of $\mathbb{A}$ (that is, of the boolean reduct $\mathbb{B}_\mathbb{A}$ of $\mathbb{A}$), the relations R_i are defined as usual, and $\widehat{A}$ is the image of the domain A of $\mathbb{A}$ under the Stone isomorphism $a \mapsto \widehat{a} := \{u \in A_* \mid a \in u\}$. We may use the notation $\mathbb{A}_+$ when referring to the underlying Kripke frame $(A_*, (R_i)_{i\in I})$ of $\mathbb{A}_*$; when it comes to concepts involving the accessibility relations we may be sloppy concerning the difference between $\mathbb{A}_+$ and $\mathbb{A}_*$. The set $\widehat{A}$ forms a basis of clopens for the Stone topology, and we will use standard facts concerning this topology without warning; a subset $X \subseteq A_*$ is said to have *measure nonzero* if it has a clopen (or, in frame terms: admissible) subset. The *closure* of a set $X \subseteq A_*$ is denoted as $\overline{X}$; recall that $\overline{X} = \bigcap\{\widehat{a} \mid a \in A, X \subseteq \widehat{a}\}$.

Given a frame $\mathbb{S} = (S, (R_i)_{i\in I})$, we define the *one step reachability relation* $R_\mathbb{S}$ as follows:

$$R_\mathbb{S} st \text{ iff for some } i \in I \text{ there are } t_1, \ldots, t_n \text{ such that } R_i s t_1 \ldots t_n \text{ and } t \text{ is one of } t_1, \ldots, t_n.$$

When no confusion arises we will write R instead of $R_\mathbb{S}$. The reflexive transitive closure $R^\omega_\mathbb{S}$ of $R_\mathbb{S}$ will be called the *reachability* relation of $\mathbb{S}$. A subset $X \subseteq S$ is *hereditary* if $s \in X$ and $R_\mathbb{S} st$ imply $t \in X$; obviously, $X \subseteq S$ is hereditary iff $R^\omega_\mathbb{S}[s] \subseteq X$ for all $s \in X$. Notice that the hereditary subsets of S correspond to the *generated subframes* of $\mathbb{S}$. An point s is called a *root* of $\mathbb{S}$ if every point is reachable from s; that is, if $R^\omega_\mathbb{S}[u] = S$. Given an algebra $\mathbb{A}$, the collection of roots of the dual structure is denoted as $I_{\mathbb{A}_*}$.

Without warning we will also employ the correspondence between (boolean) filters of $\mathbb{A}$ and closed subsets of A_*, and the correspondence between (i) congruences on $\mathbb{A}$, (ii) closed, hereditary subsets of A_* and (iii) modal filters on $\mathbb{A}$ (that is, boolean filters which are closed under induced boxes).

It is well known that subdirect irreducibility of a BAO can be characterized nicely using the notion of an *opremum* introduced by Rautenberg. An opremum of a BAO $\mathbb{A} = (\mathbb{B}_\mathbb{A}, (f_i)_{i\in I})$ is an element $o \in A$ such that $o < \top$ while for all $a \in A$ such that $a < \top$ we can find a compound diamond $\blacklozenge$ such that $o \geq \blacksquare a$. The characterization of s.i. BAOs in terms of oprema is given by the fact below (cf. KRACHT [3] for proofs and further information).

Fact 2.1 *The following are equivalent for any boolean algebra with operators $\mathbb{A}$:*

1. *$\mathbb{A}$ is subdirectly irreducible;*
2. *$\mathbb{A}$ has an opremum;*
3. *$\mathbb{A}_*$ has a largest nontrivial, closed and hereditary subset.*

Quasi-reachability

We can pretend that each compound diamond $\blacklozenge$ of a BAO $\mathbb{A}$ comes with its accessibility relation $R_\blacklozenge \subseteq A_* \times A_*$ given by $R_\blacklozenge uv$ iff $\blacklozenge a \in u$ for all $a \in v$. It is then easy to verify that the reachability relation R^ω is the union of the accessibility relations of all compound diamonds:

$$R^\omega = \bigcup_{\blacklozenge \in CD(\mathbb{A})} R_\blacklozenge .$$

In other words, R^ω can be characterized as follows:

$$R^\omega uv \text{ iff there is a compound diamond } \blacklozenge \text{ such that for all } a \in v \text{ we find } \blacklozenge a \in u. \quad (5)$$

Our definition of the *quasi*-reachability relation is obtained by swapping the universal and the existential quantifier in (5).

Definition 2.2 *Given a boolean algebra with operators $\mathbb{A}$, define the* quasi-reachability *relation $R^\star \subseteq A_* \times A_*$ as follows:*

$$R^\star uv \textit{ iff for all } a \in v \textit{ there is some compound diamond } \blacklozenge \textit{ such that } \blacklozenge a \in u. \quad (6)$$

We let $Q_{\mathbb{A}_*}$ denote the set of quasi-roots *of $\mathbb{A}_*$; that is, the collection of those ultrafilters u such that $R^\star[u] = A_*$.*

We leave it for the reader to verify that for a modal algebra $\mathbb{A} = (\mathbb{B}_\mathbb{A}, \Diamond)$, the above definition boils down to

$$R^\star uv \text{ iff for all } a \in v \text{ there is some } n \in \omega \text{ such that } \Diamond^n a \in u. \quad (2)$$

Example 2.3 It is instructive to have a somewhat closer look at Sambin's example of an s.i. algebra of which the dual Kripke frame is not rooted. First consider the frame $\mathbb{Z} = (Z, N, C)$ with Z the set of integers, N the neighbour relation (sNt iff $s = t+1$ or $s = t-1$), and C the collection of finite and cofinite subsets of Z. We leave it to the reader to verify that the algebra $\mathbb{C} = (C, \cup, -, \varnothing, \langle N \rangle)$ is subdirectly irreducible. The dual structure $\mathbb{C}_+$, based on the collection C_* of ultrafilters of $\mathbb{C}$, consists of (an isomorphic copy of) the structure (Z, N) together with a single ultrafilter ∞ containing all and only the cofinite subsets of Z. This point ∞ is reflexive, but not related to any other point, and hence, $\mathbb{C}_+$ has no roots at all.

On the other hand, using (2) the reader can easily verify that every principal ultrafilter of $\mathbb{C}$ is a *quasi-root* of $\mathbb{C}_*$.

The following proposition shows that the relation $R^\star$ has some interesting properties. In particular, it follows from item 5 below that for any ultrafilter u, the set $R^\star[u]$ is the topological closure of the subframe generated by u.

Proposition 2.4 *Let $\mathbb{A}$ be a boolean algebra with operators. The operation $R^\star$ satisfies the following properties:*

1. *$R^\star$ is transitive;*
2. *$R^\omega \subseteq R^\star$;*
3. *$R^\star[u]$ is hereditary for all ultrafilters u;*
4. *$R^\star[u]$ is closed for all ultrafilters u;*
5. *$R^\star[u] = \overline{R^\omega[u]}$ for all ultrafilters u.*

Proof. The first two items are almost immediate from the definitions and (5); taken together, they readily imply the third one. Concerning item 4, we leave it for the reader to verify that

$$R^\star[u] \;=\; \bigcap\{\widehat{a} \mid \blacksquare a \in u \text{ for all } \blacklozenge \in CD(\mathbb{A})\}. \tag{7}$$

Finally, in order to prove part 5 of the Proposition it suffices to show that $R^\star[u] \subseteq \overline{R^\omega[u]}$, since the opposite inclusion is immediate from the items 2 and 4.

Consider an arbitrary element $a \in A$ such that $R^\omega[u] \subseteq \widehat{a}$. We claim that $R^\star[u] \subseteq \widehat{a}$. Suppose for contradiction that $\blacksquare a \notin u$ for some compound diamond $\blacklozenge$. Hence, we obtain $\blacklozenge -a \in u$, so we can find some $v \in R_{\blacklozenge}[u]$ with $-a \in v$. Thus we find on the one hand, by definition of $R^\star$, that $R^\star uv$, while on the other hand $-a \in v$ implies $a \notin v$ and hence, $v \notin \widehat{a}$. Taking these facts together, we obtain the desired contradiction with our assumption that $R^\omega[u] \subseteq \widehat{a}$. Hence, we may assume that $\blacksquare a$ belongs to u for all compound diamonds $\blacklozenge$, so, by (7) we obtain $R^\star[u] \subseteq \widehat{a}$. This shows that

$$R^\star[u] \;\subseteq\; \bigcap\{\widehat{a} \mid R^\omega[u] \subseteq \widehat{a}\} \;= \overline{R^\omega[u]},$$

since a was arbitrary. QED

Recall that, for a BAO $\mathbb{A}$, the map $[R^\star] : \mathcal{P}(A_*) \to \mathcal{P}(A_*)$ is given by $[R^\star]U = \{s \in A_* \mid R^\star[s] \subseteq U\}$. This map has some nice properties that we list for future use.

Proposition 2.5 *Let $\mathbb{A}$ be a boolean algebra with operators. For any $a \in A$, we have that*

$$[R^\star]\widehat{a} \;=\; \bigcap_{\blacklozenge \in CD(\mathbb{A})} \widehat{\blacksquare a}. \tag{8}$$

Hence, $[R^\star]C$ is closed for an arbitrary closed set $C \subseteq A_*$.

Proof. Fix an element a of the BAO $\mathbb{A}$. First assume that u is an ultrafilter that belongs to $\widehat{\blacksquare a}$ for all compound diamonds $\blacklozenge$. That is, we have $\blacksquare a \in u$ for all $\blacklozenge$. Now consider an arbitrary ultrafilter v such that $R^\star uv$. By (7) we find that $a \in v$, whence $v \in \widehat{a}$. Since v was arbitrary, this gives that $u \in [R^\star]\widehat{a}$, and thus shows that $[R^\star]\widehat{a} \supseteq \bigcap_{n<\omega} \widehat{\blacksquare a}$.

For the other inclusion, suppose that u does *not* belong to the right hand side of (8). Then for some compound diamond $\blacklozenge$ we have that $\blacksquare a \notin u$; that is, $\blacklozenge{-a} \in u$. Thus we can find an ultrafilter v such that $R_\blacklozenge uv$ and $-a \in v$. Clearly then, by Proposition 2.4.2 we have that $R^\star uv$ and $v \notin \widehat{a}$, revealing that u does not belong to the left hand side of (8) either.

The second part of the Proposition is immediate from the first part and the observation that any map $[R]$ distributes over arbitrary intersections. QED

3 Results

Our first main result characterizes the simple algebras as the ones of which the dual frame is quasi-generated from each point:

Theorem 1 *Let $\mathbb{A}$ be a boolean algebra with operators. Then $\mathbb{A}$ is simple if and only if $Q_{\mathbb{A}_*} = A_*$.*

Proof. For the direction from left to right, we leave it for the reader to verify that if $s \notin Q_{\mathbb{A}_*}$, then $R^\star[s] \neq A_*$ corresponds to a non-trivial filter of $(\mathbb{A}_*)^* \cong \mathbb{A}$. This shows that if not every ultrafilter of $\mathbb{A}$ is a quasi-root of $\mathbb{A}_*$, then $\mathbb{A}$ is not simple.

For the other direction, suppose that $\mathbb{A}$ is not simple. Then $\mathbb{A}_*$ has a closed, hereditary subset $B \neq A_*$. Take an arbitrary point $s \in B$. Then $R^\omega[s] \subseteq B$ since B is hereditary, whence $R^\star[s] = \overline{R^\omega[s]} \subseteq B$ since B is closed. It follows that $R^\star[s] \neq A_*$, so s is not a quasi-root of $\mathbb{A}_*$. QED

Our second result gives a similar characterization of the subdirectly irreducible algebras.

Theorem 2 *Let $\mathbb{A}$ be a boolean algebra with operators. Then the following are equivalent:*

1. *$\mathbb{A}$ is subdirectly irreducible*
2. *$Q_{\mathbb{A}_*}$ is open and non-empty*
3. *$Q_{\mathbb{A}_*}$ has measure non-zero*
4. *there is an admissible set of quasi-roots in $\mathbb{A}_*$.*

Proof. In this proof we let $K_{\mathbb{A}_*} = A_* \setminus Q_{\mathbb{A}_*}$ denote the complement of $Q_{\mathbb{A}_*}$. The equivalence of (2), (3) and (4) is standard, so it is left to prove that the first two statements are equivalent.

For the implication ($1 \Rightarrow 2$), assume that $\mathbb{A}$ is s.i., then by Fact 2.1 we may assume that $\mathbb{A}$ has an opremum c.

We will first prove that $\mathbb{A}_+$ is quasi-rooted. Take an arbitrary point $u \in K_{\mathbb{A}_*}$; that is, we have that $R^\star[u] \neq A_*$, and since $R^\star[u]$ is closed, there must be a clopen $\widehat{a} \neq A_*$ with $R^\star[u] \subseteq \widehat{a}$. Then from (5) and $R^\omega[u] \subseteq R^\star[u] \subseteq \widehat{a}$ it follows that u belongs to $\widehat{\blacksquare a}$ for all compound diamonds $\blacklozenge$. From $\widehat{a} \neq A_*$ we infer that $a \neq \top$; so the fact that c is an opremum implies that $\bigcap\{\widehat{\blacksquare a} \mid \blacklozenge \in CD(\mathbb{A})\} \subseteq \widehat{c}$. Thus we find that $u \in \widehat{c}$. Since u was an arbitrary element of $K_{\mathbb{A}_*}$, this shows that

$$K_{\mathbb{A}_*} \subseteq \widehat{c}. \tag{9}$$

But c is an opremum, and hence, *smaller* than the top element of $\mathbb{A}$. This means that $K_{\mathbb{A}_*}$ is a *proper* subset of A_*, and thus its complement $Q_{\mathbb{A}_*}$ is non-empty.

Our second aim is to prove that $Q_{\mathbb{A}_*}$ is open, or, equivalently, that $K_{\mathbb{A}_*}$ is closed. We will first show that

$$K_{\mathbb{A}_*} = [R^\star]K_{\mathbb{A}_*}. \tag{10}$$

By reflexivity of $R^\star$ it is immediate that $[R^\star]K_{\mathbb{A}_*} \subseteq K_{\mathbb{A}_*}$. For the other inclusion, suppose that u belongs to $K_{\mathbb{A}_*}$, and take an arbitrary ultrafilter v such that $R^\star uv$. Suppose for contradiction that v is a quasi-root of $\mathbb{A}_*$, that is, suppose that every ultrafilter is $R^\star$-reachable from v. From this and the results in Proposition 2.4 it would follow immediately that u is a quasi-root of $\mathbb{A}_*$ as well, which contradicts the fact that $u \in K_{\mathbb{A}_*}$. It follows that no $R^\star$-successor of u can be a quasi-root of u; in other words, we see that $R^\star[u] \subseteq K_{\mathbb{A}_*}$, and thus that $u \in [R^\star]K_{\mathbb{A}_*}$. This proves (10).

Now we claim that in fact,

$$K_{\mathbb{A}_*} = [R^\star]\widehat{c}. \tag{11}$$

For the inclusion from left to right, first observe that it follows from (10), (9) and the monotonicity of the operation $[R^\star]$, that $K_{\mathbb{A}_*} = [R^\star]K_{\mathbb{A}_*} \subseteq [R^\star]\widehat{c}$. In order to establish the converse inclusion of (11), consider an arbitrary point u in $[R^\star]\widehat{c}$ and suppose, for contradiction, that u does *not* belong to $K_{\mathbb{A}_*}$. That is, u is a quasi-root of $\mathbb{A}_*$, so, we have that $R^\star uv$ for *every* ultrafilter v. But then $u \in [R^\star]\widehat{c}$ implies that every ultrafilter v belongs to $\widehat{c}$. This gives the desired contradiction with the fact that $\widehat{c}$, being an opremum of $(\mathbb{A}_*)^*$, must be a *proper* subset of A_*. Thus we find that indeed, $[R^\star]\widehat{c} \subseteq K_{\mathbb{A}_*}$, and we have proved (11).

Finally, observe that it immediately follows from (11) and Proposition 2.5 that $K_{\mathbb{A}_*}$ is closed.

For the converse implication, i.e., $(1 \Rightarrow 2)$, assume that $Q_{\mathbb{A}_*}$ is open and non-empty. It is not difficult to see that this implies that $K_{\mathbb{A}_*}$ is a nontrivial closed and hereditary subset of A_*. We claim that it is in fact the largest such set.

For, let $J \subset A_*$ be closed and hereditary. Suppose for contradiction that J is not contained in $K_{\mathbb{A}_*}$, then there is an ultrafilter $u \in J \setminus K_{\mathbb{A}_*}$. From $u \in J$ and the assumptions on J it easily follows that $R^\star[u] \subseteq J$. But then $R^\star[u]$ is a proper subset of A_*; from this we infer that u is not a quasi-root of $\mathbb{A}_*$; that is, we find $u \in K_{\mathbb{A}_*}$. This shows that J is a subset of $K_{\mathbb{A}_*}$ after all.

Hence, $K_{\mathbb{A}_*}$ is indeed the *largest* nontrivial, closed and hereditary subset of A_*. But then it follows from Fact 2.1 that $\mathbb{A}$ is s.i. QED

As corollaries of the last theorem we obtain some well known results showing that in some special cases, nicer characterizations are indeed possible.

We call a boolean algebra with operators *ω-transitive* if there is some compound diamond $\Diamond$ such that $\blacklozenge a \leq \Diamond a$ for all compound diamonds $\blacklozenge$ and all a in $\mathbb{A}$. (With some authors, this property goes under the name of *weak* transitivity). Recall that for an algebra $\mathbb{A}$, we let $I_{\mathbb{A}_*}$ denote the collection of roots of the dual structure.

Corollary 3.1 *Let $\mathbb{A}$ be a ω-transitive boolean algebra with operators. Then $\mathbb{A}$ is simple iff $I_{\mathbb{A}_*} = A_*$, and subdirectly irreducible iff $I_{\mathbb{A}_*}$ is non-empty and open iff there is an admissible set of roots in $\mathbb{A}_*$.*

Proof. Suppose that $\Diamond$ is a compound diamond of $\mathbb{A}$ such that $\blacklozenge a \leq \Diamond a$ for all compound diamonds $\blacklozenge$ and all a in $\mathbb{A}$. It is easy to verify that in this case we have $R^{\omega} = R_{\Diamond}$; but since $R_{\Diamond}[u]$ is closed for every ultrafilter u, it follows from Proposition 2.4.5 that $R^{\star} = R_{\Diamond} = R^{\omega}$. This means that $I_{\mathbb{A}_*} = Q_{\mathbb{A}_*}$, or in words: the roots and the quasi-roots of $\mathbb{A}_*$ *coincide*. Thus the results follows immediate from the Theorems 1 and 2. QED

Corollary 3.2 *Let $\mathbb{A}$ be a finite boolean algebra with operators. Then $\mathbb{A}$ is subdirectly irreducible iff $\mathbb{A}_*$ is rooted.*

Proof. It is easy to see that finite BAOs are ω-transitive. Hence, the result follows from Corollary 3.1 and the observation that if $\mathbb{A}$ is finite then any subset of A_* is open. QED

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