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DERIVED SETS IN EUCLIDEAN SPACES AND MODAL LOGIC

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1. Introduction

Let X be a topological space. The set P(X) of all its subsets can be considered as a closure algebra C(X); this algebra has standard Boolean operations ($\cup$, $\cap$, $-$) and the topological closure operation. The modal logic of this algebra (denoted by L(C(X)) is a normal extension of S4. In the well-known paper [1] McKinsey and Tarski proved that, in many cases, L(C(X))=S4, in particular, for all locally-Euclidean spaces.

In an appendix to their paper the authors ask about properties of derivative algebras over topological spaces. A *derivative algebra* D(X) over a space X is defined as a Boolean algebra P(X) together with the derivative operation (recall that a derivative dY of a set Y is the set of all limit points of Y). The modal logic of this algebra (L(D(X)) can be defined equivalently in "Scott - Montague style".

Let us recall this definition. Modal formulas are built from the set PV of propositional variables, classical connectives $\vee$, $\neg$, and the unary connective $\Box$. Other connectives ($\supset$, $\wedge$, $\Diamond$, **T**) are considered as abbreviations (in particular, **T** is $p\vee\neg p$, $\Diamond A$ is $\neg\Box\neg A$). We also set:

$\bar{\Box}A=\Box A\wedge A$, $\bar{\Diamond}A=A\vee\Diamond A$.

A *valuation* in a space X is a map φ : PV $\rightarrow$ P(X); the pair (X,φ) is then called a *model* in X, and the triple (X,φ,x), with $x\in X$ is a *world* in this model. The predicate "a modal formula A is true in a world (X,φ,x)" (notation: $(X,\varphi,x)\models A$, usually abbreviated to $x\models A$) is defined inductively:

1) If $A \in PV$ then $(X,\varphi,x) \models A$ iff $x \in \varphi(A)$.
2) If $A = B \vee C$ then $x \models A$ iff $x \models B$ or $x \models C$.
3) If $A = \neg B$ then $x \models A$ iff not $x \models B$.
4) If $A = \Box B$ then $x \models A$ iff there is a neighbourhood U of x in X such that $y \models B$ for any $y \in U-\{x\}$.

A formula A is called *valid* in X (notation: $X \models A$) iff A is true in any world of any model in X. Then the logic $L(D(X))$ is exactly the set of all modal formulas valid in X.

In the present paper a *modal logic* is a consistent set of modal formulas (i.e. there are formulas besides this set) containing all classical propositional tautologies and closed under three rules: substitution, modus ponens and necessitation ($\vdash A \Rightarrow \vdash \Box A$). If L is a logic, and Γ is a set of formulas, $L+\Gamma$ denotes the least modal logic containing $(L \cup \Gamma)$.

Recall also that **K4** is the least modal logic containing $\Box(p \supset q) \supset (\Box p \supset \Box q)$ and $\Box p \supset \Box\Box p$ $(p,q \in PV)$, **S4** = **K4** + $\Box p \supset p$; **D4** = **K4** + $\Diamond\top$.

It is well-known that $L(D(X))$ is always a modal logic contaning **K4**. On the other hand, it does not contain **S4** (since the formula $\Box p \supset p$ is false in (X,φ,x) provided that $\varphi(p) = X - \{x\}$).

As it was observed by Kuratowski [2], for any $n \geq 1$, $L(D(\mathbf{R}^n))$ contains **D4**. He found also the identity

(1) $d((x \cap d(-x)) \cup (-x \cap dx)) = dx \cap d(-x)$

which holds in $D(\mathbf{R}^n)$ for any $n \geq 2$ but is falsified in $D(\mathbf{R})$. In fact the essential part of (1) is the inequality

(2) $dx \cap d(-x) \leq d((x \cap d(-x)) \cup (-x \cap dx))$

since the converse holds in any derivative algebra. (2) corresponds to the modal formula

(3) $(\Diamond p \wedge \Diamond \neg p) \supset \Diamond((p \wedge \Diamond \neg p) \vee (\neg p \wedge \Diamond p))$,

and by distributivity, the latter is equivalent in **D4** to

(4) $(\Diamond p \wedge \Diamond \neg p) \supset \Diamond(\bar{\Diamond} p \wedge \bar{\Diamond} \neg p)$,

or, by duality, to

$\mathbf{G_1}$: $\Box(\bar{\Box} p \vee \bar{\Box} \neg p) \supset (\Box p \vee \Box \neg p)$.

Now a problem posed by McKinsey and Tarski [1,p.652] can be formulated in logical terms.

PROBLEM. To verify or to disprove the following statements:

(MT1) $L(D(\mathbf{R})) = \mathbf{D4}$.

(MT2) $L(D(J)) = \mathbf{D4}$. (J is Cantor's discontinuum.)

(MT3) $L(D(\mathbf{Q})) = \mathbf{D4}$.

(MT4) $L(D(\mathbf{R}^n)) = \mathbf{D4}+\mathbf{G_1}$.

(MT5) $L(D(\mathbf{R}^n)) = L(D(\mathbf{R}^2))$ for any $n>2$.

Our aim is to prove (MT2) - (MT5) and to disprove (MT1). An additional consequence of our proof is the decidability of $\mathbf{D4G_1}=\mathbf{D4}+\mathbf{G_1}$.

2. Completeness of $\mathbf{D4G_1}$

LEMMA 1 (cf. [1]). $\mathbf{D4} \subseteq L(D(X))$ for any dense-in-itself topological space X. (Recall that a space is called *dense-in-itself* iff it has no isolated points.)

We omit the proof because it is well-known that $\mathbf{K4} \subseteq L(D(X))$, and $X \models \Diamond \top$ immediately follows from the density of X.

LEMMA 2[1]. Let X be a topological space satisfying the following condition:

(5) for any open U and any $x \in U$ there is open $V \subseteq U$ such that $x \in V$ and $(V - \{x\})$ is connected.

Then $X \models \mathbf{G_1}$.

Proof. Assume the contrary, then for some world

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[1] Certainly, this fact might be known to Kuratowski in 1920.

$(X,\varphi,x) \models \Box(\bar{\Box}p \vee \bar{\Box}\neg p) \wedge \neg\Box p \wedge \neg\Box\neg p$.

For any formula A, let us set

$|A| = \{y|\ (X,\varphi,y) \models A\}$

Since $x \in |\Box(\bar{\Box}p \vee \bar{\Box}\neg p)|$ there exists an open U such that $x \in U$, $U-\{x\} \subseteq |\bar{\Box}p \vee \bar{\Box}\neg p| = I|p| \cup I|\neg p|$ (I means the interior operation in X). By (5), U contains a neighbourhood V of x such that $\dot{V}=V-\{x\}$ is connected.But $x \in |\neg\Box p| \cap |\neg\Box\neg p|$, hence $\dot{V}\cap|p|$, $\dot{V}\cap|\neg p| \neq \emptyset$. On the other hand, $\dot{V} \subseteq |\bar{\Box}p| \cup |\bar{\Box}\neg p|$ yields: $\dot{V}\cap|p| \subseteq |\bar{\Box}p|$, $\dot{V}\cap|\neg p| \subseteq |\bar{\Box}\neg p|$. Thus,V is not connected, and this is a contradiction.■

Now we shall describe Kripke semantics for G_1 . We suppose the reader to be familiar with notions of truth in a world of a Kripke model, and of validity in a Kripke frame. For a frame F, $L(F) = \{ A|\ F \models A \}$ is a modal logic called the *modal logic of* F. A class of frames C *determines* a logic λ iff $\lambda = \bigcap_{F \in C} L(F)$.

Let (W,R) be a transitive Kripke frame; we define some other relations on W:

(6) $x\bar{R}y \iff xRy \vee x=y$ (the reflexive closure of R).

(7) $x\hat{R}y \iff \exists z\ (x\bar{R}z\ \&\ y\bar{R}z)$ (the *convergence relation* in (W,R)).

(8) $\hat{R}_x = \hat{R} \cap (R(x) \times R(x))$, $x \in W$ (the convergence relation in R(x)).

(9) $\tilde{R}_x = \bigcup_{n=1}^{\infty} (\hat{R}_x)^n$ is the transitive closure of R (the *connectivity relation* in R(x)).

It is clear that $\tilde{R}_x$ is an equivalence relation.

We call a frame (W,R) *locally connected* iff

(10) $\forall x,y,z \in W\ (xRy\ \&\ xRz \Rightarrow y\tilde{R}_x z)$.

PROPOSITION 3. For any transitive Kripke frame (W,R),

$(W,R) \models G_1$ iff (W,R) is locally connnected.

<u>Proof</u>. ("If"). Assume the contrary, then for some world

(11) $(W,R,\varphi,x) \models \Box(\bar{\Box}p \vee \bar{\Box}\neg p) \wedge \neg\Box p \wedge \neg\Box\neg p$.

Let us prove that

(12) $\forall y,z \in R(x)\ (y \models \bar{\Box}p\ \&\ y\tilde{R}_x z \Rightarrow z \models \bar{\Box}p)$.

It is sufficient to show that for any $n \geq 0$

(13) $\forall y,z \in R(x)\ (y \models \bar{\Box}p\ \&\ y\hat{R}^n_x z \Rightarrow z \models \bar{\Box}p)$.

(by a definition, $\hat{R}^0_x$ is the equality relation).

The case $n=0$ is trivial, so let us suppose (13) to be true for n and check it for $(n+1)$. Suppose also $y\hat{R}^{n+1}_x z\ \&\ y \models \bar{\Box}p$, then for some t, $y\hat{R}^n_x t\ \&\ t\hat{R}z$, so for some u, $t\bar{R}u\ \&\ z\bar{R}u$:

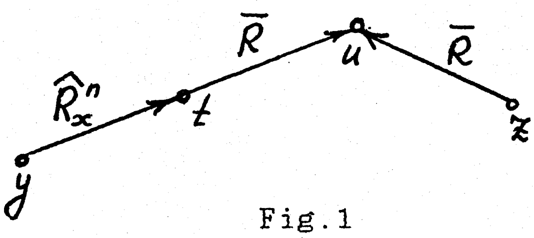


Fig.1

Since $y \models \bar{\Box}p$ we have $t \models \bar{\Box}p$ by (13), and $u \models p$ (since $t\bar{R}u$), so $z \models \bar{\Diamond}p$ (since $z\bar{R}u$). On the other hand, (11) and xRz yield $z \models \bar{\Box}p \vee \bar{\Box}\neg p$. Thus $z \models \bar{\Box}p$.

Since (W,R) is locally connected we deduce from (12) that

(14) $x \models \Diamond\bar{\Box}p \supset \Box\bar{\Box}p$

But $x \models \Diamond p \wedge \Box(\bar{\Box}p \vee \bar{\Box}\neg p)$ (by (11)), so $x \models \Diamond\bar{\Box}p$, and $x \models \Box\bar{\Box}p$ (by (14)), consequently $x \models \Box p$, in contradiction to (11).

("Only if"). Assume the contrary, then for some x,y,z we have $y,z \in R(x)$, but not $y\tilde{R}_x z$. Let $\varphi: PV \longrightarrow P(W)$ be a valuation such that $\varphi(p) = \tilde{R}_x(y)$. Then

$(W,R,\varphi,x) \models \Diamond p \wedge \Diamond\neg p$.

On the other hand, $(W,R,\varphi,x) \models \Box(\bar{\Box}p \vee \bar{\Box}\neg p)$. Really, take any $t \in R(x)$. If $t \in \varphi(p)$ then $t \in \tilde{R}_x(y)$, and $\bar{R}(t) \subseteq \tilde{R}_x(y)$ (since $\tilde{R}_x \circ \bar{R} \subseteq \tilde{R}_x$ by (7) and (9)). Hence $t \models \bar{\Box}p$.

If $t \notin \varphi(p)$ then $t \notin \tilde{R}_x(y)$, so $\bar{R}(t) \cap \tilde{R}_x(y) = \emptyset$ (since $\tilde{R}_x \circ (\bar{R})^{-1} \subseteq \tilde{R}_x$ by (7) and (9)), and $t \models \bar{\Box}\neg p$.

Thus (11) holds for this φ. ■

Our next step is to prove Kripke-completeness of $\mathbf{D4G_1} = \mathbf{D4}+\mathbf{G_1}$. This is done via weak canonical models, so let us recall corresponding definitions (cf.[3]).

$MF{\restriction}k$ will denote the set of all modal formulas whose propositional variables are among $PV{\restriction}k = \{p_1, \ldots, p_k\}$. For a modal logic L, the set $L{\restriction}k = L \cap MF{\restriction}k$ is called the *k-restriction* of L.

A definition of a *weak canonical model* $\mathfrak{M}_{L\restriction k} = (W_{L\restriction k}, R_{L\restriction k}, \varphi)$ is analogous to the non-restricted case. Namely, $W_{L\restriction k}$ is the set of all maximal L-consistent subsets of $MF{\restriction}k$ (consistency of a set x means that $\neg \bigwedge_{i=1}^{m} A_i \notin L$ whenever $A_1, \ldots, A_m \in x$);

(15) $xR_{L\restriction k}y \iff \forall A\ (\Box A \in x \Rightarrow A \in y)$;

and $\varphi: PV \longrightarrow P(W_{L\restriction k})$ is a valuation such that for any $i \geq 1$) $\varphi(p_i) = \{ x \in W_{L\restriction k} \mid p_i \in x \}$ if $i \leq k$, and $\varphi(p_i) = \emptyset$ for any $i > k$. $F_{L\restriction k} = (W_{L\restriction k}, R_{L\restriction k})$ is called a *weak canonical frame*. It is transitive for any L containing **K4**.

The following fact is known (in the non-restricted case) as the Fundamental Theorem of modal logic (cf.[4]).

PROPOSITION 4. For any $A \in MF{\restriction}k$ and any $x \in W$

1) $(\mathfrak{M}_{L\restriction k}, x) \models A$ iff $A \in x$.

2) $\mathfrak{M}_{L\restriction k} \models A$ iff $A \in L{\restriction}k$.

From now on all Kripke frames in study are transitive. Recall that a *clot* (or a *cluster*, cf.[4]) in a frame $F=(W,R)$ is either a maximal non-empty subset $C \subseteq W$ such that $C \times C \subseteq R$, or an R-irreflexive singleton. The latter is called a *degenerate* clot. A reflexive one-element clot is called *trivial*. A clot C is called *maximal* (respectively, *minimal*) in V ($V \subseteq W$) iff $R(C) \cap V \subseteq C$ (respectively, iff $R^{-1}(C) \cap V \subseteq C$). $x \in W$ is called *minimal* (*maximal*) in V iff it belongs to a minimal (maximal) clot. A *successor* of a clot C in W is a minimal clot in (R(C)-C). A transitive Kripke frame (W,R) is said to have *Zorn property* iff $\forall x\in W\ \exists y\ (xRy\ \&\ y$ is maximal in $W)$.

LEMMA 5 [5]. Any weak canonical frame of a modal logic containing **K4** has Zorn property.

Proof. Suppose $\mathbf{K4}\subseteq L$, and let $F=(W,R)$ be a weak canonical frame of L ($W = W_{L\restriction k}$, $R = R_{L\restriction k}$).

For $x,y\in W$ we set

(16) $x\preccurlyeq y \iff xRy\ \&\ \neg yRx \vee x=y$.

In the case $\mathbf{K4}\subseteq L$, R is known to be transitive, so $\preccurlyeq$ is a partial order. Now let us show that every chain in $(W,\preccurlyeq)$ has an upper bound. Indeed, let Z be such a chain. We assume that

(17) Z has no $\preccurlyeq$-maximal elements

(otherwise there is nothing to prove), and consider

(18) $S = \bigcup_{z\in Z} \{A\mid \Box A\in z\}$.

This set is L-consistent. Really, suppose $A_1,\dots,A_m\in S$, then for any i, $\Box A_i$ belongs to some $z_i\in Z$. Since Z is a chain, we may also suppose (without losing generality) that $z_1\preccurlyeq z_2\preccurlyeq\dots\preccurlyeq z_m$. By (17), Z has no maximal elements, so $R(z_m)\neq\emptyset$. Let us pick some $v\in R(z_m)$. Due to the transitivity and (15) we have z_iRv, and $A_i\in v$ $(1\le i\le m)$. Thus $\neg(\bigwedge_{i=1}^{m} A_i) \notin L$ by the consistency of v.

Since S is consistent, by Lindenbaum lemma, $S\subseteq u$ for some $u\in W$. This u is an upper bound for Z. Really, zRu for any $z\in Z$ (by (15),(18)). If uRz_0 for some $z_0\in Z$ then zRz_0 for any $z\in Z$, so $z\preccurlyeq z_0$ since Z is a chain. But this contradicts to (17). Therefore, $z\preccurlyeq u$.

Now from Zorn lemma we see that for any $x\in W$, R(x) has a maximal element, say y. This y is maximal in F .

For suppose $y \neq t$, yRt ; then not $y \prec t$ (since y is $\preccurlyeq$-maximal), and thus tRy (by (16)). But then t is in the same clot as y. Therefore the clot containing y is maximal.■

Now let a number k be fixed. For any $t \subseteq \{1,...,k\}$ we set

(19) $q(t) = \bigwedge_{i \in t} p_i \wedge \bigwedge_{1 \le i \le k,\, i \notin t} \neg p_i$

If (W,R,φ,x) is a world of a Kripke model we set

(20) $\varepsilon(x) = \{i \mid 1 \le i \le k \ \&\ (W,R,\varphi,x) \models p_i\}$, $q(x) = q(\varepsilon(x))$.

The following statement is trivial:

LEMMA 6. $(W,R,\varphi,x) \models q(t) \iff \varepsilon(x) = t$.

Two worlds (W,R,φ,x) and (W,R,φ,y) in a Kripke model are called *MF↾k-equivalent* iff $(W,R,\varphi,x) \models A \iff (W,R,\varphi,y) \models A$ for any $A \in MF{\restriction}k$. A Kripke model is called *k-distinguished* iff every two its $MF{\restriction}k$-equivalent worlds are equal. It follows immediately from proposition 4 that $\mathfrak{M}_{L{\restriction}k}$ is k-distinguished.

For any $\Delta \subseteq P(\{1,...,k\})$ we set

(21) $\alpha(\Delta) = \bigwedge_{t \in \Delta} \bar{\Diamond} q(t) \wedge \bigwedge_{1 \le t \le k,\, t \notin \Delta} \neg \bar{\Diamond} q(t)$.

If C is a clot in (W,R,φ) we set

(22) $\delta(C) = \{\varepsilon(x) \mid x \in C\}$, $\alpha(C) = \alpha(\delta(C))$.

LEMMA 7. Let C,D be maximal clots in a k-distinguished Kripke model. If $\delta(C) = \delta(D)$ then $C = D$.

Proof. We have:

$\forall u,v \in W\ (uRv \ \&\ vRu \ \&\ \varepsilon(u) = \varepsilon(v) \Rightarrow u = v)$;

indeed, it is easily seen by an induction on $A \in MF{\restriction}k$ that $u \models A$ iff $v \models A$ (provided that u,v satisfy the premise), and then $u = v$ since our model is k-distinguished.

Thus the relation $\varepsilon(a) = \varepsilon(a')$ delivers a bijective correspondence between $a \in C$ and $a' \in D$. But from $\delta(C) = \delta(D)$ we conclude (again by an induction) that a and a′ are MF↾k-equivalent. Therefore C = D.■

LEMMA 8. The set of all maximal clots in a k-distinguished Kripke model is finite.

Proof. By lemma 7, this set is equivalent to some subset of P({1,...,k}).■

LEMMA 9. Let (W,R,φ) be a transitive k-distinguished Kripke model, C be its maximal clot, and x be its maximal element. Then $(W,R,\varphi,x)\models\alpha(C)$ iff $x\in C$.

Remark. This fact seem to be well-known, nevertheless it is not mentioned e.g. in [4] or in [3]. Note also that normal forms in S5 consist of disjuncts $\alpha(\Delta)$.

Proof. "If" part is an easy consequence of lemma 6. To prove "only if" suppose $x\models\alpha(C)$. Let C′ be the clot containing x, then

$$\delta(C) = \delta(C').$$

Really, $y\in C'$ only if $x\models\bar{\Diamond}q(y)$ (by lemma 6), only if $\varepsilon(y)\in\delta(C)$ (since $x\models\alpha(C)$). Hence $\delta(C')\subseteq\delta(C)$.

Conversely, $y\in C$ only if $x\models\bar{\Diamond}q(y)$ (since $x\models\alpha(C)$), only if $\exists z\in C'$ $z\models q(y)$ (since x is maximal), only if $\varepsilon(y)\in\delta(C')$ (by lemma 6).

Finally by lemma 7, C = C′, hence $x\in C$.■

LEMMA 10. Let C be a maximal clot in a weak canonical Kripke model (W,R,φ), then for any $x\in W$, $x\models\bar{\Diamond}\alpha(C)$ iff $C\subseteq\bar{R}(x)$.

Proof. "If" follows immediately from lemma 9. To prove "only if" we apply also lemma 5.■

Recall that a Kripke frame (W,R) is called *serial* iff $R(x)\neq\emptyset$ for any $x\in W$.

THEOREM 11 (Completeness theorem). $\mathbf{D4G_1}$ is determined by the class of all transitive serial locally connected Kripke frames (such a frame will be called further a $\mathbf{D4G_1}$ *-frame*).

Proof. ("Soundness".) Every $A\in\mathbf{D4G_1}$ is valid in any $\mathbf{D4G_1}$-frame; this should be checked only for modal axioms. But axioms of **D4** are known to be valid in any transitive serial frame, and G_1 is valid by proposition 3.

("Completeness".) Assuming that $A \notin \mathbf{D4G_1}$ we have to refute A in some $\mathbf{D4G_1}$-frame. $A \in MF{\restriction}k$ for some k, and then A is not valid in $F_{D4G_1}{\restriction}k$ by proposition 4. Thus it is sufficient to show that $F_{L}{\restriction}k$ is a $\mathbf{D4G_1}$-frame for any L containing $\mathbf{D4G_1}$. The transitivity and the seriality are well-known (cf. [4]), so let us prove the local connectedness. So we consider $\mathfrak{M}_{L{\restriction}k} \underset{\sim}{=} (W,R,\varphi)$, $x \in W$, and prove that

(23) $y\tilde{R}_x z$ for any $y,z \in R(x)$.

By Zorn property (lemma 5) we can choose maximal clots $C \subseteq \bar{R}(y)$, $D \subseteq \bar{R}(z)$; and to obtain (23) it is enough to establish that

(24) $D \subseteq \tilde{R}_x(C)$.

Assume that (24) fails. Let $C_1(=C), C_2, \ldots, C_n$ be all maximal clots in $\tilde{R}_x(C)$ (their number is finite, by lemma 8). From $C \subseteq \bar{R}(y) \subseteq R(x)$ we have (by lemma 10) :

(25) $x \models \Diamond\bar{\Box}\left(\bigvee_{i=1}^{n} \bar{\Diamond}\bar{\Box}\alpha(C_i)\right)$.

From lemma 10 we also see that $\beta = \bigvee_{i=1}^{n} \bar{\Diamond}\bar{\Box}\alpha(C_i)$ is false throughout D, hence

(26) $x \models \Diamond\bar{\Box}\neg\beta$

But

(27) $x \models \Box(\bar{\Box}\beta \vee \bar{\Box}\neg\beta)$

Indeed, suppose $t \in R(x)$. If $t \in \tilde{R}_x(C)$ then for any $u \in \bar{R}(t)$ some C_i is contained in $\bar{R}(u)$ (lemma 5). Thus $u \models \bar{\Diamond}\bar{\Box}\alpha(C_i)$ (lemma 10), and $t \models \bar{\Box}\beta$.

On the other hand, if $t \notin \tilde{R}_x(C)$ then $C_i \cap \bar{R}(t) = \emptyset$ for any i, and $\forall u \in \bar{R}(t)\ u \models \neg\bar{\Diamond}\bar{\Box}\alpha(C_i)$, by lemma 10.

Thus (27) holds. Now from (25) - (27) we conclude that a substitution instance of G_1 is false in x. This contradiction proves (24). Therefore, $F_{L\restriction k}$ is locally connected.■

3. The finite model property

Our next step is to prove the finite model property for **D4G$_1$**. For this purpose we use a variant of the filtration method. To begin with, we recall some facts about filtrations (cf. [4],[6]).

Let $\mathfrak{M} = (W,R,\varphi)$ be a Kripke model, Ψ be a set of formulas closed under subformulas. Elements $x,y\in W$ are called *equivalent modulo* Ψ (in $\mathfrak{M}$) iff

(28) $\forall A\in\Psi.(\mathfrak{M},x)\models A \iff (\mathfrak{M},y)\models A$;

this is denoted by $x\equiv_{\Psi}y$. We also set

(29) $xR_{(\Psi)}y \iff \forall A\ (\Box A\in\Psi\ \&\ (\mathfrak{M},x)\models A \Rightarrow (\mathfrak{M},y)\models A)$.

Let $h\colon\ W \longrightarrow W'$ be an onto map. A model $\mathfrak{M}' = (W',R',\varphi')$ is called a *filtration of* $\mathfrak{M}$ *through* (Ψ,h) iff the following holds:

(30) $\varphi(A)=h^{-1}(\varphi'(A))$ for any $A\in PV\cap\Psi$;

and for any $x,y\in W$

(31) $h(x) = h(y) \Rightarrow x\equiv_{\Psi}y$,

(32) $xRy \Rightarrow h(x)R'h(y)$,

(33) $h(x)R'h(y) \Rightarrow xR_{(\Psi)}y$.

Remark. The construction in [6] is a special case of this one; there h is a canonical map $W \longrightarrow W/\equiv_{\Delta}$ for some Δ containing Ψ.

LEMMA 12. Let $\mathfrak{M} = (W,R,\varphi)$ be a Kripke model, $\mathfrak{M}' = (W',R',\varphi')$ be its filtration through (Ψ,h). Then for any $x\in W$, $A\in\Psi$

(34) $(\mathfrak{M},x)\models A$ iff $(\mathfrak{M}',h(x))\models A$.

Proof. This is a somewhat modified "Filtration theorem" from [4]. The proof is by an induction on the length of A. Let us consider the only non-trivial case: $A = \Box B$. Assume that (34) holds for B, and let us prove it for A.

("If"). Suppose $h(x)\models\Box B$. We have to show that $x\models\Box B$, i.e. $\forall y\in R(x)\ y\models B$. But $y\in R(x)$ only if $h(x)R'h(y)$ (32), only if $h(y)\models B$ (since $h(x)\models\Box B$), only if $y\models B$ (by (34)).

("Only if"). Suppose $x\models\Box B$. We have to prove that $h(x)\models\Box B$, that is $\forall a\in R'(h(x))\ a\models B$. But $a=h(y)$ for some y (since h is onto), and from (33) we see that $xR_{(\Psi)}y$. Therefore $y\models B$ by (29), and $a = h(y)\models B$ by (34).■

LEMMA 13. Let $h: (W,R) \longrightarrow (W',R')$ be an isotone map of Kripke frames (that is, a map satisfying (32)). Then for any $x,y,z\in W$, $y\tilde{R}_x z$ only if $h(y)\tilde{R}'_{h(x)}h(z)$.

Proof. First of all we observe that $y\hat{R}z$ only if $h(y)\hat{R}'h(z)$ (by (7),(32)). Then an easy inductive reasoning shows that $y\hat{R}^n_x z$ only if $h(y)\hat{R}'^n_{h(x)}h(z)$.■

In what follows we assume that $L = \mathbf{D4G_1}$, $\mathfrak{M}_{L\restriction k} = \mathfrak{M} = (W,R,\varphi)$. For $x,y\in W$ we set

(35) $M(x) = \{C \mid C$ is a maximal clot in $\mathfrak{M}$ & $C\subseteq\overline{R}(x)\}$,

(36) $x\sim_\Psi y \iff x\equiv_\Psi y$ & $M(x) = M(y)$.

It is clear that $(\sim_\Psi)$ is an equivalence in W. So we set

(37) $W' = W/(\sim_\Psi)$,

and let $h: W \longrightarrow W'$ be the canonical onto map.

For $a,b\in W'$ we set

(38) $a\underline{R}b \iff \exists x\in a\ \exists y\in b\ xRy$,

(39) $R' = \bigcup_{n=1}^{\infty}(\underline{R})^n$ (the transitive closure of $\underline{R}$),

and finally for $A\in PV$ we set

(40) $\varphi'(A) = h(\varphi(A))$.

LEMMA 14. If $\mathfrak{M}' = (W',R',\varphi')$ is defined by (35) - (40) then $\mathfrak{M}'$ is a filtration of $\mathfrak{M}$ through (Ψ,h).

Proof. We need to check conditions (30) - (33).

(31) holds trivially since $h(x) = h(y)$ iff $x\sim_\Psi y$, only if $x\equiv_\Psi y$ (by (36)).

The only non-trivial inclusion in (30) is:

$h^{-1}(\varphi'(A))$ $(=h^{-1}(h(\varphi(A))))\subseteq\varphi(A)$.

To show this suppose $x\in h^{-1}(h(\varphi(A)))$. Then $h(x) = h(y)$ for some $y\in\varphi(A)$, and $x\equiv_{\Psi}y$ by (31). Since $y\models A$ and $A\in\Psi$ we have $x\models A$,i.e. $x\in\varphi(A)$.

To prove (32) suppose xRy. Then $h(x)\underline{R}h(y)$ by (38), and $h(x)R'h(y)$ by (39).

To prove (33) suppose $h(x)R'h(y)$. Then $h(x)(\underline{R})^n h(y)$ for some $n>0$. Since h is onto there is a sequence $a_0, a_1, \ldots, a_n$ in W' such that $a_0=h(x)$, $a_n=h(y)$, and $\forall i\ a_i \underline{R}\ a_{i+1}$. By (38) there exist $x_i\in a_i$, $y_{i+1}\in a_{i+1}$ such that $x_i R y_{i+1}$:

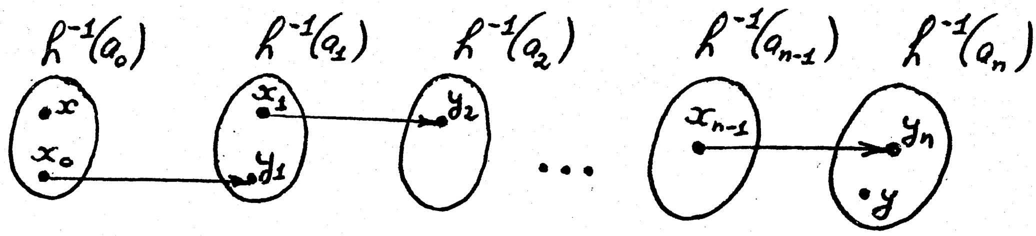


Fig.2

If $\Box A\in\Psi$ and $x\models\Box A$ then $x_0\models\Box A$ (since $x\equiv_{\Psi}x_0$ by (31)). Thus $y_1\models\Box A$ (since x_0Ry_1 and R is transitive). From $x_1\equiv_{\Psi}y_1$ we have $x_1\models\Box A$ etc. So we obtain $x_{n-1}\models\Box A$, $y_n\models A$, and $y\models A$ (since $y_n\equiv_{\Psi}y$). Therefore, $xR_{(\Psi)}y$.■

A Kripke frame F *separates* a modal formula A from a logic L iff all formulas from L are valid in F whereas A is not. A logic L has the *finite model property* (f.m.p.) iff any $A\notin L$ can be separated from L by a finite frame.

THEOREM 15. $\mathbf{D4G_1}$ has the f.m.p.

Proof. Assume that $A\notin L$, $A\in MF\restriction k$. Then A is false in some world $(\mathfrak{M}_{L\restriction k}, x)$ (proposition 4). Take $\mathfrak{M}' = (W', R', \varphi')$ as in lemma 14. The set W' is finite since a $(\sim_{\Psi})$-class of an element x is exactly characterized by its $(\equiv_{\Psi})$-class together with M(x).

But the set $W/\equiv_\Psi$ is finite (it can be imbedded into $P(\Psi)$), and $M(x)$ is a subset of some finite set (lemma 8).

Since not $x\models A$ we see that not $h(x)\models A$ (lemma 12), and to complete the proof it is enough to show that $F' = (W',R')$ is a **D4G$_1$**-frame. But R' is transitive by (39). R is serial [4], so are $\underline{R}$ (by (38)) and R' (since $\underline{R}\subseteq R'$).

Finally, let us prove the local connectedness of F'. According to (10) and (39) this means:

(41) $\forall k\geq 1\ \forall l\geq 1\ \forall a,b,c\in W\ (a\underline{R}^k b\ \&\ a\underline{R}^l c \Rightarrow b(\tilde{R}')_a c)$.

This claim will be proved by an induction on $(k+l)$.

If $k=l=1$ then there exist x_1, $x_2\in h^{-1}(a)$, $y\in h^{-1}(b)$, $z\in h^{-1}(c)$ such that x_1Ry, x_2Rz. By Zorn property (lemma 5) we can choose some maximal $t\in\overline{R}(y)$. Leaving the trivial case $x_2=t$ aside, from (36) and (35) we conclude that x_2Rt. But $F_{L\restriction k}$ is locally

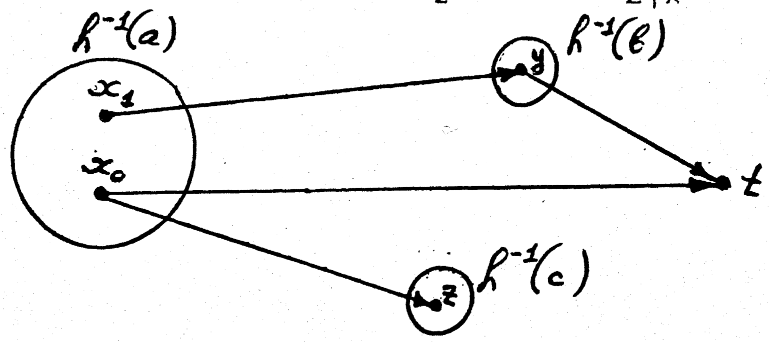


Fig.3

connected (cf. the proof of theorem 11). Thus $t\tilde{R}_{x_2} z$ whence $h(t)\tilde{R}'_a c$ (lemma 13). Consequently $b\tilde{R}'_a c$ (since $aR'bR'h(t)$).

The inductive step is rather trivial. Suppose $\max(k,l)=k>1$. Then $a\underline{R}^{k-1}d$, $d\underline{R}b$ for some d, and thus $d\tilde{R}'_a c$ by the inductive hypothesis. But we have also $dR'b$, therefore $d\tilde{R}'_a b$, and $b\tilde{R}'_a c$ since $\tilde{R}'_a$ is transitive.■

From theorems 11 and 15 we deduce

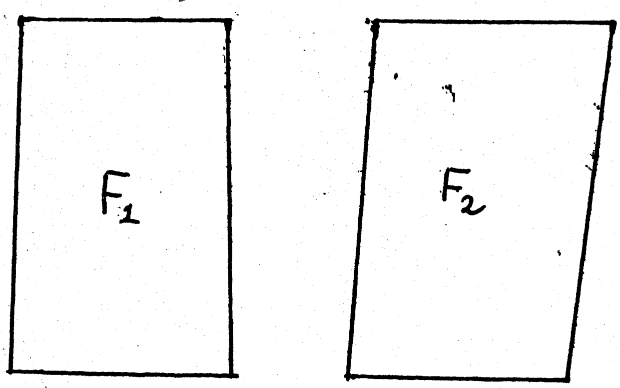
COROLLARY 16. **$D4G_1$** is determined by the class of all finite **$D4G_1$**-frames.

4. Suitable frames

Now we will describe a more convenient narrower class of finite frames for **$D4G_1$**. For this purpose we introduce some operations between Kripke frames

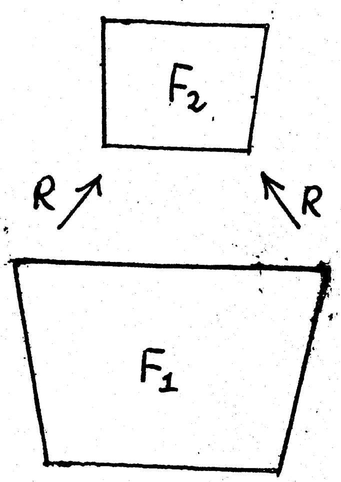
Let $F_0 = (W_0,R_0)$, $F_1 = (W_1, R_1)$ be Kripke frames. Their *disjoint sum* $(F_0 \sqcup F_1)$ is the frame (W,R) in which $W = W_0 \sqcup W_1 = (W_0\times\{0\})\cup(W_1\times\{1\})$ (the set-theoretic sum of W_0 and W_1), and $R = R_0 \sqcup R_1 = \{((x,0),(y,0))\mid xR_0y\}\cup\{((x,1),(y,1))\mid xR_1 y\}$. It is easily checked that the operation $\sqcup$ is associative up to an isomorphism. So, the disjoint sum of n frames $(F_1 \sqcup F_2 \sqcup \ldots \sqcup F_n)$ can be defined as $((F_1 \sqcup F_2) \sqcup \ldots) \sqcup F_n$.

The *ordinal sum* (F_0+F_1) is the frame (W,R) in which $W = W_0 \sqcup W_1$ and $R = \{((x,0),(y,0))\mid xR_0y\} \cup \{((x,1),(y,1))\mid xR_1y\} \cup (W_0\times\{0\})\times(W_1\times\{1\})$.



$F_1 \sqcup F_2$

Fig.4



$F_1 + F_2$

Fig.5

A map $f\colon W_0 \to W_1$ is called a *morphism* of F_0 to

F_1 iff for any $x \in W$

(42) $f(R_0(x)) = R_1(f(x))$.

An onto morphism is called a *p-morphism*; an injective morphism is called an *imbedding*. Each pair of imbeddings $j_0 : F_2 \longrightarrow F_0$, $j_1 : F_2 \longrightarrow F_1$ has an amalgam (in the categorial sense). It can be constructed as a frame $F=(W,S)$ in which $W = (W_0 \sqcup W_1)/\rho$, ρ being the least equivalence relation such that $(j_0(x),0)\rho(j_1(x),1)$ for any $x \in W_2$, and $aSb \iff \exists x \in a\ \exists y \in b\ x(R_0 \sqcup R_1)y$. Then there exist canonical imbeddings k_0, k_1 forming the commutative square

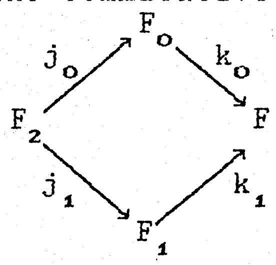


Fig.6

It is easily proved that F is transitive whenever F_0 and F_1 are.

We will use the notation $\overset{F_2}{F_0 \diagup \diagdown F_1}$ for such an amalgam, and F_0, F_1 will be usually identified with their canonical images in the amalgam. The operation of amalgam is associative up to an isomorphism, so we introduce iterated amalgams

$$\overset{G_1}{F_1 \diagup \diagdown} \overset{G_2}{F_2 \diagup \diagdown} F_3 \ \ldots \ \overset{G_{n-1}}{\diagup \diagdown F_n} .$$

Let ω^* be the set of all finite sequences of natural numbers; this is a tree ordered by the relation $\sqsubseteq$ ("to be an initial segment"). ω^* is also linearly-ordered by the lexicographic order $\preccurlyeq$. $\sqsubset, \prec$ denote corresponding strict orders. Λ denotes the empty sequence. $\alpha^\frown k$ (respectively, $k^\frown\alpha$) denotes the sequence obtained by putting the number k after (resp., before) the sequence α.

A *standard tree* is a finite substructure T of $(\omega^*, \sqsubset)$ such that for any k

(43) $\Lambda \in T$,

(44) $\alpha^\frown(k+1) \in T \Rightarrow \alpha^\frown k \in T$,

(45) $\alpha^\frown k \in T \Rightarrow \alpha \in T$.

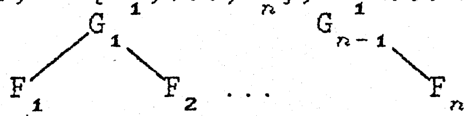
It is clear that any finite strictly ordered tree is isomorphic to some standard tree.

The restriction of a transitive frame F = (W,R) to $(R(x) \cup \{x\})$ is denoted by F^x and called the *subframe generated by* x. F itself is called *generated* iff $W = R(x) \cup \{x\}$.

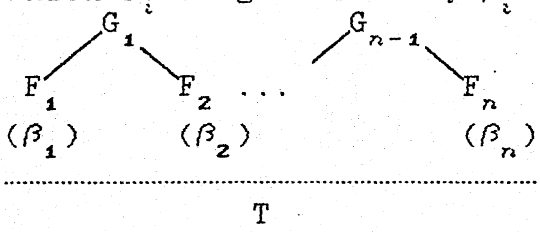
It is well-known that $L(F) \subseteq L(F^x)$ (the Generation Lemma, [4]).

Let F = (W,R) be a Kripke frame. $x \in W$ is called its *endpoint* iff $R(x) = \emptyset$. E(F) denotes the set of all endpoints in F.

Let T be a standard tree, with $E(T) = \{\alpha_1, \ldots, \alpha_n\}$, $\alpha_1 < \ldots < \alpha_n$ and let

H= 

be an iterated amalgam in which F_i is generated by β_i. Then let



be the frame obtained from the ordinal sum (T+H) by identifying every α_i with corresponding β_i (this is, so to say, an "ordinal amlagam"). Note that if β_i are replaced by some β'_i still generating F_i we will obtain an isomorphic frame, so β_i need not be indicated in the previous notation.

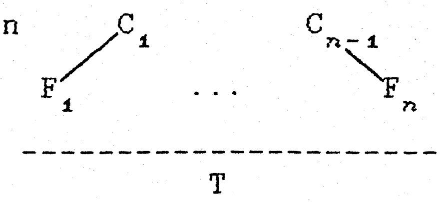
A particular case of this construction is n=1. In this case $H=F_1$, T is a finite irreflexive chain, and the resulting frame is obtained from (T+F) by identifying α_1 and β_1.

Now we are ready to give an inductive definition of a *suitable frame*:

(46) A finite non-degenerate clot is a suitable frame.

(47) If $F_1,\ldots, F_n$ are suitable $(n\geq 1)$ and C is a finite non-degenerate clot then $(C+F_1\sqcup\ldots\sqcup F_n)$ is suitable.

(48) If $F_1,\ldots,F_n$ are suitable $(n\geq 1)$ and generated by reflexive elements, $C_1,\ldots,C_{n-1}$ are clots, and T is a standard tree then



is suitable.

Our next aim is to prove that $\mathbf{D4G_1}$ is determined by suitable frames. So we will show that every finite generated $\mathbf{D4G_1}$-frame is a p-morphic image of some suitable frame.

Let us introduce some other auxilary notions. A *marked frame* is a pair (F,ρ) in which F is a generated transitive irreflexive finite frame, and ρ is a reflexive symmetric relation in E(F) (a "graph"). (F,ρ) is *regularly marked* iff ρ is a connected graph i.e. iff its transitive closure is universal on E(F). A *marked p-morphism* f: $(F_1,\rho_1)\rightarrow(F_2,\rho_2)$ of two marked frames is a p-morhism f: $F_1\rightarrow F_2$ (cf. (42)) such that for any $x,y\in F_1$

(49) $x\rho_1 y$ only if $f(x)\rho_2 f(y)$.

An *SM-tree* is a marked standard tree (T,τ) such that for any $\alpha,\beta\in T$

(50) $\alpha\tau\beta$ iff $\alpha,\beta\in E(T)$ & $(\alpha=\beta\vee\neg\exists\gamma\in E(T)(\alpha<\gamma<\beta\vee\beta<\gamma<\alpha))$.

Non-formally, $\alpha\tau\beta$ & $\alpha\neq\beta$ means that α and β become adjacent if you draw the tree T on the plane without self-intersections and with putting all endpoints on a horizontal line:

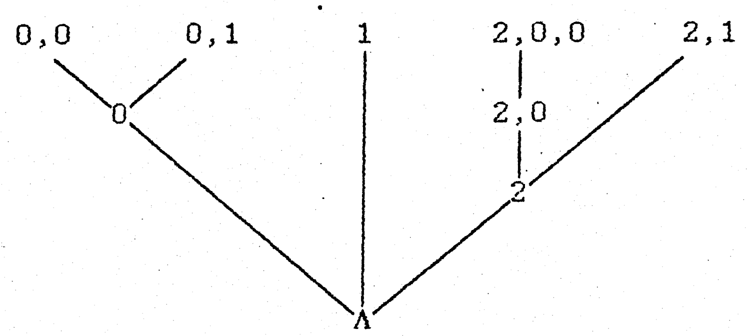


Fig.7

LEMMA 17. Let (F,ρ) be a regularly marked frame, $u,v\in E(F)$. Then there exist an SM-tree (T,τ) and a marked p-morphism $f: (T,\tau) \longrightarrow (F,\rho)$ satisfying

(51) If t_0 is the least, and t_1 is the last element in $(E(T), \preccurlyeq)$ then $f(t_0) = u$, $f(t_1) = v$.

<u>Proof.</u> By an induction on the cardinality of F.

If F is one-element we can take $T = (\{\Lambda\},=)$.

To make an inductive step let us assume that $F = (W,R)$, $W = \overline{R}(x_0)$, and X is the set all immediate successors of x_0 in F. If $y,z\in X$ we set

(52) $y\sigma z \iff \exists a\in\overline{R}(y)\ \exists b\in\overline{R}(z)\ a\rho b$.

Then σ is a connected graph on X. Indeed, suppose $y,z\in X$, $a\in\overline{R}(y)\cap E(F)$, $b\in\overline{R}(z)\cap E(F)$. Since ρ is connected, there exists a path: $a = a_1\rho a_2\rho\ldots\rho a_j = b$, and by the choice of X for any i there exists $y_i\in X\cap\overline{R}^{-1}(a_i)$. Hence we have $y\sigma y_2\sigma\ldots\sigma y_{j-1}\sigma z$, by (52).

Since σ is connected we can construct a σ-path involving all elements of X ("Ariadna thread"). Let $x_1,\ldots,x_n$ be such a path. Let (F_k,ρ_k) be the restriction of (F,ρ) to $\overline{R}(x_k)$. By (52) for each $k\in\{1,..,n\}$ we can find $v_k\in E(F_k)$, $u_{k+1}\in E(F_{k+1})$ such that $v_k \rho u_{k+1}$. We set also $u_1=u$, $v_n=v$. So F can be pictured as

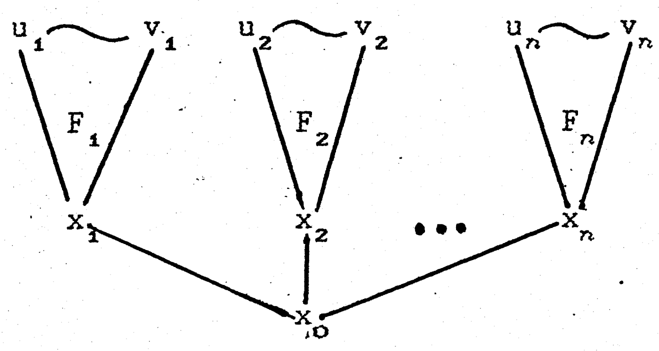


Fig.8

(But one should remember that x_k's are not necessarily distinct, and that different F_k's are not necessarily disjoint.)

Now we apply the inductive hypothesis to (F_k, ρ_k) and obtain a marked p-morphism $f_k:(T_k,\tau_k) \longrightarrow (F_k,\rho_k)$ such that

(53) If t_{0k} is the least and t_{1k} is the last in $(E(T_k),\preccurlyeq)$ then $f_k(t_{0k}) = u_k$, $f_k(t_{1k}) = v_k$.

Then we construct an SM-tree (T,τ) such that $T = \{\Lambda\} \cup \bigcup_{k=1}^{n} T_k^*$, $T_k^* = \{k^\frown\alpha \mid \alpha\in T_k\}$. (It is clear that T satisfies (44) and (45) since all T_k do.)

Finally we define $f:T \longrightarrow F$ such that

(54) $f(\Lambda) = x_0$,

(55) $f(k^\frown\alpha) = f_k(\alpha)$.

This is a required p-morphism. Really, f is onto, so x_0 satisfies (42). Since f_k are p-morphisms we have: $f(\sqsubset(k^\frown\alpha)) = f_k(\sqsubset(\alpha)) = R(f_k(\alpha)) = R(f(k^\frown\alpha))$.

To check (49) let us look at the picture of T:

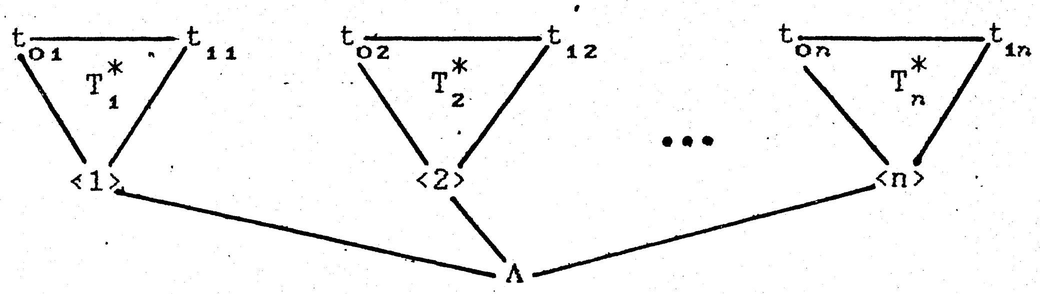


Fig.9

(Note that some of T_k^*'s can be singletons.) A routine proof shows that for $\alpha,\beta\in E(T)$:

$$\alpha\tau\beta \Rightarrow \exists k\ (\alpha,\beta\in E(T_k^*) \vee \{\alpha,\beta\}=\{t_{1k},\ t_{0(k+1)}\}).$$

Therefore $\alpha\tau\beta$ only if $f(\alpha)\rho f(\beta)$. Indeed, this is obviously true for any $\alpha,\beta\in T_k^*$ (by (55)). Otherwise $\{f(\alpha),f(\beta)\} = \{f(t_{1k}),f(t_{0(k+1)})\}=\{v_k,\ u_{k+1}\}$, hence $f(\alpha)\rho f(\beta)$ (by the choice of v_k, u_{k+1}).■

LEMMA 18. Every finite generated **D4G**$_1$-frame is a p-morphic image of some suitable frame.

<u>Proof.</u> It goes by an induction on the cardinality of a given frame $F = (W,R)$. If F is a clot there is nothing to prove, due to (46). So suppose it is not.

(i) Suppose $W = R(x)$, then the clot C containing x is non-degenerate, $C\neq W$. Let $C_1,\ldots,C_n$ be all immediate successors of C in F, $x_i\in C_i$, $F_i=F^{x_i}$. All F_i are **D4G**$_1$-frames, so let $f_i:G_i \longrightarrow F_i$ be a p-morphism of a suitable frame onto F_i. We set $G = C+(G_1\coprod\ldots\coprod G_n)$, and identify C and G_i with their images in G. Then the map $f:G \longrightarrow F$ such that

$$f(y) = \begin{cases} y \text{ if } y\in C, \\ f_i(y) \text{ if } y\in G_i, \end{cases}$$

is a p-morphism (a similar definition is in (54), (55)).

(ii) Suppose $W = R(x)\cup\{x\}$, x is irreflexive. Let V_0 be the least set satisfying

(56) $x\in V_0$,

(57) $\forall y,z\ (y\in V_o\ \&\ yRz\ \&\ \neg\exists t\ (yRt\ \&\ tRz) \Rightarrow z\in V_o)$.

In other words, (57) means that V_o contains all "strict immediate successors" of every its element. So all elements of V_o are irreflexive. We will say that a clot in F is *over* V_o if it is a successor of some endpoint of V_o. Let $X_1,\dots,X_n$ be all clots over V_o. By (57) every X_i is non-degenerate. We pick $x_i\in X_i$ and set

(58) $V = V_o\cup\{x_1,\dots,x_n\}$, $G = (V,S)$,

$S = (R\cap(V\times V)) - \{(x_i,v)\ |1\le i\le n, v\ne x_i\}$.

Thus G is generated and irreflexive, $E(G) = \{x_1,\dots,x_n\}$. Then we set

$\hat{\rho} = R\cap(E(G)\times E(G))$

(cf. (8)). This relation is obviously symmetric and reflexive; it is a connected graph since F is locally connected (cf.(10)). Therefore we can apply lemma 17 and obtain a marked p-morphism $g:(T,\tau) \longrightarrow (G,\rho)$.

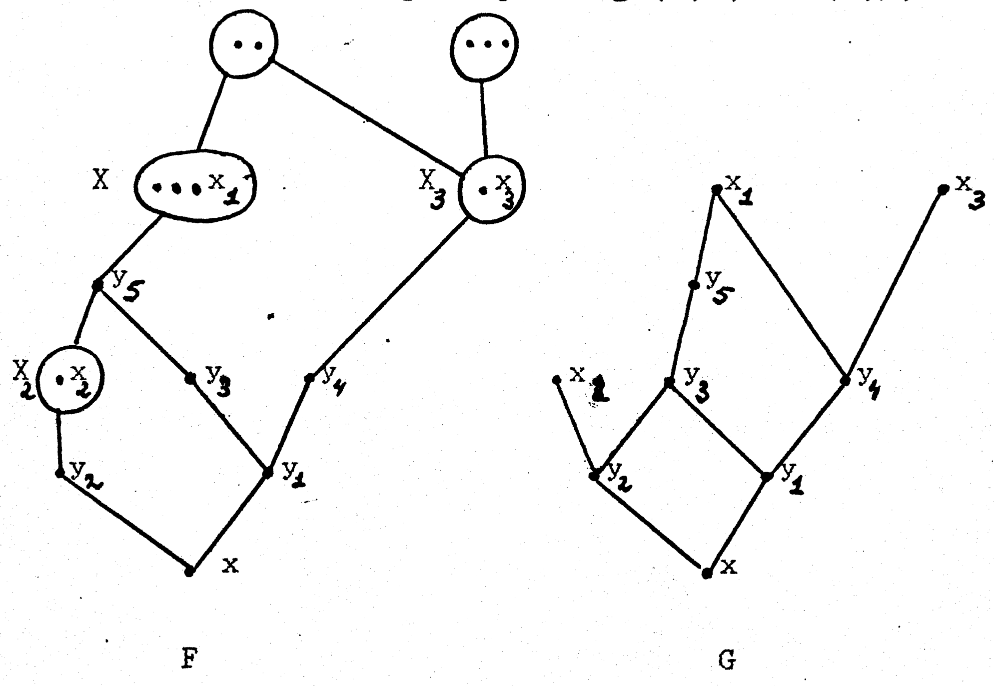


Fig.10 An example of forming G.

Each $F_i = F^{x_i}$ is a **D4G**$_1$-frame, so by the inductive hypothesis there exist a suitable Φ_i and a p-morphism $f_i : \Phi_i \longrightarrow F_i$. Suppose

$E(T) = \{e_1, \ldots, e_m\}$, $e_1 < e_2 < \ldots < e_m$,

(59) $g(e_j) = x_{k(j)} = z_j$, $\Psi_j = \Phi_{k(j)}$,

Q_j is the accessibility relation in Ψ_j.

Then $e_j \tau e_{j+1}$ and $z_j \rho z_{j+1}$ (by (49)); that is $\overline{R}(z_j) \cap \overline{R}(z_{j+1}) \neq \emptyset$ for any $j<m$. Suppose also

(60) C_j is a maximal clot in F, $C_j \subseteq \overline{R}(z_j) \cap \overline{R}(z_{j+1})$;

(61) B_j is an isomorphic copy of C_j, $c_j : B_j \longrightarrow C_j$ is a bijection.

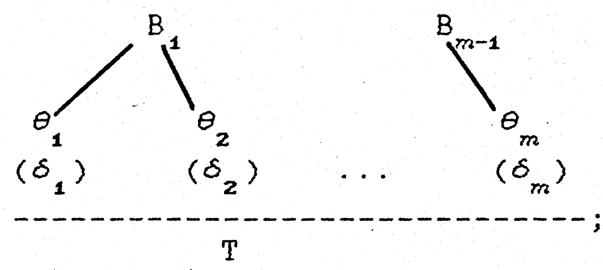
Let D_j be the least clot in Ψ_j, then $\Psi_j = D_j + \Psi_j^-$ (for some Ψ_j^-). We set

(62) $$\theta_j = \begin{cases} D_j + (B_{j-1} \bigsqcup B_j \bigsqcup \Psi_j^-) & \text{if } 1<j<m, \\ D_1 + (B_1 \bigsqcup \Psi_1^-) & \text{if } j=1, \\ D_m + (B_{m-1} \bigsqcup \Psi_m) & \text{if } j=m. \end{cases}$$

Every Ψ_j^- is a disjoint sum of suitable frames, consequently θ_j is suitable (47). As usually, Ψ_j^- and B_j are identified with their images in θ_j We choose $\delta_j \in D_j$ so that

(63) $f_{k(j)}(\delta_j) = z_j$.

Such δ_j exists since $f_{k(j)}$ is a p-morphism. Finally, we set

(64) H = 

; Q denotes the accessibility relation in H,

(65) $f : H \longrightarrow F$,

$$f(y) = \begin{cases} c_j(y) & \text{if } y \in B_j, \\ f_{k(j)}(y) & \text{if } y \in \Psi_j, \\ g(y) & \text{if } y \in T. \end{cases}$$

This definition is correct because $g(e_j) = z_j =$ $= f_{k(j)}(\delta_j)$ (by (59) and (63)). It follows from (65) that

(66) $f(\theta_j) = F^{z_j}$.

Indeed, if $1<j<m$ then $f(\theta_j) = f_{k(j)}(\Psi_j)\cup C_{j-1}\cup C_j$ (by (62),(65)) $= F^{z_j}$ since $f_{k(j)}$ is onto and by (60). Cases $j=1$ and $j=m$ bring nothing new.

To prove (42) we consider several cases.

(i) If $y\in B_j$ then $f(Q(y)) = f(B_j) = C_j$ (by (61)) $= R(f(y))$ since C_j is a maximal clot.

(ii) If $y\in\Psi_j-D_j$ then $Q(y) = Q_j(y)$, hence

$$f(Q(y)) = f_{k(j)}(Q_j(y)) = R(f_{k(j)}(y)) = R(f(y))$$

(since $f_{k(j)}$ is a p-morphism).

(iii) If $y\in D_j$, $1<j<m$, then $Q(y) = Q_j(y)\cup B_j\cup B_{j-1}$ and

$$f(Q(y)) = f_{k(j)}(Q_j(y))\cup f(B_j)\cup f(B_{j-1}) =$$

$= R(f(y))\cup C_j\cup C_{j-1} = R(f(y))$ (since $C_j\cup C_{j-1}\subseteq R(z_j)$, by (60)).

The cases (iv): $y\in D_1$,and (v): $y\in D_m$, are analogous to (iii).

(vi) Suppose $y\in T-E(T)$. Then we have

(67) $f(Q(y)\cap T) = S(g(y))$.

Really, $f(Q(y)\cap T) = f(\sqsubset(y)) = g(\sqsubset(y)) = S(g(y))$ by (65) and since g is a p-morphism.

(68) If $y\sqsubset e_j$ then $f(\theta_j)\subseteq R(g(y))$.

Indeed, $g(y)Rg(e_j) = z_j$ (since g is a p-morphism), hence $F^{z_j}\subseteq R(g(y))$, while $f(\theta_j) = F^{z_j}$ by (66).

(69) $f(Q(y))\subseteq R(g(y))$.

Indeed, $f(Q(y)\cap T) = S(g(y))$ (by (67)) $\subseteq R(g(y))$, and $Q(y)\cap\theta_j$ is either empty or θ_j. In the latter case $y\sqsubset e_j$, and $f(Q(y)\cap\theta_j) = f(\theta_j)\subseteq R(g(y))$ (by (68)). Combining all this together we come to (69).

(70) $R(g(y))\subseteq f(Q(y))$.

To show this suppose $g(y)Rt$. If $t\in V$ then $t\in S(g(y))$ (by (58)), and $t\in f(Q(y))$ (by (67)). If $t\notin V$ then consider a chain from $g(y)$ to t. Its initial segment lies in V_0 (by (57)) and afterwards it passes through some X_i (by our choice of $X_1,\dots,X_n$). Thus $g(y)Sx_iRt$, and $x_i = z_j$ for some j since g is onto. Hence $t\in F^{z_j}$, $z_j\in S(g(y)) = g(\sqsubset(y))$ (since g is a p-morphism) $= f(\sqsubset(y))\subseteq f(Q(y))$.

Now (42) follows from (69) and (70) (note that $g(y) = f(y)$).■

PROPOSITION 19. **D4G$_1$** is determined by the class of all suitable frames.

Proof. It follows from the definition that any suitable frame F is a **D4G$_1$**-frame, hence **D4G$_1$**$\subseteq L(F)$ (theorem 11). On the other hand, if $A\notin$**D4G$_1$** then $A\notin L(F)$ for some finite **D4G$_1$**-frame F. Thus $A\notin L(F^x)$ for some $x\in F$ (by the Generation Lemma [4]). F^x is a **D4G$_1$**-frame as well. Consequently, it is a p-morphic image of some suitable frame G(lemma 18), and $L(G)\subseteq L(F^x)$ by the P-morphism Lemma [4]. Therefore, $A\notin L(G)$.■

5. Topological semantics for D4G$_1$

Now let X be a topological space, F = (W,R) be a Kripke frame. An onto mapping $f:X \longrightarrow F$ is called a *d-p-morphism* iff

(71) $\forall u\in F\ df^{-1}(u) = f^{-1}(R^{-1}(u))$.

Existence of such a mapping is denoted by $X \twoheadrightarrow F$.

LEMMA 20. If F is a finite Kripke frame and $X \twoheadrightarrow F$ then $L(D(X))\subseteq L(F)$.

Proof. Let M(F) be the modal algebra of the frame F = (W,R). Recall that M(F) is the Boolean algebra P(W) of subsets together with the operation $R^{-1}:U \longmapsto R^{-1}(U)$. The map $f^*:U \longmapsto f^{-1}(U)$ is obviously a Boolean imbedding of P(W) into P(X). It is also an imbedding of M(F) into D(X) since for any $U\subseteq W$ we have

$df^{-1}(U) = d(\bigcup_{u\in U} f^{-1}(u)) = \bigcup_{u\in U} df^{-1}(u)$ (since U is finite) $= \bigcup_{u\in U} f^{-1}(R^{-1}(u))$ (by (71)) $= f^{-1}(\bigcup_{u\in U} R^{-1}(u)) =$ $= f^{-1}(R^{-1}(u))$.

Thus we obtain: $L(D(X))\subseteq L(M(F)) = L(F)$.■

LEMMA 21. Let X be a dense-in-itself separable metric space, and let B be a closed rare subset of X (i.e. B has no inner points). For any pair $m>0$, $l\geq 0$, let Φ_{ml} be the frame

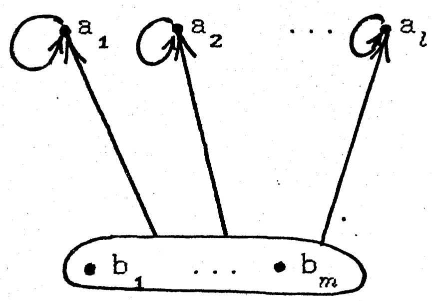


Fig.11

containing an m-element clot and l its reflexive successors. Then there exists a d-p-morphism $g:X \longrightarrow \Phi_{ml}$ such that $B\subseteq g^{-1}(b_1)$.

Remark. Tarski lemma on "dissectability" (cf.[1]) states in fact existence of a "c-p-morphism" from X onto Φ_{ml} that is, of a map f such that $cf^{-1}(u) = f^{-1}(R^{-1}(u))$ for any $u\in\Phi_{ml}$ (compare to (71)). It is clear that a d-p-morphism onto a reflexive frame is always a c-p-morphism. Thus Tarski lemma (for separable spaces) is a consequence of lemma 21.

Proof. Let $\{X_1,X_2,\ldots\}$ be a countable set of open balls forming a base for X. We will construct families of sets $(A_{ik})_{1\leq i\leq l,\, k\in\omega}$; $(B_{jk})_{1\leq j\leq m,\, k\in\omega}$ such that for any i,j,k

(72) A_{ik} is a finite union of open balls whose

closures are disjoint;

(73) $i \neq i' \Rightarrow cA_{ik} \cap cA_{i'k} = \emptyset$;

(74) $A_{ik} \subseteq A_{i,k+1}$;

(75) B_{jk} is finite;

(76) $B_{jk} \subseteq B_{j,k+1}$;

(77) $A_{ik} \cap B_{jk} = \emptyset$;

(78) $X_{k+1} \subseteq \bigcup_{i=1}^{l} A_{ik} \Rightarrow A_{i,k+1} = A_{ik}$, $B_{j,k+1} = B_{jk}$;

(79) $X_{k+1} \not\subseteq \bigcup_{i=1}^{l} A_{ik} \Rightarrow A_{i,k+1} \cap X_{k+1} \neq \emptyset$, $B_{j,k+1} \cap X_{k+1} \neq \emptyset$;

(80) $A_{ik} \cap B = \emptyset$;

(81) $B_{jk} \cap B = \emptyset$;

(82) $j \neq j' \Rightarrow B_{jk} \cap B_{j'k} = \emptyset$.

The construction goes by an induction on k.

Suppose k=0. (X-B) is infinite since X is dense-in-itself, and B is closed, $B \neq X$. Then we choose different points $v_1, \ldots, v_l \notin B$, and open balls $A_{io} \subseteq X-B$ such that $v_i \in A_{io}$, closures of all A_{io} are disjoint and $\bigcup_i cA_{io} \subset (X-B)$. (E.g. we can take $A_{io} = \{x \mid \rho(v_i,x) < 0.1\min(\rho(v_i,B), \min_{i \neq i'} (v_i,v_{i'}))\}$.) The set $(X-B) - \bigcup_{i=1}^{l} cA_{io}$ is non-empty and open, consequently it is infinite (due to the density of X), and we can find different points $w_1, \ldots, w_m$ in it. Taking $B_{jo} = \{w_j\}$ we see immediately that (72), (73), (75), (77), (80), (81) are true for k = 0.

To make the (k+1)'st step assume that all A_{ik} , B_{jk} , are constructed. We set

(83) $Y_k = \bigcup_{i=1}^{l} A_{ik}$.

Now two cases are possible.

(i) $X_{k+1} \subseteq Y_k$. Then we proceed according to (78).

(ii) $X_{k+1} \not\subseteq Y_k$. Then in fact

(84) $X_{k+1} \not\subseteq cY_k$.

For, assuming the contrary we obtain: $IcX_{k+1} \subseteq IcY_k$. But $IcX_{k+1} = X_{k+1}$ since X_{k+1} is an open ball, and $IcY_k = Y_k$ by (72), (73), (83). Hence $X_{k+1} \subseteq Y_k$ in contradiction with (ii).

Now we set

(85) $W_o = X_{k+1} - cY_k - \bigcup_{j=1}^{m} B_{jk}$;

(86) $W = W_o - B$.

Since $(X_{k+1} - cY_k)$ is open and non-empty (by (84)), such are also W_o (due to the density of X), and W (because $IB = \emptyset$).

Now we proceed as in the case k=0. We choose open balls $V_{1,k+1}, \ldots, V_{l,k+1} \subseteq W$ whose closures are disjoint and such that $\bigcup_{i=1}^{l} cV_{i,k+1} \subset W$. The set $(W - \bigcup_{i=1}^{l} cV_{i,k+1})$ being non-empty and open, is infinite, and we take points $b_{j,k+1}$, $1 \leq j \leq m$, from there. Finally, we set

(87) $B_{j,k+1} = B_{jk} \cup \{b_{j,k+1}\}$;

(88) $A_{i,k+1} = A_{ik} \cup V_{i,k+1}$.

Then the statements (74)-(76), (78) hold trivially. The first part of (72) is also trivial. The second one is proved by an induction: in the case (ii) by our construction $cV_{i,k+1} \cap cV_{i',k+1} = \emptyset$ whenever $i \neq i'$, and $cV_{i,k+1} \cap cY_k = \emptyset$ since $cV_{i,k+1} \subseteq W$ and by (85),(86).

By the construction, (73) is true for k=0. Making an inductive step in the case (ii), by (88) we obtain:

$$cA_{i,k+1} \cap cA_{i',k+1} = (cA_{ik} \cap cA_{i'k}) \cup (cA_{ik} \cap cV_{i',k+1}) \cup$$
$$\cup (cA_{i'k} \cap cV_{i,k+1}) \cup (cV_{i,k+1} \cap cV_{i',k+1}) = cA_{ik} \cap cA_{i'k} = \emptyset$$

(since all other disjuncts are empty, due to the construction; note that $cV_{i,k+1} \cup cV_{i',k+1} \subset W \subseteq X - cY_k$).

(77) is true for k=0 since $w_j \notin A_{io}$. Assuming it for k, from (87),(88) we have:

$$A_{i,k+1} \cap B_{j,k+1} = (A_{ik} \cap B_{jk}) \cup (V_{i,k+1} \cap \{b_{j,k+1}\}) \cup$$
$$\cup (A_{ik} \cap \{b_{j,k+1}\}) \cup (V_{i,k+1} \cap B_{jk}) = \emptyset$$ since $b_{j,k+1} \notin V_{i,k+1}$,

$b_{j,k+1} \in W \subseteq X-Y_k$, $V_{i,k+1} \subset W \subseteq X-B_{jk}$ (by (85),(86)).

For a proof of (79) observe that in the case (ii)

$V_{i,k+1} \subseteq W$, $b_{j,k+1} \in W$, $W \subseteq X_{k+1}$.

(80) holds for k=0 by the choice of A_{i0} . If (80) is proved for k then in the case (ii) we have:

$A_{i,k+1} \cap B = (A_{ik} \cap B) \cup (V_{i,k+1} \cap B) = V_{i,k+1} \cap B = \emptyset$

since $V_{i,k+1} \subseteq W = W_0 - B$ (by (86)).

A proof of (81) is almost the same. If k=0 then $B_{jk} \subseteq X-B$ by the choice of w_j . In the case (ii) we use (87), and $b_{j,k+1} \notin B$, by (86).

(82) holds for k=0 because $w_j \neq w_{j'}$. In the case (ii) from (87) we have

$B_{j,k+1} \cap B_{j',k+1} = B_{jk} \cap B_{j'k}$

since $b_{j,k+1}, b_{j',k+1} \in W_0$, and $W_0 \subseteq X-(B_{jk} \cup B_{j'k})$, by (85).

Now we can set (for any $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$):

(89) $A_i = \bigcup_k A_{ik}$, $B_j = \bigcup_k B_{jk}$;

(90) $B'_1 = X-(\bigcup_{i=1}^{l} A_i \cup \bigcup_{j=1}^{m} B_j)$,

and we define $g:X \longrightarrow \Phi_{ml}$ by

$$g(x) = \begin{cases} a_i & \text{if } x \in A_i , \\ b_j & \text{if } x \in B_j , \\ b_1 & \text{if } x \in B'_1 . \end{cases}$$

This definition is correct due to (73),(74),(76),(77),(82), and let us prove that g is a required d-p-morphism. To do this, we check some inclusions.

(91) $(X-\bigcup_i A_i) \subseteq dB_j$.

Really, suppose $x \notin \bigcup_i A_i$. Since $\{X_{k+1} \mid k \geq 0\}$ is an open base we have to show that $X_{k+1} \cap B_j \neq \emptyset$ whenever $x \in X_{k+1}$. But if $x \in (X_{k+1} - \bigcup_i A_i)$ then $X_{k+1} \cap B_j \neq \emptyset$ by (79).

(92) $dB_j \subseteq (X - \bigcup_i A_i)$.

This is trivial since every A_i is open ((72),(89)), so $B_j \subseteq -A_i$ implies $dB_j \subseteq d(-A_i) \subseteq -A_i$.

By the same reasonings we have

(93) $dB'_1 \subseteq (X - \bigcup_i A_i)$,

and

(94) $dA_i \subseteq (X - \bigcup_{\alpha \neq i} A_\alpha)$.

(95) $A_i \subseteq dA_i$.

This is true since A_i is open and X is dense-in-itself.

(96) $B_j \subseteq dA_i$.

A proof is analogous to that of (91): if $x \in B_j$ and $x \in X_{k+1}$ then $X_{k+1} \cap A_i \neq \emptyset$ by (79) (the premise of (79) holds because $x \in B_j \cap X_{k+1}$).

(97) $B'_1 \subseteq dA_i$

is proved in the same way.

Now observing that $g^{-1}(a_i) = A_i$, $g^{-1}(b_j) = B_j$ (provided that $j \neq 1$), $g^{-1}(b_1) = B_1 \cup B'_1$, we deduce (71) from (91)-(97). Finally, we have $B \subseteq g^{-1}(b_1)$ since $B \subseteq B'_1$ (by (80),(81),(90)).

We should also notice that in the case $l=0$ the whole construction goes the same way, but without mentioning A_{ik} .■

PROPOSITION 22. Let F be a suitable frame.

(i) Suppose $F = F^b$, b is reflexive. Suppose also that X is a spherical slice in $\mathbf{R}^n$, $n>0$:

$$X = \{x \in \mathbf{R}^n \mid r_1 \leq \|x\| \leq r_2\},\ 0 \leq r_1 < r_2 ,$$

$$Y = \{x \mid \|x\| = r_1 \text{ or } \|x\| = r_2\}.$$

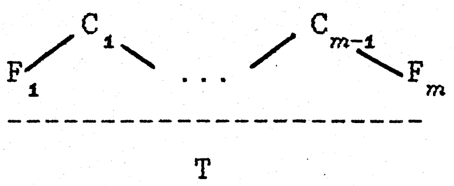
Then there exists a d-p-morphism $f: X \longrightarrow F$ such that

(98) $f(Y) = \{b\}$,

(99) the restriction $f|IX$ is also a d-p-morphism onto

F.

(ii) Suppose

F = 

is obtained by (48), $1 \le k \le m$, $F_k = F^{a_k}$, X is a non-null closed ball in $\mathbf{R}^n$, $n>0$, Y is its sphere.

Then there exists a d-p-morphism $f: X \to F$ satifying (99) and

(100) $f(Y) = \{a_k\}$.

<u>Proof.</u> Proceeding by an induction on the cardinality of F we consider cases (46)-(48).

In the case (46) lemma 21 can be applied (as for (99), we use the same argument as in the case (47) below).

(47): Suppose $F = C + (F_1 \sqcup \ldots \sqcup F_l)$, $b \in C$. If card C = m then let $g: X \to \Phi_{ml}$ be a d-p-morphism such that $Y \subseteq g^{-1}(b)$ constructed in the proof of lemma 21. Returning to this construction we observe that $A_i = g^{-1}(a_i)$ is a union of disjoint open balls (in X), say, $A_i = \bigcup_r U_{ir}$ (in fact, the number of disjuncts is infinite, but this does not matter).

Also $U_{ir} \subseteq IX$ since $Y \subseteq g^{-1}(b)$, and thus U_{ir} is an open ball in $\mathbf{R}^n$ (congruent to $\{ x \mid \|x\| \le r\}$). By the inductive hypothesis, there exist d-p-morphisms $f_{ir}: U_{ir} \to F_i$ (99).

Now a required d-p-morphism can be constructed as follows:

(101) $$f(x) = \begin{cases} g(x) & \text{if } g(x) \in C, \\ f_{ir}(x) & \text{if } x \in U_{ir}. \end{cases}$$

f is obviously onto.

(71) holds for any $u \in C$ because $f^{-1}(u) = g^{-1}(u)$,

$f^{-1}(R^{-1}(u)) = g^{-1}(C)$, and $dg^{-1}(u) = g^{-1}(C)$ (g is a d-p-morphism).

So assume that $u \in F_i = (W_i, R_i)$, and let d_{ir} be the derivative operation in U_{ir}. We have:

$$f^{-1}(u) = f_i^{-1}(u) = \bigcup_r f_{ir}^{-1}(u),$$

$$R^{-1}(u) = R_i^{-1}(u) \cup C,$$

and

(102) $f^{-1}(R^{-1}(u)) = g^{-1}(C) \cup \bigcup_r d_{ir} f_{ir}^{-1}(u)$.

Indeed, $f^{-1}(R^{-1}(u)) = f^{-1}(C) \cup f^{-1}(R_i^{-1}(u)) =$

$= g^{-1}(C) \cup \bigcup_r f_{ir}^{-1}(R_i^{-1}(u)) = g^{-1}(C) \cup \bigcup_r d_{ir} f_{ir}^{-1}(u)$,

since f_{ir} is a d-p-morphism.

Thus

(103) $f^{-1}(R^{-1}(u)) \subseteq df^{-1}(u) \cup g^{-1}(C)$.

Now let us prove that

(104) $g^{-1}(C) \subseteq df^{-1}(u)$.

Suppose $x \in g^{-1}(C)$, and let $\{X_1, X_2, \ldots\}$ be the base used in the construction from the proof of lemma 21. To obtain (104) it is sufficient to show that $X_{k+1} \cap f^{-1}(u) \neq \emptyset$ whenever $x \in X_{k+1}$ (note that $x \notin f^{-1}(u)$ since $x \in g^{-1}(C)$). But $x \in X_{k+1}$ implies $X_{k+1} \not\subseteq \bigcup_i A_i$, and $A_{i,k+1} = A_{ik} \cup V_{i,k+1}$ (88). But $V_{i,k+1}$ is U_{ir} for some r, and $f_{ir}^{-1}(u) \cap U_{ir} \neq \emptyset$ since f_{ir} is onto. By the choice of $V_{i,k+1}$ we have $V_{i,k+1} \subset X_{k+1}$. Consequently, $f^{-1}(u) \cap X_{k+1} \neq \emptyset$.

From (103) and (104) we obtain:

$f^{-1}(R^{-1}(u)) \subseteq df^{-1}(u)$,

and we have to prove the converse:

(105) $df^{-1}(u) \subseteq f^{-1}(R^{-1}(u))$.

Observing that $A_j \cap df^{-1}(u) = \emptyset$ for any $j \neq i$ (since A_j is open, and $A_j \cap f^{-1}(u) = \emptyset$) we conclude that

$df^{-1}(u) \subseteq A_i \cup g^{-1}(C)$. But

$$A_i \cap df^{-1}(u) = \bigcup_r (U_{ir} \cap df^{-1}(u)) \subseteq \bigcup_r d_{ir} f_{ir}^{-1}(u),$$ and

(105) follows from (102).

Therefore f is a d-p-morphism, and (98) holds by the definition.

To prove (99) we have to check (71) for $h = f|IX$. According to the definition,

$$h^{-1}(u) = \begin{cases} f^{-1}(u) & \text{if } u \neq b, \\ f^{-1}(b)-Y & \text{if } u=b. \end{cases}$$

Thus, $h^{-1}(R^{-1}(u)) = f^{-1}(R^{-1}(u))-Y$ for any u. On the other hand,

$$d_{IX} f^{-1}(u) = df^{-1}(u) \cap IX = df^{-1}(u)-Y.$$[12]

Therefore

(106) $h^{-1}(R^{-1}(u)) = d_{IX}\, h^{-1}(u)$

holds for any $u \neq b$, and let us consider the case $u=b$. We have:

$$h^{-1}(R^{-1}(b)) = h^{-1}(C) = g^{-1}(C)-Y = dg^{-1}(b)-Y$$

(since g is a d-p-morphism); and

$d_{IX} h^{-1}(b) = d(g^{-1}(b)-Y)-Y$. Now (106) for $u=b$ follows from the inclusion :

$$dg^{-1}(b)-Y \subseteq d(g^{-1}(b)-Y)$$

which is obvious since

$$dg^{-1}(b) = d(g^{-1}(b)-Y) \cup dY \subseteq d(g^{-1}(b)-Y) \cup Y.$$

Thus our consideration of (47) is over.

(48): In this case we have to prove the statement (ii). The frame

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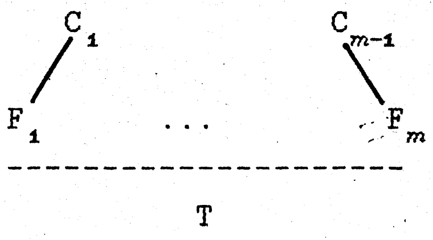
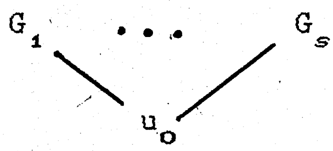
[12] d_Z denotes the derivative operation in the subspase Z of the space X.

F =



can be presented as



(i.e. $F-\{u_o\}$ is a union of subframes G_l, $1\le l\le s$, generated by succesors of u_o). Suppose $G_l=F^{y_l}$.

On the other hand, each F_j ($1\le j\le m$) is generated by some reflexive a_j ,and for any j there exists unique l_j such that $a_j\in G_{l_j}$, $1\le l_j\le s$. Also it is clear that each G_l contains some a_j ; thus

(107) $j\longmapsto l_j$ is a map from $\{1,...,m\}$ onto $\{1,...,s\}$.

Since all non-null closed balls in $\mathbf{R}^n$ are homeomorphic, we may assume that

$X = \{x|\ \|x\|\le 2\}$, $Y = \{x|\ \|x\|=2\}$.

We also set

$X_o=X$, $Y_o=Y$,

and for any i>0:

$X_i = \{x|\ \|x\|\le 1/i\}$, $Y_i = \{x|\ \|x\|=1/i\}$,

$\Delta_i = c(X_{i-1}-X_i)$.

Now looking at the zigzag diagramm

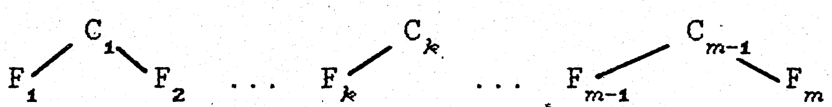


Fig.12

we construct a sequence of frames: $E_1,E_2,...$ obtained by the oscillating movement along the zigzag, beginning from F_k , C_k :

F_k , C_k ,..., F_{m-1} , C_{m-1} , F_m , C_{m-1} , F_{m-1} ,..., F_2, C_1, F_1, C_1, F_2,..., F_{m-1} , C_{m-1} , F_m , C_{m-1} ,...

(Here is a precise definition:

$$E_{i+2} = \begin{cases} F_{j+1} & \text{if } E_i=F_j \text{ , } E_{i+1}=C_j \text{ ,} \\ C_{j+1} & \text{if } E_i=C_j \text{ , } E_{i+1}=F_{j+1} \text{ , } j\neq m-1, \\ C_{j-1} & \text{if } E_i=C_j \text{ , } E_{i+1}=F_j \text{ , } j\neq 1 \\ F_{j-1} & \text{if } E_i=F_j \text{ , } E_{i+1}=C_{j-1} \text{ ,} \\ C_{m-1} & \text{if } E_i=C_{m-1} \text{ , } E_{i+1}=F_m \text{ ,} \\ C_1 & \text{if } E_i=C_1 \text{ , } E_{i+1}=F_1 \text{ .} \end{cases}$$
)

Due to the inductive hypothesis, there exist d-p-morphisms f_i such that

(108) $f_i : I\Delta_i \longrightarrow C_j$ if $E_i=C_j$,

(109) $f_i : \Delta_i \longrightarrow F_j$, $f_i(Y_i \cup Y_{i-1}) = \{a_j\}$ if $E_i=F_j$.

(Note that i is even in the first case, and odd in the second case.)

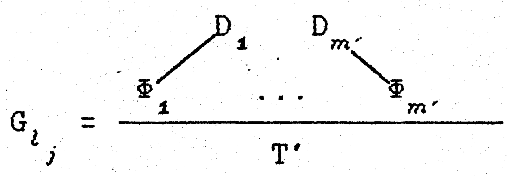
Furthermore, if $E_i=C_j$ we choose also a non-null closed ball

$\theta_i \subset I\Delta_i$,

and a d-p-morphism h_i such that

(110) $h_i : \theta_i \longrightarrow G_{l_j}$, $h_i(\theta_i - I\theta_i) = \{a_j\}$.

To construct h_i we may apply the inductive hypothesis. really, either $G_{l_j} = G_{l_j}^{a_j}$, or G_{l_j} can be presented as



(for some maximal clots $D_1,\ldots,D_{m'}$).

In the first case the statement (i) is used (we take $r_1=0$). In the second case for some j' we have

$\Phi_{j'} = \Phi_{j'}^{a_j}$

since the clot containing a_j in minimal in F-(T-E(T));

thus the statement (ii) can be used, with $F:=G_{l_j}$, $k:=j'$, $a_k:=a_j$.

Finally we define $f:X \longrightarrow F$ as follows:

$$(111)\quad f(x) = \begin{cases} u_0 & \text{if } x = 0, \\ f_i(x) & \text{if } x\in\Delta_i \text{ , i is odd,} \\ f_i(x) & \text{if } x\in(I\Delta_i-\theta_i)\text{, i is even,} \\ h_i(x) & \text{if } x\in\theta_i \text{ , i is even,} \end{cases}$$

f satisfies (100) since $f(Y) = f_1(Y_0) = \{a_k\}$ by (111) and (109).

Our aim is to prove that f is a d-p-morphism, that is (for any $x\in X$, $u\in F$)

(112) $f(x)Ru \Rightarrow x\in df^{-1}(u)$;

(113) $x\in df^{-1}(u) \Rightarrow f(x)Ru$.

Let us begin with (112), and suppose $f(x)Ru$. We analyze all possible cases (114)-(117).

(114) $x\in\Delta_i$, i is odd.

Then $f(x) = f_i(x)$ by (111), so $f(x)Ru$ implies $x\in d_{\Delta_i} f_i^{-1}(u)$ (since f_i is a d-p-morphism). Now we observe that $d_{\Delta_i} f_i^{-1}(u) \subseteq df_i^{-1}(u) \subseteq df^{-1}(u)$.

(115) $x\in(I\Delta_i-\theta_i)$, i is even.

This is a variation of the previous case.

We have $f(x) = f_i(x)$, and $f(x)Ru$ implies $x\in d_{I\Delta_i} f_i^{-1}(u)$ (since f_i is a d-p-morphism), and also $x\in df^{-1}(u)$, since $(I\Delta_i-\theta_i)$ is open and $f = f_i$ in some neighbourhood of x.

(116) $x\in\theta_i$.

If $E_i= F_j$ and $q=l_j$, then $f(x) = h_i(x)\in G_q$, and $f(x)Ru$ implies $u\in G_q$, $x\in d_{\theta_i} h_i^{-1}(u)$ (since h_i is a d-p-morphism). But then we have $x\in dh_i^{-1}(u)$, and hence $x\in df^{-1}(u)$ (since every X-neighbourhood of x contains a θ_i-neighbourhood).

(117) $x = O$.

Then for some q, $u \in G_q$, and also $q = l_j$ for some j (by (107)). By our construction the sequence E_i is periodic, hence

$E_i = C_j$

for infinitely many i's. For all such i's we have $f(\theta_i) = h_i(\theta_i) = G_q$ (by (110)), therefore

$f^{-1}(u) \cap \Delta_i \neq \emptyset$.

Thus every neighbourhood of O intersects $f^{-1}(u)$, i.e. $O \in df^{-1}(u)$.

To prove (113), we suppose $x \in df^{-1}(u)$ and again consider all possible cases.

(118) Assume that $x \in I\Delta_i$, i is odd.

Then $x \in d_{\Delta_i} f^{-1}(u) = d_{\Delta_i} f_i^{-1}(u)$ (since $f = f_i$ in Δ_i), and $f(x) = f_i(x)Ru$ (since f is a d-p-morphism).

(119) Assume that $x \in Y_i$, i is odd.

If $x \in d_{\Delta_i} f^{-1}(u)$ then $f(x)Ru$ is proved as in (118). So suppose $x \notin d_{\Delta_i} f^{-1}(u)$. Then for some X-neighbourhood V of x

$V \cap \Delta_i \cap f^{-1}(u) - \{x\} = \emptyset$

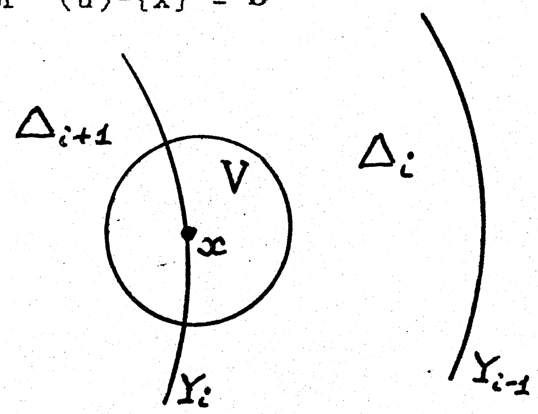


Fig.14

Since $x \in df^{-1}(u)$ we obtain:

$V \cap I\Delta_{i+1} \cap f^{-1}(u) \neq \emptyset$,

and we can choose V small enough to be disjoint with

θ_{i+1} .

If $y \in V \cap I\Delta_{i+1} \cap f^{-1}(u)$ then $u = f(y) = f_{i+1}(y) \in C_j$ (provided that $E_{i+1} = C_j$). But $f(x) = f_i(x) = a_q$ (provided that $E_i = F_q$) and it remains to show that $C_j \subseteq R(a_q)$.

But this follows immediately from the definition of the sequence E_i: if $E_i = F_q$ then either $E_{i+1} = C_q$, or $E_{i+1} = C_{q-1}$.

(120) Assume that $x \in Y_i$, i is even.

This case is analogous to (119) (there are two possibilities here: either $x \in d_{\Delta_{i+1}} f^{-1}(u)$, or not)

(121) Assume that i is even, $x \in (I\Delta_i - \theta_i)$.

Since $(I\Delta_i - \theta_i)$ is open we conclude that $x \in d_{I\Delta_i} f_i^{-1}(u)$. Thus $f(x) = f_i(x)Ru$ since f_i is a d-p-morphism.

(122) Assume that i is even, $x \in I\theta_i$.

Then $x \in d_{I\theta_i} f^{-1}(u) \subseteq d_{I\theta_i} h_i^{-1}(u)$ and $f(x) = h_i(x)Ru$ since h_i is a d-p-morphism.

(123) Assume that i is even, $x \in (\theta_i - I\theta_i)$.

This case is analogous to (119). By (110), we have $f(x) = h_i(x) = a_j$ (provided that $E_i = C_j$).

If $x \in d_{\theta_i} f^{-1}(u)$ then $f(x)Ru$ is proved as in (122).

Otherwise we can find a neighbourhood V of x such that $V \subseteq I\Delta_i$, $V \cap \theta_i \cap f^{-1}(u) - \{x\} = \emptyset$,

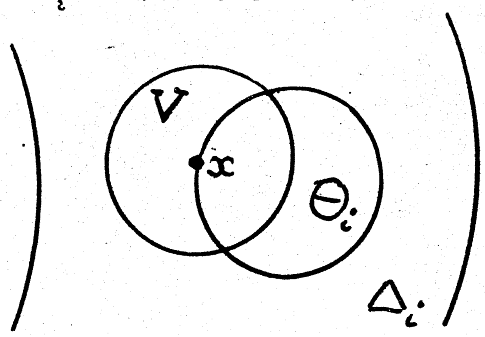


Fig.15

and we also have

$$V\cap(I\Delta_i-\theta_i)\cap f^{-1}(u) \neq \emptyset$$

since $x\in df^{-1}(u)$. Taking $y\in V \cap (I\Delta_i-\theta_i) \cap f^{-1}(u)$ we have

$$u = f(y) = f_i(y)\in C_j ,$$

and hence $f(x)(=a_j)Ru$.

The only remaining case $x = 0$ is trivial.■

THEOREM 23. (i) Let X be a topological space having an open subset homeomorphic to some $\mathbf{R}^n$, $n>0$. Then $L(D(X))\subseteq\mathbf{D4G_1}$.

(ii) If additionally X satisfies conditions of lemma 2 then $L(D(X)) = \mathbf{D4G_1}$.

Proof. If Y is an open subset of X then $L(D(X))\subseteq L(D(Y))$ since the map

$$A \longmapsto A\cap Y$$

yields a homomorphism of D(X) onto D(Y). Thus for the proof of (i) it is sufficient to show that

$$L(D(\mathbf{R}^n))\subseteq\mathbf{D4G_1} .$$

So suppose $A\notin\mathbf{D4G_1}$. Then A is falsified in some suitable frame F (proposition 19). But $\mathbf{R}^n \twoheadrightarrow F$ by proposition 22. Hence $A\notin L(D(\mathbf{R}^n))$ by lemma 20.

The statement (ii) follows now from lemma 2.■

So we have a proof of (MT4) and of (MT5).

COROLLARY 24. $L(D(X)) = \mathbf{D4G_1}$ for any topological space X locally homeomorphic to $\mathbf{R}^n$, $n>1$.

Proof. Immediately from theorem 23 and lemma 2 (since $(\mathbf{R}^n-\{0\})$ is connected).■

6. Topological semantics of D4.

Now let us verify (MT2) and (MT3).

We begin with Kripke semantics of **D4**. Transitive serial frames will be called **D4**-*frames*.

PROPOSITION 25 (cf. [4]). **D4** is determined by the class of all finite generated **D4**-frames.

D4-*quasitrees* are defined inductively:

(124) A finite non-degenerate clot is a **D4**-quasitree.

(125) If $F_1,\dots,F_n$ are **D4**-quasitrees ($n>0$) and C is a finite clot then $C+(F_1\sqcup\dots\sqcup F_n)$ is a **D4**-quasitree.

LEMMA 26. Every finite generated **D4**-frame is a p-morphic image of some **D4**-quasitree.

A proof is analogous to cases (i), (ii) in the previous proof of lemma 18 (the only difference is that C may be degenerate in the case (ii)).

Thus we obtain

PROPOSITION 27. **D4** is determined by the class of all **D4**-quasitrees.

Now recall that a topological space is called *zero-dimensional* iff its clopen (i.e. closed-open) subsets constitute a base of the topology.

PROPOSITION 28. Let X be a dense-in-itself separable zero-dimensional metric space. Then $X \twoheadrightarrow F$ for any **D4**-quasitree F.

<u>Proof.</u> Again we use an induction by the cardinality of F.

If F is a finite clot we apply lemma 21.

If $F = C + (F_0\sqcup\dots\sqcup F_{m-1})$ and C is a non-degenerate clot then we follow the proof of proposition 22 (case (47)).

If $F = C + (F_0\sqcup\dots\sqcup F_{m-1})$, $C = \{a\}$ is a degenerate clot then we construct an infinite splitiing of X as follows.

We choose a point $y\in X$ and clopens Y_1, Y_2, ... such that

$\{y\}\subset\dots\subseteq Y_{n+1}\subseteq Y_n\subseteq\dots\subseteq Y_1\subseteq Y_0 = X$,

$\rho(y,Y_n)\leq 1/n$ for any $n>0$.

Namely, if Y_n is already constructed we choose a clopen

$Y_{n+1} \subseteq Y_n \cap \{x | \rho(y,x) \leq 1/(n+1)\}$;
in any case $Y_{n+1} \neq \{y\}$ since y is non-isolated in X.

Then we select a strictly decreasing subsequence of $(Y_n)_{n\in\omega}$:

$\{y\} \subset \ldots \subset Z_{n+1} \subset Z_n \subset \ldots \subset Z_1 \subset Z_0 = X.$

Since $Z_n \subseteq Y_n$ we have

$\rho(y,Z_n) \leq 1/n$ for each $n>0$.

Thus

$\bigcap_n Z_n = \{y\}.$

So by setting $X_n = (Z_n - Z_{n+1})$ we obtain a non-trivial open splitting:

$$X-\{y\} = \bigcup_{n=0}^{\infty} X_n \ ,$$

such that

$\rho(y,X_n) \leq 1/n$ for each $n>0$,

and it is clear that every X_n is dense-in-itself and zero-dimensional.

By the inductive conjecture there exist d-p-morphisms

$f_n : X_n \longrightarrow F_{r(n)}$,

r(n) being the residue of n modulo m.

Finally we set

$$f(x) = \begin{cases} a & \text{if } x = y, \\ f_n(x) & \text{if } x \in X_n \ . \end{cases}$$

Almost the same reasonings as in proposition 22 show that f is a d-p-morphism. Really, the surjectivity of f is clear from the definition.

If $x \in X_n$ then $f(x) = f_n(x)$, and

$f(x)Ru \Leftrightarrow x \in df^{-1}(u)$

is proved as in the cases (114), (118).

If $x = y$ then $f(x)Ru$ for any $u \neq y$, so we have to prove that $y \in df^{-1}(u)$ for any $u \neq y$.

But if $u \in F_i$ then $f^{-1}(u) = \bigcup_{k=0}^{\infty} f^{-1}_{km+i}(u)$, and $f^{-1}_{km+i}(u) \cap X_{km+i} \neq \emptyset$ since f_{km+i} is onto. Thus $f^{-1}(u)$ intersects infinitely many X_n's, and also $\rho(y,X_n) \leq 1/n$. Therefore $y \in df^{-1}(u)$.■

THEOREM 29. $L(D(X)) = \mathbf{D4}$ for any dense-in-itself separable zero-dimensional metric space X.

Proof. Immediately from lemmas 1, 20 and from propositions 28, 27.■

7. The real line.

The last logic considered here is $L(D(\mathbf{R}))$. We use formulas

$$Q_i = p_i \wedge \bigwedge_{1 \leq j \leq 3,\ j \neq i} \neg p_j ,$$

$$G_2 = \Box(\bar{\Box}Q_1 \vee \bar{\Box}Q_2 \vee \bar{\Box}Q_3) \supset . \Box\neg Q_1 \vee \Box\neg Q_2 \vee \Box\neg Q_3 .$$

LEMMA 30. $G_2 \notin L(\Phi_{13})$

(Φ_{13} was defined in lemma 21.)

Proof. Take a valuation φ in Φ_{13} such that $\varphi(p_i)=\{i\}$. Then $\{x \mid x \models Q_i\} = \{a_i\}$, and $b \models \neg G_2$.■

LEMMA 31. $\mathbf{R} \models G_2$.

Proof. Suppose $x \models \neg G_2$ and set

$$|Q_i| = \{y \mid y \models Q_i\}.$$

Since $x \models \Box(\bigvee_i \bar{\Box}Q_i)$ there is an open U such that $x \in U$ and

$$U-\{x\} \subseteq I|Q_1| \cup I|Q_2| \cup I|Q_3| .$$

Furthermore, let us take an open interval V such that $\{x\} \subset V \subset U$. Then we obtain a non-trivial open splitting

$$V-\{x\} = V_1 \cup V_2 \cup V_3$$

in which

$$V_j = I|Q_j| \cap (V-\{x\}).$$

But this is impossible since $(V - \{x\})$ has only two connected components.■

From these two lemmas we deduce

PROPOSITION 31. $\mathbf{D4} \subset \mathbf{D4}+G_2 \subseteq L(D(\mathbf{R}))$.

The following two statements seem to be very probable.

CONJECTURE 1. L(D(R)) = $\mathbf{D4} + \mathbf{G}_2$.

CONJECTURE 2. Every logic $\mathbf{D4} + \mathbf{G}_n$ has the f.m.p.,

($\mathbf{G}_n$ denotes $\Box(\bigvee_{j=0}^{n} \bar{\Box} Q_j) \supset \bigvee_{j=0}^{n} \Box\neg Q_j$, $Q_j = p_j \wedge \bigwedge_{1\le i\le n,\, i\ne j} \neg p_i$).

Let us also indicate some open problems.

PROBLEM 1. To describe all logics L(D(X)) for dense-in-itself metric spaces X. In particular, is $\mathbf{D4G}_1$ the greatest of them?

PROBLEM 2. Is theorem 23 (ii) extended to the infinite dimensional case? In particular, does it hold for Hilbert space $\mathbf{l}_2$ (with the weak or with the strong topology)?

REFERENCES

[1] MCKINSEY J.C.C., TARSKI A. The algebra of topology. *Ann. Math.*,1944,v.45, 141-191.

[2] KURATOWSKI C. Sur l'operation A de l'Analysis Situs. *Fund. Math.*,1922,v.3, 182-199.

[3] FINE K. Logics containing K4, p.I. *Journal Symb. Logic*,1974,v.39,31-43.

[4] SEGERBERG K. An Essay in Classical Modal Logic, v.1-3. *Filosofiska Studier*, Uppsala,1971.

[5] ESAKIA L.L. Private communication,1975.

[6] GABBAY D.M. General filtration method for modal logic. *Journal Phil. Logic*, 1972,v.1,22-34.

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