TOPOLOGICAL AND LOGICAL EXPLORATIONS OF KRULL DIMENSION

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Abstract. Krull dimension measures the depth of the spectrum \( \text{Spec}(R) \) of a commutative ring \( R \). Since \( \text{Spec}(R) \) is a spectral space, Krull dimension can be defined for spectral spaces. Utilizing Stone duality, it can also be defined for distributive lattices. For an arbitrary topological space, the notion of Krull dimension is less useful. Isbell [23] remedied this by introducing the concept of graduated dimension. In this paper we propose an alternate concept, that of localic Krull dimension of a topological space, which has its roots in modal logic. This is done by investigating the concept of Krull dimension for closure algebras and Heyting algebras, which formalize the notions of powerset and open set algebras of topological spaces [30, 31, 35]. We compare localic Krull dimension to other well-known dimension functions, and show that it can detect topological differences between topological spaces that Krull dimension is unable to detect. We also investigate applications of localic Krull dimension to modal logic. We prove that for a \( T_1 \)-space to have a finite localic Krull dimension can be described by an appropriate generalization of the well-known concept of a nodec space. These considerations yield topological completeness and incompleteness results in modal logic that we examine in detail.

1. Introduction

For a commutative ring \( R \), let \( \text{Spec}(R) \) be the set of prime ideals of \( R \) topologized with the Zariski topology. As usual, we refer to \( \text{Spec}(R) \) as the spectrum of \( R \). The Krull dimension of \( R \) is defined as the supremum of the lengths of chains in \((\text{Spec}(R), \subseteq)\). This notion is of fundamental importance in commutative algebra and algebraic geometry (see, for example, [14, Ch. 8]). Since \( \text{Spec}(R) \) is a spectral space, where the inclusion on prime ideals is the specialization order of the Zariski topology, we can define the Krull dimension of a spectral space \( X \) as the supremum of the lengths of chains in the specialization order of \( X \). By Stone duality [36], spectral spaces are dual to distributive lattices, which paves the way to defining the Krull dimension of a distributive lattice \( L \) as the supremum of the lengths of chains in \((\text{Spec}(L), \subseteq)\), where \( \text{Spec}(L) \) is the Stone dual of \( L \). For different characterizations of the Krull dimension of distributive lattices see [4, 18, 19, 7, 8] and the references therein. If a distributive lattice \( L \) is a frame or locale (that is, a topological space without points), then a convenient characterization of the Krull dimension of \( L \) can be found in [2, Thm. 6.9].

If we define the Krull dimension of an arbitrary topological space \( X \) by means of the specialization order of \( X \), then to quote Isbell [23], the result is “spectacularly wrong for the most popular spaces, vanishing for all non-empty Hausdorff spaces; but it seems to be the only dimension of interest for the Zariski spaces of algebraic geometry.” Isbell remedied this by proposing the definition of graduated dimension. In this article we propose a different approach. We define the localic Krull dimension of a topological space \( X \) as the Krull dimension of the locale \( \Omega(X) \) of all open subsets of \( X \). As we will see, this definition turns out to be more refined. For example, every nonempty Stone space has Krull dimension and

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graduated dimension 0. On the other hand, for each \( n \) (including \( \infty \)), there is a Stone space \( X \) such that the localic Krull dimension of \( X \) is \( n \). Thus, localic Krull dimension provides more refined classification of Stone spaces, and this extends to spectral spaces and beyond.

Instead of working with the locale \( \Omega(X) \), we could work with the powerset algebra \( \wp(X) \) equipped with the closure operator or its dual interior operator. Then the locale \( \Omega(X) \) can be realized as the fixed points of the interior operator. This line of research was developed by McKinsey and Tarski in their classic paper [30]. It gives rise to the concept of a closure algebra, which is a pair \( \mathfrak{A} = (A, C) \), where \( A \) is a Boolean algebra and \( C : A \to A \) is a unary function satisfying Kuratowski’s axioms for closure.

In this article we introduce and develop the theory of Krull dimension for closure algebras. We prove that the Krull dimension of a closure algebra coincides with the Krull dimension of its open elements. Thus, the localic Krull dimension of a topological space \( X \) can alternatively be defined as the Krull dimension of the closure algebra \((\wp(X), C)\).

This approach has a number of logical applications. Closure algebras serve as algebraic models of Lewis’ well-known modal logic \( S4 \) (see, e.g., [35]). For each \( n \geq 1 \), there is a modal formula \( \varphi_n \) that is satisfied in a closure algebra \( \mathfrak{A} \) iff the Krull dimension of \( \mathfrak{A} \) is \( \leq n \). Therefore, for each \( n \geq 1 \), adding \( \varphi_n \) to \( S4 \) yields the modal logic \( S4\!,n \), which is the logic of all closure algebras of Krull dimension \( \leq n \). This yields that \( S4\!,n \) is the logic of all topological spaces whose localic Krull dimension is \( \leq n \).

We generalize the well-known concept of a nodec space to that of an \( n \)-discrete space, and prove that if \( X \) is a \( T_1 \)-space, then the localic Krull dimension of \( X \) is \( \leq n \) iff \( X \) is \( n \)-discrete. As was shown in [1], the modal logic of nodec spaces is the well-known Zeman logic \( S4\!\!.Z \). For each \( n \geq 1 \), we generalize the Zeman logic \( S4\!\!.Z \) to the \( n \)-Zeman logic \( S4\!,Z_n \), and show that \( S4\!,Z_n \) is a proper extension of \( S4\!,n \). From this we derive that \( S4\!,n \) is topologically incomplete for any class of \( T_1 \)-spaces. In fact, we show that no logic in the interval \([S4\!,n, S4\!,Z_n]\) is the logic of any class of \( T_1 \)-spaces. On the other hand, for each \( n \geq 1 \), we construct a countable crowded \( \omega \)-resolvable Tychonoff space \( Z_n \) of localic Krull dimension \( n \) such that \( S4\!,Z_n \) is the logic of \( Z_n \).

The article is organized as follows. Section 2 recalls the basic definitions, notation, and terminology used for closure algebras and Heyting algebras, as well as some connections between these classes of algebras. We also review Esakia duality for these classes of algebras. In Section 3 we define the Krull dimension of a closure algebra in two ways, externally and internally. The main result of Section 3 shows that the two definitions are equivalent, and also demonstrates multiple equivalent conditions characterizing Krull dimension of a closure algebra. In Section 4 we define external and internal Krull dimension of Heyting algebras, and show that they are equivalent. We connect the Krull dimension of Heyting algebras with that of closure algebras, and exhibit equivalent conditions characterizing the Krull dimension of a Heyting algebra.

In Section 5 we introduce the localic Krull dimension of an arbitrary topological space \( X \), and provide equivalent conditions characterizing finite (nonnegative) localic Krull dimension of \( X \), some of which require us to develop a topological version of the Jankov-Fine formulas [24, 20]. We also compare localic Krull dimension to other well-known dimension functions. In Section 6 we generalize the notion of a discrete closure algebra to that of an \( n \)-discrete closure algebra, and characterize dually when a closure algebra is \( n \)- discrete. We also generalize the well-known Zeman formula to a family of formulas we call the \( n \)-Zeman formulas \( \text{zem}_n \), and show that \( \text{zem}_n \) characterizes \( n \)-discrete closure algebras. We conclude the section by showing that each logic \( S4\!,Z_n \) obtained by adding \( \text{zem}_n \) to \( S4 \) has the finite model property. In Section 7 we introduce \( n \)-discrete topological spaces. We prove that being \( n \)-discrete is a topological property that is defined by \( \text{zem}_n \), and show that \( S4\!,Z_n \) is the logic of \( n \)-discrete
spaces. In addition, we prove that a $T_1$-space is $n$-discrete iff its localic Krull dimension is $\leq n$, thus yielding topological incompleteness results. The section culminates by showing that each logic $S4.Z_n$ is the logic of a single countable crowded $\omega$-resolvable Tychonoff space of localic Krull dimension $n$.

2. Closure algebras and Heyting algebras

**Definition 2.1.** A closure operator on a Boolean algebra $A$ is a unary function $C: A \to A$ satisfying Kuratowski’s axioms:

- $a \leq C(a)$;
- $CC(a) \leq C(a)$;
- $C(a \lor b) = C(a) \lor C(b)$;
- $C(0) = 0$.

A closure algebra is a pair $\mathfrak{A} = (A, C)$, where $A$ is a Boolean algebra and $C$ is a closure operator on $A$. As usual, if $C$ is a closure operator on $A$, then its dual interior operator is defined by $I(a) = -C(-a)$ for each $a \in A$.

A typical example of a closure algebra is the powerset algebra of a topological space $X$; that is, the pair $\mathfrak{A}_X = (\mathcal{P}(X), C_X)$, where $\mathcal{P}(X)$ is the powerset of $X$ and $C_X$ is closure in $X$. By the McKinsey-Tarski representation theorem [30], every closure algebra is represented as a subalgebra of $\mathfrak{A}_X$ for some topological space $X$.

There is another representation of closure algebras due to Jónsson-Tarski [25] and Kripke [28], which is central to modal logic. Recall that a Kripke frame is a pair $\mathfrak{F} = (W, R)$, where $W$ is a set and $R$ is a binary relation on $W$. Kripke frames provide relational semantics of modal logic [3, 6]. Those Kripke frames where the relation is reflexive and transitive provide relational semantics of Lewis’ well-known modal system $S4$. This is why Kripke frames with reflexive and transitive relations are often referred to as $S4$-frames. Given an $S4$-frame $\mathfrak{F} = (W, R)$ and $A \subseteq W$, let

$$R^{-1}[A] = \{w \in W \mid \exists a \in A \text{ with } wRa\}.$$ 

Then the powerset algebra $\mathfrak{A}_\mathfrak{F} = (\mathcal{P}(W), C_R)$ is a closure algebra, where $C_R(A) = R^{-1}[A]$. Moreover, every closure algebra is represented as a subalgebra of $\mathfrak{A}_\mathfrak{F}$ for some $S4$-frame $\mathfrak{F}$.

There is a close connection between the McKinsey-Tarski and Kripke representations. Suppose $\mathfrak{F} = (W, R)$ is an $S4$-frame. Call $U \subseteq W$ and $R$-upset if $w \in U$ and $wRv$ imply $v \in U$ ($R$-downsets are defined dually). Let $\tau_R$ be the collection of all $R$-upsets of $\mathfrak{F}$. Then $\tau_R$ is a topology on $W$ in which every $w \in W$ has a least open neighborhood

$$R[w] := \{v \in W \mid wRv\}.$$ 

Such topological spaces are called Alexandroff spaces, and can alternatively be described as the topological spaces in which intersections of arbitrary families of opens are open. Thus, $S4$-frames correspond to Alexandroff spaces.

In [16] Esakia put together Stone duality for Boolean algebras with Kripke representation of closure algebras to obtain a full duality for closure algebras. By Esakia duality, the category of closure algebras is dually equivalent to the category of Esakia spaces (aka descriptive $S4$-frames).

**Definition 2.2.** A Stone space is a zero-dimensional compact Hausdorff space, and an Esakia space is an $S4$-frame $\mathfrak{F} = (W, R)$ such that $W$ is equipped with a Stone topology satisfying

- $R[w]$ is closed;
- $U$ clopen implies $R^{-1}[U]$ is clopen.
The dual Esakia space of a closure algebra $\mathfrak{A}$ is the pair $\mathfrak{A}_* = (W, R)$, where $W$ is the Stone space of $A$ and

$$wrv \iff (\forall a \in \mathfrak{A})(a \in v \Rightarrow C(a) \in w).$$

The dual closure algebra of an Esakia space $\mathfrak{F} = (W, R)$ is the closure algebra $\mathfrak{F}^* = (\text{Clop}(W), C_R)$, where $\text{Clop}(W)$ is the Boolean algebra of clopen subsets of $W$ and

$$C_R(U) = R^{-1}[U]$$

and hence $I_R(U) = W \setminus R^{-1}[W \setminus U]$.

Then $\beta : \mathfrak{A} \to \mathfrak{A}_*$ and $\varepsilon : \mathfrak{E} \to \mathfrak{E}_*$ are isomorphisms, where

$$\beta(a) = \{w \in W \mid a \in w\}$$

and $\varepsilon(w) = \{U \in \text{Clop}(W) \mid w \in U\}$.

In the finite case, the topology on an Esakia space becomes discrete, and we identify finite Esakia spaces with finite $\mathbf{S}_4$-frames.

Let $\mathfrak{A}$ be a closure algebra and $\mathfrak{A}_* = (W, R)$ be the Esakia space of $\mathfrak{A}$. As is customary, we adopt topological terminology and call $a \in \mathfrak{A}$ closed if $a = C(a)$, open if $a = I(a)$, dense if $C(a) = 1$, and nowhere dense if $IC(a) = 0$. The following is well known (and easy to see):

- $a$ is closed iff $\beta(a)$ is a clopen $R$-downset in $\mathfrak{A}_*$;
- $a$ is open iff $\beta(a)$ is a clopen $R$-upset in $\mathfrak{A}_*$;
- $a$ is dense iff $C_R \beta(a) = W$;
- $a$ is nowhere dense iff $I_R C_R \beta(a) = \emptyset$.

The relativization of $\mathfrak{A}$ to $a \in \mathfrak{A}$ is the closure algebra $\mathfrak{A}_a$ whose underlying set is the interval $[0, a]$, the meet and join in $\mathfrak{A}_a$ coincide with those in $\mathfrak{A}$, the complement of $b \in \mathfrak{A}_a$ is given by $a - b$, and the closure of $b$ is given by $C_a(b) = a \cup C(b)$. It follows that the interior of $b \in \mathfrak{A}_a$ is given by $I_a(b) = a \cap I(a \to b)$. If $\mathfrak{A} = \mathfrak{A}_X$ is the powerset algebra of a topological space $X$ and $Y \subseteq X$, then the relativization of $\mathfrak{A}$ to $Y$ is the powerset algebra $\mathfrak{A}_Y$ of the subspace $Y$ of $X$.

The relativization $\mathfrak{A}_a$ is realized dually as the restriction of $R$ to the powerset algebra $\mathfrak{A}_a$. In order to describe dually a connection between nowhere dense elements and relativizations, we recall the notion of an $R$-maximal point.

**Definition 2.3.** Let $\mathfrak{F} = (W, R)$ be an $\mathbf{S}_4$-frame, $U \subseteq W$, and $w \in U$. Then $w$ is an $R$-maximal point of $U$ provided $wRu$ and $u \in U$ imply $uRw$. We denote the set of $R$-maximal points of $\mathfrak{F}$ by $\max_R(\mathfrak{F})$.

It is well known (see, e.g., [17, Sec. III.2]) that in an Esakia space $\mathfrak{F} = (W, R)$, the set $\max_R(\mathfrak{F})$ is a closed $R$-upset, and for each $w \in W$ there is $v \in \max_R(\mathfrak{F})$ such that $wRv$.

**Lemma 2.4.** Let $\mathfrak{A}$ be a closure algebra and $\mathfrak{A}_*$ be its Esakia space. Suppose $a \in \mathfrak{A}$ and $d \in \mathfrak{A}_a$. Then $d$ is nowhere dense in $\mathfrak{A}_a$ iff $\beta(d) \cap \max_Q \beta(a) = \emptyset$, where $Q$ is the restriction of $R$ to $\beta(a)$.

**Proof.** Since $\mathfrak{A}_a$ is an Esakia space and $\beta(a)$ is clopen in $\mathfrak{A}_*$, it is well known (see, e.g., [17, Sec. III.2]) that $\mathfrak{F} = (\beta(a), Q)$ is also an Esakia space. As $\max_Q \beta(a)$ is a $Q$-upset of $\beta(a)$, the condition $\beta(d) \cap \max_Q \beta(a) = \emptyset$ is equivalent to $C_Q [\beta(d)] \cap \max_Q \beta(a) = \emptyset$, which in turn is equivalent to $I_Q C_Q [\beta(d)] \cap \max_Q \beta(a) = \emptyset$. Since $I_Q C_Q [\beta(d)]$ is a $Q$-upset of $\beta(a)$, the last condition is equivalent to $I_Q C_Q [\beta(d)] = \emptyset$. Therefore, $\beta(d) \cap \max_Q \beta(a) = \emptyset$ iff $\beta(I_a C_a d) = \emptyset$, which is equivalent to $d$ being nowhere dense in $\mathfrak{A}_a$. \hfill $\Box$

We next turn to Heyting algebras, which are closely related to closure algebras [31, 35].

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1. Despite subscript being used to denote both a relativization of a closure algebra $\mathfrak{A}$ and the powerset algebra of a space $X$, there is no ambiguity when $\mathfrak{A} = \mathfrak{A}_X$ because $(\mathfrak{A}_X)_Y = \mathfrak{A}_Y$. 

**Definition 2.5.** A Heyting algebra is a bounded implicative lattice; that is, a bounded distributive lattice such that $\land$ has a residual $\to$ satisfying

$$x \leq a \to b \iff a \land x \leq b.$$ 

If $\mathfrak{A}$ is a closure algebra, then $\mathfrak{H}(\mathfrak{A}) := \{1a \mid a \in \mathfrak{A}\}$ is a Heyting algebra. Conversely, if $\mathfrak{H}$ is a Heyting algebra, then the free Boolean extension $\mathfrak{B}(\mathfrak{H})$ of $\mathfrak{H}$ can be equipped with a closure operator $C$ so that $\mathfrak{A}(\mathfrak{H}) := (\mathfrak{B}(\mathfrak{H}), C)$ is a closure algebra, $\mathfrak{H}$ is isomorphic to $\mathfrak{H}(\mathfrak{A}(\mathfrak{H}))$, and $\mathfrak{A}(\mathfrak{H}(\mathfrak{A}))$ is isomorphic to a subalgebra of $\mathfrak{A}$ (see, e.g., [35, Sec. IV.1 and IV.3] or [17, Sec. II.2 and II.5]).

As with closure algebras, a typical example of a Heyting algebra is the Heyting algebra $\mathfrak{H}_X$ of all opens of a topological space $X$, and every Heyting algebra is represented as a subalgebra of $\mathfrak{H}_X$ for some topological space $X$ [31, 35].

Another representation of Heyting algebras is by means of $R$-upsets of $S4$-frames, but since $R$-upsets do not distinguish between points that are $R$-related to each other, we may restrict ourselves to those $S4$-frames that are in addition antisymmetric. This extends to Esakia duality between Heyting algebras and partially ordered Esakia spaces [16].

Define an equivalence relation on an $S4$-frame $\mathfrak{H} = (W, R)$ by setting

$$w \sim v \iff wRv \text{ and } vRw.$$ 

As is customary, we call equivalence classes of $\sim$ $R$-clusters.

If $\mathfrak{A}$ is a closure algebra and $\mathfrak{A}_a$ is the dual of $\mathfrak{A}$, then the dual $\mathfrak{H}(\mathfrak{A})_a$ of $\mathfrak{H}(\mathfrak{A})$ is obtained by modding out $\mathfrak{A}_a$ by the equivalence relation $\sim$. Conversely, if $\mathfrak{H}$ is a Heyting algebra, then the dual $\mathfrak{A}(\mathfrak{H})_a$ of $\mathfrak{A}(\mathfrak{H})$ is isomorphic to the dual $\mathfrak{H}_a$ of $\mathfrak{H}$ (see, e.g., [17, Sec. III.4]).

We conclude this preliminary section by a brief discussion of relativizations of Heyting algebras. Let $\mathfrak{H}$ be a Heyting algebra and $a \in \mathfrak{H}$. The relativization of $\mathfrak{H}$ with respect to $a$ is the Heyting algebra $\mathfrak{H}_a$ whose underlying set is the interval $[a, 1]$ and the meet, join, and implication in $\mathfrak{H}_a$ coincide with those in $\mathfrak{H}$. If $\mathfrak{H} = \mathfrak{H}_X$ is the Heyting algebra of all opens of a topological space $X$ and $U$ is an open subset of $X$, then the relativization of $\mathfrak{H}$ with respect to $U$ is isomorphic to the Heyting algebra of all opens of the subspace $X \setminus U$.

### 3. Krull Dimension of Closure Algebras

In this section we define the Krull dimension of a closure algebra $\mathfrak{A}$ via chains in the Esakia space of $\mathfrak{A}$. We relate this definition to the concept of depth of an Esakia space, which plays an important role in modal logic. The main result of the section is a pointfree characterization of the Krull dimension of $\mathfrak{A}$ without accessing the Esakia space of $\mathfrak{A}$.

For an $S4$-frame $\mathfrak{H} = (W, R)$, we write $wRv$ provided $wRv$ and $vRw$. We call a finite sequence $\{w_i \in W \mid i < n\}$ an $R$-chain provided $w_i R w_{i+1}$ for all $i$, and define the length of the $R$-chain $\{w_i \in W \mid i < n\}$ to be $n - 1$. Note that we allow the empty $R$-chain which has length $-1$.

**Definition 3.1.** Let $\mathfrak{A}$ be a closure algebra. Define the Krull dimension $\kappa \dim(\mathfrak{A})$ of $\mathfrak{A}$ as the supremum of the lengths of $R$-chains in $\mathfrak{A}_a$. If the supremum is not finite, then we write $\kappa \dim(\mathfrak{A}) = \infty$.

The definition of the length of an $R$-chain that we have adopted has its roots in algebra. Modal logicians have used a similar concept of depth of a frame $\mathfrak{H} = (W, R)$. But in modal logic the length of an $R$-chain $\{w_i \in W \mid i < n\}$ is typically defined to be $n$. This notion of length is always one more than the notion of length in algebra. The difference is whether we count the number of $R$-links in the $R$-chain (as algebraists do) or the number of points in the $R$-chain (as modal logicians do). Therefore, the Krull dimension of $\mathfrak{A}$ is one less
than the depth of $\mathfrak{A}_s$ (provided the Krull dimension of $\mathfrak{A}$ is finite). Thus, $\text{kd}(\mathfrak{A}) = n$ iff $\text{depth}(\mathfrak{A}_s) = n + 1$ for $n \in \omega$. It is expressible by a modal formula whether $\text{depth}(\mathfrak{A}_s)$ is bounded by $n$.

**Definition 3.2.** For $n \geq 1$, consider the formulas:

$$\text{bd}_1 = \Box \square p_1 \rightarrow p_1,$$

$$\text{bd}_{n+1} = \Box (\square p_{n+1} \land \neg \text{bd}_n) \rightarrow p_{n+1}.$$

The modal language is interpreted in a closure algebra $\mathfrak{A} = (A, C)$ by assigning to each propositional letter an element of $\mathfrak{A}$, letting the classical connectives disjunction and negation be the Boolean join and complement in $\mathfrak{A}$, and letting the modal diamond be the closure $C$. For a modal formula $\varphi$, we say $\varphi$ is *satisfiable* in $\mathfrak{A}$ provided $\varphi$ is interpreted as 1 for some assignment of the propositional letters. We say $\varphi$ is *valid* in $\mathfrak{A}$, written $\mathfrak{A} \models \varphi$, whenever $\varphi$ is 1 under all assignments of the letters. The following lemma is well known (see, e.g., [6, Prop. 3.44]).

**Lemma 3.3.** Let $\mathfrak{A}$ be a nontrivial closure algebra and $n \geq 1$. Then $\text{depth}(\mathfrak{A}_s) \leq n$ iff $\mathfrak{A} \models \text{bd}_n$.

We next describe when $\text{kd}(\mathfrak{A}) \leq 0$. Recall that $\mathfrak{A}$ is *trivial* if $0 = 1$, it is *discrete* if $C$ is the identity function, and it is an *$S5$-algebra* (or *monadic algebra*) if $a \leq IC(a)$ for all $a \in A$. If $\mathfrak{A}_s$ is the Esakia space of $\mathfrak{A}$ then it is well known that $\mathfrak{A}$ is trivial iff $\mathfrak{A}_s = \emptyset$, that $\mathfrak{A}$ is discrete iff $R$ is the identity, and $\mathfrak{A}$ is an *$S5$-algebra* iff $R$ is an equivalence relation.

**Lemma 3.4.** Let $\mathfrak{A}$ be a closure algebra.

1. $\text{kd}(\mathfrak{A}) = -1$ iff $\mathfrak{A}$ is the trivial algebra.
2. $\text{kd}(\mathfrak{A}) \leq 0$ iff $\mathfrak{A}$ is an $S5$-algebra.
3. $\text{kd}(\mathfrak{A}) = 0$ iff $\mathfrak{A}$ is a nontrivial $S5$-algebra.
4. If $\mathfrak{A}$ is discrete, then $\text{kd}(\mathfrak{A}) \leq 0$.

**Proof.** (1) Suppose $\mathfrak{A}$ is trivial. Then $\mathfrak{A}_s = \emptyset$, so the only $R$-chain in $\mathfrak{A}_s$ is the empty chain whose length is $-1$. Therefore, $\text{kd}(\mathfrak{A}) = -1$. Conversely, if $\text{kd}(\mathfrak{A}) = -1$, then every $R$-chain in $\mathfrak{A}_s$ has length $-1$, and hence is the empty chain. Thus, $\mathfrak{A}_s = \emptyset$, and so $\mathfrak{A}$ is the trivial algebra.

(2) Suppose $\mathfrak{A}$ is an $S5$-algebra. Then $R$ is an equivalence relation, so there are no $w, v \in \mathfrak{A}_s$ with $wRv$. Therefore, every $R$-chain in $\mathfrak{A}_s$ has length $\leq 0$. Thus, $\text{kd}(\mathfrak{A}) \leq 0$. Conversely, suppose $\text{kd}(\mathfrak{A}) \leq 0$. Then every $R$-chain in $\mathfrak{A}_s$ has length $\leq 0$. Therefore, if $xRy$, then it cannot be the case that $yRx$. Thus, $R$ is symmetric, and so $\mathfrak{A}$ is an $S5$-algebra.

(3) This follows from (1) and (2).

(4) This follows from (2) since every discrete algebra is an $S5$-algebra. \qed

**Remark 3.5.**

1. Since not every $S5$-algebra is discrete, the converse of Lemma 3.4(4) does not hold.
2. Suppose $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}_X$ for some topological space $X$. If $\mathfrak{A}$ consists of clopen subsets of $X$, then $\mathfrak{A}$ is discrete, and hence $\text{kd}(\mathfrak{A}) \leq 0$.

By Lemma 3.4, whether the Krull dimension of $\mathfrak{A}$ is $\leq 0$ can be determined internally in $\mathfrak{A}$, without accessing $\mathfrak{A}_s$. The goal of the remainder of this section is to develop a pointfree description of the Krull dimension of $\mathfrak{A}$ that does not require the Esakia space of $\mathfrak{A}$. In fact, we will prove that $\text{kd}(\mathfrak{A})$ can be defined recursively as follows.

**Definition 3.6.** The *Krull dimension* $\text{kd}(\mathfrak{A})$ of a closure algebra $\mathfrak{A}$ can be defined as follows:
Proof. Then $d$ is nowhere dense in $\mathfrak{A}_a$, then $d$ is nowhere dense in $\mathfrak{A}$.

Definition 3.8. For $a_1, \ldots, a_{n+2} \in \mathfrak{A}$, define:

\[
\begin{align*}
  d_1 &= \text{Cl}a_1 - a_1 & e_1 &= \text{Cl}(Ia_2 \wedge d_1), \\
  \vdots & & \vdots \\
  d_{n+1} &= \text{Cl}(Ia_{n+1} \wedge d_n) - a_{n+1} & e_{n+1} &= \text{Cl}(Ia_{n+2} \wedge d_{n+1}).
\end{align*}
\]

Clearly $d_{n+2} = \text{Cl}(Ia_{n+2} \wedge d_{n+1}) - a_{n+2} = e_{n+1} - a_{n+2}$, so $d_{n+2} \leq e_{n+1}$. It is also straightforward to see that if we interpret $p_i$ as $a_i$, then since $\square$ is interpreted as $I$ and $\Diamond$ as $\text{Cl}$, the formula $\neg bd_n$ is interpreted as $d_n$, and the antecedent of $bd_{n+1}$ as $e_n$.

Lemma 3.9. Let $\mathfrak{A}$ be a closure algebra, $a_1, \ldots, a_{n+2} \in \mathfrak{A}$, and $d_i, e_i$ be defined as in Definition 3.8.

\begin{enumerate}
  \item $d_1$ and $e_1$ are nowhere dense in $\mathfrak{A}$.
  \item $Ia_{n+2} \wedge d_{n+1}$ is nowhere dense in $\mathfrak{A}_{e_n}$.
  \item $e_{n+1}$ and $d_{n+2}$ are nowhere dense in $\mathfrak{A}_{e_n}$.
\end{enumerate}

Proof. (1) We have

\[
\text{IC}(d_1) = \text{IC}((\text{Cl}a_1 - a_1) \leq I(\text{Cl}a_1 - Ia_1) = \text{IC}a_1 - \text{Cl}a_1 \leq \text{Cl}a_1 - \text{Cl}a_1 = 0.
\]

Therefore, $d_1$ is nowhere dense in $\mathfrak{A}$. This yields that $Ia_2 \wedge d_1$ is nowhere dense in $\mathfrak{A}$. Thus, $e_1 = \text{Cl}(Ia_2 \wedge d_1)$ is nowhere dense in $\mathfrak{A}$.

(2) Since $e_n$ is closed in $\mathfrak{A}$, we have $C_{e_n}(a) = C(a)$ for all $a \leq e_n$. To see that $Ia_{n+2} \wedge d_{n+1}$ is nowhere dense in $\mathfrak{A}_{e_n}$, let $u$ be open in $\mathfrak{A}_{e_n}$ with $u \leq \text{Cl}(Ia_{n+2} \wedge d_{n+1})$. We set $u' = u \wedge Ia_{n+1}$. Then $u'$ is open in $\mathfrak{A}_{e_n}$ and $u' \leq a_{n+1}$, so

\[
u' \wedge Ia_{n+2} \wedge d_{n+1} = u' \wedge Ia_{n+2} \wedge (e_n - a_{n+1}) \leq u' \wedge (e_n - a_{n+1}) = u' - a_{n+1} = 0.
\]

Therefore, $u' \wedge \text{Cl}(Ia_{n+2} \wedge d_{n+1}) = 0$. This together with $u' \leq u \leq \text{Cl}(Ia_{n+2} \wedge d_{n+1})$ yields that $u' = 0$. Thus, $u \wedge Ia_{n+1} = 0$, and so $u \wedge Ia_{n+1} \wedge d_{n+1} = 0$. But $Ia_{n+1} \wedge d_{n+1}$ is dense in $\mathfrak{A}_{e_n}$, giving that $u = 0$. Consequently, $Ia_{n+2} \wedge d_{n+1}$ is nowhere dense in $\mathfrak{A}_{e_n}$.

(3) By (2), $Ia_{n+2} \wedge d_{n+1}$ is nowhere dense in $\mathfrak{A}_{e_n}$. Therefore, $e_{n+1} = \text{Cl}(Ia_{n+2} \wedge d_{n+1})$ is nowhere dense in $\mathfrak{A}_{e_n}$. Thus, $d_{n+2} = e_{n+1} - a_{n+2}$ is nowhere dense in $\mathfrak{A}_{e_n}$.

The next lemma concerns the internal Krull dimension of a closure algebra.

Lemma 3.10. Let $\mathfrak{A}$ be a closure algebra.
(1) For $a \in \mathfrak{A}$, we have $\text{kdim}(\mathfrak{A}_a) \leq \text{kdim}(\mathfrak{A})$.

(2) $\text{kdim}(\mathfrak{A}) \leq n$ iff $\text{kdim}(\mathfrak{A}_d) \leq n - 1$ for every closed nowhere dense $d \in \mathfrak{A}$.

Proof. (1) If $\text{kdim}(\mathfrak{A}) = \infty$, then there is nothing to prove. Suppose $\text{kdim}(\mathfrak{A}) = n$. Let $d \in \mathfrak{A}_a$ be nowhere dense in $\mathfrak{A}_a$. By Lemma 3.7, $d$ is nowhere dense in $\mathfrak{A}$. Since $\text{kdim}(\mathfrak{A}) = n$, we see that $\text{kdim}(\mathfrak{A}_d) \leq n - 1$. Because $(\mathfrak{A}_a)_d = \mathfrak{A}_d$, we conclude that $\text{kdim}(\mathfrak{A}_a) \leq n$. Thus, $\text{kdim}(\mathfrak{A}_a) \leq \text{kdim}(\mathfrak{A})$.

(2) One implication is trivial. For the other, let $d$ be nowhere dense in $\mathfrak{A}$. Then $\mathfrak{C}(d)$ is closed and nowhere dense in $\mathfrak{A}$. Therefore, $\text{kdim}(\mathfrak{A}_{\mathfrak{C}(d)}) \leq n - 1$. Thus, (1) yields

$$\text{kdim}(\mathfrak{A}_d) = \text{kdim}((\mathfrak{A}_{\mathfrak{C}(d)})) \leq \text{kdim}(\mathfrak{A}_{\mathfrak{C}(d)}) \leq n - 1.$$  

Consequently, $\text{kdim}(\mathfrak{A}) \leq n$. \hfill \Box

We next recall the notion of an Esakia morphism between Esakia spaces.

**Definition 3.11.** Suppose $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (V, Q)$ are Esakia spaces.

(1) A map $f : W \to V$ is a $p$-morphism provided $\mathfrak{C}_{R}f^{-1}(v) = f^{-1}(\mathfrak{C}_{Q}\{v\})$ for all $v \in V$.

(2) An Esakia morphism is a continuous $p$-morphism $f : W \to V$.

It is well known (see, e.g., [17, Sec. IV.3]) that Esakia morphisms correspond dually to closure algebra homomorphisms; that is, $h : \mathfrak{A} \to \mathfrak{B}$ is a closure algebra homomorphism iff $h_{\ast} : \mathfrak{B}_{\ast} \to \mathfrak{A}_{\ast}$ is an Esakia morphism, where $h_{\ast}(w) = h^{-1}(w)$. Moreover, $h$ is 1-1 (resp. onto) iff $h_{\ast}$ is onto (resp. 1-1).

The modal language is interpreted in an Esakia space $\mathfrak{F}$ by interpreting the modal formulas in the dual closure algebra $\mathfrak{F}^\ast$. Consequently, a modal formula $\varphi$ is satisfiable (resp. valid, written $\mathfrak{F} \models \varphi$) in $\mathfrak{F}$ exactly when $\varphi$ is satisfiable (resp. valid) in $\mathfrak{F}^\ast$.

We call $\mathfrak{F}$ rooted if there is $r \in W$ with $W = R[r]$. We refer to $r$ as a root of $\mathfrak{F}$. In general, $r$ is not unique. Let $\mathfrak{F} = (W, R)$ be a finite rooted $\mathsf{S4}$-frame. It is well known [24, 20] that with $\mathfrak{F}$ we can associate the Jankov-Fine formula $\chi_{\mathfrak{F}}$, which satisfies the following property:

$\chi_{\mathfrak{F}}$ is satisfiable in an Esakia space $\mathfrak{G}$ iff there is an Esakia space $\mathfrak{E}$ and Esakia morphisms $\mathfrak{F} \xleftarrow{f} \mathfrak{E} \xrightarrow{g} \mathfrak{G}$ such that $f$ is onto and $g$ is 1-1.

Let $\mathfrak{F}_n = (W_n, R)$ be the $n$-element chain, where $W_n = \{w_0, \ldots, w_{n-1}\}$ and $w_i R w_j$ iff $j \leq i$; see Figure 1.

\begin{center}
\begin{tikzpicture}
\node (w0) at (0,0) {$w_0$};
\node (w1) at (0,-1) {$w_1$};
\node (wn1) at (0,-3) {$w_{n-1}$};
\node (wn2) at (0,-2) {$w_{n-2}$};
\draw (w0) -- (w1);
\node at (0,-1.5) {\ldots};
\end{tikzpicture}
\end{center}

**Figure 1.** The $n$-element chain.

We are ready to characterize the internal Krull dimension of a closure algebra.

**Theorem 3.12.** Let $\mathfrak{A}$ be a nontrivial closure algebra and $n \geq 1$. The following are equivalent:

(1) $\text{kdim}(\mathfrak{A}) \leq n - 1$.

(2) There does not exist a sequence $e_0, \ldots, e_n$ of nonzero closed elements of $\mathfrak{A}$ such that $e_0 = 1$ and $e_{i+1}$ is nowhere dense in $\mathfrak{A}_{e_i}$ for each $i \in \{0, \ldots, n - 1\}$. 
(3) \( \mathfrak{A} \models \text{bd}_n \).
(4) \( \text{depth}(\mathfrak{A}_n) \leq n \).
(5) \( \mathfrak{A} \models \neg \chi_{\mathfrak{A}_{n+1}} \).
(6) \( \mathfrak{B}_{n+1} \) is not isomorphic to a subalgebra of a homomorphic image of \( \mathfrak{A} \).
(7) There do not exist an Esakia space \( \mathfrak{G} \) and Esakia morphisms \( \mathfrak{B}_{n+1} \xleftarrow{f} \mathfrak{G} \xrightarrow{g} \mathfrak{A}_n \) such that \( f \) is onto and \( g \) is 1-1.
(8) \( \mathfrak{B}_{n+1} \) is not isomorphic to a subalgebra of \( \mathfrak{A} \).
(9) \( \mathfrak{B}_{n+1} \) is not an image of \( \mathfrak{A}_n \) under an onto Esakia morphism.

Proof. (1)\( \Rightarrow \) (2): Induction on \( n \). Let \( n = 1 \). Since \( \mathfrak{A} \) is nontrivial, \( \text{kdim}(\mathfrak{A}) \leq 0 \) yields \( \text{kdim}(\mathfrak{A}) = 0 \). Therefore, for any nowhere dense \( d \) in \( \mathfrak{A} \), we have \( \text{kdim}(\mathfrak{A}_d) = -1 \), so \( \mathfrak{A}_d \) is trivial, and hence \( d = 0 \). Thus, \( \mathfrak{A} \) has no nonzero closed nowhere dense elements, as required. Next let \( n > 1 \) and \( \text{kdim}(\mathfrak{A}) \leq n - 1 \). Suppose there is a sequence \( e_0, \ldots, e_n \) of nonzero closed elements in \( \mathfrak{A} \) such that \( e_0 = 1 \) and \( e_{i+1} \) is nowhere dense in \( \mathfrak{A}_{e_i} \) for each \( i \in \{0, \ldots, n-1\} \). Then \( e_1, \ldots, e_n \) is a sequence of nonzero closed elements of \( \mathfrak{A}_{e_1} \) such that \( e_{i+1} \) is nowhere dense in \( \mathfrak{A}_{e_i} \) for each \( i \in \{1, \ldots, n-1\} \). By the induction hypothesis, applied to \( \mathfrak{A}_{e_1} \), we have \( \text{kdim}(\mathfrak{A}_{e_1}) > n - 1 \). Since \( e_1 \) is nowhere dense in \( \mathfrak{A} \) with \( \text{kdim}(\mathfrak{A}_{e_1}) > n - 1 \), we conclude that \( \text{kdim}(\mathfrak{A}) > n \). This contradicts (1).

(2)\( \Rightarrow \) (3): If \( \mathfrak{A} \not\models \text{bd}_n \), then there exist \( a_1, \ldots, a_n \in \mathfrak{A} \) such that \( d_n \neq 0 \), where \( d_n \) is defined as in Definition 3.8. Put \( e_0 = 1 \) and \( e_{n+1} = 1 \). Let \( e_1, \ldots, e_n \) be defined as in Definition 3.8. Then

\[
e_n = C(\mathfrak{A}_{a_{n+1}} \land d_n) = C(\mathfrak{A}_1 \land d_n) = C(d_n) \geq d_n \neq 0.
\]

Therefore, \( e_0, \ldots, e_n \) is a sequence of nonzero closed elements in \( \mathfrak{A} \) such that \( e_0 = 1 \) and, by Lemma 3.9, \( e_{i+1} \) is nowhere dense in \( \mathfrak{A}_{e_i} \) for each \( i \in \{0, \ldots, n-1\} \).

(3)\( \Rightarrow \) (1): Suppose that \( \text{kdim}(\mathfrak{A}) > n - 1 \). We define a decreasing sequence \( b_0, \ldots, b_n \) of closed elements in \( \mathfrak{A} \) such that \( b_{i+1} \) is nowhere dense in \( \mathfrak{A}_{b_i} \) and \( \text{kdim}(\mathfrak{A}_{b_{i+1}}) > (n - 1) - (i + 1) \). Set \( b_0 = 1 \). If \( b_i \) is already defined with \( \text{kdim}(\mathfrak{A}_{b_i}) > (n - 1) - i \), then by Lemma 3.10, there is a closed nowhere dense \( b_{i+1} \) of \( \mathfrak{A}_{b_i} \) such that \( \text{kdim}(\mathfrak{A}_{b_{i+1}}) > (n - 1) - (i + 1) \). Noting that \( \text{kdim}(\mathfrak{A}_{b_n}) > (n - 1) - n = -1 \), it follows that \( \mathfrak{A}_{b_n} \) is not trivial, and hence \( b_n \neq 0 \).

Let \( a_i = -b_i \) and let \( d_i \) be defined as in Definition 3.8. We show that for each \( i \) we have \( b_i = d_i \). If \( i = 1 \), then since \( b_1 \) is nowhere dense in \( \mathfrak{A} \), we have

\[
b_1 = 1 \land b_1 = (-\mathfrak{I}C b_1) \land b_1 = \mathfrak{I}C(-b_1) \land -(b_1) = \mathfrak{I}C a_1 = a_1 = d_1.
\]

Next suppose that \( b_i = d_i \), and show that \( b_{i+1} = d_{i+1} \). Since \( a_{i+1} \) is open in \( \mathfrak{A} \), \( b_{i+1} \) is nowhere dense in \( \mathfrak{A}_{b_i} \), and \( b_i \) is closed in \( \mathfrak{A} \), we have

\[
b_{i+1} = b_i \land b_{i+1} = C(b_i - b_{i+1}) \land b_{i+1} = C(b_i - b_{i+1}) - (b_{i+1})
\]

\[
= C(a_{i+1} \land b_i) - a_{i+1} = C(\mathfrak{I}a_{i+1} \land d_i) - a_{i+1} = d_{i+1}.
\]

Thus, \( d_n = b_n \neq 0 \). Since \( \neg \text{bd}_n \) is interpreted in \( \mathfrak{A} \) as \( d_n \), we conclude that \( \mathfrak{A} \) refutes \( \text{bd}_n \).

(3)\( \Leftrightarrow \) (4)\( \Leftrightarrow \) (8): This is well known; see, e.g., [29, Lem. 2].

(5)\( \Leftrightarrow \) (7): This is the Jankov-Fine Theorem.

(6)\( \Leftrightarrow \) (7): This follows from Esakia duality.

(6)\( \Rightarrow \) (8): This is obvious.

(8)\( \Leftrightarrow \) (9): This follows from Esakia duality.

(4)\( \Rightarrow \) (7): This is obvious since onto Esakia morphisms do not increase the depth. \( \square \)

As an immediate consequence, we obtain:

**Corollary 3.13.** The internal and external Krull dimensions of a closure algebra coincide, and so Definitions 3.1 and 3.6 are equivalent.
4. KRULL DIMENSION OF HEYTING ALGEBRAS

We next study Krull dimension of Heyting algebras. As with closure algebras, we first define Krull dimension of Heyting algebras externally and then provide an equivalent internal definition of it. We also show that Krull dimensions of a closure algebra $A$ and the associated Heyting algebra $H(\mathfrak{A})$ of opens of $\mathfrak{A}$ coincide.

**Definition 4.1.** Let $\mathfrak{H}$ be a Heyting algebra. Define the *Krull dimension* $\kdim(\mathfrak{H})$ of $\mathfrak{H}$ as the supremum of the lengths of $R$-chains in $\mathfrak{H}_*$. If the supremum is not finite, then we write $\kdim(\mathfrak{H}) = \infty$.

**Lemma 4.2.**

1. If $A$ is a closure algebra, then $\kdim(A) = \kdim(H(\mathfrak{A}))$.
2. If $\mathfrak{H}$ is a Heyting algebra, then $\kdim(H) = \kdim(A(\mathfrak{H}))$.

*Proof.* (1) Since $H(\mathfrak{A})_*$ is obtained from $\mathfrak{A}_*$ by modding out $R$-clusters, we see that the corresponding $R$-chains in $\mathfrak{A}_*$ and $H(\mathfrak{A})_*$ have the same length. Thus, $\kdim(A) = \kdim(H(\mathfrak{A}))$.

(2) This is obvious since $H_*$ is isomorphic to $(A(\mathfrak{H}))(\mathfrak{A})$.

As with closure algebras, the concept of Krull dimension of a Heyting algebra $\mathfrak{H}$ is closely related to that of the depth of $\mathfrak{H}$. It is well known that whether the depth of $\mathfrak{H}$ is $\leq n$ is described by the following formulas in the language of intuitionistic logic.

**Definition 4.3.** For $n \geq 1$, consider the formulas:

\[
\begin{align*}
ibd_1 &= p_1 \lor \neg p_1, \\
ibd_{n+1} &= p_{n+1} \lor (p_{n+1} \to \ibd_n).
\end{align*}
\]

The intuitionistic language is interpreted in a Heyting algebra $\mathfrak{H}$ by assigning to propositional letters elements of $\mathfrak{H}$ and by interpreting conjunction, disjunction, implication, and negation as the corresponding operations of $\mathfrak{H}$. The next lemma is well known (see, e.g., [6, Prop. 2.38]).

**Lemma 4.4.** Let $\mathfrak{H}$ be a nontrivial Heyting algebra and $n \geq 1$. Then $\text{depth}(\mathfrak{H}_*) \leq n$ iff $\mathfrak{H} \models \ibd_n$.

To characterize the Krull dimension of a Heyting algebra internally, we require some preparation. We call an element $a$ of a Heyting algebra $\mathfrak{H}$ dense if $\neg a = 0$.

**Lemma 4.5.** Let $\mathfrak{H}$ be a Heyting algebra, $a \in \mathfrak{H}$, and $b \in \mathfrak{H}_a$. If $b$ is dense in $\mathfrak{H}_a$, then $b$ is dense in $\mathfrak{H}$.

*Proof.* Since $b$ is dense in $\mathfrak{H}_a$ and $a$ is the bottom of $\mathfrak{H}_a$, we have $b \rightarrow a = a$. Therefore, $\neg b = b \rightarrow 0 \leq b \rightarrow a = a$. On the other hand, $a \leq b$ implies $\neg b \leq \neg a$. Thus, $\neg b \leq a \land \neg a = 0$, and hence $b$ is dense in $\mathfrak{H}$.

**Lemma 4.6.** Let $\mathfrak{A}$ be a closure algebra and let $a \in \mathfrak{A}$ be open. Then $a$ is dense in $\mathfrak{H}(\mathfrak{A})$ iff $-a$ is nowhere dense in $\mathfrak{A}$.

*Proof.* Since $a$ is open, $-a$ is closed. Therefore,

\[
\begin{align*}
\text{a is dense in } \mathfrak{H}(\mathfrak{A}) & \text{ iff } -a = 0 \\
& \text{ iff } I(-a) = 0 \\
& \text{ iff } IC(-a) = 0 \\
& \text{ iff } -a \text{ is nowhere dense}.
\end{align*}
\]
We are ready to give an internal recursive definition of the Krull dimension of a Heyting algebra.

**Definition 4.7.** The Krull dimension $\kdim(\mathcal{H})$ of a Heyting algebra $\mathcal{H}$ can be defined as follows:

- $\kdim(\mathcal{H}) = -1$ if $\mathcal{H}$ is the trivial algebra,
- $\kdim(\mathcal{H}) \leq n$ if $\kdim(\mathcal{H}_b) \leq n - 1$ for every dense $b \in \mathcal{H}$,
- $\kdim(\mathcal{H}) = n$ if $\kdim(\mathcal{H}) \leq n$ and $\kdim(\mathcal{H}) \not\leq n - 1$,
- $\kdim(\mathcal{H}) = \infty$ if $\kdim(\mathcal{H}) \not\leq n$ for any $n = -1, 0, 1, 2, \ldots$.

The next two results concern the internal definition of the Krull dimension.

**Lemma 4.8.** Let $\mathcal{H}$ be a Heyting algebra and let $a \in \mathcal{H}$. Then $\kdim(\mathcal{H}_a) \leq \kdim(\mathcal{H})$.

**Proof.** If $\kdim(\mathcal{H}) = \infty$, then there is nothing to prove. Suppose $\kdim(\mathcal{H}) = n$. Let $b \in \mathcal{H}_a$ be dense in $\mathcal{H}_a$. By Lemma 4.5, $b$ is dense in $\mathcal{H}$. Since $\kdim(\mathcal{H}) = n$, we see that $\kdim(\mathcal{H}_b) \leq n - 1$. Because $(\mathcal{H}_a)_b = \mathcal{H}_b$, we conclude that $\kdim(\mathcal{H}_a) \leq n$. Thus, $\kdim(\mathcal{H}_a) \leq \kdim(\mathcal{H})$. □

**Theorem 4.9.**

1. If $\mathfrak{A}$ is a closure algebra, then $\kdim(\mathfrak{A}) = \kdim(\mathcal{H}(\mathfrak{A}))$.
2. If $\mathcal{H}$ is a Heyting algebra, then $\kdim(\mathcal{H}) = \kdim(\mathcal{H}(\mathfrak{A}))$.

**Proof.** (1) By Theorem 3.12, $\kdim(\mathfrak{A}) \geq n$ iff there is a sequence $e_0, \ldots, e_n$ of nonzero closed elements of $\mathfrak{A}$ such that $e_0 = 1$ and $e_{i+1}$ is nowhere dense in $\mathfrak{A}_{e_i}$ for each $i \in \{0, \ldots, n - 1\}$.

By [2, Thm. 6.9], $\kdim(\mathcal{H}(\mathfrak{A})) \geq n$ iff there is a sequence $1 > b_1 > \cdots > b_n > 0$ in $\mathcal{H}(\mathfrak{A})$ such that $b_{i-1}$ is dense in $\mathcal{H}(\mathfrak{A})_{b_i}$ for each $i \in \{1, \ldots, n\}$. The two conditions are equivalent by Lemma 4.6. The result follows.

(2) Since $\mathcal{H}$ is isomorphic to $(\mathfrak{H}(\mathcal{H}(\mathfrak{A})))$, we have $\kdim(\mathcal{H}) = \kdim(\mathcal{H}(\mathfrak{A}(\mathcal{H})))$. By (1), $\kdim(\mathcal{H}(\mathfrak{A}(\mathcal{H}))) = \kdim(\mathcal{H}(\mathfrak{A}))$. Thus, $\kdim(\mathcal{H}) = \kdim(\mathcal{H}(\mathfrak{A}))$. □

As a consequence we obtain:

**Corollary 4.10.** The external and internal definitions of the Krull dimension of a Heyting algebra coincide, so Definitions 4.1 and 4.7 are equivalent.

**Proof.** Apply Corollary 3.13, Lemma 4.2, and Theorem 4.9. □

Let $\mathcal{L}_n$ be the $(n + 1)$-element linear Heyting algebra. Then $(\mathcal{L}_n)_*$ is isomorphic to the $n$-element chain $\mathfrak{F}_n$ shown in Figure 1. Let $\chi(\mathcal{L}_n)$ be the Jankov formula of $\mathcal{L}_n$. Another immediate consequence of our results is the following:

**Corollary 4.11.** Let $\mathcal{H}$ be a nontrivial Heyting algebra and $n \geq 1$. The following are equivalent:

1. $\kdim(\mathcal{H}) \leq n - 1$.
2. There does not exist a sequence $1 > b_1 > \cdots > b_n > 0$ in $\mathcal{H}$ such that $b_{i-1}$ is dense in $\mathcal{H}_{b_i}$ for each $i \in \{1, \ldots, n\}$.
3. $\mathcal{H} \models \text{id}_{b_n}$.
4. $\text{depth}(\mathcal{H}) \leq n$.
5. $\mathcal{H} \models \neg \chi(\mathcal{L}_{n+1})$.
6. $\mathcal{L}_{n+1}$ is not isomorphic to a subalgebra of a homomorphic image of $\mathcal{H}$.
7. $\mathcal{L}_{n+1}$ is not isomorphic to a subalgebra of $\mathcal{H}$. 


5. Localic Krull Dimension of Topological Spaces

As we pointed out in the introduction, it is inadequate to define the Krull dimension of a topological space $X$ as the supremum of the lengths of chains in the specialization order of $X$. The previous two sections suggest that a more adequate definition would result by working with the Krull dimension of either $\mathfrak{A}_X$ or $\mathfrak{H}_X$. Since $\mathfrak{H}_X$ is the Heyting algebra of opens of $\mathfrak{A}_X$, by Lemma 4.2 (or Theorem 4.9), $\text{ldim}(\mathfrak{A}_X) = \text{ldim}(\mathfrak{H}_X)$. Because $\mathfrak{H}_X$ is the locale of opens of $X$, we refer to this new concept as the localic Krull dimension of $X$.

**Definition 5.1.** Define the localic Krull dimension $\text{ldim}(X)$ of a topological space $X$ as the Krull dimension of $\mathfrak{H}_X$ (equivalently, as the Krull dimension of $\mathfrak{A}_X$); that is, $\text{ldim}(X) = \text{kdim}(\mathfrak{H}_X) = \text{kdim}(\mathfrak{A}_X)$.

**Remark 5.2.** It is immediate from the results of the previous section that the localic Krull dimension of a topological space $X$ can be defined recursively as follows:

- $\text{ldim}(X) = -1$ if $X = \emptyset$,
- $\text{ldim}(X) \leq n$ if $\text{ldim}(D) \leq n − 1$ for every nowhere dense subset $D$ of $X$,
- $\text{ldim}(X) = n$ if $\text{ldim}(X) \leq n$ and $\text{ldim}(X) \nleq n − 1$,
- $\text{ldim}(X) = \infty$ if $\text{ldim}(X) \nleq n$ for any $n = -1, 0, 1, 2, \ldots$.

**Lemma 5.3.** If $Y$ is a subspace of $X$, then $\text{ldim}(Y) \leq \text{ldim}(X)$.

**Proof.** By Lemma 3.10(1), $\text{ldim}(Y) = \text{kdim}(\mathfrak{A}_Y) \leq \text{kdim}(\mathfrak{A}_X) = \text{ldim}(X)$. \hfill $\Box$

**Lemma 5.4.** $\text{ldim}(X) \leq n$ iff for every closed nowhere dense subset $D$ of $X$ we have $\text{ldim}(D) \leq n − 1$.

**Proof.** Apply Lemma 3.10(2). \hfill $\Box$

To obtain an analogue of Theorem 3.12 for localic Krull dimension, we require an analogue of the Jankov-Fine theorem for topological spaces. Let $\mathfrak{F} = (W, R)$ be a finite rooted $\mathbf{S}^4$-frame and choose any enumeration of $W = \{w_i \mid i < n\}$ in which $w_0$ is a root of $\mathfrak{F}$. We recall [20] that the Jankov-Fine formula $\chi_{\mathfrak{F}}$ associated with $\mathfrak{F}$ is the conjunction of the following formulas:

1. $p_0$,
2. $\Box(p_0 \lor \cdots \lor p_{n-1})$,
3. $\Box(p_i \rightarrow \neg p_j)$ for distinct $i, j < n$,
4. $\Box(p_i \rightarrow \Diamond p_j)$ whenever $w_i Rw_j$, and
5. $\Box(p_i \rightarrow \neg \Diamond p_j)$ whenever $w_i Rw_j$.

The modal language is interpreted in a topological space $X$ by interpreting it in the powerset algebra $\mathfrak{A}_X$. Consequently, a modal formula $\varphi$ is satisfiable (resp. valid, written $X \models \varphi$) in $X$ exactly when $\varphi$ is satisfiable (resp. valid) in $\mathfrak{A}_X$. If $\varphi$ is satisfiable at $x \in X$, then we write $x \models \varphi$. An interior map between topological spaces $X, Y$ is a continuous open map $f : X \to Y$. We call $Y$ an interior image of $X$ if there is an onto interior map $f : X \to Y$. The next lemma generalizes [20, Lem. 1] to topological spaces.

**Lemma 5.5.** Let $X$ be a topological space. Then $\chi_{\mathfrak{F}}$ is satisfiable in $X$ iff $\mathfrak{F}$ is an interior image of an open subspace of $X$.

**Proof.** First suppose that $\mathfrak{F}$ is an interior image of an open subspace $U$ of $X$, say via $f : U \to \mathfrak{F}$. Let $p_i$ be interpreted as $A_i := f^{-1}(w_i)$ when $i < n$ and as $A_i := \emptyset$ when $i \geq n$. Since $A_0 = f^{-1}(w_0) \neq \emptyset$, there is $x \in U$ with $x \models p_0$. We show that $x \models \chi_{\mathfrak{F}}$. As $A_0 \cup \cdots \cup A_{n-1} = U$ and $x \in U$, we see that $x \models \Box(p_0 \lor \cdots \lor p_{n-1})$. Suppose $i \neq j$. Because $A_i \cap A_j = \emptyset$, we see that $x \models \Box(p_i \rightarrow \neg p_j)$. Suppose $w_i Rw_j$. Then $w_i \in C_R\{w_j\}$, so since $f$ is interior,
\( A_i = f^{-1}(w_i) \subseteq f^{-1}C_R\{w_j\} = C_Uf^{-1}(w_j) = CA_j \), where \( C_U \) denotes closure in the subspace \( U \). Therefore, \( x \models \Box(p_i \rightarrow \Diamond p_j) \). Finally, suppose \( w_i R w_j \). Then \( \{w_i\} \cap C_R\{w_j\} = \emptyset \). As \( f \) is interior, this yields \( f^{-1}(w_i) \cap C_Uf^{-1}(w_j) = \emptyset \). Thus, \( A_i \cap C_UA_j = \emptyset \), which gives \( x \models \Box(p_i \rightarrow \neg \Diamond p_j) \). Consequently, \( \chi_{(n)} \) is satisfiable at \( x \) in \( X \).

Conversely suppose that \( \chi_{(n)} \) is satisfied at some \( x \in X \) by interpreting \( p_i \) as \( A_i \subseteq X \). Set

\[
U = \bigcup_{i<n} A_i \cap \bigcap_{0 \leq i \neq j < n} I((X \setminus A_i) \cup (X \setminus A_j)) \cap \bigcap_{w_i R w_j} I((X \setminus A_i) \cup CA_j) \cap \bigcap_{w_i R w_j} I((X \setminus A_i) \cup (X \setminus CA_j))
\]

Then \( U \) is open and nonempty since \( x \in A_0 \cap U \). Define \( f : U \to \mathfrak{X} \) by setting \( f(y) = w_i \) provided \( y \in A_i \) (for \( i < n \)). To see that \( f \) is well defined, let \( y \in A_i \cap A_j \). Then \( y \notin X \setminus C(A_i \cap A_j) = I((X \setminus A_i) \cup (X \setminus A_j)) \). Therefore, it follows from the definition of \( U \) that \( i = j \), and so \( f \) is well defined.

To see that \( f \) is onto, since \( w_0 \) is a root of \( \mathfrak{X} \), we have \( w_0 R w_j \), and so \( U \subseteq (X \setminus A_0) \cup CA_j \) for all \( j < n \). Recalling that \( x \in A_0 \cap U \), we get \( x \in CA_j \) for each \( j < n \). As \( U \) is open and contains \( x \), we have \( U \cap A_j = \emptyset \) for each \( j < n \). Thus, \( f \) is onto.

Finally, to see that \( f \) is interior, it is sufficient to show that \( f^{-1}(C_R\{w_j\}) = C_Uf^{-1}(w_j) \) for each \( j < n \). Suppose \( y \in f^{-1}(C_R\{w_j\}) \). Then \( f(y) R w_j \). Assuming \( f(y) = w_i \), we have \( y \in A_i \) and \( y \in (X \setminus A_i) \cup CA_j \), giving \( y \in CA_j \). So \( y \in C_UA_j = C_Uf^{-1}(w_j) \). Conversely, suppose \( y \notin f^{-1}(C_R\{w_j\}) \). Then \( f(y) R w_j \). Assuming \( f(y) = w_i \), we have \( y \in A_i \) and \( y \in (X \setminus A_i) \cup (X \setminus CA_j) \), yielding \( y \in X \setminus CA_j \). Thus, \( y \notin CA_j \), and hence \( y \notin C_UA_j = C_Uf^{-1}(w_j) \). Consequently, \( f \) is interior, and hence \( \mathfrak{X} \) is an interior image of an open subspace of \( X \).

The next theorem is an analogue of Theorem 3.12 for localic Krull dimension, and is the main result of this section.

**Theorem 5.6.** Let \( X \neq \emptyset \) and \( n \geq 1 \). Let \( \mathfrak{X}_{n+1} \) be the \((n+1)\)-element chain. The following are equivalent:

1. \( \mathrm{idim}(X) \leq n - 1 \).
2. There does not exist a sequence \( E_0, \ldots, E_n \) of nonempty closed subsets of \( X \) such that \( E_0 = X \) and \( E_{i+1} \) is nowhere dense in \( E_i \) for each \( i \in \{0, \ldots, n - 1\} \).
3. \( X \models \mathrm{bd}_n \).
4. \( X \models \neg \chi_{\mathfrak{X}_{n+1}} \).
5. \( \mathfrak{X}_{n+1} \) is not an interior image of any open subspace of \( X \).
6. \( \mathfrak{X}_{n+1} \) is not an interior image of \( X \).

**Proof.** (1)\( \leftrightarrow \) (2)\( \leftrightarrow \) (3): This follows from the equivalence of Items (1), (2), and (3) of Theorem 3.12, Definition 5.1, the correspondence between relativizations and subspaces, and the fact that \( X \) and \( \mathfrak{X}_X \) validate exactly the same modal formulas.

(4)\( \leftrightarrow \) (5): We have \( X \models \neg \chi_{\mathfrak{X}_{n+1}} \) iff \( \chi_{\mathfrak{X}_{n+1}} \) is not satisfiable in \( X \). This, by Lemma 5.5, is equivalent to \( \mathfrak{X}_{n+1} \) not being an interior image of any open subspace of \( X \).

(5)\( \Rightarrow \) (6): This is obvious.

(6)\( \Rightarrow \) (2): Suppose there is a sequence \( E_0, \ldots, E_n \) of nonempty closed subsets of \( X \) such that \( E_0 = X \) and \( E_{i+1} \) is nowhere dense in \( E_i \) for each \( i \in \{0, \ldots, n - 1\} \). We show that \( \mathfrak{X}_{n+1} \) is an interior image of \( X \). Let \( E_{n+1} = \emptyset \). Define \( f : X \to W_{n+1} \) by \( f(x) = w_i \) if \( x \in E_i \setminus E_{i+1} \) for \( i \leq n \). Clearly \( f \) is well-defined and onto since \( \{E_i \setminus E_{i-1} \mid i \leq n \} \) is a partition of \( X \). Moreover, \( C(E_i \setminus E_{i+1}) = E_i \) since \( E_i \) is closed in \( X \) and \( E_{i+1} \) is nowhere dense in \( E_i \) for
\[ f^{-1}(C_R \{w_i\}) = f^{-1}\left(\{w_1, \ldots, w_n\}\right) = \bigcup_{j=i}^{n} (E_j \setminus E_{j+1}) = E_i = C(E_i \setminus E_{i+1}) = Cf^{-1}(w_i). \]

Consequently, \( f \) is an onto interior map, and hence \( \mathfrak{F}_{n+1} \) is an interior image of \( X \).

(1) \( \Rightarrow \) (5): Suppose there is an open subspace \( Y \) of \( X \) and an onto interior map \( f : Y \to \mathfrak{F}_{n+1} \). For each \( i < n + 1 \), put \( E_i = f^{-1}(C_R \{w_i\}) \neq \emptyset \). Since \( f \) is interior, each \( E_i \) is closed in \( Y \). Therefore, for \( i < n \), we have

\[ \mathbf{I}_{E_i} \mathbf{C}_{E_i}(E_{i+1}) = \mathbf{I}_{E_i}(E_{i+1}) = E_i \setminus \mathbf{C}_{E_i}(E_i \setminus E_{i+1}) = E_i \setminus \mathbf{C}_Y(E_i \setminus E_{i+1}) = E_i \setminus C_Y f^{-1}(w_i) = E_i \setminus f^{-1}(C_R \{w_i\}) = E_i \setminus E_i = \emptyset. \]

So \( E_{i+1} \) is nowhere dense in \( E_i \). Moreover, \( E_0 = f^{-1}(C_R \{w_0\}) = Y \). Thus, there is a sequence \( E_0, \ldots, E_n \) of nonempty closed subsets of \( Y \) such that \( E_0 = Y \) and \( E_{i+1} \) is nowhere dense in \( E_i \) for each \( i \in \{0, \ldots, n-1\} \). Since Items (1) and (2) are equivalent, \( \text{ldim}(Y) > n - 1 \). By Lemma 5.3, \( \text{ldim}(X) \geq \text{ldim}(Y) > n - 1 \).

We next compare localic Krull dimension to other well-known dimension functions. We recall that if \( X \) is a regular space, then the Isbell dimension of \( X \) is denoted by \( \text{ind}(X) \), if \( X \) is a Tychonoff space, then the Čech-Lebesgue dimension of \( X \) is denoted by \( \text{dim}(X) \), and if \( X \) is a normal space, then the Brouwer-Čech dimension of \( X \) is denoted by \( \text{Ind}(X) \) (see, e.g., [15, Ch. 7] for a detailed account of these three dimension functions). Also, for a spectral space \( X \), let \( \text{gdim}(X) \) denote the Krull dimension of \( X \), and for a \( T_0 \)-space \( X \), let \( \text{gdim}(X) \) denote Isbell’s graduated dimension of \( X \) [23].

Lemma 5.7. Let \( X \) be a topological space.

1. If \( X \) is a spectral space, then \( \text{ldim}(X) \leq \text{ldim}(X) \).
2. If \( X \) is a \( T_0 \)-space, then \( \text{gdim}(X) \leq \text{ldim}(X) \).
3. If \( X \) is a regular space, then \( \text{ind}(X) \leq \text{ldim}(X) \).
4. If \( X \) is a normal space, then \( \text{Ind}(X) \leq \text{ldim}(X) \) and \( \text{dim}(X) \leq \text{ldim}(X) \).

Proof. (1) The Krull dimension of a spectral space \( X \) can be defined as the supremum of the lengths of chains in the specialization order \( R \) of \( X \). Since \( \varepsilon : X \to (\mathfrak{F}_X)_* \), is an \( R \)-embedding, the supremum of the lengths of chains in the specialization order of \( X \) can be no larger than the supremum of the lengths of chains in \( (\mathfrak{F}_X)_* \). The result follows.

(2) Recall that Isbell’s graduated dimension of a \( T_0 \)-space \( X \) is the least \( n \) such that some lattice basis of \( \mathfrak{F}_X \) is a directed union of finite topologies of Krull dimension \( n \). Suppose the Isbell dimension of \( X \) is \( n \). The lattice of all opens \( \mathfrak{F}_X \) is a directed union of finite topologies \( \tau_i \) since the variety of distributive lattices is locally finite. Because the Krull dimension of each \( \tau_i \) is \( \geq n \), we see that \( \text{ldim}(X) \geq n \), as desired.

(3) Induction on \( n \geq -1 \). The base case is clear since \( \text{ind}(X) = -1 \) if \( X = \emptyset \), which happens iff \( \text{ldim}(X) = -1 \). For the inductive step, suppose \( \text{ldim}(X) = n \). If \( Y \) is closed and nowhere dense in \( X \), then \( \text{ldim}(Y) \leq n - 1 \). By the inductive hypothesis, \( \text{ind}(Y) \leq n - 1 \). Because the boundary of an open set is (closed and) nowhere dense in \( X \), it follows that the boundary \( B \) of an open subset of \( X \) has \( \text{ind}(B) \leq n - 1 \). Thus, \( \text{ind}(X) \leq n \).

(4) Let \( X \) be normal. Replacing each occurrence of \( \text{ind} \) in the proof of (3) by \( \text{Ind} \) yields \( \text{Ind}(X) \leq \text{ldim}(X) \). By [15, Thm. 7.2.8], \( \text{dim}(X) \leq \text{Ind}(X) \leq \text{ldim}(X) \).

Remark 5.8. It remains open whether \( \text{dim}(X) \leq \text{ldim}(X) \) for any Tychonoff space \( X \).

We next calculate the localic Krull dimension of some well-known spaces.

Example 5.9.
(1) It follows from the celebrated McKinsey-Tarski theorem [30] that every finite rooted $S4$-frame is an interior image of any separable crowded metrizable space. Let $\mathbb{R}$, $C$, and $\mathbb{Q}$ denote the real line, the Cantor discontinuum, and the rational line, respectively. It follows from Theorem 5.6 that each of $\mathbb{R}, C, \mathbb{Q}$ has infinite localic Krull dimension.

(2) Let $n \geq 1$. It is well known that the $n$-element chain is an interior image of the ordinal $\omega^n$, and that the $(n + 1)$-element chain is not an interior image of $\omega^n$. By Theorem 5.6, $ldim(\omega^n) = n - 1$.

(3) A reasoning similar to (2) yields that $ldim(\omega^n + 1) = n$ and $ldim(\omega^\omega + 1) = \infty$. Since these ordinals are compact, and hence Stone spaces, we obtain the examples alluded to in the introduction.

For $T_1$-spaces there is an alternate description of localic Krull dimension, which is based on an appropriate generalization of the concept of a nodec space. In the next section we first generalize the concept of a discrete closure algebra to that of an $n$-discrete closure algebra, and show that $n$-discrete closure algebras can be characterized by an appropriate generalization of the well-known Zeman formula in modal logic. We then use these results in Section 7 to give an alternate description of localic Krull dimension for $T_1$-spaces.

### 6. $n$-DISCRETE ALGEBRAS AND $n$-ZEMAN FORMULAS

**Definition 6.1.** Let $\mathcal{A}$ be a closure algebra.

1. Call $\mathcal{A}$ 0-discrete if $\mathcal{A}$ is discrete.
2. For $n \geq 1$, call $\mathcal{A}$ $n$-discrete if $\mathcal{A}_a$ is $(n - 1)$-discrete for each nowhere dense $a \in \mathcal{A}$.

Let $\mathfrak{F} = (W, R)$ be an Esakia space. Set

$$M_i = \max R(W) \text{ and } M_{n+1} = \max R\left(W \setminus \bigcup_{i=1}^{n} M_i\right) \text{ for } n \geq 1.$$ 

Note that $M_1$ is always closed, and if $n > 1$, then $M_n$ could be empty.

We will freely use the well-known fact that if $U, V$ are disjoint closed subsets of $W$ such that $U$ is an $R$-upset and $V$ is an $R$-downset, then there is a clopen $R$-upset containing $U$ and disjoint from $V$.

**Definition 6.2.** Let $\mathfrak{F} = (W, R)$ be an Esakia space and let $U \subseteq W$.

1. We call $w \in U$ an $R$-minimal point of $U$ if $u \in U$ and $uRw$ imply $wRu$.
2. We call $w$ a strictly minimal point of $U$ if $u \in U$ and $uRw$ imply $u = w$.

Clearly every strictly minimal point is $R$-minimal, but the converse is not true in general. We let $\min R(\mathfrak{F})$ denote the set of $R$-minimal points and $\min(\mathfrak{F})$ the set of strictly minimal points of $\mathfrak{F}$.

**Theorem 6.3.** A closure algebra $\mathcal{A}$ is $n$-discrete iff depth($\mathcal{A}_a$) $\leq n + 1$ and $M_{n+1} \subseteq \min(\mathcal{A}_a)$.

**Proof.** Suppose that $\mathcal{A}$ is $n$-discrete. Let $\mathcal{A}_a = (W, R)$. First we show that depth($\mathcal{A}_a$) $\leq n + 1$. If not, then there exist $w_0, \ldots, w_{n+1} \in W$ such that $w_{i+1} \in R w_i$ for $i \leq n$. We build inductively a decreasing sequence of clopen $R$-downsets $A_0, \ldots, A_{n+1}$ of $\mathcal{A}_a$ such that $w_i \notin A_{i+1}$, $w_{i+1} \in A_{i+1}$, and $A_{i+1} \cap \max R(A_i) = \emptyset$. Let $A_0 = W$. Suppose $A_i$ is already constructed. Since $w_{i+1} \in R w_i$, we have $w_{i+1} \notin R[\max R(A_i) \cup \{w_i\}]$. Therefore, there is a clopen $R$-downset $A_{i+1}$ such that $A_{i+1} \subseteq A_i$, $w_{i+1} \in A_{i+1}$, and $A_{i+1} \cap R[\max R(A_i) \cup \{w_i\}] = \emptyset$. Let $a_0, \ldots, a_{n+1} \in \mathcal{A}$ be such that $\beta(a_i) = A_i$ for $i \leq n + 1$. Since $A_{i+1} \cap \max R(A_i) = \emptyset$, Lemma 2.4 yields that $a_{i+1}$ is nowhere dense in $\mathcal{A}_{a_i}$. Because $\mathcal{A}$ is $n$-discrete, $\mathcal{A}_{a_i}$ is $(n - i)$-discrete for each $i \leq n$. Proof.
Since depth(\(\mathfrak{A}_{a_n}\)) is 0-discrete, and hence discrete. As \(a_{n+1} \in \mathfrak{A}_{a_n}\), we must have \(a_{n+1} = 0\). But this is a contradiction since \(w_{n+1} \in A_{n+1} = \beta(a_{n+1})\). Thus, depth(\(\mathfrak{A}_a\)) \(\leq n+1\).

Next we show that \(M_{n+1} \subseteq \min(\mathfrak{A}_a)\). If not, then there are distinct \(w, v \in M_{n+1}\) such that \(wRv\) and \(vRw\). Let \(w_0, \ldots, w_{n-1} \in W\) be such that \(w_i \in M_{i+1}\) for \(i \leq n-1\) and \(w_{n+1} \in \bar{R}w_i\) for \(i < n-1\). Set \(w_n := w\) and \(w_{n+1} := v\). Then \(w_n \bar{R}w_{n-1}\). Let \(A_0, \ldots, A_n\) be the clopen \(R\)-downsets of \(\mathfrak{A}_a\) constructed above for the sequence \(w_0, \ldots, w_n\). Since \(w_n \not\in M_{n+1}\) there is clopen \(A_{n+1}\) of \(\mathfrak{A}_a\) such that \(w_n \not\in A_{n+1}\), \(w_{n+1} \in A_{n+1}\), and \(A_{n+1}\) can be chosen so that \(A_{n+1} \cap \max_R(A_n) = \emptyset\). Let \(a_0, \ldots, a_n \in \mathfrak{A}_a\) be such that \(\beta(a_i) = A_i\). The same argument as above yields that \(a_{i+1}\) is nowhere dense in \(\mathfrak{A}_a\), for \(i \leq n\). Since \(\mathfrak{A}_a\) is \(n\)-discrete, \(\mathfrak{A}_{a_n}\) is discrete, so \(a_{n+1} = 0\), contradicting to \(w_{n+1} \in A_{n+1} = \beta(a_{n+1})\). Thus, \(M_{n+1} \subseteq \min(\mathfrak{A}_a)\), as desired.

Conversely, suppose that \(\mathfrak{A}_a\) is not \(n\)-discrete and depth(\(\mathfrak{A}_a\)) \(\leq n+1\). We show that \(M_{n+1} \not\subseteq \min(\mathfrak{A}_a)\). Since \(\mathfrak{A}_a\) is not \(n\)-discrete, there is a sequence of closed elements \(a_0, \ldots, a_n \in \mathfrak{A}_a\) such that \(a_0 = 1\), \(a_{i+1}\) is nowhere dense in \(\mathfrak{A}_{a_i}\) for \(i < n\), and \(\mathfrak{A}_{a_n}\) is not discrete. Let \(A_i := \beta(a_i)\) for \(i \leq n\). Then each \(A_i\) is a clopen \(R\)-downset and Lemma 2.4 gives \(A_{i+1} \cap \max_R(A_i) = \emptyset\) for \(i < n\). As \(\mathfrak{A}_{a_n}\) is not discrete, there is \(a \in \mathfrak{A}_{a_n}\) such that \(a \not\in \mathfrak{A}_a\). Therefore, there is \(w \in \mathfrak{C}_R(\beta(a)) \setminus \beta(a)\). Thus, there is \(v \in \beta(a)\) such that \(wRv\). Clearly \(w, v\) are distinct. We build \(w_0, \ldots, w_{n+1}\) as follows. Set \(w_{n+1} := w\) and \(w_n := v\). As \(a \leq a_n\), we see that \(w_n \in A_n\). Suppose \(w_i\) has already been chosen in \(A_i\) for \(1 \leq i \leq n\). Since \(A_i \subseteq A_{i-1}\), there is \(w_{i-1} \in \max_R(A_{i-1})\) such that \(w_i \bar{R}w_{i-1}\). As \(a_i\) is nowhere dense in \(\mathfrak{A}_{a_{i-1}}\), we have \(w_i \not\in \max_R(A_{i-1})\), so \(w_i \bar{R}w_{i-1}\). Therefore,

\[
\cdots \bar{R}w_1 \bar{R}w_0.
\]

Since depth(\(\mathfrak{A}_a\)) \(\leq n + 1\), we must have \(w_n \bar{R}w_{n+1}\). Thus, \(w_n, w_{n+1} \in M_{n+1}\), and hence \(M_{n+1} \not\subseteq \min(\mathfrak{A}_a)\).

In order to axiomatize \(n\)-discrete closure algebras, we generalize the Zeman formula

\[
zem = \Box \Diamond \Box p \rightarrow (p \rightarrow \Box p)
\]

as follows.

**Definition 6.4.** For \(n \geq 1\), let

\[
zem_n = \Box (\Box p_{n+1} \rightarrow \Diamond d_n) \rightarrow (p_{n+1} \rightarrow \Box p_{n+1}).
\]

We call \(zem_n\) the \(n\)-Zeman formula.

As we saw in Section 3, if we interpret \(p_i\) as \(a_i \in \mathfrak{A}\), then \(-\Diamond d_n\) is interpreted as \(d_n\). Therefore, \(\Box (\Box p_{n+1} \rightarrow \Diamond d_n)\) is interpreted as \(-\mathfrak{C}(\mathfrak{I}d_{n+1} \cap d_n) = -e_n\).

**Lemma 6.5.** Suppose \(\{b_n \mid n \in \omega\}\) is a family of closed elements of \(\mathfrak{A}\) such that \(b_0 = 1\) and \(b_{n+1}\) is nowhere dense in \(\mathfrak{A}_{b_n}\). For \(n \geq 1\), set \(a_n = -b_n\). Then \(e_n = d_n = b_n\), where \(e_n\) and \(d_n\) are defined as in Definition 3.8.

**Proof.** We show by induction on \(n \geq 1\) that \(d_n = b_n\). If \(n = 1\), then as \(b_1\) is closed and nowhere dense in \(\mathfrak{A}\), we see that \(a_1\) is closed and dense in \(\mathfrak{A}\). So

\[
d_1 = \mathfrak{C}a_1 - a_1 = \mathfrak{C}a_1 - a_1 = 1 - a_1 = b_1.
\]

For the inductive step, notice that \(b_n - b_{n+1}\) is dense in \(\mathfrak{A}_{b_n}\) because \(b_{n+1}\) is nowhere dense in \(\mathfrak{A}_{b_n}\). Therefore, \(\mathfrak{C}(b_n - b_{n+1}) = b_n\). Since \(a_{n+1}\) is open in \(\mathfrak{A}\), we have

\[
d_{n+1} = \mathfrak{C}(Ia_{n+1} \cap d_n) - a_{n+1} = \mathfrak{C}(a_{n+1} \cap b_n) - a_{n+1}
\]

\[
= \mathfrak{C}(b_n - b_{n+1}) - a_{n+1} = b_n - a_{n+1} = b_n \wedge b_{n+1} = b_{n+1}.
\]
Finally,

\[ e_n = C(Ia_{n+1} \land d_n) = C(a_{n+1} \land b_n) = C(b_n - b_{n+1}) = b_n. \]

\[ \square \]

Suppose \( \mathfrak{A} \) is a closure algebra and \( \mathfrak{A}_* = (W, R) \) is its Esakia space. For \( a_1, \ldots, a_{n+1} \in \mathfrak{A} \), let \( d_n \) and \( e_n \) be defined as in Definition 3.8. Set \( D_n = \beta(d_n) \) and \( E_n = \beta(e_n) \). If we interpret \( p_i \) as \( \beta(a_i) \), then \( D_n \) is the interpretation of \( \neg \text{bd}_n \) and \( W \setminus E_n \) is the interpretation of \( \Box (\Box p_{n+1} \rightarrow \text{bd}_n) \).

Lemma 6.6. Suppose \( \mathfrak{A} \) is a closure algebra and \( \mathfrak{A}_* = (W, R) \) is the Esakia space of \( \mathfrak{A} \). Let \( n \geq 1, a_1, \ldots, a_n \in \mathfrak{A} \), \( d_n \) is as in Definition 3.8, and \( D_n = \beta(d_n) \). Then \( (\bigcup_{i=1}^n M_i) \cap D_n = \emptyset \).

Proof. Set \( A_i = \beta(a_i) \). First suppose that \( n = 1 \) and \( w \in M_1 \cap D_1 \). Then \( w \in M_1 \) and \( w \in D_1 = C_R(I_R(A_1) \setminus A_1) \). Therefore, there is \( v \in I_R(A_1) \) with \( wRv \). Since \( w \in M_1 \) and \( wRv \), we see that \( vRw \). Thus, as \( v \in I_R(A_1) \), we have \( w \in A_1 \). The obtained contradiction proves that \( M_1 \cap D_1 = \emptyset \).

Next suppose that \( (\bigcup_{i=1}^n M_i) \cap D_n = \emptyset \) and \( w \in (\bigcup_{i=1}^{n+1} M_i) \cap D_{n+1} \). Then \( w \in \bigcup_{i=1}^{n+1} M_i \) and \( w \in C_R(I_R(A_{n+1}) \cap D_n) \). Therefore, there is \( v \in I_R(A_{n+1}) \cap D_n \) with \( wRv \). From \( v \in D_n \) it follows that \( v \notin \bigcup_{i=1}^n M_i \), so \( v \notin M_{n+1} \). But then \( w \in M_{n+1} \), so \( vRw \). This together with \( v \in I_R(A_{n+1}) \) yields \( w \in A_{n+1} \), a contradiction. Thus, \( (\bigcup_{i=1}^{n+1} M_i) \cap D_{n+1} = \emptyset \). \( \square \)

Therefore, Lemma 6.6 yields that \( \bigcup_{i=1}^n M_i \) is always contained in the interpretation of \( \text{bd}_n \).

Theorem 6.7. Let \( \mathfrak{A} \) be a nondiscrete closure algebra, \( \mathfrak{A}_* = (W, R) \) be its Esakia space, and \( n \geq 1 \). Then \( \mathfrak{A} \) is n-discrete iff \( \mathfrak{A} \models \text{zem}_n \).

Proof. By Theorem 6.3, \( \mathfrak{A} \) is n-discrete iff \( \text{depth}(\mathfrak{A}_*) \leq n + 1 \) and \( M_{n+1} \subseteq \text{min}(\mathfrak{A}_*) \). Thus, it is sufficient to prove that \( \mathfrak{A} \models \text{zem}_n \) iff \( \text{depth}(\mathfrak{A}_*) \leq n + 1 \) and \( M_{n+1} \subseteq \text{min}(\mathfrak{A}_*) \).

Suppose that \( \mathfrak{A} \models \text{zem}_n \). We first show that \( \text{depth}(\mathfrak{A}_*) \leq n + 1 \). If \( \text{depth}(\mathfrak{A}_*) > n + 1 \), then there are \( w_0, \ldots, w_{n+1} \in X \) such that \( w_{i+1} \not= w_i \) for each \( i \leq n \). We build \( B_0, \ldots, B_{n+1} \subseteq X \) as follows. Set \( B_0 = X \). Suppose \( B_i \) is already built. Since \( w_{i+1} \not= w_i \), we have \( w_{i+1} \not= R [\max_R(B_i) \cup \lbrace w_i \rbrace] \). Therefore, as \( \mathfrak{A}_* \) is an Esakia space, there is an \( R \)-downset \( B_{i+1} \subseteq W \) such that \( B_{i+1} \subseteq B_i \), \( w_{i+1} \in B_{i+1} \), and \( B_{i+1} \cap R [\max_R(B_i) \cup \lbrace w_i \rbrace] = \emptyset \). For \( i = 1, \ldots, n \), let \( A_i = X \setminus B_i \) and let \( A_{n+1} = X \setminus (B_n \setminus B_{n+1}) \). Let \( b_i \) be such that \( \beta(b_i) = B_i \), and set \( A_i = \beta(a_i) \), \( D_i = \beta(d_i) \), and \( E_i = \beta(e_i) \).

We claim that if we interpret \( p_i \) as \( A_i \), then \( \mathfrak{A}_* \not\models \text{zem}_n \). For this it is sufficient to show that \( I_R(E_n \cup A_{n+1}) \cap A_{n+1} \not\subseteq I_R(A_{n+1}) \). Clearly \( w_{n+1} \in A_{n+1} \). By Lemma 6.5, \( e_n = b_n \), and hence \( E_n = B_n \). Since

\[ w_{n+1} \in X = I_R(X) = I_R(A_{n+1} \cup (X \setminus A_{n+1})) = I_R(A_{n+1} \cup (B_n \setminus B_{n+1})) \subseteq I_R(A_{n+1} \cup C_R(B_n \setminus B_{n+1})) = I_R(A_{n+1} \cup B_n) = I_R(A_{n+1} \cup E_n), \]

we have that \( w_{n+1} \in I_R(E_n \cup A_{n+1}) \cap A_{n+1} \). On the other hand, since \( w_{n+1} \not= w_n \), we see that \( w_{n+1} \not\in I_R(A_{n+1}) \). Thus, \( I_R(E_n \cup A_{n+1}) \cap A_{n+1} \not\subseteq I_R(A_{n+1}) \), and hence \( \mathfrak{A}_* \not\models \text{zem}_n \). The obtained contradiction proves that \( \text{depth}(\mathfrak{A}_*) \leq n + 1 \).

Next we show that \( M_{n+1} \subseteq \text{min}(\mathfrak{A}_*) \). If not, then there are distinct \( w, v \in M_{n+1} \) such that \( wRv \) and \( vRw \). Since \( w \in M_{n+1} \), there are \( w_0, \ldots, w_{n-1} \in X \) such that \( w_i \in M_{i+1} \), \( w_{i+1} \not= w_i \), and \( w_{i+1} \not= w_{i+1} \). Let \( w_n := w \) and \( w_{n+1} := v \). Build \( B_0, \ldots, B_{n+1} \subseteq W \) as above. Since \( w_n \not= w_{n+1} \), there is \( B_{n+1} \subseteq W \) such that \( w_n \not\in B_{n+1} \) and \( w_{n+1} \in B_{n+1} \), and \( B_{n+1} \) can be selected so that \( \max_R(B_n) \cap B_{n+1} = \emptyset \). For \( i = 1, \ldots, n \), let \( A_i = W \setminus B_i \) and let \( A_{n+1} = W \setminus (B_n \setminus B_{n+1}) \). The same argument as above yields that \( I_R(E_n \cup A_{n+1}) \cap A_{n+1} \not\subseteq I_R(A_{n+1}) \). Therefore, \( \mathfrak{A}_* \not\models \text{zem}_n \), a contradiction. This proves that if \( \mathfrak{A} \models \text{zem}_n \), then \( \text{depth}(\mathfrak{A}_*) \leq n + 1 \) and \( M_{n+1} \subseteq \text{min}(\mathfrak{A}_*) \).
Conversely, suppose that \( \text{depth}(\mathfrak{A}_n) \leq n + 1 \) and \( M_{n+1} \subseteq \text{min}(\mathfrak{A}_n) \). Let \( p_1, \ldots, p_{n+1} \) be interpreted as \( a_1, \ldots, a_{n+1} \in \mathfrak{A} \). Let \( A_i = \beta(a_i) \), \( D_i = \beta(d_i) \), and \( E_i = \beta(e_i) \). Then in \( \mathfrak{A}_n \) the formula \( \text{bd}_n \) is interpreted as \( W \setminus D_n \) and \( \Box(p_{n+1} \rightarrow \text{bd}_n) \) is interpreted as \( W \setminus E_n \). Therefore, to see that \( \mathfrak{A} \) satisfies \( z_{em,n} \), it is sufficient to show that \( I_R(E_n \cup A_{n+1}) \cap A_{n+1} \subseteq I_R(A_{n+1}) \).

Suppose \( w \in I_R(E_n \cup A_{n+1}) \cap A_{n+1} \) and \( wRv \). Then \( w \in A_{n+1} \) and \( v \in E_n \cup A_{n+1} \). If \( v \in E_n \), then \( v \in C_R(I_R A_{n+1} \cap D_n) \). So there is \( u \in I_R A_{n+1} \cap D_n \) such that \( vRu \). By Lemma 6.6, \( u \notin \bigcup_{i=1}^{n} M_i \). Thus, \( u \in M_{n+1} \subseteq \text{min}(\mathfrak{A}_n) \), yielding \( w = v = u \). Consequently, \( v \in A_{n+1} \), and hence \( w \in I_R(A_{n+1}) \), as desired.

Let \( S_{4,n} := S_4 + \text{bd}_n \) and \( S_{4,Z,n} := S_4 + z_{em,n} \). It is well known (see, e.g., [6, Thm. 8.85]) that every normal extension of \( S_{4,n} \) has the finite model property.

**Theorem 6.8.**

1. \( S_{4,n+1} \not\subseteq S_{4,Z,n} \).
2. \( S_{4,Z,n} \) has the finite model property.

**Proof.** (1) Suppose \( \mathfrak{A} \models S_{4,Z,n} \). By Theorem 6.7, \( \text{depth}(\mathfrak{A}_n) \leq n + 1 \). Therefore, by Theorem 3.12, \( \mathfrak{A} \models S_{4,n+1} \). Thus, \( S_{4,n+1} \subseteq S_{4,Z,n} \). To see that the inclusion is proper, consider the finite \( S_4 \)-frame \( \mathfrak{F} \) depicted in Figure 2. Since depth(\( \mathfrak{F} \)) = \( n + 1 \), we see that \( \mathfrak{F} \) satisfies \( S_{4,n+1} \). On the other hand, as \( M_{n+1} = \{r_1, r_2\} \not\subseteq \varnothing = \text{min}(\mathfrak{F}) \), the frame \( \mathfrak{F} \) does not satisfy \( S_{4,Z,n} \).

![Figure 2. An \( S_{4,n+1} \)-frame \( \mathfrak{F} \) that is not an \( S_{4,Z,n} \)-frame.](image)

(2) Follows from (1) since every normal extension of \( S_{4,n} \) has the finite model property. \( \square \)

7. \( n \)-discrete spaces and topological completeness of \( S_{4,Z,n} \)

In this section we use the results of the previous section to generalize the concept of a discrete space to that of an \( n \)-discrete space. We show that a \( T_1 \)-space \( X \) is 1-discrete iff \( X \) is nodec, and more generally, \( X \) is \( n \)-discrete iff \( \text{ldim}(X) \leq n \). This yields a number of topological incompleteness results in modal logic. The main result of the section is the construction of a countable crowded \( \omega \)-resolvable Tychonoff space \( Z_n \) of localic Krull dimension \( n \) such that \( S_{4,Z,n} \) is the logic of \( Z_n \).

**Definition 7.1.** A nonempty topological space \( X \) is called \( n \)-discrete provided \( \mathfrak{A}_X \) is \( n \)-discrete.

**Remark 7.2.** It follows that \( X \) is 0-discrete iff \( X \) is discrete, and for \( n \geq 1 \), \( X \) is \( n \)-discrete iff every nowhere dense subset of \( X \) is \( (n - 1) \)-discrete.

**Remark 7.3.** Let \( X \) be a nonempty nondiscrete space. By Theorem 6.7, \( X \) is \( n \)-discrete iff \( X \models z_{em,n} \).
Remark 7.4. Let $X$ be a nonempty Alexandroff space and let $R$ be the specialization order of $X$. Then $\mathfrak{S} := (X, R)$ is an $S4$-frame. Since $X$ and $\mathfrak{S}$ satisfy the same modal formulas, by Remark 7.3, $X$ is $n$-discrete iff $\mathfrak{S} \models \text{zem}_n$. Moreover, a slightly simplified version of the proof of Theorem 6.7 yields that $\mathfrak{S} \models \text{zem}_n$ iff $\text{depth}(\mathfrak{S}) \leq n + 1$ and $M_{n+1} \subseteq \min(\mathfrak{S})$. This, in particular, implies that each $S4.Z_n$ is a canonical logic. In fact, since by Theorem 6.8(2), $S4.Z_n$ has the finite model property, $S4.Z_n$ is the logic of finite $n$-discrete Alexandroff spaces.

Recall that a space is nodec if every nowhere dense set is closed. It is well known (see, e.g., [10]) that a space is nodec iff every nowhere dense set is closed and discrete. As follows from [1], the Zeman formula $\text{zem}$ defines the class of nodec spaces, and $S4.Z = S4 + \text{zem}$ is the logic of the class of nodec spaces.

Lemma 7.5. Let $X$ be a $T_1$-space.

1. $\text{ldim}(X) \leq 0$ iff $X$ is discrete.
2. $\text{ldim}(X) \leq 1$ iff $X$ is nodec.

Proof. (1) If $X$ is discrete, then the only nowhere dense subset of $X$ is $\emptyset$. Therefore, $\text{ldim}(X) \leq 0$. Conversely, if $X$ is not discrete, then there is $x \in X$ such that $\{x\}$ is not open, so $I\{x\} = \emptyset$. Since $X$ is $T_1$, we see that $IC\{x\} = I\{x\} = \emptyset$, so $\{x\}$ is nowhere dense. Thus, $\text{ldim}(X) > 0$.

(2) Suppose $X$ is nodec. Let $D$ be an arbitrary nowhere dense subspace of $X$. Then $D$ is closed and discrete. Since $X$ is $T_1$, so is $D$. Therefore, by (1), $\text{ldim}(D) \leq 0$. Thus, $\text{ldim}(X) \leq 1$. Conversely, if $X$ is not nodec, then there is a nowhere dense subspace $D$ of $X$ which is not closed. Therefore, $CD$ is a nowhere dense subspace of $X$ which is not discrete. By (1), $\text{ldim}(CD) > 0$. Thus, $\text{ldim}(X) > 1$.


Proof. By Remark 7.4 and Lemma 7.5, $S4.Z_1$ is the logic nodec spaces. By [1, Thm. 4.6], the same is true of $S4.Z$. Thus, $S4.Z = S4.Z_1$.

Theorem 7.7. Let $X$ be a $T_1$-space and $n \in \omega$. Then $\text{ldim}(X) \leq n$ iff $X$ is $n$-discrete.

Proof. By induction on $n$. The case $n = 0$ is Lemma 7.5(1). Suppose for every $T_1$-space $Y$, we have $Y$ is $n$-discrete iff $\text{ldim}(Y) \leq n$. We show that $X$ is $(n+1)$-discrete iff $\text{ldim}(X) \leq n+1$. We have $\text{ldim}(X) \leq n+1$ iff $\text{ldim}(Y) \leq n$ for every nowhere dense subspace $Y$ of $X$. Since a subspace of a $T_1$-space is a $T_1$-space, by inductive hypothesis, this is equivalent to every nowhere dense subspace $Y$ of $X$ being $n$-discrete. But this is equivalent to $X$ being $(n+1)$-discrete.

Corollary 7.8. For $n \geq 1$, no logic in the interval $[S4_{n+1}, S4.Z_n)$ is the logic of any class of $T_1$-spaces.

Proof. Suppose $L \in [S4_{n+1}, S4.Z_n)$ and $K$ is a class of $T_1$-spaces. If $L$ is the logic of $K$, then for each $X \in K$, we have $X \models L$. Therefore, since $S4_{n+1} \subseteq L$, we have $X \models bd_{n+1}$. By Theorem 5.6, $\text{ldim}(X) \leq n$. As $X$ is $T_1$, by Theorem 7.7, $X$ is $n$-discrete. By Remark 7.3, $X \models \text{zem}_n$. Thus, $S4.Z_n \subseteq L$, a contradiction. Consequently, $L$ is not the logic of any class of $T_1$-spaces.

On the other hand, for each $n \geq 1$ we construct a countable crowded $\omega$-resolvable Tychonoff space $Z_n$ of localic Krull dimension $n$ such that $S4.Z_n$ is the logic of $Z_n$. The basic building block for the construction is a countable crowded $\omega$-resolvable Tychonoff nodec space $Y$ such that the remainder $Y^* = \beta Y \setminus Y$ contains a subspace homeomorphic to $\beta \omega$ which consists entirely of remote points of $Y$. In Section 7.1 we explain why such a building block $Y$ exists, in Section 7.2 we build the spaces $Z_n$ from $Y$, and in Section 7.3 we prove that $S4.Z_n$ is the logic of $Z_n$. 

7.1. The basic building block. Let $X$ be a topological space. We recall (see Juhász [26, 27]) that a $\pi$-base of $X$ is a collection $\mathcal{B}$ of nonempty open subsets of $X$ such that every nonempty open subset of $X$ contains a member of $\mathcal{B}$. The $\pi$-weight $\pi(X)$ of $X$ is the smallest cardinality of such a family. We will be interested in Tychonoff spaces of countable $\pi$-weight.

For a compact Hausdorff space $X$, let $EX$ be the Gleason cover of $X$ [21, 34]. It is well known that $EX$ is constructed as the Stone space of the Boolean algebra of regular open subsets of $X$, and hence $EX$ is an extremally disconnected compact Hausdorff space, where we recall that a space is extremally disconnected if the closure of each open set is open.

If $\nabla \in EX$, then $\bigcap \{C_X(U) \mid U \in \nabla\}$ is a singleton of $X$, which we denote by $p_X(\nabla)$. This defines a map $p_X : EX \to X$. It is well known that $p_X$ is an irreducible map; that is, $p_X$ is an onto continuous map such that for every proper closed subset $F$ of $EX$, the image $p_X(F)$ is a proper closed subset of $X$. Since $p_X$ is evidently closed, this yields that $F \subseteq EX$ is nowhere dense iff $p_X(F) \subseteq X$ is nowhere dense, and that $\pi(X) = \pi(EX)$.

Let $Z$ be a subspace of $X$. A point $x \in X \setminus Z$ is remote from $Z$ provided $x \notin C_X(D)$ for every nowhere dense subset $D$ of $Z$. Observe that if $x$ is remote from $Z$, then $x$ is remote from every subspace of $Z$. The following simple lemma was used in [32, 12] for constructing various examples.

**Lemma 7.9.** For a $T_1$-space $X$, if every $x \in X$ is remote from $X \setminus \{x\}$, then $X$ is nodec.

**Proof.** Let $D$ be a nowhere dense subset of $X$ and $x \notin D$. Since $X$ is a $T_1$-space, $D$ is a nowhere dense subset of $X \setminus \{x\}$. Therefore, as $x$ is remote from $X \setminus \{x\}$, we see that $x \notin C(D)$. Thus, $X$ is nodec. $\square$

Suppose $X$ is a Tychonoff space. A remote point of $X$ is a point $p \in \beta X \setminus X$ that is remote from $X$. In the context of Čech-Stone compactifications, remote points are very well studied in the literature. In particular, we have:

**Theorem 7.10.** [5, 9] If $X$ is a nonpseudocompact Tychonoff space with countable $\pi$-weight, then the remainder $X^* := \beta X \setminus X$ contains a point that is remote from $X$.

Here we recall that a Tychonoff space $X$ is pseudocompact if every continuous real-valued function on $X$ is bounded. This result was generalized to products of such spaces in [11].

Let $I$ be the closed unit interval and let $EI$ be the Gleason cover of $I$. For $t \in I$, let $X = EI \setminus p_I^{-1}(\{t\})$. Since $X$ is a dense subspace of $EI$, it is $C^*$-embedded in $EI$ (see, e.g., [37, Prop. 10.47]), meaning that every bounded continuous real-valued function on $X$ extends to $EI$. Therefore, by [37, Thm. 1.46], $\beta X = EI$. It is also clear that $X$ is a nonpseudocompact Tychonoff space with countable $\pi$-weight. Thus, by Theorem 7.10, there is a point $x_t \in p_I^{-1}(\{t\})$ that is remote from $X$.

Let $D$ be any countable dense subset of $I$ (e.g., $D = I \cap \mathbb{Q}$). We set $Y := \{x_t \mid t \in D\}$.

**Lemma 7.11.** [32, 12] $Y$ is a countable crowded extremally disconnected $\omega$-resolvable nodec space that is of countable $\pi$-weight.

Here we recall (see, e.g., [13]) that a partition $\mathcal{P}$ of a space $X$ is dense if each $D \in \mathcal{P}$ is dense in $X$, and that $X$ is $\kappa$-resolvable if it has a dense partition of size $\kappa$. We now isolate the crucial property of $Y$ that makes our construction in Section 7.2 work.

**Proposition 7.12.** $Y$ has a compact set of remote points that is homeomorphic to $\beta \omega$.

**Proof.** Since $Y$ is countable, we can pick a nonempty closed $G_\delta$-subset $S$ of $\beta Y$ such that $Y \cap S = \emptyset$. Put $T = \beta Y \setminus S$. By [37, Thm. 1.49], $\beta T = \beta Y$ and $T^* = S$. By [9, Thm. 11.1],...
we can choose a countably infinite discrete set \( D \) consisting entirely of remote points of \( T \) every limit point of which is also a remote point of \( T \). Observe that every point from \( D \) is remote from \( Y \) since \( Y \) is a subspace of \( T \). We show that \( D \) is \( C^* \)-embedded in \( \beta Y \) by utilizing a technique of [33]. Since \( D \subseteq T^* = S \) and \( S \) is closed, \( CD \subseteq S \). Because \( Y \subseteq \beta Y \setminus S \), we see that \( C(D) \cap Y \subseteq C(D) \setminus S = \emptyset \). Therefore, \( D \) is closed in the subspace \( D \cup Y \), which is normal since it is countable. By the Tietze Extension Theorem (see, e.g.,[15, Thm. 2.1.8]), \( D \) is \( C^* \)-embedded in \( D \cup Y \), and so \( D \) is \( C^* \)-embedded in \( \beta Y \). This, by [37, Thm. 1.46], yields that \( C(D) = \beta D \), and hence \( Y \) has a compact set of remote points that is homeomorphic to \( \beta \omega \).

\[ \square \]

7.2. The spaces \( Z_n \). Let \( \mathcal{F} = (W, R) \) be a rooted \( S4 \)-frame. We call \( \mathcal{F} \) a tree if \( R \) is a partial order and \((\forall w, u, v \in W)(u, v \in C_R\{w\} \Rightarrow uRv \text{ or } vRu\)) We will always denote the root of a tree \( \mathcal{F} \) by \( r \), the \( R \)-maximal points of \( \mathcal{F} \) by \( \text{max}(\mathcal{F}) \), and call \( v \) a child of \( w \) provided \( wRv \) and from \( wRuRv \) it follows that \( w = u \) or \( u = v \). For \( n \geq 1 \), let \( \mathcal{F}_n \) denote the tree of depth \( n \) in which all non-\( R \)-maximal points have \( \omega \) children.

We call a cluster in \( \mathcal{F} \) trivial if it is a singleton, and proper otherwise. The skeleton of \( \mathcal{F} \) is the partially ordered \( S4 \)-frame obtained by modding out the clusters of \( \mathcal{F} \). We call \( \mathcal{F} \) a quasi-tree if the skeleton of \( \mathcal{F} \) is a tree. A cluster of a quasi-tree \( \mathcal{F} \) is maximal if all its points are \( R \)-maximal, and it is the root cluster if it contains a root of \( \mathcal{F} \).

Let \( \mathcal{P} \) be a partition of a space \( X \). We call \( \mathcal{P} \) clopen provided each \( A \in \mathcal{P} \) is clopen in \( X \).

For a cardinal \( \kappa \), we consider the \( \kappa \)-fork depicted in Figure 3.

**Figure 3.** The \( \kappa \)-fork.

**Lemma 7.13.** The \( \kappa \)-fork is an interior image of a space \( X \) iff there are a closed nowhere dense subset \( N \) of \( X \) and a clopen partition \( \mathcal{P} = \{A_\lambda \mid \lambda < \kappa\} \) of the subspace \( X \setminus N \) such that \( CA = A \cup N \) for each \( A \in \mathcal{P} \).

**Proof.** Let \( \mathcal{F} = (W, R) \) be the \( \kappa \)-fork. First suppose that \( f : X \to W \) is an onto interior map. Let \( N = f^{-1}(r) \) and \( A_\lambda = f^{-1}(w_\lambda) \). Then

\[ C_N = C f^{-1}(r) = f^{-1} C_R(\{r\}) = f^{-1}(r) = N \]

and

\[ I_C N = I N = I f^{-1}(r) = f^{-1} I_R(\{r\}) = f^{-1}(\emptyset) = \emptyset. \]

Therefore, \( N \) is closed and nowhere dense in \( X \). Clearly \( \mathcal{P} = \{A_\lambda \mid \lambda < \kappa\} \) is a partition of \( X \setminus N \). Moreover, since each \( \{w_\lambda\} \) is simultaneously an \( R \)-upset and an \( R \)-downset in the subframe \( W \setminus \{r\} \), each \( A_\lambda \) is clopen in \( X \setminus N \). Finally,

\[ CA_\lambda = C f^{-1}(w_\lambda) = f^{-1}(C_R\{w_\lambda\}) = f^{-1}(\{w_\lambda, r\}) = A_\lambda \cup N. \]

Next suppose that there are a closed nowhere dense subset \( N \) of \( X \) and a clopen partition \( \mathcal{P} = \{A_\lambda \mid \lambda < \kappa\} \) of the subspace \( X \setminus N \) such that \( CA = A \cup N \) for each \( A \in \mathcal{P} \). Define
$f : X \to W$ by setting
\[ f(x) = \begin{cases} r & \text{if } x \in N \\ w_\lambda & \text{if } x \in A_\lambda \end{cases} \]

It is clear that $f$ is a well-defined onto map. Moreover,

$$f^{-1}(C_R\{r\}) = f^{-1}(r) = N = CN = Cf^{-1}(r)$$

and

$$f^{-1}(C_R\{w_\lambda\}) = f^{-1}(\{w_\lambda, r\}) = A_\lambda \cup N = CA_\lambda = Cf^{-1}(w_\lambda).$$

Thus, $f$ is interior. \qed

We assume the reader is familiar with the construction of attaching spaces or adjunction space (see, e.g., [22, pp. 12–14] or [38, pp. 65–66]). Given an indexed family of spaces $X_i$ and subspaces $Y_i \subseteq X_i$, along with continuous maps $f_i : Y_i \to Z$, one can form an adjunction space which is a quotient of the topological sum $\bigoplus_{i \in I} X_i$ in which the only nontrivial equivalence classes are

\[ \{(y_i, y_j) \mid i, j \in I, \ y_i \in Y_i, \ y_j \in Y_j, \ f_i(y_i) = f_j(y_j)\}. \]

When $Z$ is a singleton, the adjunction space is often referred to as the \textit{wedge sum}.

Given an equivalence relation $\equiv$ on a set $X$, let $[x]$ be the equivalence class of $x \in X$. We call $U \subseteq X$ \textit{saturated} provided that $x \in U$ implies $[x] \subseteq U$. Recall that open (resp. closed) sets in a quotient space $X/\equiv$ correspond to saturated open (resp. closed) sets in $X$.

Using $Y$ we recursively build the family of spaces $\{Z_n \mid n \geq 1\}$ such that each $Z_n$ is a subspace of $Z_{n+1}$ and there is an onto interior mapping $\alpha_n : Z_n \to \mathcal{T}_{n+1}$.

\textbf{Base case} ($n = 1$): Let $\{Y_n \mid n \in \omega\}$ be a pairwise disjoint family of spaces such that there is a homeomorphism $h_n : Y \to Y_n$ for each $n \in \omega$. Fix $y \in Y$ and set $y_n = h_n(y)$. Let $Z_1$ be the wedge sum of $\{(Y_n, y_n) \mid n \in \omega\}$. We identify each $Y_n \setminus \{y_n\}$ with its image in $Z_1$ and refer to the point $\{y_n \mid n \in \omega\}$ in $Z_1$ using the symbol $y$; see Figure 4. Since $\mathcal{T}_2$ is the $\omega$-fork and $\{y\}$ is a closed nowhere dense subset of $Z_1$ such that $\{Y_n \setminus \{y_n\} \mid n \in \omega\}$ is a clopen partition of $Z_1 \setminus \{y\}$ satisfying $y \in C_{Z_1}(Y_n \setminus \{y_n\})$, it follows from Lemma 7.13 that there is an onto interior mapping $\alpha_1 : Z_1 \to \mathcal{T}_2$ such that $\alpha_1^{-1}(r) = \{y\}$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (y0) at (0,0) {$y_0$};
  \node (y1) at (1,0) {$y_1$};
  \node (y2) at (2,0) {$y_2$};
  \node (y) at (1.5,-1) {$y$};
  \node (z1) at (3,0) {$Z_1$};
  \draw [->] (y0) -- (y1) -- (y2) -- (y); \draw (y) -- (z1);
\end{tikzpicture}
\caption{Realizing $Z_1$ as a wedge sum of the $Y_i$.}
\end{figure}

\textbf{Recursive step} ($n \geq 1$): Suppose $Z_n$ with the above properties is already built. Identify $\mathcal{T}_{n+1}$ with the subframe $\mathcal{T}_{n+2} \setminus \text{max}(\mathcal{T}_{n+2})$. Enumerate $\text{max}(\mathcal{T}_{n+1})$ as $\{w_i \mid i \in \omega\}$. Label points in $\text{max}(\mathcal{T}_{n+2})$ as $w_{i,j}$ where $w_{i,j}$ is the $i^{th}$ child of $w_i$. Let $\alpha_n : Z_n \to \mathcal{T}_{n+1}$ be an onto interior map such that $(\alpha_n)^{-1}(r) = \{y\}$ where $y$ is the point in the base case defining $Z_1$. Set $X_i = (\alpha_n)^{-1}(C_{R\{w_i\}})$; see Figure 5.

Since $X_i$ is countable, there is a continuous bijection $f : \omega \to X_i$ which extends to a continuous onto map $g : \beta\omega \to \beta X_i$. Up to homeomorphism, $\beta\omega$ is a subspace of $\beta Y$ such that each point in $\beta\omega$ is a remote point of $Y$. Consider the quotient space $Q_i$ of $\beta Y$ obtained by the equivalence relation whose only nontrivial equivalence classes are the fibers.
of $g$, namely $g^{-1}(x)$ for each $x \in \beta X_i$. By [15, Thm. 2.4.13] the quotient mapping of $\beta Y$ onto $Q_i$ is closed. Intuitively, $Q_i$ is obtained from $\beta Y$ by replacing the copy of $\beta \omega$ that ‘is remote from $Y’$ by $\beta X_i$. We identify $Y$, $\beta X_i$, and $X_i$ with their respective images in $Q_i$, see Figure 6. For a nowhere dense subset $N$ of $Y$, we have $C_{\beta Y}(N) \cap \beta \omega = \emptyset$, so $C_{\beta Y}(N)$ is saturated, and hence $C_{Q_i}(N) \cap \beta X_i = \emptyset$.

Viewing $Y \cup X_i$ as a subspace of $Q_i$, the subsets $Y$ and $X_i$ are complements of each other, $Y$ is dense, and $X_i$ is closed and nowhere dense. Let $A_i$ be the adjunction space of $\omega$ copies of $Y \cup X_i$ glued through the identity map on the copies of $X_i$. That is, up to homeomorphism, $A_i$ is the quotient of the topological sum $\bigcup_{m \in \omega} (Y \cup X_i) \times \{m\}$ under the equivalence relation whose nontrivial equivalence classes are $\{(x,m) \mid m \in \omega\}$ for each $x \in X_i$; see Figure 7.

To facilitate defining $\alpha_{n+1} : Z_{n+1} \to \Sigma_{n+2}$ we denote the $\omega$ copies of $Y$ in $A_i$ by $Y_{i,j}$ where $j \in \omega$. We also identify $X_i$ with its homeomorphic copy in $A_i$. The quotient mapping from $\bigoplus_{j \in \omega} Y_{i,j} \cup X_i$ onto $A_i$ is closed. Thus, in $A_i$, we have that $\bigcup_{j \in \omega} Y_{i,j}$ and $X_i$ are complements of each other, $\bigcup_{j \in \omega} Y_{i,j}$ is dense, and $X_i$ is closed and nowhere dense.

We define $Z_{n+1}$ as the adjunction space of the $A_i$ for $i \in \omega$ through the following gluing. For each $A_i$, consider the inclusion mapping $I_i : X_i \to Z_n$. Glue through the equivalence relation whose nontrivial equivalence classes are $\{(x_i, x_j) \mid x_i \in X_i, x_j \in X_j, I_i(x_i) = I_j(x_j)\}$. Intuitively the gluing is through identifying points in $X_i$ and $X_j$ that are equal in $Z_n$; see Figure 8. Identify the $Y_{i,j}$, $X_i$, and $Z_n$ with their images in $Z_{n+1}$. Observe that $Y_{i,j}$ is open in $Y_{i,j} \cup X_i$ and saturated in $\bigoplus_{j \in \omega} (Y_{i,j} \cup X_i)$, hence open in $A_i$. Similarly, $Y_{i,j}$ is saturated in
By the inductive hypothesis, \( \text{ldim}(n) \) for each \( n \) be nowhere dense in \( X \). Then \( Y \).

**Proof.** Induction on \( n \).

**Base case** \((n = 0)\): \( \text{ldim}(Y_i) = 0 \). Let \( N \) be nowhere dense in \( X \).

Then \( N_i = N \cap Y_i \) is nowhere dense in \( Y_i \). Therefore, \( \text{ldim}(N_i) = -1 \), and so \( N_i = \emptyset \). Thus, \( N = \emptyset \). From this it follows that \( \text{ldim}(N) = -1 \), and hence \( \text{ldim}(X) = 0 \).

**Inductive step** \((n \geq 0)\): Suppose for any family of spaces \( \{Y'_i\mid i \in \omega\} \), if \( 0 \leq \text{ldim}(Y'_i) \leq n \) for each \( i \), then \( \text{ldim}(\bigoplus_{i \in \omega} Y'_i) \leq n \). Assume \( 0 \leq \text{ldim}(Y_i) \leq n + 1 \) for each \( i \in \omega \). Let \( N \) be nowhere dense in \( X \). Then \( Y'_i = N \cap Y_i \) is nowhere dense in \( Y_i \). Therefore, \( \text{ldim}(Y'_i) \leq n \).

By the inductive hypothesis, \( \text{ldim}(N) \leq n \). Thus, \( \text{ldim}(X) \leq n + 1 \). \( \square \)
Lemma 7.15. For $n \geq 1$, $\text{ldim}(Z_n) = n$.

Proof. Since $\mathfrak{S}_{n+1}$ is an interior image of $Z_n$, the $(n+1)$-element chain is an interior image of $Z_n$. By Theorem 5.6, $\text{ldim}(Z_n) \geq n$. We show that $\text{ldim}(Z_n) \leq n$ by induction on $n \geq 1$.

Base case ($n = 1$): Let $N$ be nowhere dense in $Z_1$. Set $N_i = N \cap Y_i$ for each $i \in \omega$. Then $N_i$ is nowhere dense in $Z_1$. Noting $Y_i$ is a closed subspace of $Z_1$ homeomorphic to $Y$ (which is a crowded $T_1$-space), it follows that $N_i$ is nowhere dense in $Y_i$. Because $Y$ is nodec, $Y_i$ is nodec, and so $N_i$ is closed in $Y_i$. Let $N'$ be the union of the $N_i$ in the topological sum of the $Y_i$ which is the preimage of the adjunction space $Z_1$. Then $N'$ is closed in the sum. Since $N'$ is the preimage of $N$, we see that $N$ is closed in $Z_1$. Therefore, $Z_i$ is nodec. Because $Z_1$ is a $T_1$-space, it follows from Lemma 7.5(2) that $\text{ldim}(Z_1) \leq 1$.

Inductive step ($n \geq 1$): Assume $\text{ldim}(Z_n) = n$. Since $Z_{n+1}$ was constructed in three stages, our proof is also in three stages. First we show that $\text{ldim}(Y \cup X_i) \leq n + 1$, next that $\text{ldim}(A_i) \leq n + 1$, and finally that $\text{ldim}(Z_{n+1}) \leq n + 1$.

Stage 1: Since $\text{ldim}(Z_n) = n$ and each $X_i \subseteq Z_n$, by Lemma 5.3, $\text{ldim}(X_i) \leq n$. Also, the $(n+1)$-element chain is an interior image of $X_i$, giving that $\text{ldim}(X_i) \geq n$. Thus, $\text{ldim}(X_i) = n$.

Let $N$ be nowhere dense in $Y \cup X_i$, and set $M = N \cap Y$. Then $M$ is nowhere dense in $Y \cup X_i$. Let $U$ be an open subset of $Y$ contained in $C_YM$. Since $Y$ is open in $Y \cup X_i$, we have that $U$ is open in $Y \cup X_i$ and is contained in $C_YM \subseteq CM$. Because $M$ is nowhere dense in $Y \cup X_i$, we obtain $U = \emptyset$, and so $M$ is nowhere dense in $Y$. Since $Y$ is nodec, $M$ is closed and discrete in $Y$. By the construction of $Y \cup X_i$, each $x \in X_i$ is the image of a set of points each remote from $Y$, and hence $CM \cap X_i = \emptyset$. Thus, $CM \subseteq Y$, from which it follows that $C_YM = CM$. Therefore, since $M$ is closed in $Y$, it is closed in $Y \cup X_i$. Consequently, $M$ is closed in $N$. In fact, $M$ is clopen in $N$ since $Y$ is open and $M = N \cap Y$. Therefore, $N$ is the disjoint union of $M$ and $N \cap X_i$. As $M$ is discrete, $\text{ldim}(M) \leq 0$. Also, since $N \cap X_i$ is a subspace of $X_i$, we have $\text{ldim}(N \cap X_i) \leq \text{ldim}(X_i) = n$. By Lemma 7.14, $\text{ldim}(N) \leq n$. Thus, $\text{ldim}(Y \cup X_i) \leq n + 1$.

Stage 2: Let $N$ be nowhere dense in $A_i$. Set $N_j = N \cap Y_{i,j}$. Recalling that $Y_{i,j} \cup X_i$ is homeomorphic to $Y \cup X_i$ by replacing $M$ by $N_j$ and $Y \cup X_i$ by $Y_{i,j} \cup X_i$ in the proof of Stage 1, we see that $N_j$ is closed in $Y_{i,j} \cup X_i$ and $N_j \cap X_i = \emptyset$ for all $j \in \omega$. Therefore, $\bigcup_{j \in \omega} N_j$ is closed in the topological sum $\bigoplus_{j \in \omega}(Y_{i,j} \cup X_i)$. Since $\bigcup_{j \in \omega} N_j$ is also saturated in $\bigoplus_{j \in \omega}(Y_{i,j} \cup X_i)$, it is closed in $A_i$, and hence closed in $N$. Also, $\bigcup_{j \in \omega} N_j = N \cap \bigcup_{j \in \omega} Y_{i,j}$ is open in $N$ since $\bigcup_{j \in \omega} Y_{i,j}$ is open in $A_i$. Therefore, $N$ is the disjoint union of $N \cap X_i$ and $\bigcup_{j \in \omega} N_j$. By Lemma 7.14, $\text{ldim}\left(\bigcup_{j \in \omega} N_j\right) \leq 1 \leq n$ since $\text{ldim}(N_j) \leq \text{ldim}(Y_{i,j}) \leq 1$. Also $\text{ldim}(N \cap X_i) \leq \text{ldim}(X_i) = n$, so utilizing Lemma 7.14 again yields $\text{ldim}(N) \leq n$. Thus, $\text{ldim}(A_i) \leq n + 1$.

Stage 3: Let $N$ be nowhere dense in $Z_{n+1}$. Set $N_i = (N \cap A_i) \setminus X_i$. By recognizing that $N_i$ is realized within the discussion of Stage 2 as $\bigcup_{j \in \omega} N_{i,j}$, we see that each $N_i$ is closed in $A_i$, and hence $\bigcup_{i \in \omega} N_i$ is closed in $\bigoplus_{i \in \omega} A_i$. Moreover, $\bigcup_{i \in \omega} N_{i}$ is saturated, and so $\bigcup_{i \in \omega} N_{i}$ is closed in $Z_{n+1}$. Therefore, $\bigcup_{i \in \omega} N_{i}$ is also closed in $N$. But $\bigcup_{i \in \omega} N_{i} = N \cap (Z_{n+1} \setminus Z_n)$, so $\bigcup_{i \in \omega} N_{i}$ is open in $N$. Thus, $N$ is the disjoint union of $N \cap Z_n$ and $\bigcup_{i \in \omega} N_{i}$. Since $\text{ldim}(N_i) \leq \text{ldim}(A_i \setminus X_i) = \text{ldim}\left(\bigoplus_{j \in \omega} Y_{i,j}\right) \leq 1$, Lemma 7.14 yields that $\text{ldim}\left(\bigcup_{i \in \omega} N_{i}\right) \leq 1 \leq n$. Also $\text{ldim}(N \cap Z_n) \leq \text{ldim}(Z_n) = n$, so by Lemma 7.14, $\text{ldim}(N) \leq n$. Consequently, $\text{ldim}(Z_{n+1}) \leq n + 1$.  

7.3. Completeness. Since $S4.Z_n$ has the finite model property, $S4.Z_n$ is the logic of finite uniquely rooted $S4$-frames $\mathfrak{F}$ of depth $\leq n + 1$. Since each such $\mathfrak{F}$ can be unraveled into a
uniquely rooted finite quasi-tree $\mathfrak{T}$ whose depth is $\leq n + 1$, we see that $S4.\mathcal{Z}_n$ is the logic of uniquely rooted finite quasi-trees $\mathfrak{T}$ of depth $\leq n + 1$.

Let $\mathcal{Q}_n$ be the quasi-tree whose skeleton is $\mathfrak{T}_n$ and in which the root cluster is the only trivial cluster and all other clusters are countably infinite. Clearly identifying the clusters yields an onto p-morphism $p_n : \mathcal{Q}_n \to \mathfrak{T}_n$. Because every uniquely rooted finite quasi-tree of depth $\leq n + 1$ is an interior image of $\mathcal{Q}_{n+1}$, we see that $S4.\mathcal{Z}_n$ is the logic of $\mathcal{Q}_{n+1}$. Since we will utilize this fact, we state it as a lemma.

**Lemma 7.16.** $S4.\mathcal{Z}_n$ is the logic of $\mathcal{Q}_{n+1}$.

Since $\dim(Z_n) = n$ and $Z_n$ is $T_1$, we see that $Z_n \models S4.\mathcal{Z}_n$. Therefore, to show that $S4.\mathcal{Z}_n$ is the logic of $Z_n$, in view of Lemma 7.16, it is sufficient to prove that $\mathcal{Q}_{n+1}$ is an interior image of $Z_n$. The idea of the proof is to ‘fatten’ the mapping $\alpha_n : Z_n \to \mathfrak{T}_{n+1}$ to a mapping $Z_n \to \mathcal{Q}_{n+1}$. Let $\mathcal{C}_\kappa$ be the $\kappa$-cluster as depicted in Figure 9.

![Figure 9. The $\kappa$-cluster.](image)

**Lemma 7.17.** A space $X$ is $\kappa$-resolvable iff $\mathcal{C}_\kappa$ is an interior image of $X$.

**Proof.** First suppose that $X$ is $\kappa$-resolvable. Then there is a dense partition $\{D_\lambda : \lambda < \kappa\}$ of $X$. Define $f : X \to \mathcal{C}_\kappa$ by $f(x) = w_\lambda$ if $x \in D_\lambda$. Clearly $f$ is a well-defined onto map. Moreover, for each $\lambda < \kappa$, we have:

$$\text{Cf}^{-1}(w_\lambda) = \text{C}(D_\lambda) = X = f^{-1}(\{w_\lambda : \lambda < \kappa\}) = f^{-1}(\text{CR}\{w_\lambda\}).$$

Thus, $f$ is an interior map.

Conversely, let $f : X \to \mathcal{C}_\kappa$ be an onto interior map. Then $\{f^{-1}(w_\lambda) : \lambda < \kappa\}$ is a partition of $X$ such that

$$\text{Cf}^{-1}(w_\lambda) = f^{-1}(\text{CR}\{w_\lambda\}) = f^{-1}(\{w_\lambda : \lambda < \kappa\}) = X.$$

Thus, $\{f^{-1}(w_\lambda) : \lambda < \kappa\}$ is a dense partition of $X$, and hence $X$ is $\kappa$-resolvable. \[\square\]

**Theorem 7.18.** For each $n \geq 1$, $S4.\mathcal{Z}_n$ is the logic of $Z_n$.

**Proof.** As we already pointed out, in view of Lemma 7.16, it is sufficient to show that $\mathcal{Q}_{n+1}$ is an interior image of $Z_n$. The proof is by induction on $n$.

Let $n = 1$. Let $\mathcal{C}_1$ be the maximal cluster in $\mathcal{Q}_2$ whose $p_2$-image is $w_i \in \text{max}(\mathfrak{T}_2)$ (here we are using the enumeration of $\text{max}(\mathfrak{T}_2)$ as it appears in the recursive step of defining the $Z_n$). So $\mathcal{C}_1 = p_2^{-1}(w_i)$. Since each $Y_i \setminus \{y_i\}$ is an open subspace of $Y_i$, $Y_i$ is homeomorphic to $Y$, and $Y$ is $\omega$-resolvable, we see that each $Y_i \setminus \{y_i\}$ is $\omega$-resolvable. As $Y_i \setminus \{y_i\}$ is homeomorphic to the subspace $Y_i \setminus \{y\}$ of $Z_1$, by Lemma 7.17, there is an onto interior map $f_i : Y_i \setminus \{y\} \to \mathcal{C}_i$. Define $f : Z_1 \to \mathcal{Q}_2$ by

$$f(z) = \begin{cases} f_i(z) & \text{if } z \in Y_i \setminus \{y\} \\ r & \text{if } z = y \end{cases}$$

Since $\{Y_i \setminus \{y\} \mid i \in \omega\} \cup \{y\}$ is a partition of $Z_1$ and each $f_i$ is onto, $f$ is a well-defined onto map. Let $w \in \mathcal{Q}_2$. Suppose $w \in \mathcal{C}_i$ for some $i \in \omega$. Then

$$f^{-1}(\text{CR}\{w\}) = f^{-1}(\mathcal{C}_i \cup \{r\}) = f_i^{-1}(\mathcal{C}_i) \cup \{y\} = (Y_i \setminus \{y\}) \cup \{y\} = \text{C}_Z(Y_i \setminus \{y\}) = \text{C}_Z f^{-1}(w).$$
Otherwise \( w \) is the root, and so
\[
f^{-1}(C_R\{w\}) = f^{-1}(w) = \{y\} = C_{Z_1}\{y\} = C_{Z_1}f^{-1}(w).
\]

Thus, \( f : Z_1 \to Q_2 \) is an onto interior map.

Let \( n \geq 1 \). Suppose \( g : Z_n \to Q_{n+1} \) is an onto interior map. Identify \( Q_{n+2} \) with the subframe \( Q_{n+2} \setminus \max_R(\{w\}) \). Let \( w_{i,j} \in \max(\mathcal{F}_{n+2}) \) be the \( j \)-th child of \( w_i \in \max(\mathcal{F}_{n+1}) \) (as in the recursive step of building the \( Z_n \)). Let \( \mathcal{C}_{i,j} \) be the maximal cluster in \( Q_{n+2} \) whose \( p_{n+2}' \)-image is \( w_{i,j} \). So \( \mathcal{C}_{i,j} = p_{n+2}'(w_{i,j}) \). Also, let \( \mathcal{C}_i \) be the maximal cluster in \( Q_{n+1} \) whose \( p_{n+2}' \)-image is \( w_i \in \max(\mathcal{F}_{n+1}) \). So \( \mathcal{C}_i = p_{n+2}'(w_i) \). Since each subspace \( Y_{i,j} \) of \( Z_{n+1} \) is homeomorphic to \( Y \), we see that \( Y_{i,j} \) is \( \omega \)-resolvable. By Lemma 7.17, there is an onto interior map \( f_{i,j} : Y_{i,j} \to \mathcal{C}_{i,j} \). Define \( f : Z_{n+1} \to Q_{n+2} \) by
\[
f(z) = \begin{cases} f_{i,j}(z) & \text{if } z \in Y_{i,j} \\ g(z) & \text{if } z \in Z_n \end{cases}
\]

Since \( \{Y_{i,j} \mid i,j \in \omega \} \cup \{Z_n\} \) is a partition of \( Z_{n+1} \) and the \( f_{i,j} \) and \( g \) are onto, \( f \) is a well-defined onto map. Let \( w \in Q_{n+2} \). Suppose \( w \in \mathcal{C}_{i,j} \) for some \( i,j \in \omega \). Because \( Z_n \) is closed in \( Z_{n+1} \), both \( g \) and \( f_{i,j} \) are interior maps, and \( g^{-1}C_R(\mathcal{C}_i) = X_i \), we have
\[
f^{-1}(C_R(\{w\}) = f^{-1}(\mathcal{C}_{i,j} \cup C_R(\mathcal{C}_i)) = f_{i,j}^{-1}(\mathcal{C}_{i,j}) \cup g^{-1}C_R(\mathcal{C}_i) = Y_{i,j} \cup X_i = C_{Z_{n+1}}(Y_{i,j}) = C_{Z_n}(C_{Y_{i,j}}(f_{i,j}^{-1}(w))) = C_{Z_{n+1}}f^{-1}(w).
\]

Otherwise \( w \in Q_{n+1} \), and so
\[
f^{-1}(C_R(\{w\}) = g^{-1}(C_R(\{w\}) = C_{Z_n}g^{-1}(w) = C_{Z_{n+1}}f^{-1}(w).
\]

Thus, \( f : Z_{n+1} \to Q_{n+2} \) is an onto interior map.

As an immediate consequence, we obtain:

**Corollary 7.19.** For each \( n \geq 1 \), \( S4.Z_n \) is the logic of a countable crowded \( \omega \)-resolvable Tychonoff space of localic Krull dimension \( n \).

Moreover, since \( S4.Z = S4.Z_1 \), we obtain the following topological completeness for the Zeman logic:

**Corollary 7.20.** \( S4.Z \) is the logic of a countable crowded \( \omega \)-resolvable Tychonoff nodec space.

That \( S4.Z \) is the logic of nodec spaces was shown in [1, Thm. 4.6], but the proof required the use of Alexandroff nodec spaces. The above corollary strengthens this result considerably by providing a topologically “nice” nodec space whose logic is \( S4.Z \).

**References**


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