PARTIAL CONSERVATIVITY
AND MODAL LOGICS

K.N. Ignatiev
ITLI Prepublication Series
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Received January 1991
PARTIAL CONSERVATIVITY AND MODAL LOGICS.

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ABSTRACT: PA is Peano arithmetic. Let $\Gamma$ be an arbitrary decidable set of arithmetical formulas; the $\Pi^0_n$-formula $\alpha \rightarrow^\Gamma \beta$ (where $\alpha, \beta$ range over codes of arithmetical sentences), is a formalization of the assertion that the theory $PA + \beta$ is $\Gamma$-conservative over $PA + \alpha$, i.e. any sentence $\gamma$ in $\Gamma$ which is provable in $PA + \beta$ is also provable in $PA + \alpha$. We extend Solovay's modal analysis of the formalized provability predicate of $PA$ to the formalized conservative relation. Namely, for $\Gamma = \Pi^0_n$, $n \geq 2$, $\Gamma = \Sigma^0_n$, $n \geq 3$, we give an axiomatization and a decision procedure for the class of those modal formulas that express arithmetically valid principles of $\Gamma$-conservativity.

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§1. Introduction.

In [3] Guaspari considered a notion of $\Gamma$-conservativity, (where $\Gamma=\Sigma_n$, $\Pi_n$, $n \geq 1$): theory $T_2$ is $\Gamma$-conservative over theory $T_1$ if any $\Gamma$-sentence provable in $T_2$ is also provable in $T_1$ ("theory" means "r.e. theory in arithmetical language containing Peano Arithmetic PA"). Guaspari proved some "obvious" properties of partial conservativity: for example, each consistent theory $T$ has a proper extension which is $\Gamma$-conservative over $T$, and the set of all sentences $A$ such that PA+A is $\Gamma$-conservative over PA is not r.e. .

Guaspari also noted, using Orey-Hálk results ( [1], [2] ), that $\Pi_1$-conservativity (over reflexive theories like PA, ZF, etc.) coincides with relative interpretability, and gave a model- characterization of the notion of relative interpretability.

Later Lindström ( [4],[5] ) proved that the set of all sentences $A$ s.t. PA+A is $\Gamma$-conservative over PA is $\Pi_2^0$-complete (here $\Gamma=\Sigma_1$ ). Bennett ( [6] ) proved some other facts connected with the partial conservativity.

The notion of $\Gamma$-conservativity (as well as relative interpretability) can be used for an arithmetical interpretation of modal language with unary modal operator $\Box$ and binary modal operator $\Rightarrow$, where $\Box$ is translated as provability in PA, and $A \Rightarrow B$ is translated as "PA+B is $\Gamma$-conservative over PA+A" (or "PA+A interprets the theory PA+B" ). Provability logic for $\Gamma$-conservativity (respectively, provability logic for relative interpretability) is, by definition, a set of all modal formulas, any arithmetical translation of which is provable in PA; we denote this set by $\text{CL}(\Gamma)$.

In [9],[10] it was proved, (with a use of some results from [7],[8] ) that the modal system ILM is the logic of relative interpretability over PA; thus, ILM is also the logic of $\Pi_1$-conservativity (in [11] this result was extended generalized to more rich class of basic theories).

The aim of this paper is to give an axiomatization and a Kripke-like semantic for the logic of $\Pi_n-, \Sigma_n$-conservativity for $n \geq 3$ (this logics coincide). In appendix we also prove the
arithmetical completeness of the logic of $\Pi_2$-conservativity.

If we consider axioms of ILM:

A1. $A \supset B \rightarrow (\Box A \supset \Box B)$
A2. $A \supset C \land B \supset C \rightarrow (A \lor B) \supset C$
A3. $\Box (A \supset B) \rightarrow A \supset B$
A4. $A \supset B \land B \supset C \rightarrow A \supset C$
A5. $\Diamond A \supset A$
M. $A \supset B \rightarrow (A \land \Box C) \supset (B \land C)$

from the point of view of the translation $\supset$ as $\Gamma$-conservativity, we can note that axioms A2, A3, A4 are always arithmetically valid, and axioms A1, A5, M express, respectively, that $\forall \gamma \in \Gamma$, $\gamma \vdash \Box A \supset \Box B \rightarrow \gamma$; $\forall \gamma \in \Gamma$, $\exists \psi \exists \epsilon \in \Pi_1$ $\gamma \psi \epsilon \in \Gamma$. Thus, one can see that for $\Gamma = \Pi_\gamma$, $\Sigma_n$, $n \geq 2$, only A5 fails. On the other hand, for such $\Gamma$, the principle M has an obvious generalization: for example, for $\Gamma = \Pi_2$ it is

Sb. $A \supset B \rightarrow (A \land (C \supset D)) \supset (B \land (C \supset D))$.

Unfortunately, the absence of A5 create some difficulties in the proof of modal completeness theorem for conservativity logics.

In modal completeness proof we use Veltman models ( [7] ), Visser's simplified models ( [8] ), and some technical concepts from [7]; in proving the arithmetical completeness we use some ideas from [5] and [10].

We believe that the technique developed in this paper will be useful in investigation of other logics of partial conservativity.

We are special thanks to V. Shavrukov for his substantial support and helpful discussions.


Definition 2.1. The modal language $\mathcal{L}(\sigma,\supset)$ consists of an infinite set of propositional variables $p, q, \ldots$; boolean connectives $\neg, \lor, \land, \rightarrow, \leftrightarrow$; and two modal operators: unary operator ' $\sigma$ ' and binary operator ' $\supset$ '; ' $\Box$ ' is abbreviation for ' $\neg \sigma \neg$ '.

We write "modal formula" instead of "$\mathcal{L}(\sigma,\supset)$-formula".

Definition 2.2. The logic CL is the minimal set of modal formulas containing the following axioms and closed under following
rules:

L0. All tautologies of propositional logic.

L1. \( \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \)

L2. \( \Box A \rightarrow \Box \Box A \)

L3. \( \Box (\Box A \rightarrow A) \rightarrow \Box A \)

A1. \( A \rightarrow B \rightarrow (\Box A \rightarrow \Box B) \)

A2. \( A \rightarrow C \land B \rightarrow C \rightarrow (A \lor B) \rightarrow C \)

A3. \( \Box (A \rightarrow B) \rightarrow A \rightarrow B \)

A4. \( A \rightarrow B \land B \rightarrow C \rightarrow A \rightarrow C \)

R1. \( \vdash A, \vdash A \rightarrow B \rightarrow \vdash B \) (modus ponens)

R2. \( \vdash A \rightarrow \vdash \Box A \) (necessitation)

Definition 2.3. The logic CLM is the minimal set of modal formulas closed under R1,R2 and containing L0-L3, A1-A4, plus

M. \( A \rightarrow B \rightarrow (A \land \Box C) \rightarrow (B \land \Box C) \)

Definition 2.4. The logic SCL is the minimal set of modal formulas closed under R1,R2 and containing L0-L3, A1-A4, plus

Sa. \( A \rightarrow B \rightarrow (A \land \Box\neg C) \rightarrow (B \land \Box\neg C) \)

Sb. \( A \rightarrow B \rightarrow (A \land\neg C) \rightarrow (B \land\neg C) \)

Proposition 2.5. SCL \( \supset \) CLM.

Proof. CL \( \vdash \Box A \rightarrow (\Box \neg A) \rightarrow 1 \) by A1, A3; hence, CL+Sa \( \vdash \) M.

Let \( \text{Proof}_{PA}(n,x) \) be the \( \Delta_0 \)-arithmetic formula representing the relation: \( n \) is the Gödelnumber of the PA-proof of the formula with Gödelnumber \( x \); \( \text{Prov}_{PA}(x) := \exists n \text{ Proof}_{PA}(n,x) \); \( \neg PA \) Q will stand for \( \text{Prov}_{PA}(\neg Q) \).

Definition 2.6. Let \( \Gamma \) be a decidable set of arithmetical formulas closed under disjunctions such that \( 1 \in \Gamma \), and \( \Gamma(x) \) be \( \Delta^PA_1 \)-formula representing the relation "\( x = \neg Q \land Q \in \Gamma \)"; then we define:

\( A \to_{\Gamma} B := \forall Q \ (\Gamma(\neg Q) \land \neg PA_1 (B \to Q) \to PA_1 (A \to Q)) \)

It is clear that \( A \to_{\Gamma} B \in PA \)

Definition 2.7. Let \( \Gamma \) be as above. An arithmetical interpretation \( f_{\Gamma} \) is a mapping of \( \Box (\neg, \rightarrow) \)-formulas to arithmetical sentences which commutes with the boolean connectives and translates \( \Box \) as provability in \( PA \) and \( \rightarrow \) as \( \Gamma \)-conservativity:

\( f_{\Gamma}(\Box A) := \neg PA_1 f_{\Gamma}(A) \)

\( f_{\Gamma}(A \to B) := f_{\Gamma}(A) \to f_{\Gamma}(B) \)
Definition 2.8. Let $\Gamma$ be as above. We define provability logic for $\Gamma$-conservativity $\text{CL}(\Gamma)$ and minimal provability logic for $\Gamma$-conservativity $\text{CL}^+(\Gamma)$ (shortly, logic of $\Gamma$-conservativity and minimal logic of $\Gamma$-conservativity) as follows:

$$\text{CL}(\Gamma):=\{ A \in \mathcal{L}(\mathcal{N},\mathcal{U}) \mid \text{for any arithmetical interpretation } f_{\Gamma}, \quad \text{PA} \vdash f_{\Gamma}(A) \}$$

$$\text{CL}^+(\Gamma):=\{ A \mid \Gamma \vdash f_{\Gamma}(A) \}$$

(It is supposed that $\Gamma'$ as well as $\Gamma$ is closed under disjunctions)

Theorem 1. For $n \geq 3$,

$$\text{CL}(\Pi_n) = \text{CL}(\Sigma_n) = \text{CL}^+(\Pi_n) = \text{CL}^+(\Sigma_n) = \text{SCL}.$$ 

Theorem 2. $\text{CL} = \text{CL}^+(\{1\})$.

Theorem 3. $\text{CLM} = \text{CL}^+(\Pi_1)$.

§3. Kripke Semantics.

Definition 3.1. A model $\mathcal{K} = \langle K, R, \{S_x\}_x \rangle$, $\vdash$ contains a nonempty set $K$, binary relation $R$, a family of binary relations $\{S_x\}$ for each $x \in K$ and a forcing relation $\Vdash$ such that:

1. $R$ is transitive and (converse) wellfounded;
2. $\Vdash$ commutes with boolean connectives;
3. $\vdash \neg A \iff \forall y (xRy \rightarrow y \Vdash A)$
   $$x \Vdash A \rightarrow B \iff \forall y (xRy \wedge y \Vdash A \rightarrow \exists z (yS_z \wedge xRz \wedge z \Vdash B))$$

We say that model $\mathcal{K}$ is simplified iff $S_x$ does not depend on $x$, and that modal formula $A$ is valid in $\mathcal{K}$, $(\mathcal{K} \models A)$, iff $\forall x \in K, x \Vdash A$.

Proposition 3.2. A formula $\Box A \rightarrow (\neg A) \rightarrow \perp$ is valid in every model $\mathcal{K}$.

Definition 3.3.

1. A $\text{CL}$-model is a model $\mathcal{K} = \langle K, R, \{S_x\}_x \rangle$, $\vdash$ such that for each $x$, $S_x$ is transitive and reflexive.
2. A $\text{CLM}$-model is a $\text{CL}$-model $\mathcal{K} = \langle K, R, \{S_x\}_x \rangle$, $\vdash$ such that for any $x, y, z, t \in K$, $yS_x zRt \Rightarrow yRt$.
3. A $\text{SCL}$-model is a $\text{CLM}$-model $\mathcal{K} = \langle K, R, \{S_x\}_x \rangle$, $\vdash$ such that for any $x, y, z$ $yS_x z \Rightarrow K = K \wedge S_y \wedge \neg yS_y$, where for any $u \in K$, $K_u$ denotes \{v | uRv\}.

Definition 3.4.
1. A simplified CL-model is a simplified model $K = \langle K, R, S, \vdash \rangle$ such that:
   1. $K$ is finite or countable.
   2. $S$ is an equivalence relation.

2. A simplified CLM-model is a simplified model $K = \langle K, R, S, \vdash \rangle$ such that:
   1. $K$ is finite or countable.
   2. $S$ is reflexive and transitive.
   3. $xSyRz \Rightarrow xRz$.
   4. There exists a natural number $N$ and a mapping $\mu: K \rightarrow \{1, 2, \ldots, N\}$ such that:
      a) $xSy \Rightarrow \mu(x) = \mu(y)$;
      b) $xRy \Rightarrow \mu(x) < \mu(y)$.

3. A simplified SCL-model is a simplified model $K = \langle K, R, S, \vdash \rangle$ such that:
   1. $K$ is finite.
   2. $S$ is an equivalence relation.
   3. $xSyRz \Rightarrow xRz$.

Lemma 3.5 (Soundness). For $l=CL, CLM, SCL$ respectively, for any modal formula $A$ if $l \vdash A$ then $A$ is valid in each $l$-model $K$.

Proof. Entirely routine.

Theorem 4. (First modal completeness theorem).
For $l=CL, CLM, SCL$ respectively, for any modal formula $A$ if $l \vdash A$, then $A$ is not valid in some finite $l$-model $K$.

Theorem 5. (Second modal completeness theorem).
For $l=CL, CLM, SCL$ respectively, for any modal formula $A$ if $l \vdash A$, then $A$ is not valid in some simplified $l$-model $K$.

In this paragraph let $1$ be an arbitrary extension of $\text{CL}$, closed under modus ponens and necessitation.

Our goal is to prove a modal completeness of $1$ with respect to some class of $\text{CL}$-models (i.e. $S^X$ must be reflexive and transitive). We will act by the usual way: to use $1$-consistent subsets of an "adequate" set $\Phi$ as nodes of a countermodel; it is supposed that $\Phi$ contains all subformulas of the refuting formula, is closed under negations, subformulas and some special operations which will be explained later.

**Definition 4.1.** An adequate set of formulas is a finite set $\Phi$ which fulfills the following conditions:
1. $\Phi$ is closed under subformulas.
2. If $A \in \Phi$ and $A$ is not a negation, then $\neg A \in \Phi$.
3. $\vdash \Phi$.
4. If $A$ as well as $B$ is an antecedent or a consequent of some $\rightarrow$-formula in $\Phi$, then $A \rightarrow B \in \Phi$.
5. If $A \rightarrow B, A \rightarrow B_1, A \rightarrow B_2, \ldots, A \rightarrow B_n \in \Phi$, then $\vdash (A \rightarrow (A \rightarrow B_1 \rightarrow B_2 \rightarrow \ldots \rightarrow B_n)) \in \Phi$.

**Definition 4.2.** $\Phi := \{ A \mid A \rightarrow X \in \Phi \) \) or $X \rightarrow A \in \Phi$ for some $X \}.$

**Proposition 4.3.** Each finite set of formula can be extended to an adequate set $\Phi$.

In the following reasoning we consider a fixed adequate set $\Phi$.

**Definition 4.4.** $W := \{ x \in \Phi \mid x \text{ is maximal 1-consistent set} \}$

Further it will be necessary to define a binary relation $<$ on $W$ (we will write '$<' instead of '$R'$ for convenience) and a family of binary relations $(S^x), x \in W$, such that

\begin{align*}
&\text{(*) for any } A \in \Phi, x \in W, x \rightarrow A \Leftrightarrow A \in x. \\
&\text{(Where as usual we have defined } x \rightarrow p \Leftrightarrow p \in x).)
\end{align*}

By 1-consistency of $x$ and propositions 2.5, 3.2, it is enough to prove condition (*) for formulas of the form $B \rightarrow C$ and $\neg (B \rightarrow C)$.

The definition of '$<$' is natural:

**Definition 4.5.** For $x, y \in W$

\begin{enumerate}
\item $x < y \iff 1) \forall \sigma D e \Phi: D e \sigma x \Rightarrow D, e D e y.$
\item $\exists e \sigma D e \Phi: e D e x, e D e y.$
\end{enumerate}

(y is a successor of $x$)

In the definition of $S^X$ the following concept of "C-critical successor" is essential:
Definition 4.6. For \( x, y \in W \), \( C \in \mathfrak{F} \)
\[
x <_C y \iff 1) \; x < y.
\]
(\( y \) is a \( C \)-critical successor of \( x \))  \hspace{1em} 2) \( \forall A \supset C \in x \; \neg A \vDash y \).

To explain why \( C \)-critical successors must be used we note that

Proposition 4.7.
\[
CL \vdash \neg (A \supset C) \land B \supset C \ldots \land B_n \supset C \rightarrow \neg ( (A \land B_1 \land \ldots \land B_n) \supset (C \lor B_1 \lor \ldots \lor B_n))
\]

Proof. Let \( B := B_1 \lor \ldots \lor B_n \). We have:
\[
CL \vdash B_1 \supset C \ldots \land B_n \supset C \rightarrow B \supset C
\]
\[
CL \vdash B \supset C \rightarrow (A \land B) \supset (C \lor B)
\]
\[
CL \vdash B \supset C \land (A \land B) \supset (C \lor B) \rightarrow A \supset (C \lor B) \hspace{1em} (A2)
\]
\[
\rightarrow (C \lor B) \supset C \hspace{1em} (A2)
\]
\[
\rightarrow A \supset C \hspace{1em} (A4)
\]
\[
CL \vdash B \supset C \land \neg (A \supset C) \rightarrow \neg ( (A \land B) \supset (C \lor B)).
\]
QED.

Suppose it is necessary to provide \( x \vdash \neg (A \supset C) \) in the model
that we want to construct. Let \( \langle B_1 \supset C, \ldots, B_n \supset C \rangle := \langle x \supset C | x \supset C \in x \rangle \). By
proposition 4.7, because \( x \) is \( 1 \)-consistent,
\[
x \vdash \neg ( (A \land B_1 \land \ldots \land B_n) \supset (C \lor B_1 \lor \ldots \lor B_n))
\]
Thus, by definition 4.6, we obtain:

\[
(\ast \ast) \; \exists y: 1) \; x <_C y, \; y \vdash A
\]
\[
2) \forall z \; y \vdash x, z, x < z \rightarrow a) \; x < z
\]
\[
b) \; z \supset C.
\]
(\( \text{In fact, a) implies b), because by condition 4 of definition 4.1, }\]
\( \supset C \in \mathfrak{F} \))

Condition \((\ast \ast) 1)\) can always be satisfied:

Lemma 4.8. Let \( x \in W \), \( \neg (A \supset C) \in x \). Then there exists a \( C \)-critical
successor \( y \) of \( x \) such that \( A \vDash y \).

Proof. Assume that \( x \) has the form:
\[
x = (\neg (A \supset C) ; B_1 \supset C, \ldots, B_n \supset C ; \diamond D_1, \ldots, \diamond D_m, \ldots)
\]
(\( \text{Since } \supset \in \mathfrak{F} \) and \( \supset C \in x \) and \( n > 0 \)).

Consider a set
\[
y' := (A ; \neg B_1, \ldots, \neg B_n ; \diamond D_1, \ldots, \diamond D_m ; D_1, \ldots, D_m ; \diamond (A \supset B_1 \lor \ldots \lor B_n))
\]
By condition 5 of the definition of an adequate set, \( y' \in \mathfrak{F} \). We show
that \( y' \) is \( 1 \)-consistent. Suppose not. We have:
\[
1 \vdash \diamond D_1 \land \ldots \land \diamond D_m \land D_1 \land \ldots D_m \rightarrow (\diamond (A \supset B_1 \lor \ldots \lor B_n) \rightarrow (A \supset B_1 \lor \ldots \lor B_n)).
\]
Using the necessitation rule, Löb's axiom (L3), and L2, we obtain:
$1 \vdash \diamond D_1 \land \ldots \land \diamond D_m \rightarrow \Box (A \rightarrow B_1 \lor \ldots \lor B_n)$.

On the other hand, by A2,

$1 \vdash \Box x \rightarrow (B_1 \lor \ldots \lor B_n) > C$.

Thus, by A3, A4 we obtain:

$1 \vdash \Box x \rightarrow (A > C)$,

i.e. $x$ is inconsistent. Contradiction.

Now it is sufficient to put $y' := \text{maximal } l\text{-consistent extension of } y'$. QED.

As to the definition of $S_x$, it is clear from (**) that ideally we were to demand $S$ to maintain the status of $C$-critical successor for any $C$ (it means that if $x \prec_C y$, $y \equiv x, z$, then $x \prec_C y$). Unfortunately it is impossible in a general case, however, it becomes possible, if $C$ is fixed.

Lemma 4.9 Let $x, y \in W$, $A > B \equiv x$, $A \equiv y$ and $y$ is $C$-critical successor of $x$, where $C \equiv x$. Then there exists a $C$-critical successor $z$ of $x$ such that $B \equiv z$.

Proof. By condition 4 of the definition of an adequate set, $B > C \equiv x$. Therefore, $\neg (B > C) \equiv x$ or $B \equiv C \equiv x$. In the last case by A4 $A > C \equiv x$ and, since $x \prec_C y, \neg A \equiv y$; it contradicts $l$-consistency of $y$. Hence, $\neg (B > C) \equiv x$. Now we use lemma 4.8. QED.

To counteract these difficulties we will, following [7], multiply the nodes of $W$ such that for any node there would be only one $C$ such that the relations $S_x$ are to maintain the $C$-critical status of $x$ for just this $C$.

This idea along with lemmas 4.8, 4.9 is sufficient for the proof of the modal completeness of CL, which will be shown in the following paragraph.

§5. Modal Completeness of CL.

Proof of theorem 4 for l=CL.

Fix a modal formula $\phi$ such that $CL \vdash \phi$, an adequate set $\Phi$ such that $\phi \in \Phi$, a set $W$ of maximal $CL$-consistent subsets of $\Phi$ and an element $x_0$ of $W$ such that $\neg \phi \in x_0$. Now we define a countermodel for $\phi$ (we use some concepts from the previous paragraph):

Definition 5.1. Let $x \in W$. The depth of $x$ is the maximal $n$ such
that there exists a chain:
\[ x = y_0 < y_1 < \ldots < y_n. \]

**Proposition 5.2.**

a). Let \( x < y \). Then \( \text{depth}(x) > \text{depth}(y) \).

b). Let \( (\circ D | \circ D \in x) \subseteq (\circ D | \circ D \in y) \). Then \( \text{depth}(x) \geq \text{depth}(y) \).

**Definition 5.3.** Let \( K := \langle K, R, (S_x)_x \rangle \), where \( K := \{ \langle x, \tau \rangle, \text{ where } x \in W, \tau \text{ is a sequence of formulas from } \Phi, \text{ and } |\tau| \leq \text{depth}(x) \} \)

\[ \langle x, \tau \rangle > \langle x', \tau' \rangle \quad \iff \quad x < x' \text{ and } \tau_1 \subseteq \tau_2, \]

(\( \text{i.e. } \tau_1 \text{ is a proper initial segment of } \tau_2 \))

\[ \langle x, \tau \rangle < \langle y, \tau \rangle \quad \iff \quad \text{if } \tau \supseteq \tau_0 \ast E \text{ and } y_0 \in x_1, \text{ then } \]

\[ \tau_2 \supseteq \tau_0 \ast E \text{ and } y_0 \in x_2, \]

\[ \langle x, \tau \rangle \vdash p \quad \iff \quad p \in x. \]

Of course, \( S_{\langle x, \tau \rangle} \) is reflexive and transitive, hence we have defined a finite CL-model.

**Lemma 5.4.** For any \( A \in \Phi \) and \( \langle x, \tau \rangle \in K \)

\[ \langle x, \tau \rangle \vdash A \iff A \in x. \]

**Proof.** Induction on the structure of \( A \). We only need to consider the case \( A = B \circ C \).

1. Suppose \( \neg (B \circ C) \in x \). By lemma 4.8, there exists \( y \) such that \( x < y \) and \( B \circ y \). By the induction hypothesis, \( \langle y, \tau \ast < C \rangle \vdash B \). (Note that by proposition 5.2 a), \( \langle y, \tau \ast < C \rangle \in K \). On the other hand, let \( \langle y, \tau \ast < C \rangle < \langle x, \tau \rangle < \langle z, \sigma \rangle, \langle x, \tau \rangle > \langle z, \sigma \rangle \).

By the definition of \( S_{\langle x, \tau \rangle}, x \notin z \), hence, (because \( C \circ C \in x \)), \( \neg C \in z \) and \( \langle z, \sigma \rangle \vdash C \).

2. Suppose \( B \circ C \in x \), and \( \langle y, \sigma \rangle \vdash B \), where \( \langle y, \tau \rangle > \langle y, \sigma \rangle \). By the induction hypothesis, \( B \circ y \). Since \( \tau \circ \sigma \), there exists a unique \( E \) s.t. \( \sigma \geq \tau \ast < E \). There are two cases to consider:

   **Case 1.** \( x < E \). Then, by lemma 4.9, there exists \( z \) such that \( x < E z \) and \( C \in z \). So, we have: \( \langle z, \tau \ast < E \rangle \in K, \langle y, \sigma \rangle < \langle x, \tau \rangle < \langle z, \tau \ast < E \rangle, \langle z, \tau \ast < E \rangle \vdash C \).

   **Case 2.** \( x < E \) does not hold. Note that each successor is \( 1 \)-critical successor. So, we can apply lemma 4.9 to obtain \( z \) s.t. \( x < z, C \in z \) and use the construction from case 1 with any \( \Phi \)-formula instead of \( E \). QED.

Now it is enough to note that by the last lemma and definition
of $x_0$, $<x_0,<>\not\models \phi$. This completes the proof.

To finish the paragraph we discuss the principal difficulties arising in consideration of richer logics of conservativity and the basic ways to overcome them.

First, to provide the condition $yS \not\models zRt \Rightarrow yRt$ it is necessary to demand the condition $\tau_1 \subseteq \tau_2$ for $S$ to be fulfilled. So, the problem arises how to provide for the pair $<z,\sigma>$ obtained in part 2 of the proof of lemma 5.4 to belong to $K$, i.e. for the depth of $z$ not to be more that the depth of $y$. The natural way to solve this problem — to transfer all boxes from $y$ to $z$ — cannot be applied because the necessary adequate set became infinite.

We propose the following way to approach this problem. Imagine that our model is graduated by levels (by the length of $\tau$). Now restrict the validity of the C-critical-maintain condition in the definition of $S$ only to the level immediately above $<x,\tau>$. It is enough to preserve the reasoning in part 1 of the proof of lemma 5.4, because the "counterexample" constructed there lays immediately above the node $<x,\tau>$. On the other hand, one can easily see that the difficulty mentioned above (sufficiently small depth of the node $z$) does not arise if we deal with $<y,\sigma>$ laying immediately above $<x,\tau>$. Considering the higher levels one needn’t to worry about C-critical status, and so it becomes possible to transfer a sufficient quantity of boxes from $y$ to $z$.

Second, for the condition $zS_x y$, $zRt \Rightarrow yRt$ to be fulfilled, we cannot use the above method (to transfer the sufficient quantity of boxes’ negations from $y$ to $z$), because adequate sets expand too rapidly. The solution of this problem (described in §7) formally uses Visser’s construction of obtaining the simplified models (cf.[8]), but really it is the idea that "counterexample" for the formula $A \Rightarrow B$ (i.e., if $x \not\models A \Rightarrow B$, such $y$ that $xRy$, $y \not\models A$, $\forall z(yS_z z, xRz \rightarrow z \not\models B)$) must not be an end of an $S$-arrow.
§6. Modal Completeness of CLM.

Definition 6.1. A CLM-adequate set of modal formulas is a finite set \( \Phi \) s.t. there exist set \( \Phi_\rightarrow \) and set \( \Phi_\leftarrow \) containing only boxed formulas which satisfy the following conditions:

1. \( \Phi_\rightarrow = \{ A \mid \text{A} \rightarrow \text{X} \in \Phi \text{ or } \text{X} \rightarrow \text{A} \in \Phi \text{ for some X} \}; \)
2. \( \Phi_\leftarrow \subseteq \Phi; \)
3. \( \Phi \) is closed under negations and subformulas (in sense of definition 4.1);
4. if \( A, B \in \Phi_\rightarrow \), then \( A \rightarrow B \in \Phi \);
5. \( \neg A \in \Phi_\leftarrow \);
6. if \( A, B_1, B_2, \ldots, B_n \in \Phi_\rightarrow \), then \( \square(\text{A} \rightarrow \text{B}_1 \lor \ldots \lor \text{B}_n) \in \Phi \);
7. if \( A \in \Phi_\rightarrow \), \( \square \text{D}_1, \ldots, \square \text{D}_n \in \Phi \), then there is a formula \( A' \) which is GL-equivalent to \( A \land \text{D}_1 \land \ldots \land \text{D}_n \) such that \( \square A' \in \Phi_\leftarrow \);
8. if \( B \in \Phi_\rightarrow \), \( \square \text{D} \in \Phi_\leftarrow \), then there is a formula \( B' \) which is GL-equivalent to \( B \land \text{D} \) such that \( B' \in \Phi_\rightarrow \).

Note that each CLM-adequate set is an adequate set.

We will show that every finite set of modal formulas is contained in some CLM-adequate set.

Definition 6.2. Consider an arbitrary set of modal formulas \( X \). The LR-closure of \( X \) is the minimal pair \( \langle L, R \rangle \) (with respect to each component) of sets of modal formulas such that \( L \supseteq X \) and:

\[
\begin{align*}
(L) & \forall \text{A} \in L, \forall \text{B} \in L, \text{A} \rightarrow \text{B} \in L; \\
(R) & \forall \text{A} \in L, \forall \text{B}, \text{C}_1, \ldots, \text{C}_n \in L, \text{A} \rightarrow (B \rightarrow \text{C}_1 \lor \ldots \lor \text{C}_n) \in R.
\end{align*}
\]

Lemma 6.3. Let \( X \) be a finite set of modal formulas, and \( \langle L, R \rangle \) be the LR-closure of \( X \). Then \( L \) as well as \( R \) consists of finitely many equivalence classes with respect to GL-provable equivalence.

Proof. We can assume that \( X = \{ \bot, p_1, \ldots, p_m \} \). The proof proceeds by induction on \( m \). First, we note that formula \( D \) belongs to \( R \) if \( D \) has the form:

\[
(*) \quad D = A \land \square(B^{(1)} \rightarrow C_1^{(1)} \lor \ldots \lor C_n^{(1)}) \land \ldots \land \square(B^{(k)} \rightarrow C_1^{(k)} \lor \ldots \lor C_n^{(k)}),
\]

where \( A, B^{(i)}, C_j^{(i)} \in L \), and \( D \) belongs to \( L \), iff

\[
(**) \quad D = p_1 \land \square B_1 \land \ldots \land \square B_n, \text{ or } D = \bot,
\]

where \( p_1 \in X \) and \( B_1, \ldots, B_n \in L \cup R \).

By \((*)\), it is sufficient to prove that \( L \) consists of finitely many equivalence classes. By \((**)\), it is enough to prove that there modulo to GL-provable equivalence there are only finitely many
formulas of the form \( \sigma \top B \) with \( B \in \mathcal{LJR} \). By (*) and (**), \( B \) has the form

\[
B = p_i \land \sigma D
\]

where \( D \) is a boolean combinations of formulas from \( \mathcal{LJR} \). But it is well-known that

\[
\mathcal{GL} \vdash \sigma(\neg p_i \lor \neg D) \leftrightarrow \sigma(\neg p_i \lor \neg D'),
\]

where \( D' \) is obtained from \( D \) by replacing all the occurrences of \( p_i \) in \( D \) by \( 1 \). By the induction hypothesis, there are only finitely many such \( D' \) modulo to \( \mathcal{GL} \)-equivalence. QED.

**Corollary 6.4.** Let \( X \) be a finite set of modal formulas. Then there exist finite sets \( \bar{L} \) and \( \bar{R} \) such that \( \bar{L} \models X \) and:

- (L) \( \forall A \in \bar{L}, B \in \mathcal{LJR} \exists A' \in \bar{L} : \mathcal{GL} \vdash A' \leftrightarrow A \land \sigma B \);
- (R) \( \forall A \in \mathcal{LJR}, B \in \bar{L} \exists A' \in \bar{L} : \mathcal{GL} \vdash A' \leftrightarrow A \land \sigma (B \rightarrow C_1 \lor \ldots \lor C_n) \);

if \( \sigma D \) is a subformula of some formula from \( \mathcal{LJR} \), then

- either \( D = B \rightarrow C_1 \lor \ldots \lor C_n \) for \( B, C_1, \ldots, C_n \in \bar{L} \);

- or if \( D \sigma E \) is a subformula of some formula from \( \mathcal{LJR} \), then \( D \rightarrow E \) is a subformula of some formula from \( X \).

**Proof.** We define two sequences: \( L_0, L_1, \ldots, L_n, \ldots, R_0, R_1, \ldots, R_n, \ldots \):

\[
L_0 := X, R_0 := \emptyset ;
\]

\[
L_{n+1} := L \cup \{ D = A \land \sigma B : \text{ s.t. there is no formula in } L_n \text{ GL-equivalent to } D, \text{ where } A \in L_n, B \in \mathcal{LJR} \};
\]

\[
R_{n+1} := R \cup \{ D = A \land \sigma (B \rightarrow C_1 \lor \ldots \lor C_k) : \text{ s.t. there is no formula in } R_n \text{ GL-equivalent to } D, \text{ where } A \in \mathcal{LJR}, B, C_1, \ldots, C_k \in L_n \}.
\]

Define now \( \bar{L} := \bigcup_{n=0}^{\infty} L_n, \bar{R} := \bigcup_{n=0}^{\infty} R_n \). By lemma 6.3, \( \bar{L} \) and \( \bar{R} \) are finite and have all necessary properties.

**Lemma 6.5.** Each finite set \( \Phi_0 \) of modal formulas can be extended to a CLM-decompositional set \( \Phi \).

**Proof.** We can assume that \( \top \in \Phi_0 \). Let \( \Phi \) be the closure of \( \Phi_0 \) under subformulas and negations, \( \bar{L} \) and \( \bar{R} \) be sets defined in corollary 6.4 (where \( X = \Phi_0 \)). We define:

\[
\Phi_0 := \bar{L};
\]

\[
\Phi := (\neg A \land \sigma \in \mathcal{LJR});
\]

\[
\Phi_2 := (\neg A \land \sigma \in \mathcal{LJR}) \cup A \rightarrow B \cup A \land \sigma (A \rightarrow B \lor \ldots \lor B_n) \cup A, B_1, \ldots, B_n \in \bar{L};
\]

\( \Phi \) is closure of \( \Phi_2 \) under subformulas and negations.

Now we check conditions 1-8 of the definition 6.1.
1. Obviously, $A \vDash \Phi$ implies $A \vDash A \vDash \Phi$. Conversely, if $A \vDash A \vDash \Phi$ or $A \vDash A \vDash \Phi$ then by corollary 6.4(\textnumero) $A \vDash \Phi_1$ (because $\Phi_1$ is closed under subformulas).

2, 3, 4, 5, 6 are trivial.

7. Suppose $\Box D_1, \ldots, \Box D_n \vDash \Phi$. For any $i \leq n$ by corollary 6.4(\textnumero) one of three cases holds:
   a) $D_i = A$ for $A \vDash \Box \Box \Box R$;
   b) $D_i = B \rightarrow C_1 \lor \ldots \lor C_n$, $B, C_1, \ldots, C_n \vDash \Box$;
   c) $D_i \vDash \Phi$.

Case c) can easily reduced to case a), because $D_i \vDash \Phi$ implies $\neg D_i \vDash \Phi_1 \subseteq \Box$. Let now $(\Box D_1, \ldots, \Box D_k)$ be the set of boxes satisfying condition (a) and $(\Box D_{k+1}, \ldots, \Box D_n)$ be the set of boxes satisfying condition (b). Let also $A \vDash \Phi$. By 6.4(L),

there exists $A' \vDash \Box \Box \Box$ such that $GL \vdash A' \leftrightarrow A' \land \Box D_1 \land \ldots \land \Box D_k$.

By 6.4(R),

there exists $A'' \vDash \Box$ such that $GL \vdash A'' \leftrightarrow A'' \land \Box D_{k+1} \land \ldots \land \Box D_n$.

By the definition of $\Phi$,

8. If $B \vDash \Box \Box \Box$ and $\Box D = \Box A$ for $A \vDash \Box \Box \Box R$, then by corollary 6.4(L) there exists $B' \vDash \Phi$ such that $GL \vdash B' \leftrightarrow B' \land \Box D$.

This completes proof of lemma 6.5.

Proof of theorem 4 for $l = CLM$.

As usual, we fix a modal formula $\phi$ such that $CLM \vDash \phi$, an $CLM$-adequate set $\Phi$ (with the corresponding sets $\Phi_1, \Phi_2$) such that $\phi \vDash \Phi$, the set $W$ of maximal $CLM$-consistent subsets of $\Phi$ and an element $x_0$ of $W$ such that $\neg \phi \vDash x_0$.

Definition 6.6. For $x, y \in W$

\[ x \triangleleft y \iff \Box D e \Phi_1 \land D e x \rightarrow \Box D e y; \]

\[ x \triangleleft y \iff \Box D e \Phi_1 \land D e x \rightarrow D, \Box D e y. \]

Proposition 6.7. $\triangleleft^W$, $\triangleleft^W$ are transitive and $x \triangleleft^W y \triangleleft^W z \Rightarrow x \triangleleft^W z$.

Definition 6.8. Let $K = \langle K, R, S, \rightarrow \rangle$, where

$K = \langle x, \tau \rangle$, where $x e W$, $\tau$ is a sequence of formulas from $\Phi$,

$|\tau| \leq \text{depth}(x_0) - \text{depth}(x)$;

\[ <x_1, \tau_1>_R <x_2, \tau_2> : \iff x_1 \triangleleft^W x_2 \land \tau_1 \subseteq \tau_2; \]

\[ <x_1, \tau_1>_S <y_0, \tau_0> <x_2, \tau_2> : \iff \]

1) $\tau_1 = \tau_2$;

2) $x_1 \triangleleft^W x_2$;

3) if $\tau_1 = \tau_2 = \tau \ast E$ and $y_0 \triangleleft^W x$, then $y_0 \triangleleft^W x_2$.  

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\(<x,\tau> \vdash p \iff \phi x\).

**Proposition 6.9.** \(K\) is a CLM-model.

**Proof.** Use proposition 6.7.

**Proposition 6.10.** Let \(x \in W\) and \(\text{D} \in \phi\) (i.e. \(\neg \text{D} \in \phi\)). Then there exists \(y \in W\) such that \(x < y\), \(\text{D} \in \phi\).

**Proof.** Use lemma 4.9 for \(A \Rightarrow B \Rightarrow C \Rightarrow D \Rightarrow 1\).

**Lemma 6.11.** Let \(A\) be a subformula of \(\phi\). Then for any \(<x,\tau> \in K\)

\(<x,\tau> \vdash A \iff A \in \phi\).

**Proof.** Let \(A = B \Rightarrow C\). Of course, \(B, C \in \phi\).

1. Let \(\neg (B \Rightarrow C) \in \phi\). By lemma 4.8, there exists \(y\) such that \(x < y\) and \(B \in \phi\). Of course, \(<y,\tau > \in \phi\), and if \(<y,\tau > \in \phi\), and if \(<y,\tau > \in \phi\), then by definition of \(S_{<x,\tau >} x < z\), hence \(\neg C \in \phi\).

2. Let \(B \Rightarrow C \in \phi\), and \(<y,\sigma> \vdash B\), where \(<x,\tau > \in K\). We consider two cases:

   **Case 1.** \(\sigma = \tau \in \phi\) and \(x < y\).

   Let \(\{\text{D}_1, \ldots, \text{D}_n\} : = \gamma \in \phi\). By condition 8 of the definition of a CLM-adequate set there exist \(B', C'\) which are GL-equivalent to \(B \land \text{D}_1 \land \ldots \land \text{D}_n\) and \(C \land \text{D}_1 \land \ldots \land \text{D}_n\) respectively such that \(B' \Rightarrow C' \in \phi\).

   Evidently, (by axiom \(M\)) \(B' \Rightarrow C' \in \phi\), \(B' \Rightarrow y\). Therefore by lemma 4.9 there exists \(z\) such that \(x < z\) and \(C' \in \phi\). So, we have \(z, \sigma > \in K\)

   (because \(|\sigma| = |\tau| + 1\) and \(\text{depth}(z) < \text{depth}(x)\)), \(y < z\) and \(z \vdash C\)

   (because \(z \vdash C < \text{D}_1 \land \ldots \land \text{D}_n\)) and thus \(<y, \sigma > \in K\).

   **Case 2.** \(|\sigma| > |\tau| + 2\) or \(x \geq y\) does not hold.

   Let \(\{\text{D}_1, \ldots, \text{D}_n\} : = \{\text{D} \mid \text{D} \in \phi\} \gamma\). By condition 7 of the definition of CLM-adequate set, there exist \(B', C'\) which are GL-equivalent to \(B \land \text{D}_1 \land \ldots \land \text{D}_n\) and \(C \land \text{D}_1 \land \ldots \land \text{D}_n\) respectively such that \(\neg B', \neg C' \in \phi\).

   By axioms \((A1), (M), \text{CLM} \vdash X \rightarrow (\text{D} \rightarrow \text{C})\).

   We show that \(\neg B' \in \phi\). Suppose not. Then \(\neg B' \in \phi\) and \(x < y\) implies \(\neg B' \in \phi\) (because \(\neg B' \in \phi\)). Contradiction. Thus, \(\neg B' \in \phi\) and hence \(\text{D} \in \phi\). By proposition 6.10, there exists \(z\) such that \(C' \in \phi\) and \(x < z\).

   By proposition 5.2b), \(\text{depth}(z) \leq \text{depth}(y)\), therefore \(<z, \sigma > \in K\). So, we have: \(<y, \sigma > S_{<x,\tau >} x < z, \sigma >, <x,\tau > R < z, \sigma >, <z, \sigma > \vdash C\). QED.

As usual, we note that \(<x_0, \gamma > \vdash \phi\). This completes the proof of theorem 4 for \(1 = \text{CLM}\).

Note that any simplified SCL-model is also finite SCL-model. Thus, it is enough to prove modal completeness of SCL w.r.t. simplified models.

First, we define for each CL-model a "pattern" which will be used for definition of corresponded simplified model (this definitions will be different for different considered logics).

**Definition 7.1.** Let \( K=(K,R,(S_x), \rightarrow) \) be a CL-model. Assume that \( 0 \notin K \). A **heir of the first type** of \( K \) is the tuple
\[ \mathcal{H} = \langle H, c, \langle, R(\cdot), \text{end}(\cdot) \rangle, \]
where
\[ H:=\langle \Gamma, \Delta \rangle: 1) \Gamma \) is a finite sequence of elements from \( K; \)
\[ 2) \Delta \) is a finite sequence of elements from \( K \cup \{0\}; \]
\[ 3) |\Delta| = |\Gamma| - 1; \]
\[ \text{So, let } \Gamma = \langle x_0,\ldots,x_n \rangle, \Delta = \langle y_0,\ldots,y_{n-1} \rangle. \]
\[ 4) \forall i < n \quad y_i = 0 \Rightarrow x_i R x_{i+1}, \]
\[ y_i \neq 0 \Rightarrow x_i S y_i \]
\[ \langle \Gamma_1, \Delta_1 \rangle \subseteq \langle \Gamma_2, \Delta_2 \rangle \quad \iff \quad \Gamma_1 \subseteq \Gamma_2 \land \Delta_1 \subseteq \Delta_2 \]
\[ \langle \Gamma_1, \Delta_1 \rangle \cong \langle \Gamma_2, \Delta_2 \rangle \quad \iff \quad 1) \langle \Gamma_1, \Delta_1 \rangle \subseteq \langle \Gamma_2, \Delta_2 \rangle; \]
\[ \text{So, let } \Gamma_1 = \langle x_0,\ldots,x_n \rangle, \Delta_1 = \langle y_0,\ldots,y_{n-1} \rangle \]
\[ \Gamma_2 = \langle x_0,\ldots,x_m \rangle, \Delta_2 = \langle y_0,\ldots,y_{m-1} \rangle \]
\[ m > n \]
\[ 2) \exists i \quad (n \leq i < m \wedge y_i = 0 \land \forall j \quad (i < j < m \rightarrow \exists k \quad (y_j = x_k \wedge n \leq k \leq i)) \)
\[ R(\langle x_0,\ldots,x_n, y_0,\ldots,y_{n-1} \rangle) \equiv \langle x_0,\ldots,x_k, y_0,\ldots,y_{k-1} \rangle, \]
where \( k = \min \{ j \mid \forall i \quad (j \leq i < n \rightarrow y_i \neq 0) \} \)
\[ \text{end}(\langle x_0,\ldots,x_n, y_0,\ldots,y_{n-1} \rangle) = x_n. \]
A **heir of the second type** of the \( K \) is the tuple
\[ \mathcal{H}_1 = \langle H_1, c_1, \langle, R_1(\cdot), \text{end}_1(\cdot) \rangle, \]
where
\[ H_1 := \{ t \in H \mid \forall x \subseteq t \forall y (R(x) \subseteq y \rightarrow \text{end}(x) \neq \text{end}(y)) \}, \]
and \( c_1, \langle, c_1, R_1(\cdot), \text{end}_1(\cdot) \) are restrictions of \( c, \langle, R(\cdot), \text{end}(\cdot) \) on \( H_1 \).
\( (\subseteq \) is the reflexive closure of \( \subset \).

We left to the reader verification of the following simple facts (it is supposed that \( \mathcal{H} = \langle H, c, \langle, R(\cdot), \text{end}(\cdot) \rangle \) is a heir of an arbitrary type of \( K=(K,R,\rightarrow) \):
**Proposition 7.2.** (For any \( x, y, z, \ldots \in H \))

a) \( c \) is transitive and irreflexive;
b) \( R(R(x)) \neq R(x) \);
c) \( R(x) \subseteq x \);
d) \( \exists R(x) = x \);
e) \( x < y \rightarrow x \leq y \);
f) \( x \leq y < z \rightarrow x < z \);
g) \( < \) is transitive and irreflexive;
h) \( x < y, R(y) \leq z \leq y \rightarrow x < z \).

**Proposition 7.3.** If \( K \) is a CLM-model, then

a) the heir of \( K \) of the 2-nd type is finite;
b) \( \forall x, y \in H \quad x < y \Rightarrow \text{end}(x) \neq \text{end}(y) \).

**Proof.** a) Let \( H \) be the heir of \( K \) of the second type; \( \langle \Gamma, \Delta \rangle \in H \). We claim that for any \( i \neq j \), \( x_i \neq x_j \), hence there are only finitely many such \( \Gamma \); on the other hand, since \( |\Delta| = |\Gamma| - 1 \), for each sequence \( \Gamma = < x_0, \ldots, x_n > \) there are only finitely many such \( \Delta \) that \( \langle \Gamma, \Delta \rangle \in H \).

Suppose not: \( \langle \Gamma, \Delta \rangle = < x_0, \ldots, x_n >, y_0, \ldots, y_{n-1} > \in H_1 \), \( i < j \), \( x_i = x_j \). Let \( x = < x_0, \ldots, x_j >, y_0, \ldots, y_{j-1} > \). Then \( \theta = \langle \Gamma, \Delta \rangle \), \( y = < x_0, \ldots, x_1 >, y_0, \ldots, y_{i-1} >, < y_0, \ldots, y_{k-1} > \); \( \langle x_0, \ldots, x_k, y_0, \ldots, y_{k-1} \rangle = R(x) \).

**Case 1.** \( k < i \). Then \( x \leq t \), \( R(x) \leq y < x \), \( \text{end}(y) = \text{end}(x) \). It contradicts the definition of \( H \).

**Case 2.** \( k > i \). Then by definition of \( R(\cdot) \), \( y_{k-1} = 0 \), hence

\[
x_0 \cdots x_{k-1} \cdots x_{k-2} \cdots x_{k-1} \cdots x_1 \cdots x_{j-1} x_j = x_i,
\]

where \( Q_s = (R \text{ or } S_{y_s}) \). Because \( K \) is a CLM-model, \( xQ_s yRz \) implies \( xRz \), and thus \( x_{k-1} y \). Contradiction.

b) Let \( y = < x_0, \ldots, x_n >, y_0, \ldots, y_{n-1} > \), \( x = < x_0, \ldots, x_k >, y_0, \ldots, y_{k-1} > \), \( x < y \). We must show that \( xR_k x_n \). Indeed, by the definition of \(< \), for some \( i, k \leq i < n \),

\[
x_k Q_{k-1} Q_{k-2} \cdots x_1 R_k x_1 S_{y_{i+1}} \cdots S_{y_{n-1}} x_n
\]

By the property of CLM-models \( xR_k x_1 \). If \( i+1 = n \), we have done; else, there is \( j, n \leq j < i \), such that \( y_{n-1} = x_j \), and \( xR_j x_n \); as above, we obtain \( xR_k x_n \).

**Lemma 7.4.** Let \( \text{end}(x)Ry \). Then there exists \( y \) such that

1) \( \text{end}(y) = y \);
2) \( x < y \).

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3) \( R(y) = y; \)

4) for any \( z \) \( ( R(z) = y \wedge x < z \rightarrow \text{end}(y) S_{\text{end}(x)} \text{end}(z) \wedge \end{end}(x) \text{Rend}(z) ) \).

**Proof.** Let \( x = \langle x_0, \ldots, x_n, y_0, \ldots, y_{n-1} \rangle, x \in \text{Ry}. \) We define \( y = \langle x_0, \ldots, x_n, y, y_0, \ldots, y_{n-1}, 0 \rangle. \) The properties 1), 2), 3) are trivial. We check property 4): fix \( z \) such that \( R(z) = y, x < z; \) let \( z = \langle x_0, \ldots, x_n, y, z_0, z_1, \ldots, z_l, y_0, \ldots, y_{n-1}, 0, y_{n+1}, \ldots, y_{n+l+1} \rangle. \)

Since \( R(z) = y, y_{n+1}, \ldots, y_{n+l+1} \neq 0; \) because \( x < z, \) we have \( i = n \) in the definition of \( < \) and thus for any \( j > n \) \( y_j = x_n, \) i.e.

\[
y S_{x_0} S_{x_1} \ldots S_{x_l} z \quad x R_{z_0} \ldots, x R_{z_l}
\]

By the transitivity of \( S_{x_n}, y S_{x_n} z \) and \( x R_{z_1}. \) QED.

**Lemma 7.5.** Let \( x < y, \) \( end(y) S_{\text{end}(x)} z, \) \( end(x) R z. \) Then there exists \( z \) such that:

1) \( end(z) = z; \)
2) \( x < z; \)
3) \( R(y) = R(z); \)
4) \( y S z \) or \( z S y. \)

If our heir has the 1-st type, we also can require

4*) \( y S z. \)

**Proof.** Let \( y = \langle x_0, \ldots, x_n, y_0, \ldots, y_{n-1} \rangle, \) \( x = \langle x_0, \ldots, x_k, y_0, \ldots, y_{k-1} \rangle, \) \( k < n \), \( x < y. \) Let also

\[
\langle x_0, \ldots, x_j, y_0, \ldots, y_{j-1} \rangle := R(y).
\]

( Note that \( k < j \leq n \).) Fix \( z \) s.t. \( x S_n z, x R z. \)

Case 1. \( z \notin \{ x_j, \ldots, x_n \} \) or our heir has the 1-st type.
Then \( z = \langle x_0, \ldots, x_n, z, y_0, \ldots, y_{n-1}, x_k \rangle \) has all necessary properties.

Case 2. \( z = x_s, j \leq s \leq n, \) and our heir has the 2-nd type.
Then it is sufficient to define \( z = \langle x_0, \ldots, x_s, y_0, \ldots, y_{s-1} \rangle. \)
Because \( R(y) S z \subseteq y \) and \( x < y, \) by proposition 7.2 h), \( x < z. \)

Lemma is proved.

We have proved all necessary properties of a heirs and continue with the following definition:

**Definition 7.6.** Let \( l = CL, CLM, SCL \) and \( \phi \) be a modal formula such that \( l \models \phi; \) \( l \)-countermodel for \( \phi \) is

a) ( for \( l = CL, CLM \) ) finite \( l \)-model \( K \) such that \( K \models \phi; \)

b) ( for \( l = SCL \) ) finite CLM-model \( K \) such that \( K \models \phi \) and for
any subformula $\phi$ of the type $A \land B$ or $\neg D$, for any $x, y, z \in K$, if $y S_x z$, then $y$ and $z$ agree on this subformula.

We will reduce a 1-countermodel for $\phi$ to a simplified 1-countermodel for $\phi$; however, before it we must show that if $SCL \models \phi$, then there exists a $SCL$-countermodel for $\phi$.

**Definition 7.7.** For any modal formula $\phi$ let $X(\phi)$ be the set of all $\models$-subformulas of $\phi$ and all formulas of the form $(\neg D) \models 1$, where $\neg D$ is a subformula of $\phi$; $\neg X$ is the set of negations of all formulas from $X$.

**Definition 7.8.** For any modal formula $\phi$

$$S(\phi) := \neg E \land (A \rightarrow (A \land D) \land B \land D),$$

where $\neg E := E \land \neg E$.

**Lemma 7.9.** If $SCL \models \phi$, then there exists a $SCL$-countermodel for $\phi$.

**Proof.** Assume that $SCL \models \phi$, then $CLM \models S(\phi) \models \phi$; let $K = \langle X, R, (S_x, \neg S) \rangle$, $\models$ be a finite $CLM$-countermodel for $S(\phi) \models \phi$. We can assume that $x_0 \ni S(\phi) \models \phi$ and $K = \{ x_0 \} \cup x | x_0 \in X \cup X, A \models B \in X \}$. Define $S$ as an equivalence relation on $K$: $xSy$ iff $x$ and $y$ agree on each formula from $X$. We claim that the model $\langle X, R, (S_x \cap S), \models \rangle$ (where $\models$ coincides with $\models$ on propositional variables) has all properties required.

It is sufficient to show that the restriction of $S_x$ to $S_x \cap S$ preserves forcing of formulas from $X(\phi)$. Indeed, let $x \ni A \land B, xRy, y \ni A$, and $D$ be a conjunction of all formulas from $X(\phi)$ and its negations which are true in $y$. Since $K \models S(\phi), x \ni (A \land D) \land (B \land D)$ and there exists $z$ s.t. $y S_x z, xRz, z \ni B \land D$; evidently, $y S_x z$. QED.

**Definition 7.10.** Suppose that $1 = CL, CLM, SCL$; $\phi$ is a modal formula such that $1 \models \phi$; $K$ is a 1-countermodel for $\phi$; $K, K_1$ are heirs of $K$. We define a simplified 1-model $K' = \langle K', R', S', \models \rangle$ by the following table:

<table>
<thead>
<tr>
<th>1</th>
<th>$K'$</th>
<th>$xR'y$</th>
<th>$xS'y$</th>
<th>$x \models p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CL</td>
<td>$H$</td>
<td>$x &lt; y \land \neg \text{end}(x) \land \text{end}(y)$</td>
<td>$R(x) \models R(y)$</td>
<td>$\text{end}(x) \models p$</td>
</tr>
<tr>
<td>CLM</td>
<td>$H$</td>
<td>$x &lt; y$</td>
<td>$R(x) \models R(y) \land \neg x \models y$</td>
<td>in any case.</td>
</tr>
<tr>
<td>SCL</td>
<td>$H_1$</td>
<td>$R(x) &lt; y$</td>
<td>$R(x) \models R(y)$</td>
<td></td>
</tr>
</tbody>
</table>
Proposition 7.11. $K'$ is, indeed, a simplified $l$-model.

Proof.

$l=CL$.

1. $R'$ is transitive and wellfounded: by proposition 7.2g)
2. $S'$ is an equivalence relation. It is trivial.

$l=CIM$.

1. $R'$ is transitive and wellfounded: by propositions 7.2g) and 7.3b).
2. $S'$ is reflexive and transitive. It is trivial.
3. $xS'\, yR'\, z \Rightarrow xR'\, z$: by proposition 7.2f)
4. There exists a natural number $N$ and a mapping $\mu:K' \rightarrow \{1, \ldots, N\}$ such that
   a) $xS'\, y \Rightarrow \mu(x)=\mu(y)$;
   b) $xR'\, y \Rightarrow \mu(x)<\mu(y)$;

   Define for $x=\langle x_0, \ldots, x_n, y_0, \ldots, y_{n-1} \rangle$

   $\mu(x):=|\{i:y_i=0\}|+1$.

$l=SCL$.

1. $K'$ is finite: by proposition 7.3a).
2. $R'$ is transitive.

   Let $R(x)<y$ and $R(y)<z$. By proposition 7.2h), $R(x)<R(y)$.

   By proposition 7.2g), $R(x)<z$.
3. $R'$ is irreflexive: by proposition 7.2 d).
4. $S'$ is an equivalence relation. It is trivial.
5. $xS'\, yR'\, z \Rightarrow xR'\, z$. It is trivial.

Lemma 7.12. For any $xK'$ and modal formula $A$ ( in the case $l=SCL$ it is necessary to require that $A$ is a subformula of $\phi$ ),

   $\vdash x \leftrightarrow end(x) \vdash A$.

Proof. Induction on the complexity of $A$. We consider the case $A=B\triangleright C$. Let $x:=end(x)$; $t:=R(x)$, if $l=SCL$, $t:=x$ otherwise; $t:=end(t)$. In any case ( by definition 7.6b ) , $t \vdash A$ $\Leftrightarrow x \vdash A$.

1. Suppose that $x \vdash B\triangleright C$. We will show that $t \vdash B\triangleright C$. Indeed, let $t\, R\, y$, $y \vdash B$. We use lemma 7.4 to obtain $yK'$; by the induction hypothesis, $y \vdash B$, hence there is $z$ such that $z \vdash C$, $t\, R\, z$, $y\, S\, z$. In any case, $y=R(y)=R(z)$ and $t<z$ ( for $l=SCL$ it follows from $t=R(t)$ ), hence by a property of $y$ ( claim 4) of lemma 7.4 ), $y\, S\, z$, $t\, R\, z$, where $z:=end(z)$. By the induction hypothesis, $z \vdash C$. Thus, we proved that $t \vdash B\triangleright C$. 

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2. Suppose that $t \vdash B \supset C$. We will show that $x \vdash B \supset C$. Indeed, let $xR'y$, $(y := end(y))$, $y \vdash B$. One can see (using proposition 7.3b) that $tRy$; by the induction hypothesis, $y \vdash B$, hence there is $z$ s.t. $tRz$, $yS_t z$, $z \vdash C$. Now we can use lemma 7.5 and obtain $z$ s.t. $z \vdash C$ ( $end(z) = z$ and induction hypothesis ), $yS'z$ (in the case $1=CLM$ we use property 4 of $z$ from lemma 7.5 ), $xR'z$ (we use that $tRz$ and $t < z$). Thus, $x \vdash B \supset C$.

Thus, if $x \vDash \phi$, then $< < x > , < > \vDash \phi$ and $K' \vDash \phi$. This complete the proof of the second modal completeness theorem.

**Corollary 7.13.** For any modal formula $\phi$,

$$SCL \vdash \phi \iff CLM \vdash S(\phi) \rightarrow \phi.$$  

**Remark.** The author doesn’t know whether we can use only finite simplified model for $1=CL$, but for $1=CLM$ we cannot. Indeed, the following formula

$$(\Diamond \top) \supset (\Diamond \top \land (p \supset q)) \land (\Diamond \top) \supset (\Diamond \top \land (p \supset q)) \rightarrow \Box 1$$

is not provable in CLM:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>R</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>R</td>
<td>S</td>
</tr>
<tr>
<td>p</td>
<td>R</td>
<td>S</td>
</tr>
<tr>
<td>q</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

and is valid in every finite simplified CLM-model.
§8. Arithmetical Completeness of SCL.

We will prove theorem 1 in more general formulation:

Theorem 8.1. Let $\Gamma$ be decidable set of arithmetical formulas closed under conjunctions such that $\Sigma_2 \cup \Pi_2 \subseteq \Sigma_N$ for some $N$. Then SCL is the provability logic for $\Gamma$-conservativity ( i.e. SCL = $\text{CL}(\Gamma) = \text{CL}^+(\Gamma)$ )

Proof. Arithmetical soundness of SCL is evident. Assume that SCL $\vdash \phi$, and let $K = \langle K, R, S, \vdash \rangle$ be a simplified SCL-countermodel for $\phi$. Without a loss of generality we can assume that

(*) \[ \exists w \in K \quad \forall x \in K \quad (x = w \lor wRx) \]

(**) \[ \exists u \in K \quad (u = w \land u \vdash \phi) \]

Define now a usual Solovay function $h$:

Definition 8.2.

\[ h(0) = w; \]

\[ \text{if } \text{Proof}_{PA}(n, [l \neq z]) \text{ and } h(n)Rz \]

\[ \text{then } h(n+1) := z; \]

\[ \text{else } h(n+1) := h(n). \]

"$l = z$" stands for ( $\Sigma_2$-formula ) $\lim_{n \to \infty} h(n) = z$.

and establish its usual properties:

Lemma 8.3. ( PA proves that )

1. There exists unique $z$ such that $l = z$.
2. If $xRy$, then $l = x \to \neg \circ_{PA} l \neq y$.
3. If $x \neq w$, then $l = x \to \circ_{PA} (l = y \to xRy)$.

In the following considerations we will use inside PA notion "truth" for some formulas defined by their Gödelnumbers. It is admissible, because one can check ( using assumption $\Gamma \subseteq \Sigma_N$ ) that complexity of all such formulas bounded by $\Pi_N$.

Definition 8.4. For all $z \in K$ we define a formula $L = z$ ( by Diagonal Lemma ):

\[ L = z : \iff \text{"For any } x \text{ such that } l = x \]

\[ \text{if there exists } n \text{ such that } \]

\[ \text{Proof}_{PA}(n, [L = z \to Q]); \]

\[ Q \in \Gamma, \text{ Q is false;} \]

\[ xSz, \text{ h(n)Rz} \]

\[ \text{then our } z \text{ must be minimal with respect to such } n \]

\[ \text{else } z = x.\]

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Lemma 8.5. (PA proves that)

1. There exists unique z such that L=z.
2. If xRy,xRz,ySz, then \( L=x \rightarrow L=y \rightarrow L=z \).
3. If \( x \neq w \), then \( L=x \rightarrow \neg \varphi_{PA} (L=y \rightarrow xRy) \).
4. If \( xRy \), then \( L=x \rightarrow \neg \varphi_{PA} L \neq y \).

Proof.

1. is trivial.

2. Reason in PA:

   Let \( L=x \) and \( \varphi_{PA} (L=z \rightarrow Q) \), where \( Q \in \Gamma \). Let also \( l=t \), and

   \( \varphi_{PA} (n, [L=z \rightarrow Q]) \), where \( n \) so great that \( h(n)=t \);

   evidently, \( \varphi_{PA} (h(n)=t) \).

   Reason in PA+L=y:

   Let \( l=v \), then \( vSy \), hence \( vSz \). Suppose that \( Q \) is false, and note that \( t=h(n)Rz \).

   Why \( L \neq z \)? By definition 8.4, we have the only reason: \( \exists m \leq n \) \( \varphi_{PA} (m, [L=y \rightarrow Q]) \), where \( Q \) is false. Of course, it implies that \( L=y \rightarrow Q \) is true and \( L \neq y \). Contradiction. Thus, \( Q \) is true.

   So, \( \varphi_{PA} (L=y \rightarrow Q) \).

3. Reason in PA:

   Let \( L=x \neq w \), and suppose \( l=v \), where \( vSx \) and \( h(n)=v \). By claim 3 of lemma 8.3, \( \varphi_{PA} (l=t \rightarrow vRt) \).

   Reason in PA:

   Let \( L=y \) and \( l=t \). We have \( vRt \), \( tSy \) and \( xRt \); hence, we can assume \( t \neq y \).

   By definition 8.4, fix a number \( m \) and a sentence \( Q \in \Gamma \) s.t.:

   \( \varphi_{PA} (m, [L=y \rightarrow Q]) \);

   \( Q \) is false;

   \( h(m)Ry \).

   Consider two cases:

   Case 1. \( m \leq n \). Then, as above, \( L=y \) is false. Contradiction.

   Case 2. \( m > n \). Then \( h(m)=h(n)=v \) or \( vRh(m) \);

   hence, \( vRy \) and \( xRy \).

   Thus, we proved that \( xRy \).

   So, \( \varphi_{PA} (L=y \rightarrow xRy) \).
4. Reason in PA+L=x.

Let \( L=t \). Assume that \( \neg PA(L \equiv y) \). We claim that \( \neg PA(l \equiv y) \).
Indeed, consider an arbitrary \( z \) such that \( xRy, ySz \). By claim 2 of present lemma, \( L = z \rightarrow t = y \), hence \( \neg PA(L \equiv z) \). We have:

\[
\neg PA(L = z \wedge ySz \rightarrow \neg xRz) \quad (a \text{ as we proved above})
\]

\[
PA(L = z \rightarrow xRz) \quad (\text{by claim 3 of present lemma})
\]

Thus, \( \neg PA(L = z \rightarrow \neg ySz) \), hence by definition 8.4 \( \neg PA(l \equiv y) \).
But \( tSxRy \), hence \( tRy \) and we have a contradiction with claim 2 of lemma 8.3. So, our assumption ( \( \neg PA(L \equiv y) \) ) is false.

This completes the proof of lemma 8.5.

Now we can define an arithmetical interpretation \( f_\Gamma \):

\[
f_\Gamma(p) := \exists z \ (L = z \wedge z \models p).
\]

**Lemma 8.6.** Let \( x = w \) and \( A \) be an arbitrary modal formula. Then

\[
\begin{align*}
x \models A & \Rightarrow PA \models L = x \rightarrow f_\Gamma(A) \\
x \models A & \Rightarrow PA \models L = x \rightarrow \neg f_\Gamma(A)
\end{align*}
\]

**Proof.** As usual, we consider only the case \( A = B \models C \).

1. Suppose \( x \models B \models C \). Reason in PA+L=x:

Let \( \neg PA(f_\Gamma(C) \rightarrow Q) \), where \( Q \in \Gamma \).

Reason in PA+\( f_\Gamma(B) \):

Let \( L = y \). By the induction hypothesis for \( B \), \( y \models B \). By claim 3 of lemma 8.5, \( xRy \), hence there exists \( z \) such that \( ySz, xRz, z \models C \).

Here we interrupt our reasoning and note that by the induction hypothesis for \( C \), \( \neg PA(L = z \rightarrow f_\Gamma(C)) \), hence \( \neg PA(L = z \rightarrow Q) \); by claim 2 of lemma 8.5, \( \neg PA(L = y \rightarrow Q) \).

We continue reasoning in PA+\( f_\Gamma(B) \):

Since \( L = y, Q \) is true.

So, we have proved \( \neg PA(f_\Gamma(B) \rightarrow Q) \), therefore \( f_\Gamma(B) \models f_\Gamma(C) \).
( Note that in this proof the finiteness of \( K \) is essential. )

2. Suppose \( x \models B \models C \), then there exists \( y \) such that \( xRy, y \models B, \forall z(ySz \wedge xRz \rightarrow z \models C) \). Reason in PA+L=x:

Let \( Q \) denote the arithmetical formula \( \exists v(l = v \wedge ySv) \). Of course, \( Q \in \Sigma_2 \).

Note that by definition 8.4, \( PA \models Q \leftrightarrow \exists v(L = v \wedge ySv) \); by the induction hypothesis (for \( C \) ) \( PA \models Q \rightarrow \neg f_\Gamma(C) \).

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Assume that $f_{\Gamma}(B) \geq_{\Gamma} f_{\Gamma}(C)$. Since $\varphi_{PA}(f_{\Gamma}(C) \rightarrow \neg Q)$ and $\Gamma \models \Pi_2$, $\varphi_{PA}(f_{\Gamma}(B) \rightarrow \neg Q)$. By the induction hypothesis (for $B$), $\varphi_{PA}(L=\neg_{\Gamma} f_{\Gamma}(B))$. Thus, $\varphi_{PA}(L=\neg_{\Gamma} f_{\Gamma}(B))$. It contradicts claim 4 of lemma 8.5. Thus, $\neg f_{\Gamma}(B) \geq_{\Gamma} f_{\Gamma}(C)$.

Using standard PA-soundness argument one can obtain a following proposition:

**Proposition 8.7.** $L=w$ is true (i.e. $=L=w$).

**Corollary 8.8.** For any $x \in K$ PA does not prove $L=\neg x$.

**Proof.** If $x=\neg$, it is trivial. If $wRx$, claim 4 of lemma 8.5 shows that $L=w \rightarrow \neg \varphi_{PA}(L=\neg x)$ is true and by the previous proposition $\varphi_{PA}(L=\neg x)$ is false. QED.

Recall now that $u=w$ and $u \vdash \phi$ (see (***) in the beginning of this paragraph). By lemma 8.6, if $PA \vdash f_{\Gamma}(\phi)$, then $PA \vdash L=\neg u$. It contradicts corollary 8.8. Thus, $PA \vdash f_{\Gamma}(\phi)$.

This completes the proof of theorem 1.

In conclusion of this paragraph we consider a "truth variant" of provability logic for $\Gamma$-conservativity.

**Definition 8.10.** The logic $\text{SCL}^\omega$ is the minimal set of modal formulas closed under modus ponens and containing all theorems of $\text{SCL}$ and all formulas of the form $\varphi A \rightarrow A$.

**Lemma 8.11.** Assume that $\text{SCL}^\omega \vdash \phi$. Then there exists a simplified $\text{SCL}$-model $K = <K,R,S, \rightarrow>$ and a node $w \in K$ such that:

1. $\forall x \in K \ wRx \vee wSx$;
2. $w \vdash \phi$;
3. if $B \vdash C$ is a subformula of $\phi$, $w \vdash B \rightarrow C$, and for some $x$ $wSx$ and $x \vdash B$, then there exists $y$ such that $wSy$ and $y \vdash C$;
4. if $\varphi D$ is a subformula of $\phi$ and $w \vdash \varphi D$, then for any $x \in K \ x \vdash D$.

**Proof.** Assume $\text{SCL}^\omega \vdash \phi$. Let $X$ be the set of all formulas of the form $\varphi A \rightarrow A$, then by definition of $\text{SCL}^\omega$, $X \cup \{\neg \phi\}$ is $\text{SCL}$-consistent (i.e. every finite subset of $X \cup \{\neg \phi\}$ is $\text{SCL}$-consistent). Define a set $\Phi$ as the maximal $\text{SCL}$-consistent extension of $X \cup \{\neg \phi\}$, and $\psi$ as the conjunction of all subformulas of $\phi$ and their negations which belong to $\Phi$. Obviously, $\psi \in \Phi$ and $\psi \land \neg \psi \in \Phi$ (because $(\neg \psi \rightarrow \neg \psi) \in X$), hence $\text{SCL} \vdash (\psi \land \neg \psi)$ and there exist a simplified $\text{SCL}$-model $K = <K,R,S, \rightarrow>$ and nodes $v,w \in K$ such that $K \models (v) \cup (x \rightarrow vRx), vRw, v \vdash \psi, w \vdash \neg \psi$.

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We claim that SCL-model $K$ and node $w$ satisfy conditions 2, 3, 4 from the formulation of present lemma. Indeed, $\neg \phi$ is a conjunct of $\psi$, hence $w \vdash \neg \phi$. We check condition 3 (condition 4 is analogous). Let $w \models B \rightarrow C$ and $x \models B$, $wSx$. Since $B \rightarrow C$ is a subformula of $\phi$, $B \rightarrow C$ is a conjunct of $\psi$ and $v \models B \rightarrow C$. Obviously, $vRx$ and hence there exists $y$ such that $wSy$, $y \models C$.

Thus, the model $K' = \langle K', R', S', \models \rangle$, where $K' = \{ x \in K | wRx \text{ or } wSx \}$ and $R'$, $S'$, $\models$ are restrictions of $R, S, \models$ on $K'$, has all necessary properties.

**Remark.** As in the proof of modal completeness of SCL, we note that if $\mathfrak{D}$ is a subformula of $\phi$, $(\neg \mathfrak{D}) \vdash \bot$ needs not to be among subformulas of $\phi$. But applying condition 4 of the previous lemma, we claim that we can assume that $(\neg \mathfrak{D}) \vdash \bot$ is a subformula of $\phi$ whenever $\mathfrak{D} \vdash \bot$ is. So, in the following reasoning, as usual, we will consider only $\vdash$-subformulas of $\phi$.

**Theorem 8.12.** (Arithmetical completeness of $SCL^\omega$).

Let $\Gamma$ be as above (see theorem 8.1). For any modal formula $\phi$

$SCL^\omega \vdash \phi$ iff for any arithmetical interpretation $f_\Gamma = f_\Gamma(\phi)$.

**Proof.** Arithmetical soundness of $SCL^\omega$ is evident.

Let $SCL^\omega \vdash \phi$. Fix a simplified SCL-model described in lemma 8.11. Define a binary relation $R'$ on $K$ by the following:

$$xR'y \iff xRy \lor (xSwSy).$$

**Proposition 8.13.** If $B \rightarrow C$ and $\mathfrak{D}$ are subformulas of $\phi$, then for any $x \in K$

$$x \models B \rightarrow C \iff \forall y (xR'y \land y \models B \rightarrow \exists z (xR'z \land ySz \land z \models C)),$$

$$x \models \neg \mathfrak{D} \iff \forall y (xR'y \rightarrow y \models \neg \mathfrak{D}).$$

**Proof.** Using conditions 3, 4 from lemma 8.11.

We change definition 8.4 by replacing $R$ with $R'$. (So, $L = w$ does not imply $L = w$). The proof is to be changed as follows:

**Lemma 8.14.** (PA proves that)

1. There exists an unique $z$ s.t. $L = z$.
2. If $xR'y, ySz, xR'z$ then $L = x \rightarrow L = y \models \bot, L = z$.
3. For any $x$, $L = x \rightarrow \mathfrak{D}(L = y \rightarrow xR'y)$.
4. For any $x, y$, if $xRy$, then $L = x \rightarrow \neg \mathfrak{D}(L = y)$.

**Proof**

1. is trivial.

2. The only interesting case is $wSxSySz$. Reason in PA+$L = y$: 26
Let \( \text{Proof}_{PA}(n, [L=z \rightarrow Q]) \), \( Q \in \Gamma \); of course, \( l = w \), \( \text{PA}_w(h(n) = w) \).

Reason in \( \text{PA} + L = y \):

Assume that \( Q \) is false. Because \( h(n) = wR'z \), the only reason why \( L = z \) is \( \exists m \in n \) \( \text{Proof}_{PA}(m, [L = y \rightarrow Q]) \).

Like earlier, \( L = y \) is false. Contradiction. So, \( Q \) is true.

Thus, \( \text{PA}_w(L = y \rightarrow Q) \), hence \( L = y \models L = z \).

3. It is a trivial modification of claim 3 of lemma 8.5.

4. Proof does not differ from the proof of lemma 8.5.

We define Solovay-like interpretation \( f_\Gamma \) as above.

**Lemma 8.15.** Let \( a \) be a subformula of \( \phi \), \( x \in K \). Then

\[
\begin{array}{l}
x \vdash A \Rightarrow \text{PA} \vdash L = x \rightarrow f_\Gamma(A), \\
x \nvdash A \Rightarrow \text{PA} \vdash L = x \rightarrow \neg f_\Gamma(A).
\end{array}
\]

**Proof.** We assume that \( A = B \models C \).

In the case \( x \vdash B \models C \) proof does not differ from the proof of the similar case of lemma 8.6.

In the case \( x \nvdash B \models C \) one must replace \( R \) by \( R' \) in the proof of lemma 8.6 and use claims 2, 3 of lemma 8.14 and proposition 8.13.

The proof of proposition 8.7 is not changed. Therefore, since \( \text{PA} \vdash L = w \rightarrow \neg f_\Gamma(\phi) \) and \( L = w \) is true, \( f_\Gamma(\phi) \) is false.

We have proved arithmetical completeness of \( \text{SCL}_\omega \).

**Remark.** In fact, we proved that the existence of a \( \text{SCL}-\)model described in lemma 8.11, is equivalent to \( \text{SCL}_\omega \models \phi \). Thus, the logic \( \text{SCL}_\omega \) is decidable.

**Examples:**

1. Consider a formula \( c(\neg p \rightarrow cp) \rightarrow cp \). We claim that this formula is not derivable in \( \text{SCL}_\omega \). Indeed, there is a "countermodel" for this formula (in sense of lemma 8.11):

\[
\begin{array}{c}
S \\
\downarrow \\
p \\
\downarrow \\
\downarrow \\
R \quad R \\
w \quad p
\end{array}
\]

(Note that we must define \( w \vdash p \), because \( w \vdash \neg p \) and \( w \vdash \top \).)

Thus, we proved that there exists a sentence \( Q \) such that (for fixed \( \Gamma \))
(*) \( \text{PA} \vdash Q, \) but \( \text{PA} \vdash \top \Rightarrow Q. \)

2. If we want to find a false sentence \( Q \) satisfying (*), we are to consider a formula
\[ \neg p \land \neg (\top \Rightarrow p) \rightarrow \neg p \]
and a following countermodel:

We have: \( w \vdash \top \Rightarrow p, \top \vdash \top \) and \( x \vdash p. \)

§9. Arithmetical completeness of CL and CLM.

Proof of theorem 2.

An arithmetical soundness is evident. We only need to prove a completeness.

Assume that \( CL \vdash \phi. \) By the second modal completeness theorem, there exists a simplified CL-model \( K=\langle K, R, S, \vdash \rangle \) such that:

1). There is \( w, u \in K \) such that:
\begin{align*}
\forall x \in K & \ wR x \lor w = x; \\
u & = w; \quad u \vdash \phi.
\end{align*}

This condition is not strong enough to work in \( \text{PA} \) with \( K \), because \( K \) may be infinite. So, we need also the following properties of \( K \) (these properties can be checked by inspection of the modal completeness proof):

2). The relations \( R, S, \vdash \) are p.r.; moreover, the p.r. definitions of \( R, S, \vdash \) are provable in \( \text{PA} \).

3). All properties of \( K \) as a simplified CL-model are provable in \( \text{PA} \) (more precisely, we must demand, instead of "\( R \) is wellfounded", that there exists \( n \in \omega \) such that ( \( \text{PA} \) proves that ) \( K \) does not consist any \( R \)-chain of the length more that \( n \)).

Let for \( X \subseteq K \) \([X]\) denote \( \{ y \in K | \exists x \in X : x Sy \} \).

4). If \( X \subseteq K \) is finite and \( C \) is a modal formula, then the arithmetical formula \( A(x, X) := \exists z (xRx \land z \in [X] \lor z \vdash C) \) is \( \text{PA} \)-equivalent to
\[A_0\text{-formula.}\]

We define a usual Solovay function \( h \) (see definition 8.2). For an infinite \( K \) we need the following lemma (instead of lemma 8.3):

**Lemma 9.1.** (PA proves that.)

1. There exists a unique \( x \) such that \( l=x \).
2. \( l=x \rightarrow \forall y (xRy \rightarrow \square_{PA} l=y) \);
3. \( l=x \neq w \rightarrow \square_{PA} (l=y \rightarrow xRy) \).

For any finite \( X \subseteq K \) we define also a formula \( \gamma_X := \exists x (l=x \wedge x \notin X) \) (i.e. \( \gamma_X := \exists x (l=x \wedge x \notin X) \)). Let \( \Gamma \) be the set of formulas

\[ \Gamma := \{ \gamma_X \mid X \subseteq K, X \text{ is finite} \} \cup \{1\}. \]

Of course, \( PA \vdash \gamma_X \wedge \gamma_Y \rightarrow \gamma_{X \cup Y} \); \( PA \vdash \gamma_X \vee \gamma_Y \rightarrow \gamma_{X \cap Y} \). So, working in PA we can assume that \( \Gamma \) is closed under disjunctions.

We introduce an arithmetical interpretation \( f_\Gamma \) as usual:

\[ f_\Gamma (p) := \exists x (l=x \wedge x \vdash p). \]

We will prove that \( PA \vdash f_\Gamma (\phi) \).

**Lemma 9.2.** For any modal formula \( A \) PA proves that for any \( x \neq w \)

\[ l=x \rightarrow (x \vdash A \leftrightarrow f_\Gamma (A)). \]

**Proof.** We consider the case \( A=B\triangleright C \). Reason in PA:

Let \( l=x \), where \( x \neq w \).

1. Suppose \( x \vdash B\triangleright C \). Consider an arbitrary finite subset \( X \) of \( K \) and assume that \( \square_{PA} (f_\Gamma (C) \rightarrow \gamma_X) \). We claim that

\[ (*) \quad \text{there is no } z \in K \text{ s.t. } xRz, z \in X, z \vdash C. \]

Indeed, if such \( z \) exists, then by the induction hypothesis for \( C \), \( \square_{PA} (l=z \rightarrow f_\Gamma (C)) \), hence \( \square_{PA} (l=z \rightarrow \gamma_X) \) and \( \square_{PA} (l=z) \).

It contradicts claim 2 of lemma 9.1. Thus, by property 4) of model \( K \) PA proves \((*)\).

We reason in \( PA+f_\Gamma (B) \):

Let \( l=y \). By claim 3 of lemma 9.1, \( xRy \). We knew that \( x \vdash B \triangleright C \). By the induction hypothesis for \( B \), \( y \vdash B \). So, there exists \( z \) s.t. \( z \vdash C, xRz, ySz \).

By \((*)\), \( z \notin X \), hence \( y \notin X \). So, we proved that \( l \notin X \), i.e. \( \gamma_X \).

Thus, we proved that \( \square_{PA} (f_\Gamma (C) \rightarrow \gamma_X) \) implies \( \square_{PA} (f_\Gamma (B) \rightarrow \gamma_X) \) and left to the reader to prove that \( \square_{PA} (f_\Gamma (C) \rightarrow 1) \) implies \( \square_{PA} (f_\Gamma (B) \rightarrow 1) \).

2. Suppose \( x \vdash B \triangleright C \). Then there exists \( y \) such that \( xRy \),
y \rightarrow B,

(**) there is no zeK s.t. xRz, ySz, z \rightarrow C.

Let Y:=\{y\}. Obviously, x\in[Y] \equiv xSy. As above, PA \hdash (**) .

Claim 1. \sqcap_{PA} (f_{\Gamma}(C) \rightarrow y_{\gamma}). Proof: Reason in PA+\gamma_{\gamma}.

Let l=z, then ySz; of course, xRz; by (**), t \rightarrow C;
by the induction hypothesis for C, \gamma_{f_{\Gamma}(C)}.

Claim 2. \sqcap_{PA} (f_{\Gamma}(B) \rightarrow y_{\gamma}). Suppose not. Then ( by the
induction hypothesis for B ) \sqcap_{PA} (l=y \rightarrow y_{\gamma}), i.e. \sqcap_{PA} (l \neq y).

Contradiction.

So, since y_{\gamma} \in \Gamma, f_{\Gamma}(B) \rightarrow \gamma f_{\Gamma}(C) does not hold.

As usual, we conclude that PA \hdash l=u \rightarrow \gamma f_{\Gamma}(\phi) and PA \vdash f_{\Gamma}(\phi).

Proof of theorem 3
First, we recollect several facts connected with the logic of
\Pi_{1}\text{-conservativity ( see also } \S 1):\)

Theorem 9.3. ( Hájek-Guaspari, cf.[2],[3] ). If T_{1}, T_{2} are
r.e. extensions of PA in the language of PA, then T_{2} is
\Pi_{1}\text{-conservative over } T_{1}\text{ iff } T_{1}\text{ interprets } T_{2}.

Definition 9.4. The logic ILM is given by all axioms of CLM
(i.e. L0-L3, A1-A4, M , see } \S 2 ) plus

A5. \sqcap A \rightarrow A

and usual inference rules ( modus ponens and necessitation ).

Definition 9.5. A simplified ILM-model K is a simplified model
<K,R,S, \rightarrow> which fulfills the following conditions:

1. K is finite or countable.
2. S\geq R.
3. xSyRz \Rightarrow xRz.

Theorem 9.6. ( Visser [8], cf. also [7] ). Logic ILM is
complete w.r.t. simplified ILM-models.

Theorem 9.7. ( Shavrukov [9], Berarducci [10] ). ILM is
provability logic for relative interpretability ( over PA ). So, by
theorem 9.3., ILM is also the logic of \Pi_{1}\text{-conservativity.}

We begin to prove the theorem. As above, we only need to prove
a completeness of CLM.

Assume that CLM \vdash \phi and K=<K,R,S, \rightarrow> is simplified CLM-model
s.t.K \vdash \phi. We remind that by definition 3.4.2 there exists a mapping
\mu: K \rightarrow \{1,\ldots,N\} such that \forall x,y \in K ( xSy \rightarrow \mu(x) = \mu(y), xRy \rightarrow
\(\mu(x) < \mu(y)\). Let \(S = \{q_1, \ldots, q_n\}\) be a set of new propositional variables which are not contained in \(\phi\).

**Definition 9.8.** A simplified model \(K^*\) is \(<K^*, R^*, S^*, \models^*\rangle\), where

\[
\begin{align*}
K^* &:= K; \\
R^* &:= R; \\
x S^* y &:= x S y \lor \exists z(x R z S y); \\
x \models^* p_i, \text{ where } p_i \in S, &:= x \models p_i; \\
x \models^* q_i, \text{ where } q_i \in S, &:= \mu(x) \neq i.
\end{align*}
\]

One can easily check that \(K^*\) is a simplified ILM-model.

**Definition 9.9.** For any modal formula \(A\) we define a translation \(A^*\) by the following way:

1. for any propositional variable \(p\) \(p^* := p\);
2. \(\models\) commutes with boolean connectives and \(\Box\);
3. \((A \Rightarrow B)^* := \bigwedge_{i_1, \ldots, i_k \in S} (A^* \wedge q_{i_1} \wedge \ldots \wedge q_{i_k}) \Rightarrow (B^* \wedge q_{i_1} \wedge \ldots \wedge q_{i_k})\).

**Lemma 9.10.** For any \(x \in K\) and modal formula \(A\) which does not contain \(q_1, \ldots, q_n\),

\[
x \models A \iff x \models^* A^*.
\]

**Proof.** We consider the case \(A = B \Rightarrow C\).

1. Suppose \(x \models B \Rightarrow C\). We show that for any \(q_1, \ldots, q_k \in S\)

\[
x \models (B^* \wedge q_{i_1} \wedge \ldots \wedge q_{i_k}) \Rightarrow (C^* \wedge q_{i_1} \wedge \ldots \wedge q_{i_k}).
\]

Indeed, assume that \(x \models B\), then there is \(y \models B \wedge q_{i_1} \wedge \ldots \wedge q_{i_k}\). By the induction hypothesis, \(y \models B\), then there is \(z \models C\). Since \(\mu(y) = \mu(z)\), \(\forall q_i \in S \quad y \models^* q_i \iff z \models^* q_i\), hence \(z \models^* C \wedge q_{i_1} \wedge \ldots \wedge q_{i_k}\). By definition of \(S^*\), \(y \models S^* z\).

2. Suppose \(x \models B \Rightarrow C\). Then there exists \(y \models B\), \(\forall x \models B\), \(\forall z(x R z, y S z \rightarrow z \models C)\). Let \(n := \mu(y)\). We claim that

\[
x \models^* (B^* \wedge q_n) \Rightarrow (C^* \wedge q_n),
\]

hence \(x \models^* (B \Rightarrow C)^*\). Suppose not. Since \(y \equiv B \wedge q_n\) (because \(y \models B\) and \(\mu(y) = n\)), there is \(z \models C \wedge q_n\). Since \(z \equiv^* q_n, \mu(z) = n = \mu(y)\), and by definition of \(S^*\) and properties of \(\mu, y S^* z\). Thus, \(y S^* z, x R z, z \models C\). Contradiction.

**Lemma 9.10** implies that \(K^* \models \phi^*\), hence by theorems 9.6, 9.7 there exists an arithmetical interpretation \(f^*_{\prod^1}\) s.t. \(PA \models f^*_{\prod^1}(\phi^*)\).

Define a set \(\Gamma\) as closure under disjunctions of the set.
\[(f_{\Pi_1}(q_1), \ldots, f_{\Pi_1}(q_n)) \cup \Pi_1.\]

To conclude the proof it remains to check the following simple fact:

**Lemma 9.11.** For any modal formula \(A\),

\[PA \vdash f_{\Pi_1}(A^*) \leftrightarrow f_\Gamma(A),\]

where \(f_{\Pi_1}\) is defined above, \(f_\Gamma\) coincides with \(f_{\Pi_1}\) on propositional variables.

**Proof.** Induction on \(A\); for \(A = B \rightarrow C\) we use the following proposition:

**Proposition 9.11.1.** Let \(\Gamma\) be an arbitrary set of arithmetical formulas (as usual, we assume that \(\Gamma\) is closed under disjunctions), \(\Delta\) be an arbitrary finite set of arithmetical formulas. Define \(\Gamma'\) as closure of \(\Gamma \cup \Delta\) under disjunctions, i.e.

\[\Gamma' = (\forall q_1 \forall \ldots \forall q_n | \gamma \in \Gamma, q_1, \ldots, q_n \in \Delta).\]

Then for any arithmetical formulas \(A, B\)

\[PA \vdash A \rightarrow B \leftrightarrow \bigwedge_{\gamma} (A \land q_1 \land \ldots \land q_n) \rightarrow (B \land q_1 \land \ldots \land q_n).\]

**Proof** is trivial.

So, \(PA \vdash f_\Gamma(\phi)\).

---

§10. Conclusion Remarks.

**Generalization.** We can consider any r.e. theory \(T\) as an "internal" theory instead of \(PA\) (i.e. translate \(A \rightarrow B\) as "\(T + B\) is \(\Gamma\)-conservative over \(T + A\)", and \(\sigma A\) as "\(A\) is provable in \(T\)" ); if \(T\) is \(\Sigma_1\)-sound (i.e. each \(\Sigma_1\)-sentence which is provable in \(T\) is true ) theorem 1 holds (in fact, it is sufficient to demand \(T \vdash \sigma_1^n\) for any \(n\)).

**Unsolved problems.**

1. The main unsolved problem in this area is, of course, the logic of \(\Sigma_1, \Sigma_2\)-conservativity. The logic of \(\Sigma_1\)-conservativity is, evidently, an extension of \(CL + M^*\), where \(M^* := A \rightarrow B \rightarrow (A \land C) \rightarrow (B \land C)\), but it is a proper extension (for example, \(CL(\Sigma_1) \vdash \sigma_1 \rightarrow (\sigma_2 \land \sigma_1) \rightarrow \sigma_1\)). So, we have not now any interesting hypothesis about axiomatisation of \(CL(\Sigma_1)\).

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As to $\Sigma_2$-conservativity, the author does not knew any principle which extend $CL+M^*+Sa$. However, there is a non-trivial truth principle:

$$A\rightarrow B \land \diamond (B \rightarrow C\rightarrow D) \rightarrow (A \rightarrow C\rightarrow D).$$

(This principle is also valid for $\Sigma_n$-conservativity for any $n$, but for $n\geq 3$ it is derivable from $Sb$ and $\diamond A \rightarrow A$).

3. Does theorem 1 holds for $T=\Delta_0^0 + EXP$, etc.? (Note that the proof of claim 1 of lemma 8.5 uses $T\vdash \neg \Sigma_n^0$, where $\Gamma=\Pi_n^0$).

4. What is the truth provability logic for $\Pi_2$-conservativity (i.e. the set of all modal formulas whose arithmetical interpretations are truth)?

5. Is it sufficient to consider only finite simplified models for $CL$?

6. Has the logic of $\Pi_2$-conservativity a simple finite models? (A variant: is there a modal formula $\phi$ such that for any $n$ $CL(\Pi_2) \vdash \phi \rightarrow \neg \diamond^{n-1}$, but $CL(\Pi_2) \nvdash \phi$?).
APPENDIX

The Logic of $\Pi_2$-conservativity.

**Definition** A.1. The logic SbCLM is given by axioms L0-L3, A1-A4,M,Sb and usual inference rules (modus ponens and necessitation).

**Theorem** A.2. SbCLM is the logic of $\Pi_2$-conservativity (i.e. SbCLM=CL($\Pi_2$)=CL$(\Pi_2^+)$).

**Proof.** Arithmetical soundness of SbCLM is evident. Fix an arbitrary formula $\phi$ s.t. SbCLM$\vdash \phi$.

**Definition** A.3.

\[ X := (A \rightarrow B | A \rightarrow B \leq \phi) \cup (\neg D \vdash A | \neg D \leq \phi) ; \]

\[ \text{Sb}(\phi) := \neg \bigwedge \left( \begin{array}{c}
A \rightarrow B \\
A \rightarrow B \rightarrow (A \wedge C_1 \wedge \ldots \wedge C_n) \\
A \rightarrow B \in \mathcal{X} \\
C_1, \ldots, C_n \in \mathcal{X}
\end{array} \right). \]

(cf. definitions 7.7, 7.8)

**Lemma** A.4. There exists CLM-model $\mathcal{K}=\langle K, R, \{S_x\} \rangle$, $\models$ such that:

1) $K$ is finite;

2) for any $A \rightarrow B \in \mathcal{X}$, $x, y, z \in K$, if $yx_z \in \mathcal{X}$ and $z \models A \rightarrow B$, then $y \models A \rightarrow B$;

3) $K \models \phi$.

**Proof.** One can see that CLM$\models$Sb$(\phi) \rightarrow \phi$. Let $\mathcal{K}=\langle K, R, \{S_x\} \rangle$, $\models$ be a finite CLM-model with a bottom node $x_0$ such that $x_0 \models \neg \text{Sb}(\phi) \wedge \phi$. We define a binary relation $S$ on $K$: $xSy$ iff for any $A \rightarrow B \in \mathcal{X}$ if $y \models A \rightarrow B$, then $x \models A \rightarrow B$. The model $\mathcal{K}_1 := \langle K, R, \{S_x \cap S\} \rangle$, $\models$ has all properties required (cf. proof of lemma 7.9).\footnote{\textit{Proof.} Fix an CLM-model $\mathcal{K}=\langle K, R, \{S_x\} \rangle$, $\models$ defined in lemma A.4. We define a binary relation $S_1$ as a transitive closure of $US_1$ ( $x \in K$ ), and equivalence relation $S := S_1^* (i.e. xSy :\Rightarrow \exists x_1 x \wedge y_1 y \wedge S_1 x \Rightarrow y_1 y \wedge S_1 x \Rightarrow y_1 y \wedge S_1 x )$.}

**Lemma** A.5. There exists a simplified CLM-model $\mathcal{K}=\langle K, R, S, \models \rangle$ and a set $K_0 \subseteq K$ such that:

1) $K$ is finite;

2) for any $A \rightarrow B \in \mathcal{X}$, $x, y \in K$, if $xSy$, and $y \models A \rightarrow B$, then $x \models A \rightarrow B$;

3) a) $ySx$, $yRz$, $z \in K_0 \Rightarrow xRz$

   b) if $A \rightarrow B \in \mathcal{X}$, $x \in K$, $x \models A \rightarrow B$, then there is $y \in K_0$ such that

   \[ xRy, y \models A, \forall z (xRz \wedge ySz \rightarrow z \models B) \]

   c) $x, y \in K_0$, $xSy \Rightarrow x=y$

4) $K \models \phi$.

**Proof** Fix an CLM-model $\mathcal{K}=\langle K, R, \{S_x\} \rangle$, $\models$ defined in lemma A.4. We define a binary relation $S_1$ as a transitive closure of $US_1$ ( $x \in K$ ), and equivalence relation $S := S_1^*$ (i.e. $xSy :\Rightarrow \exists x_1 x \wedge y_1 y \wedge S_1 x \Rightarrow y_1 y \wedge S_1 x \Rightarrow y_1 y \wedge S_1 x$).
a) \( xSy \Rightarrow \forall z ( xRz \leftrightarrow yRz ) \);
b) \( xSy \Rightarrow \forall C \in X ( x \leftarrow C \iff y \leftarrow C ) \);
c) \( xRy, yS_1x \) is impossible.

**Proof**

a) use definition of CLM-model;
b) use property 2 of \( K \) (from lemma A.4);
c) use definition of CLM-model (cf. proof of proposition 7.3).

**Definition A.5.2.**

\( K= <H, C, \cdot, R(\cdot), end(\cdot)> \) is the heir of the second type of \( K \) (see definition 7.1)

for any \( x \in H \), \( S(x) := \min \{ y \leq x \mid end(x) \subseteq end(y) \} \)

\( K'= <K', R', S', \leftarrow > \) is the simplified model:

\( K' := H \)

\( xR'y \iff S(x) \subseteq y \)

\( xS'y \iff R(x)=R(y) \land S(x) \subseteq S(y) \)

\( x \leftarrow p \iff end(x) \leftarrow p \)

\( K_0 \) is subset of \( K' \):

\( K_0 := \{ x \in K' \mid R(x)=x \} \).

**Lemma A.5.3**

a) \( end(S(x)) \subseteq end(x) \);
b) \( S(S(x)) = S(x) \);
c) \( xR'y \Rightarrow end(x) \subseteq end(y) \);
d) \( S(x) \supseteq R(x) \);
e) \( R' \) is transitive;
f) \( K' \) is finite;
g) \( K' \) is a finite simplified CLM-model;
h) \( xS'y \land C \in X \land end(y) \leftarrow C \Rightarrow end(x) \leftarrow C \);
i) \( yS'x, yR'z, z \in K_0 \Rightarrow xR'z \);
j) \( R(x)=R(y) \land x \subseteq y \Rightarrow xS'y \);
k) \( R(x)=R(y) \land x \subseteq y \land end(y)S_1end(x) \Rightarrow yS'x \).

**Proof**

a) immediately from the definition of \( S(\cdot) \);
b) is trivial;
c) use a) and propositions A.5.1a) and 7.3b);
d) use definition of \( R(\cdot) \) and proposition A.5.1c);
e) if \( S(x) \subseteq y, S(y) \subseteq z \), then by d) and proposition 7.2h), \( S(x) \subseteq S(y) \); by proposition 7.2g), \( S(x) \subseteq z \);

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f) use proposition 7.3a);
g) use e), f) and some obvious properties of \( R' \) and \( S' \);
h) let \( \text{end}(y) \Rightarrow C \). By a) and proposition A.5.1.b.
\[ \text{end}(S(y)) \Rightarrow C. \] Since \( R(y) \subseteq S(x) \subseteq S(y) \), \( \text{end}(S(x)) \Rightarrow C \); and as above
\[ \text{end}(x) \Rightarrow C. \]
i) use the following simple property of heirs:
\[ (*) \text{ if } R(z)=z, y<z, R(x)=R(y), \text{ then } x<z. \]
j) 1) immediately from the definition of \( S' \).

**Lemma A.5.4.** If \( A \) is a subformula of \( \phi \), and \( x \in K' \), then \( x \Rightarrow A \iff \text{end}(x) \Rightarrow A. \)

**Proof.** We consider the case \( A \lor \lnot C; \) \( x: = \text{end}(x); \) \( t: = S(x); \)
\( t: = \text{end}(t) \). By lemma A.5.3a) and proposition A.5.3b), \( t \Rightarrow \lnot C \iff x \Rightarrow \lnot \lnot C. \)

1. Suppose \( x \Rightarrow \lnot C \). We will show that \( t \Rightarrow \lnot C \). Indeed, let 
\( tRy, y \Rightarrow B \). We use lemma 7.4 to obtain \( y \in K' \) (moreover, \( y \in K_0 \)); by
induction hypothesis, \( y \Rightarrow B \); since \( S(x) = t \prec y \), there is \( z \) s.t. \( z \Rightarrow C, \)
\( xR'z, yS'z \). By definition of \( S' \), \( R(z)=R(y)=y, \) and by definition of \( R' \) and lemma A.5.3b), \( t=S(t) \prec z \); lemma 7.4 implies that \( yS'z, tRz, \)
where \( z: = \text{end}(z) \). By induction hypothesis, \( z \Rightarrow C \). Thus, we proved
that \( t \Rightarrow \lnot C. \) (In fact, we proved that \( t \Rightarrow \lnot C \) implies \( \exists y \in K_0 \) (\( xR'y, \)
\( y \Rightarrow B, \forall z(xR'z \land yS'z \rightarrow z \Rightarrow C) \) ).

2. Suppose \( t \Rightarrow \lnot \lnot C \). We will show that \( x \Rightarrow \lnot \lnot C \). Indeed, let 
\( xR'y, y= \text{end}(y), y \Rightarrow B. \) By definition of \( R' \) and lemma A.5.3c), \( tRy. \)
Since \( y \Rightarrow B \), there is \( z \) such that \( tRz, yS'z, z \Rightarrow C. \) Now we can use
lemma 7.5 and obtain \( z \) such that \( z \Rightarrow C, R(y)=R(z), xR'z \) (because
\( S(x) = t \prec z \)). By lemma 7.5, we consider two cases:

Case 1. \( z \geq y. \) Then by lemma A.5.3k), \( yS'z. \)

Case 2. \( y \geq z. \) Then by lemma A.5.3l), because \( yS'z, \) we
obtain \( yS'z. \)

Thus, in any case \( yS'z, \) and we proved that \( x \Rightarrow \lnot \lnot C. \) QED.

We have proved that \( K' \models \phi. \) Set \( K_0 \) was defined in definition
A.5.2; as to properties a)-c) of \( K_0 \), a) was be stated in lemma
A.5.3.i); b) can be easily obtained from part 1 of the proof of
lemma A.5.4, and c) is trivial. It remained to show that for any
\( x, y \in K' \), if \( xS'y \) and \( y \Rightarrow C, \) then \( x \Rightarrow C \), where \( C \in X. \) Indeed, by lemma
A.5.4 \( \text{end}(y) \Rightarrow C \Rightarrow y \Rightarrow C, \) \( \text{end}(x) \Rightarrow C \Rightarrow x \Rightarrow C; \) now it is enough
to use lemma A.5.3h).
Lemma A.5 is thus proved.

We continue with the arithmetical part of our proof.

We will use the following fact which is a simple modification of Goldfarb's theorem:

Lemma A.6. Let $T_0, \ldots, T_n$ be r.e. extensions of PA, $S(x) \in \Sigma^2_1$. Then there exists a formula $\sigma(x) \in \Sigma^1_1$ such that PA + "for any $i$ $T_i$ is consistent" proves that for any $x$

1) if $S(x)$, then PA $\vdash \sigma(x)$;

2) if not $S(x)$, then for any $i$ $T_i \vdash \sigma(x)$.

Proof (sketch). Let $S(x) = \exists y \delta(x, y)$ where $\delta(x, y) \in A_0$. The following formula (defined by Diagonal Lemma),

$$\sigma(x) := \exists y(\delta(x, y) \wedge \forall z < y \forall i \vdash \text{Proof}_i(z, [\sigma(x)])$$

has all necessary properties.

Fix a model $K' = \langle K', R', S', \vdash \rangle$ and a set $K_0$ which fulfills conditions of Lemma A.5; we can suppose that $v \models \phi$, where $v$ is a bottom node of $K'$. Let $w$ be a new node, and $K := K \cup \{w\}$, $R := R \cup \{<w, x> | x \in K'\}$, $S := S \cup \{<w, w>\}$, $\vdash$ is an arbitrary extension of $\models$ on $K$, $K_0 := K_0 \cup \{w, v\}$.

We claim that the model $K$ satisfies all conditions of lemma A.5. The only interesting case is 3b).

Let $x \in K_0$, $A = B \in X$, $x \models A \rightarrow B$.

Case 1. $x = w$. Use the property 3b) of $K'$ from Lemma A.5.

Case 2. $x = w$, $v \models A \rightarrow B$. We can put $y := v$.

Case 3. $x = w$, $v \models A \rightarrow B$. One can see that $v \models A \rightarrow B$; so, we can reason as in case 1.

Consider a set $K_0$ as a submodel of $K$ and define a Solovay function $h$ on this submodel; thus, we will use lemma 8.3 w.r.t. $x, y, \ldots \in K_0$, $l := \text{lim } h$; of course, $l \in K_0$.

Definition A.7.

1) $\text{Tr}(\cdot)$ is $\Sigma^2$-definition of truth for $\Sigma^2$-formulas;

2) let $\text{Tr}(x) := \exists y \text{tr}(y, x)$, where $\text{tr}(\cdot, \cdot) \in \Pi^1_1$; it is supposed that

$$\text{PA} \vdash \text{Tr}(x) \rightarrow \exists^n y \text{ tr}(y, x);$$

3) $(Q, n, m)$ will stand for

$$\exists A \in \Sigma^2_1 (\text{Proof}_{PA}(n, [A \rightarrow Q^1]) \land \text{tr}(m, [A^1])).$$

(This formula is $\Pi^1_1$, because the first quantifier can be bounded.
by p.r. function on n).

**Proposition A.8.** \( \forall Q \forall n_0. PA \vdash \exists m \exists n<m(Q,n,m) \rightarrow Q. \)

**Proof.** It is enough to show that 
\[ \forall Q \forall n. PA \vdash \exists m(Q,n,m) \rightarrow Q, \]
or
\[ \forall Q \forall n. PA \vdash \forall A \exists \Sigma_2 \left( \text{Proof}_PA(P,A \rightarrow Q^l) \land \exists m \text{tr}(m,[A\rightarrow l]) \rightarrow Q \right). \]
We noted above that the quantifier on A is bounded by n, hence we can assume that this quantifier is "external"; on the other hand, \( \exists m \text{tr}(m,[A\rightarrow l]) \leftrightarrow A. \) Thus, we must show that 
\[ \forall Q \forall n. \forall A \in \Sigma_2 \vdash \text{Proof}_PA(P,A \rightarrow Q^l) \rightarrow (A \rightarrow Q). \]
But it is trivial.

**Definition A.9.** We will define (in PA) a function \( H: \omega \rightarrow K \times (\omega+1) \) (\( H_0 \) and \( H_1 \) will stand for left and right components of \( H \)) and a constants \( L \) and \( R_k \) by induction on \( \mu(l) \); we will define them such that the formula \( Q(m,n,z) := H(m) = \langle z,n \rangle \) will be \( \Sigma_2 \).

**Basis.** If \( l = w \), \( H(m) = \langle w, w \rangle \).

**Induction.** Assume we have defined \( H \) in the case \( \mu(l) < n \).

1. Let \( A_\times(y) := (l = x \land y \in \text{Range}(H_0)) \), where \( \mu(x) < n \); by the induction hypothesis, \( A_\times(\cdot) \in \Pi_2 \). Let \( A_\times(y) = \forall m \ S_\times(m,y) \), where \( S_\times(\cdot,\cdot) \in \Sigma_1 \) and it is supposed that \( PA \vdash (\neg A_\times(y) \rightarrow \exists m \ S_\times(m,y)) \).

We use lemma A.6 (where \( \{T_0, \ldots, T_n\} := (PA+L=Z|xRz,z\in K_0) \)) to define a formula \( \sigma_\times(m,y) \) such that

\[ PA + \forall z (xRz \land z \in K_0 \rightarrow \text{Proof}_PA(L=z)) \] proves that for any \( y,m \)

1. if \( \neg S_\times(m,y) \), then \( \forall z \in K_0 (xRz \rightarrow \neg \text{Proof}_PA(L=z \rightarrow \sigma_\times(m,y))) \),
2. if \( S_\times(m,y) \), then \( \neg \text{Proof}_PA \sigma_\times(m,y) \).

or, by the definition of \( A_\times(\cdot) \),

\[ PA + \forall z (xRz \land z \in K_0 \rightarrow \text{Proof}_PA(L=z)) \] proves that for any \( y \)

1. if \( l = x \) and \( y \in \text{Range}(H_0) \), then
   \[ \exists m \forall z \in K_0 (xRz \rightarrow \neg \text{Proof}_PA(L=z \rightarrow \sigma_\times(m,y))) \],
2. if \( l \neq x \) or \( y \notin \text{Range}(H_0) \), then \( \forall m \text{Proof}_PA \sigma_\times(m,y) \).

2. \( H(0) := \langle x, \alpha \rangle \), where

   - if \( \text{PA}(L=x) \), then
     \[ \alpha := \min(k | \text{Proof}_PA(k, [L \neq x]^l)) \]
   - else
     \[ \alpha := \omega. \]

3. For any \( m \geq 0 \),
if there exists a pair \(<z,k>\) such that
1) \(H_{0}(m)Sz\);
2) \(H_{1}(m)>k\);
3) \(h(k)Rz\);
4) \(\{L\neq z,k,m\}\);
5) \(\forall i<k \forall b \ (h(i)Sb \rightarrow \sigma_{h(i)}(i,b) \vee bRz)\)
then
\(H(m+1) := \{z_0,k_0\},\)
where \(<z_0,k_0>\) is the minimal pair \(<z,k>\)
satisfying conditions 1)-5) (w.r.t. k).

else
\(H(m+1) := H(m)\).

4. \(L := \lim_{m \rightarrow \infty} H_{0}(m)\).
5. \(Rk := \lim_{m \rightarrow \infty} H_{1}(m)\).

One can see using condition 2) from the definition of H, that H can
"jump" only finitely many times, hence these limits exist. Note
also that by condition 1) for any \(m \mu(H_{0}(m)) = \mu(l) = n,\) and by
condition 3) \(\mu(h(k)) < n;\) thus, we can use the formula \(\sigma_{h(i)}\), where
\(i \leq k,\) in the definition of H.

Lemma A.10 (PA )

a) if \(L = y, \ y \in K_{0},\) then \(l = y;\)
b) for any \(n, \ \Box_{PA}(Rk > n);\)
c) if \(xRy, \ y \in K_{0},\) then \(l = x \rightarrow \Box_{PA}L \neq y;\)
d) \(l = x \neq w \rightarrow \Box_{PA}(L = y \rightarrow xRy).\)

Proof.

a) Use property c) of \(K_{0}\) (lemma A.5 ), and an obvious fact
\(1SL.\)

b) Fix \(n \geq 0\) and reason in PA:
Let \(l = t.\) Suppose that \(Rk \leq n\) and \(m_0 := \min(m|H(m) = L,Rk>).\)
Case 1. \(m_0 = 0.\) By the definition of \(H, \ \Box_{PA}(Rk, ['L = t']),\)

hence \(L \neq t.\) Contradiction.

Case 2. \(m_0 > 0.\) By the definition of \(H, \ \{L \neq x,Rk,m_0\};\) by
proposition A.8, \(L \neq x.\) Contradiction.

Thus, \(Rk > n.\)

c) One can see from the definition of \(H\) that (PA proves
that) for any \(y\)
\[(\ast) \quad l = y \land \text{Proof}_{PA}(n, [L \not= y]) \rightarrow Rk \leq n.\]

We have:
\[
\begin{align*}
\text{PA} \vdash \text{Proof}_{PA}(n, [L \not= y]) & \rightarrow \Box_{PA}(\text{Proof}_{PA}(n, [L \not= y])) \\
& \rightarrow \Box_{PA}(l = y \rightarrow Rk \leq n) & \text{by (\ast)} \\
& \rightarrow \Box_{PA}(l \not= y) & \text{by claim b)} \\
\text{PA} \vdash \Box_{PA}(L \not= y) & \rightarrow \Box_{PA}(l \not= y) & \text{by claim 2 of lemma 8.3.} \\
\text{PA} \vdash l = x \rightarrow \neg \Box_{PA}(L \not= y) & & \text{by claim 2 of lemma 8.3.}
\end{align*}
\]

d) Let \(l = x = h(n)\). Reason in PA:

Let \(L = y\)

\begin{itemize}
\item\textbf{Case 1. } \(l = y\). Then by claim 3 of lemma 8.3, \(xRy\).
\item\textbf{Case 2. } \(l \not= y\). Then by definition of H, \(h(Rk)Ry\); by claim a), \(Rk \succ H\), hence

\[
x = h(H)Rh(Rk) \quad \text{or} \quad h(H) = h(Rk);\]
\end{itemize}

in both cases \(xRy\).

**Corollary A.11.** For any \(x \in K\),
\[
\text{PA} + l = x \quad \text{proves that for any } y
\]

1. if \(y \in \text{Range}(H_0)\), then

\[
\exists^m \forall z \in K_0 \quad \text{(xRz} \rightarrow \neg \Box_{PA}(L = z \rightarrow \sigma_x(z, y)))
\]

2. if \(y \not\in \text{Range}(H_0)\), then \(\forall m \Box_{PA} \sigma_x(m, y)\).

**Proof.** Use definition of \(\sigma_x\) and lemma A.10.c).

We define arithmetical interpretation \(f\) (henceforward \(f\) denote \(f_{\Pi^2_2}\), \(\triangleright\) (in the arithmetical context) denote \(\triangleright_{\Pi^2_2}\)) as usual:

\[
f(p) := L \not= p.
\]

**Lemma A.12.** For any subformula A of \(\phi\) and \(x \not= w\),

\[
\begin{align*}
\text{PA} & \vdash A \rightarrow L = x \rightarrow f(A) \\
\text{PA} & \vdash A \rightarrow L = x \rightarrow \neg f(A).
\end{align*}
\]

**Proof.** We consider the case \(A = B \triangleright C \in X\).

1. Suppose \(x \vdash B \triangleright C\). Reason in \(\text{PA} + L = x\).

Let \(l = t\); of course, \(tSx\) and \(t \not= w\).

Let \(F := \text{Range}(H_0)\), \(m_0 := \min(m | H(m) = \langle L, Rk \rangle)\) and for any \(a, b \in F\)

\[
a \prec b \iff \max(i \leq m_0 | H_0(i) = a) < \max(i \leq m_0 | H_0(i) = b).
\]

Set \(F\) is finite, and we can work with \(F\) inside \(\text{PA}\), using the following properties of \(F\):

1. \(\prec\) is a linear order with maximal element \(x\);
2. \(x \prec y \Rightarrow xSy\);
3. \(t \in F\).
(But, of course, PA does not know that $F=\text{Range}(H_0)$).

Note also that by corollary A.11, if $b \in F$, then

\[ (*) \quad \forall m \sigma_{PA}(m,b) \]

Suppose that $a_{PA}(f(C) \rightarrow Q), Q \in \Pi_2$.

Reason in $PA + f(B)$:

Let $L=y$. Assume that $Q$ is false, i.e. for some $m \tr(m, [\neg Q])$; by definition A.7, we can choose $m$ so great that $H(m) = <y, Rk>$.

By the induction hypothesis, $y \not\in B$.

Let $u$ be the maximal element of the set $\{ a \in F | a \R y \}$ w.r.t. order $<$ (such $u$ exists, because by lemma A.10d), $tRy$, and $teF$). Since $u \not< x$, $uSx$, (by the properties 1–3 of $F$, see above), by the property 2 of $K$, (lemma A.5) we have $u \not\in B \cap C$; on the other hand, by the definition of $u$, $uRy$. Thus, there exists $z$ such that $uRz$, $ySz$, $z \not\in C$. Note also that $tSuRz$ implies $tRz$.

Interrupting our reasoning, we note that $a_{PA}(L= z \rightarrow f(C))$, hence $a_{PA}(L= z \rightarrow Q)$. Let $n$ be so large that $h(n) = t$ and $Proof_{PA}(n, [\neg Q \rightarrow L= z])$. We continue:

By definition A.7, $(L \not= z, n, m)$. We claim that the pair $<z, n>$ satisfies conditions 1)–5) from the definition of $H$; it implies that $H(m+1) = <z', n'>$, where $n' < n$. It is a contradiction, because $Rk > n$.

Indeed, using $Rk > n$, we check these conditions:

1), 2), 3) are trivial, because $ySz$, $Rk > n$ and $h(n) = tRz$.

4): see above.

5). Fix $i < n$, $b \in K$, where $h(i)Sb$ (note that $h(n) = t$).

Case 1. $h(i) \not= t$. Therefore, $h(i)Rt$ and, since $teK_0$, $h(i)Sb$ implies $bRt$; but we have $tRy$, hence $bRy$.

Case 2. $h(i) = t$, $b \not\in F$. Then by $(*) \sigma_t(i,b)$.

Case 3. $h(i) = t$, $b \in F$, $l = y$ (hence, $y \in K_0$).

Then by the property of $K_0$ (lemma A.5) $tRy$ and $tSb$ imply $bRy$.

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Case 4. \( h(i) = t, b \in F, l \neq y, b < u \). Then \( bSu \) and \( uRy \) imply \( bRy \).

Case 5. \( h(i) = t, b \in F, l \neq y, u < b \). We claim that \( \sigma_t(i, b) \). Suppose not. Because \( i < Rk, L = y \neq l \) implies \( bRy \) (it is enough to apply condition 5) from the definition of \( H \) to the "jump" \( H \) to \( \langle y, Rk \rangle \). However, it contradicts the definition of \( u \).

Thus, in any case \( \sigma_t(i, b) \) or \( bRy \).

We proved that \( Q \) is true.

Thus, \( \wp_{PA}(f(B) \rightarrow Q) \), hence \( f(B) \rightarrow f(C) \).

2. Suppose \( x \models B \Rightarrow C \); by the property of \( K_0 \), there is \( y \) such that \( y \in K_0 \), \( y \models B \), \( xRy \), \( \forall z(xRz \land ySz \rightarrow z \models C) \).

Reason in \( PA + L = x \).

Let \( l = t, tSx \). Since \( x \in \text{Range}(H_0) \), by corollary A.11, there exists \( m \) such that \( \wp_{PA}(L = y \rightarrow \sigma_t(m, x)) \) and \( h(m) = t \). Let \( Q := l = y \rightarrow \sigma_t(m, x) \). Of course, \( Q \in \Pi_2^2 \).

Claim 1. \( \wp_{PA}(f(B) \rightarrow Q) \). Proof: Suppose not. Because \( y \models B \), \( \wp_{PA}(L = y \rightarrow f(B)) \) and \( \wp_{PA}(L = y \rightarrow Q) \), hence by lemma A.10.a \( \wp_{PA}(L = y \rightarrow \sigma_t(m, x)) \). Contradiction.

Claim 2. \( \wp_{PA}(f(C) \rightarrow Q) \). Proof: Reason in \( PA + \neg Q \):

Let \( L = z \). Since \( l = y \), \( ySz \). We claim that \( xRz \); hence, by the definition of \( y \), \( z \models C \), and by the induction hypothesis, \( \neg f(C) \).

Since \( xRy \), we can assume that \( y \neq z \). Consider a "jump" \( H \) to \( \langle z, Rk \rangle \). Since \( \neg \sigma_t(m, x) \), \( t = h(m) \), \( tSx \) and \( Rk > m \), by the condition 5) from the definition of \( H \), we have \( xRz \). QED.

We proved that \( f(B) \rightarrow f(C) \) does not hold.

As usual, we finish our proof by \( PA \vdash L = v \rightarrow f(\phi) \), hence \( PA \vdash f(\phi) \) (we used that \( v \in K_0 \), hence by lemma A.10.c \( PA \vdash L = v \)).

This concludes the proof of the arithmetical completeness of \( SbCLM \). As in the case \( SCL^\omega \), we in fact have proved some more:

Corollary A.13. For any modal formula \( \phi \)

\[
SbCLM \vdash \phi \iff CLM \vdash Sb(\phi) \rightarrow \phi.
\]

In particular, the logic \( SbCLM \) is decidable.
REFERENCES


11. Hájek P., Montagna F. ILM is the logic of $\Pi_1$-conservativity, preprint, Siena, 1989.
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