Axioms for card games

Hans van Ditmarsch *

March 27, 2000

Abstract

We axiomatize two different game states for card games, the state where cards have been dealt over players but where they haven’t picked up their cards from the table yet, and the state where they have picked up their cards. The first is mainly interesting for its use in indirect description proofs. The second is extensively illustrated by the example of three players and three cards. We prove that the axiomatizations describe the respective models underlying the game states, in the technical sense that all other models are bisimilar to them. We show that our results correspond to those of fixed point computations of the description of modal models.

1 Introduction

A dealing of cards over players defines the initial state of a knowledge game. We represent that state by a pointed multiagent S5 model. All the players’ knowledge is encoded in this model by way of the accessibility relations for the players between dealings that are relevant given the actual dealing of cards. Why is this the correct model for the initial state of the game?

We answer this question as follows. First, we axiomatize the knowledge that players have about the game and about each other, and we show that our preferred model is indeed a model of this theory. Second, we show that this model is the ‘only’ model of the theory, because all other models are bisimilar to it. This strengthens our conviction that we have both the right model and the right theory for the state of the game under consideration.

The axiomatization of a given finite modal model can also be computed in a standard way by means of a fixed point construction. We relate our results, that are derived from analyzing agent behaviour, to those from applying this technique on multiagent S5 models.

*University of Groningen, Cognitive Science and Engineering, Grote Kruisstraat 2/1, 9712 TS Groningen, the Netherlands, hans@cs.rug.nl, http://tcs2.ppsw.rug.nl/~hans/  
†Apart from the input of my supervisors Gerard Renardel and Johan van Benthem, I am much in debt to both Wiebe van der Hoek and Barteld Kooi for their willingness to discuss previous versions innumerable times.
We proceed as follows: based on an informal analysis of agent properties, first in section 2 we present the theory $33^+$, describing the initial state for the knowledge game for three persons and three cards. We show that $\text{hexa}$ is a model of $33^+$. We proceed by proving some dependencies among the axioms, by presenting different versions for some axioms, and by presenting models of restrictions of $33^+$ that are ‘countermodels’ in the sense that they clearly do not model the initial game state we attempt to describe. This also serves as a further justification of our informal analysis. We then present a compact but equivalent version of $33^+$, that we name $33^-$, or just $33$. Yet another alternative is to characterize the model $\text{hexa}$ by the (exclusive) disjunction of a partial description of its worlds. We show the equivalence of that formula to $33$. Next, we prove that $33$ describes $\text{hexa}$: we show that all S5EC$_{3}$ models of $33$ are bisimilar to $\text{hexa}$.

In section 3 we then continue with the general case: the knowledge game for a given parameter dealing $d$ from $m$ cards to $n$ players. The intended $S5EC_{n}$ model $I_{d}$ for its initial game state is also described in [vD00k]. At first sight, it seems less clear what its axiomatization is: although some of the axioms from $33^+$ have obvious generalizations, this is not obvious for the agents’ ignorance. We illustrate the difficulties by presenting rejected candidate axioms. It turns out that we can characterize ignorance in three different ways, that are all equivalent to each other. The resulting axioms make up the theory $\text{kgames}_{d}^+$. We also present a shortened version $\text{kgames}_{d}^-$. We then prove that each model of $\text{kgames}_{d}^-$ is bisimilar to the intended model $I_{d}$.

We can also describe game models indirectly by relating different models with actions. See [vD00a]. Because bisimulation is invariant under action execution, it simplifies our bisimulation proof obligations. We refer to it in section 4 and give details in [vD00b].

In section 4 we discuss the game situation where the cards have been dealt but where players haven’t picked up and looked into their own cards. It has a simpler intended model $\text{pre}I_{d}$ and a simpler axiomatization $\text{prekgames}_{d}^-$. Again, we prove that $\text{pre}I_{d}$ is unique. The model $I_{d}$ results from $\text{pre}I_{d}$ by executing the action of ‘turning cards’, thus providing the indirect proof that was mentioned above.

In section 5 we discuss other issues. We compute the descriptions of $I_{d}$ and $\text{pre}I_{d}$ by a fixed point construction for finite models. We discuss hypercubes, models for distributed systems that seem much related to models for card games. Finally, we present some ideas on the belief revision that seems necessary for the efficient computation of the axiomatizations of other game states, resulting from action execution. For that, we need pre- and postconditions of actions.

2
1.1 Logical preliminaries

We use logical notions and terminology as in [MvdH95]. An $S5_n$ model is an $S5$ model for a set of $n$ agents. An $S5EC_n$ model is a multiagent $S5$ model plus access computed for general and common knowledge operators.

The $S5_n$ proof system ($S5EC_n$ proof system) is the axiomatic proof system consisting of the $S5$ axiom schemata and rules for all agents (and for all modal operators). We write $\varphi \vdash \psi$ for $\vdash \varphi \Rightarrow \psi$. We use soundness and completeness of these systems without restriction, see [MvdH95] for proofs. Instead of axiomatic proof we generally use a more informal natural deduction style of proof. It allows for a more natural presentation of cases, and it introduces modal operators by the derivation rule $\varphi_1, \ldots, \varphi_m \vdash \psi \Rightarrow \Box \varphi_1, \ldots, \Box \varphi_m \vdash \Box \psi$.

Unless specifically stated otherwise, we assume that $\models$ denotes $\models_{S5EC_n}$ and $\vdash$ denotes $\vdash_{S5EC_n}$. In section 2 we more specifically assume $\models$ denotes $\models_{S5EC_n}$ and $\vdash$ denotes $\vdash_{S5EC_n}$.

Axioms

In our epistemic language, we distinguish axiom schemata, constraints (‘axioms’) and contingencies. Any instance of an axiom scheme, such as $K_1 \varphi \rightarrow \varphi$, is an axiom. Differently put, these instances are closed under uniform substitution. A constraint, such as $r_1 \rightarrow K_1 r_1$ (for ‘if player 1 holds the red card, he knows it’) is also an axiom. However, constraints are not closed under uniform substitution: $r_2 \rightarrow K_1 r_2$ is not an axiom, because player 1 doesn’t know the card of player 2 in the initial state of the game. Instead of constraints, we still call them axioms, as long as it is understood that they are not instances of schemata. Formulas that are neither axioms nor deductive consequences of axioms are contingencies. Contingencies may hold in specific worlds of a model only, like $r_1$ if player 1 holds red in the actual dealing of cards.

Common knowledge

It is not only the case that player 1 holds (at least) a card $-r_1 \lor w_1 \lor b_1 -$, but this is also commonly known $-C_{123} (r_1 \lor w_1 \lor b_1) -$. How explicit do we need to be about such common knowledge? Mostly, it suffices to leave it explicit. Axioms are commonly known, because we have necessitation for common knowledge operators. Indeed it will be the case, that not just the axiom (constraint) but also knowledge of it, is essential in order to prove equivalences and dependencies among axioms. From a semantic point of view, observe that axioms hold in all worlds of a model. Therefore, from a given world, they hold in all $(\cup_{a \in A} \sim_a)^*$-accessible worlds. Therefore they are commonly known in that world.

Exclusive disjunction

We sometimes use ‘exclusive or’ $\lor$ and therefore define it here, as an $n$-ary operation, for each $n \geq 2$:
\(\bigvee_{i=1}^{n} p_i := (p_1 \land \neg p_2 \land \ldots \land \neg p_n) \lor (\neg p_1 \land p_2 \land \ldots \land \neg p_n) \lor \ldots \lor (\neg p_1 \land \neg p_2 \land \ldots \land p_n)\)

Instead of \(\bigvee_{i=1}^{n} p_i\) we also write \(p_1 \lor \ldots \lor p_n\). Observe that if one defines exclusive disjunction as a binary operation only, we do not get the desired truth-functionality for the \(n\)-ary case. Although \((p_1 \lor p_2) \lor p_3\) is equivalent to \(p_1 \lor (p_2 \lor p_3)\), neither of those is equivalent to \(\bigvee_{i=1}^{3} p_i\).

Model and state descriptions

Our motivation for this investigation was the following: we described a model for a game state but had some doubts on whether it the right model and some doubts on whether it is unique. To remove those doubts, we axiomatize game states and investigate how the resulting theories correspond to our preferred model. It then turns out that they describe the model, in the standard logical sense that all models of the theory are bisimilar to the preferred model, see e.g. [vB98, BM96]. As the preferred model is finite, we can compute its description in epistemic logic with common knowledge operators, in a straightforward way. These issues are discussed separately in section 5.1. We should not forget, however, that we started with doubts about both the models and the theories, that validates our approach of first axiomatizing agent behaviour and subsequently reducing those axioms because of interdependencies.

2 Axioms for three players each holding one card

In this section, we present the theory \(33^+\), and its shortened but equivalent version \(33\), that describe the \(S5_3\) (\(S5EC_3\) model \(\text{hexa}\), see figure 1 (reflexive arrows are not drawn in the figure). The knowledge state \((\text{hexa}, \text{rub})\) has been introduced in [vD00c]. It is a (pointed) model of the initial state of the knowledge game for 3 players (1, 2, 3) and three cards \((r, w, b)\), where 1 holds red, 2 holds white and 3 holds blue. Observe that any further refinement of access in \(\text{hexa}\), symmetrically for all agents, results in the fully refined model consisting of six singleton worlds.

![Figure 1: The model hexa for three players each holding a card](image_url)
2.1 The theory $33^+$

What information do the players have about this initial game state? They know how many cards there are, namely three. They know that the cards are all different, namely one red, one white and one blue. That all cards are different, means that the dealing of cards over players is a function. They know that each of them holds one card. Beyond that, if they hold a card, they know it, and if they don’t hold a card, they also know that they do not hold it. They don’t know anything else, and there seem to be two sides of that ignorance. First, a player doesn’t know that another player holds a specific card. Second, he also doesn’t know another player not to hold a specific card, unless it is his own card: in other words: apart from his own card, a player can imagine any card to be in possession of another player.

The theory $33^+$ is the set of axioms (constraints) formalizing this information. The axioms are listed in table 1. We call it $33$ because there are 3 players and 3 cards. We index it with large because we will later present an equivalent but smaller version of the theory. In $33^+$, the terms in (sansserif) roman are to be interpreted as the conjunction the set of sentences following them, e.g. $\text{see}33 = \bigwedge_{a \in A} \bigwedge_{e \in C} (c_a \rightarrow K_e c_a)$. Further, $33^+$ is the conjunction of all its axioms. We may also think of common knowledge $C_{123} 33^+$ as the requested formalization, and we implicitly assume distribution of the common knowledge operator $C_{123}$ over conjunction. In that way, e.g., we derive $C_{123} (r_1 \rightarrow K_1 r_1)$ (it is commonly known, that if player 1 holds the red card, he knows it).

First we show that $\text{hexa}$ satisfies $33^+$. Then we relate the intended meaning of the axioms to their formulation, and present some alternative formulations of axioms. We demonstrate the axioms’ independence by showing that for all axioms, the theory without that axiom has a countermodel of that axiom. This also provides circumstantial evidence, so to speak, that our preferred model is likely to be the right model.

2.2 Hexa is a model of $33^+$

**Fact 1**

$\text{hexa} \models 33^+$

For all conjuncts $\varphi$ of all axioms of $33^+$ we have to show $\text{hexa} \models \varphi$, i.e. for all worlds $w \in \text{hexa}$, $\text{hexa}, w \models \varphi$. Because of symmetry in the model, it suffices to show that, e.g., $\text{hexa}, rwb \models \varphi$. For a proof, see the appendix on page 29. Having proven that $\text{hexa} \models 33^+$, we have also proven that $\text{hexa} \models C_{123} 33^+$: because of the definition of the interpretation of common knowledge, and because access for $C_{123}$ on $\text{hexa}$ is universal, $\text{hexa} \models C_{123} 33^+$ is equivalent to $\forall w \in \text{hexa} : \text{hexa}, w \models 33^+$. 

5
<table>
<thead>
<tr>
<th>Statement</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1 \rightarrow K_1 r_1$</td>
<td>players see their own cards</td>
</tr>
<tr>
<td>$w_1 \rightarrow K_1 w_1$</td>
<td></td>
</tr>
<tr>
<td>$b_1 \rightarrow K_1 b_1$</td>
<td></td>
</tr>
<tr>
<td>$r_2 \rightarrow K_2 r_2$</td>
<td></td>
</tr>
<tr>
<td>$w_2 \rightarrow K_2 w_2$</td>
<td></td>
</tr>
<tr>
<td>$b_2 \rightarrow K_2 b_2$</td>
<td></td>
</tr>
<tr>
<td>$r_3 \rightarrow K_3 r_3$</td>
<td></td>
</tr>
<tr>
<td>$w_3 \rightarrow K_3 w_3$</td>
<td></td>
</tr>
<tr>
<td>$b_3 \rightarrow K_3 b_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg r_1 \rightarrow K_1 \neg r_1$</td>
<td>players only see their own cards</td>
</tr>
<tr>
<td>$\neg w_1 \rightarrow K_1 \neg w_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg b_1 \rightarrow K_1 \neg b_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg r_2 \rightarrow K_2 \neg r_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg w_2 \rightarrow K_2 \neg w_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg b_2 \rightarrow K_2 \neg b_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg r_3 \rightarrow K_3 \neg r_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg w_3 \rightarrow K_3 \neg w_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg b_3 \rightarrow K_3 \neg b_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg (r_1 \wedge r_2)$</td>
<td>there is at most one card of each colour</td>
</tr>
<tr>
<td>$\neg (w_1 \wedge w_2)$</td>
<td></td>
</tr>
<tr>
<td>$\neg (r_1 \wedge r_3)$</td>
<td></td>
</tr>
<tr>
<td>$\neg (w_1 \wedge w_3)$</td>
<td></td>
</tr>
<tr>
<td>$\neg (r_2 \wedge r_3)$</td>
<td></td>
</tr>
<tr>
<td>$\neg (w_2 \wedge w_3)$</td>
<td></td>
</tr>
<tr>
<td>$\neg (r_1 \wedge w_1 \wedge b_1)$</td>
<td>there is at least one card per player</td>
</tr>
<tr>
<td>$\neg (r_2 \wedge w_2 \wedge b_2)$</td>
<td></td>
</tr>
<tr>
<td>$\neg (r_3 \wedge w_3 \wedge b_3)$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_2 r_1$</td>
<td>players don’t know others’ cards</td>
</tr>
<tr>
<td>$\neg K_3 r_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_1 r_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 w_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_2 w_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 w_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_1 b_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 b_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_2 b_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 b_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_2 w_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 w_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_1 w_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 w_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_2 w_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 w_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_2 b_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 b_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_1 b_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 b_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_2 b_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 b_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_2 \neg r_1$</td>
<td>players can imagine others to hold other cards</td>
</tr>
<tr>
<td>$\neg K_3 \neg r_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_1 \neg r_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 \neg w_2$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_2 \neg w_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg K_3 \neg b_3$</td>
<td></td>
</tr>
<tr>
<td>$\neg w_1 \rightarrow \neg K_2 \neg w_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg w_2 \rightarrow \neg K_3 \neg w_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg w_3 \rightarrow \neg K_1 \neg w_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg b_1 \rightarrow \neg K_1 \neg b_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg b_2 \rightarrow \neg K_2 \neg b_1$</td>
<td></td>
</tr>
<tr>
<td>$\neg b_3 \rightarrow \neg K_3 \neg b_1$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: The theory 33+
2.3 Meaning of and dependencies between axioms

Some proofs are provided in the running text. Other proofs are found in the appendix, on page 29. We remind the reader that we sometimes prove $\varphi \vdash \psi$ from the equivalent, in $\text{S5EC}_n$, $C_{123}\varphi \vdash \psi$.

2.3.1 see33 and dontsee33

Theorem see33 says that in all worlds accessible to a player he should hold the same cards. Theorem dontsee33 says that in all worlds accessible to a player he should hold the same cards not: if a player does not hold a card, he doesn’t hold that card in all worlds accessible to him. Both dontsee33 and see33 separately follow from the other axioms of $33^+$. We will retain see33 and delete dontsee33.

Informal proof:

$$33^+ \vdash \text{dontsee33} \vdash \text{dontsee33}.$$  
We prove the case $\neg r_1 \rightarrow K_1 \neg r_1$. Suppose $\neg r_1$. From $\neg r_1$ and $r_1 \lor w_1 \lor b_1$ follows $w_1 \lor b_1$.

From $w_1$ and see33 follows $K_1 w_1$. From atleast and atmost (or equivalently: from dealings33, as defined below) follows $w_1 \rightarrow \neg r_1$. From $w_1$ and $w_1 \rightarrow \neg r_1$ follows $\neg r_1$. Therefore, from $K_1 w_1$ and $K_1(w_1 \rightarrow \neg r_1)$ (which holds because $33^+$ and therefore also dealings33 are commonly known) follows $K_1 \neg r_1$.

Similarly as for $w_1$, from $b_1$ and see33 follows $K_1 b_1$, and from $b_1$ and $b_1 \rightarrow \neg r_1$ follows $\neg r_1$. Continuing as before, we derive $K_1 \neg r_1$.

Therefore $w_1 \lor b_1 \rightarrow K_1 \neg r_1$, and therefore $\neg r_1 \rightarrow K_1 \neg r_1$. \hfill $\square$

$$33^+ \vdash \text{see33} \vdash \text{see33};$$ see the appendix.

2.3.2 Atmost33 and atleast33

Axiom atmost33 says that the same card cannot be held by two different players. This is obvious, as a dealing $d$ is a function from the set of cards $C$ to the set of players $A$. As the propositional language we use to describe game states doesn’t have functions, we have to be explicit about it: atmost33 states that a dealing is a function. It even states that a dealing is a partial function, as it doesn’t require all cards to be dealt. We can express that a dealing is a total function, by the proposition that each card is held by exactly one person:

$$\text{function33} := (r_1 \lor r_2 \lor r_3) \land (w_1 \lor w_2 \lor w_3) \land (b_1 \lor b_2 \lor b_3).$$

It holds that atmost33, atleast33 $\vdash$ function33. For a proof see the appendix.

Axiom at least33 says that every player holds at least one card. We actually wanted to express that every player holds exactly one card:

$$\text{exactly33} := (r_1 \land w_1) \land (r_2 \land w_2) \land (r_3 \land w_3)$$
Because of \textit{atmost33}, \textit{atleast33} \models \textit{exactly33} the former is already sufficient. For a proof see the appendix.

The axiom \textit{dealings33} expresses that exactly one of six different dealings of cards can be the case. Let:
\[
\delta_{abc} = a_1 \land \neg b_1 \land \neg c_1 \land \neg a_2 \land b_2 \land \neg c_2 \land \neg a_3 \land \neg b_3 \land c_3
\]

then:
\[
dealings33 := \delta_{rwb} \lor \delta_{rhr} \lor \delta_{whr} \lor \delta_{brw} \lor \delta_{bbr} \lor \delta_{rwr}
\]

\textit{Dealings33} is equivalent to \textit{atleast33} and \textit{atmost33}. For proofs of \textit{dealings33} \models \textit{atleast33} \land \textit{atmost33} and \textit{atleast33} \land \textit{atmost33} \models \textit{dealings33}, see the appendix.

2.3.3 \textbf{Don'tknowthat33 and don'tknownot33}

Axiom \textit{don'tknowthat33} says that a player doesn't know another player's card. We might weaken it with the precondition that the first player doesn't hold that card himself, because that more properly expresses what an agent knows:
\[
don'tknowother33 := \bigwedge_{a \neq b \in A} \bigwedge_{c \in C} (\neg c_a \to \neg K_a c_b).
\]

From this weaker axiom \textit{don'tknowother33} we can deduce \textit{don'tknowthat33}: for a proof of \textit{atmost33}, \textit{don'tknowother33} \models \textit{don'tknowthat33}, see the appendix. Of course, the reverse, \textit{don'tknowthat33} \models \textit{don'tknowother33}, holds trivially. We prefer \textit{don'tknowthat33} over \textit{don'tknowother33}, because it is shorter.

Axiom \textit{don'tknownot33} says that a player can imagine another player to hold any card that he doesn't hold himself. In this case the antecedent is essential, as one cannot imagine other players to have the same cards as oneself: \[\bigwedge_{a \neq b \in A} \bigwedge_{c \in C} M_a c_b\] obviously doesn't hold.

Both \textit{don'tknowthat33} and \textit{don'tknownot33} separately follow from the other axioms. For proofs, see the appendix. We will retain \textit{don'tknowthat33} and delete \textit{don'tknownot33}.

2.4 \textbf{Nonintended models for restrictions of the theory}

To understand why all axioms are indispensable for formalizing the game state of three persons each holding a card, it is instructive to present countermodels of restrictions of the theory \textit{33+}. Because these models clearly do not model that state, this strengthens our case for the preferred model \textit{hexa}.

In this subsection we also use the following notation for card dealings: let \(C, D, E\) be subsets of cards, then \(C|D|E\) is the dealing where player 1 holds all
cards in $C$, player 2 all in $D$, and player 3 all in $E$. For convenience we write a set as the string of its elements, where an empty string is represented by $\varepsilon$. Thus $r|\text{wbr}|\varepsilon$ is the dealing of cards where player 1 holds a red card, player 2 a red, a white and a blue card, and player 3 holds no card.

**See33**

Figure 2 (where $rwb$ stands for $r|w|b$, etc.) is a model of $33 - \text{see33} - \text{dontsee33}$. This model $M$ doesn’t satisfy $\text{see33}$, as $M,rwb \not\models r_1 \rightarrow K_1 r_1$. None of the players can distinguish between any of the six dealings. Incidentally, it is the model of the state of the game where the cards have already been dealt but where the players haven’t looked up their own cards yet. See also section 4. Other models of $33 - \text{see33} - \text{dontsee33}$ are those resulting from a permutation of access for 1, 2 and 3 in the model hexa (see figure 1).

```

\begin{tikzpicture}
  \node (rwb) at (0,0) {$rwb$};
  \node (r123) at (-1,-1) {$r_1, 2, 3$};
  \node (w123) at (1,-1) {$w_1, 2, 3$};
  \node (wbr123) at (0,-2) {$wbr, 1, 2, 3$};
  \node (rbr123) at (0,-3) {$rbr, 1, 2, 3$};
  \node (brw123) at (0,-4) {$brw, 1, 2, 3$};
  \draw[->] (rwb) -- (r123);
  \draw[->] (rwb) -- (w123);
  \draw[->] (rwb) -- (wbr123);
  \draw[->] (r123) -- (rbr123);
  \draw[->] (w123) -- (rbr123);
  \draw[->] (w123) -- (brw123);
\end{tikzpicture}
```

Figure 2: Nobody has looked in his cards (universal access for all agents). Assume transitivity of links.

**Atmost33**

Without $\text{atmost33}$ there are more dealings relevant ($\{1, 2, 3\}$-accessible) to a given dealing. In the theory $33^+$, delete $\text{atmost33}$ and replace $\text{atleast33}$ for the stronger axiom exactly33. The theory $33^+ - \text{atmost33} - \text{atleast33} + \text{exactly33}$ has a model $M_{27}$ containing 27 worlds, where a world is characterized by any distribution of three cards of any three colours red, white and blue over three players (thus there are $3 \cdot 3 \cdot 3 = 27$ possibilities). E.g. $r|r|w$ is such a world, where 1 and 2 hold a red card and three holds a white card. Observe that $M_{27} \not\models \text{atmost33}$, as $M_{27}, r|r|w \not\models -(r_1 \land r_2)$.

The theory $33^+ - \text{atmost33}$ reveals an implicit constraint that is imposed by the language. That theory has a model $M'$ containing much more worlds than $M_{27}$, namely any world corresponding to a dealing of between three and nine cards of any of three colours red, white and blue, where every player holds at least one card, but with the restriction that a player cannot hold more than one card of the same colour. The last is, because our language cannot express that a player holds more than one card of the same colour! Some cases in more detail:
Total of four cards: There are $3^4$ worlds where one of the three players holds two different cards and the others hold one card.\footnote{There are three different combinations of two different cards for a player: $rw$, $bw$, $br$. Any of the three players can hold two cards. Thus the total is: $3 \cdot 3 \cdot 3 + 3 \cdot 3 \cdot 3 + 3 \cdot 3 \cdot 3 = 81$.}

Total of five cards: We have to consider both the case where one player holds three different cards and the others both one, and the case where there are two players holding two different cards. There is only one combination of three different cards: $rwb$. Thus the total is $3^3 \cdot 1 + 3 \cdot (3 \cdot 3 \cdot 3) = 90$ possibilities.

Total of six, seven, eight, and nine cards: similarly to four and five cards.

\textbf{Atleast33}

Without \textbf{atleast33} there are $27 + 18 + 9 + 1 = 55$ different dealings of at most three different cards over three players.\footnote{27 combinations of three cards, e.g. $rwb|\not{r}w$ and $r|\not{r}wob$, 18 of two, 9 of one, and 1 of zero.} Given the model $M_{65}$ with all these dealings as worlds, $M_{65} \models \text{atleast33}$ as $M_{65}, rwb|\not{r}w \models r_2 \lor u_2 \lor b_2$. Somewhat similarly to above, a model of special interest is the model $M''$ containing only 27 worlds (different from those in $M_{27}$, above!), for all the different dealings of \textit{exactly} three different cards (as formalized by \textbf{function33}), although not necessarily dealt to different players. Model $M_{65}$ is not a model of this slightly strengthened theory $33^+ - \text{atleast33} - \text{atmost33} + \text{function33}$, whereas model $M''$ is.

\textbf{Dontknowthat33}

Figure 3 is a model $M$ of $33^+ - (\text{dontknowthat33} + \text{dontknownot33})$. Incidentally, this is the model resulting from the action of player 1 showing player 2 his red card, given a request for his card. Obviously, $M \not\models \text{dontknowthat33}$, as $M, rwb \not\models \neg K_2 p_1$. Player 2 has become less ignorant. Actually, any model resulting of any game action on hexa results in a decrease of ignorance, as measured in terms of the dealings still imaginable for a player, and therefore satisfies $33^+ - (\text{dontknowthat33} + \text{dontknownot33})$.

![Diagram](attachment:image.png)

Figure 3: Any 3/3 game state satisfies $33 - (\text{dontknowthat33} + \text{dontknownot33})$
2.5 The theory 33

Given the equivalences proven in the previous subsection, we define the theory 33 as the conjunction of the three axioms in table 2. Obviously, $\vdash 33 \leftrightarrow 33^+$.

Observe that $\vdash \text{atmost33} \land \text{atleast33} \leftrightarrow \text{dealings33}$, as shown before. An initial state $(\text{hexa}, d)$ of the game for 3 persons and 3 cards is described by the conjunction its atomic description and common knowledge of the theory 33. For example, $\delta_{rub} \land C_{123}$ describes the initial state of the game where 1 holds red, 2 holds white and 3 holds blue.

\[
\text{see33} := \bigwedge_{a \in \{1,2,3\}} \bigwedge_{c \in \{r,w,b\}} (c_a \rightarrow K_a c_a)
\]
\[
\text{dealings33} := \delta_{rub} \lor \delta_{rwb} \lor \delta_{wrb} \lor \delta_{brw} \lor \delta_{brw}
\]
\[
\text{don'tknowthat33} := \bigwedge_{a \neq b \in \{1,2,3\}} \bigwedge_{c \in \{r,w,b\}} \neg K_a c_b
\]

| Table 2: Theory 33 |

In the previous subsection we have shown that we cannot substantially weaken the theory (by deleting axioms), because it then would model structures of different game states. In the next subsection 2.6, we show that we do not need to strengthen the theory, because in a technical sense \textit{hexa} already is its only model. Together, this shows that we have chosen the right model, and the right axioms, for describing the game state of three players each holding a card. Before we continue with that subsection, we digress on other versions of 33 that are equivalent to it:

2.5.1 Summing up worlds

Instead of formalizing observed agent properties, we might describe the model \textit{hexa} as the (exclusive) disjunction of the modal description of its worlds. Let $\delta_{xyz}$ be the formula describing the valuation corresponding to the dealing of cards $xyz$, as before. Let $\delta^1_{xyz} := x_1 \land \neg y_1 \land \neg z_1$, $\delta^2_{xyz} := \neg x_2 \land y_2 \land \neg z_2$, $\delta^3_{xyz} := \neg x_3 \land \neg y_3 \land z_3$. Now define:

\[
\sigma_{xyz} := K_1 \delta^1_{xyz} \land K_2 \delta^2_{xyz} \land K_3 \delta^3_{xyz} \land M_1 \delta_{zzz} \land M_2 \delta_{yyz} \land M_3 \delta_{xyz}
\]
\[
\sigma^{33} := \sigma_{rub} \lor \sigma_{rwb} \lor \sigma_{wrb} \lor \sigma_{brw} \lor \sigma_{brw} \lor \sigma_{brw}
\]

In $\sigma^{33}$ we do not have to make exclusive disjunction explicit, as the disjuncts exclude each other anyway. It is now easy to prove (see the appendix) that both $\sigma^{33} \vdash 33$ and $33 \vdash \sigma^{33}$. The axiomatization of \textit{hexa} as summing up worlds relates to the „state descriptions“ in [vB98] and [BM96]. In subsection 5.1 we will discuss the general procedure for computing these descriptions from given finite models, such as \textit{hexa}.  

11
2.6 Any S5 model of theory 33 is bisimilar to hexa

Proposition 1
Let \( M = (W, \{\sim_1, \sim_2, \sim_3\}, V) \) be an S5_3 model of 33, i.e. \( M \models 33 \). Then \( M \) is bisimilar to hexa.

As we also present a more general version of this proposition, namely for any number of players and cards, we have moved the proof of the underlying proposition to the appendix. As compared to the proof for the general proposition, the underlying proof is more explicit in what axioms are (only) needed in what direction of the bisimulation: see 33 is essential in the forth part of the proof, don't know that 33 is essential in the back part of the proof.

3 Axioms for players holding cards

In the previous section we have axiomatized the initial state of a game for three players each holding a card. In this section we generalize our results for any number of players and cards. Let \( d \in A^C \) be a dealing of a (nonempty) finite set \( C \) of cards over a finite set \( A \) of more than two players.

Model of the initial state of a knowledge game

In \([vD00c]\) we presented a pointed S5_n model for the information state of a game where these cards have been dealt and where everybody has (only) looked in his cards (to every combination of an agent \( a \) and a card \( c \) corresponds an atomic proposition \( e_a \), for ‘\( a \) holds \( c \)’):

\[
I_d = \langle D_d, (\sim_a)_{a \in A}, V \rangle
\]

where:

\[
D_d = \{ d \in A^C | \forall a \in A : |d^{-1}(a)| = |d^{-1}(a)| \}
\]

\[
\forall d_1, d_2 \in D_d : \forall a \in A : d_1 \sim_a d_2 \iff d_1^{-1}(a) = d_2^{-1}(a)
\]

\[
\forall c \in C : \forall a \in A : \forall d \in D_d : V(c) = 1 \iff d(c) = a, \text{ and else } 0
\]

Previous to the presentation of the axiomatization, we introduce two central concepts: type of a dealing, and description of a dealing.

Type of a dealing

Parameters for the axiomatization are: the number of agents, the number of cards, and the actual dealing of cards. So in an indirect manner, as a dealing is a function from cards to agents, only that dealing of cards. We actually need less than that, namely only the number of cards that each player holds in a dealing \( d \). We call this the type of \( d \).

The type \(\text{type}(d)\) of a dealing \( d \) is the sequence \( |d^{-1}(1)|, |d^{-1}(2)|, \ldots, |d^{-1}(n)| \). Just as for dealings, we use vertical bars as separators. For an example the type
of the dealing where 1 holds red, 2 holds white and 3 holds blue: \( \text{type}(rwbl) = \text{type}(r|w|b) = 1|1|1 \). Another example: \( \text{type}(ab|cde|f|g|bh) = 2|3|1|0|2 \). Observe that for any dealing \( d \), \( D_d = \{ d' \mid \text{type}(d') = \text{type}(d) \} \), in words: \( D_d \) is the set of dealings of type \( \text{type}(d) \).

**Description of a dealing**

Let \( d \in A^C \) be a dealing of cards, let \( P \) be the set of all atomic propositions \( c_a \), with \( a \in A \) and \( c \in C \). Define, for all \( c_a \in P \):

\[
\begin{align*}
\text{sign}_d(c_a) &= c_a & \text{if } d(c) = a \\
\text{sign}_d(c_a) &= \neg c_a & \text{if } d(c) \neq a
\end{align*}
\]

then:

\[
\delta_d = \bigwedge_{c_a \in P} \text{sign}_d(c_a)
\]

The formula \( \delta_d \) is called the description of the dealing of cards \( d \) or, in a model such where worlds are dealings, the atomic description of the world \( d \). The generalization of the axiom dealing33 will be:

\[
\text{dealings} := \bigvee_{d' \in D_d} \delta_{d'}
\]

We further define:

\[
\delta_d^a = \bigwedge_{c \in C} \text{sign}_d(c_a)
\]

This formula is called the description of the cards of player \( a \). In \( S5_n + \) dealings we can derive some equivalences that will appear to be useful in the continuation:

\[
\forall d \in A^C : \delta_d \iff \bigwedge_{a \in A} \delta_d^a
\]

\[
\forall d \in A^C : \delta_d^a \iff \bigvee_{d' \sim d} \delta_{d'}
\]

**3.1 The theory kgames\(_d^+\)**

As in the previous section, we are looking for the \( S5EC_n \) axiomatization of the model underlying this initial state (where \( n = |A| \)). Also as before, we profess uncertainty on the adequacy of \( L_d \) as a model for this state, and therefore discuss equivalences among and various weaker versions of the axioms, and the countermodels they have.

Our first try at axiomatization is to generalize the axioms in table 1. We have to keep in mind that the generalization of 'player \( a \) holding a (one) card' is
‘player $a$ holding $[d^{-1}(a)]$ cards’. We will extensively comment on the generalization of ignorance. Table 3 presents the propositional axiomatization $\text{kgames}_d^+$. The formula $\delta_d \land C A \text{kgames}_d^+$ describes the knowledge of the players in the initial state of a game for $d$. Unless confusion would otherwise result, instead of $\text{kgames}_d^+$ or $\text{kgames}_{Y^d(d)}^+$ we sometimes write $\text{kgames}_d^+$. Similarly, axioms from dealings are sometimes given an index $d$. Further we will write, unless confusion results: $\theta a := [d^{-1}(a)]$, the number of cards of agent $a \in A$ in dealing $d$. In axiom $\text{don't know that}$ of Table 3 we write $\theta -a b$ for $|C| - \theta a - \theta b$ (‘the number of cards not held by $a$ or $b$).

$\text{see} := \land_{a \in A} \land_{c \in C} (c_{a} \rightarrow K_a c_{a})$

$\text{don't see} := \land_{a \in A} \land_{c \in C} (\neg c_{a} \rightarrow K_a \neg c_{a})$

all cards are different

(there is at most one card of each colour)

$\text{atmost} := \land_{a \neq b \in A} \land_{c \in C} \neg(c_{a} \land c_{b})$

each player has (at least) a known number of cards

$\text{atleast} := \land_{a \in A} \lor_{c \neq c_{1} \ldots \neq c_{t} \in C} \land_{i=1}^{t} c_{a_{i}}$

players don’t know the cards of others

$\text{don't know that} := \land_{a \neq b \in A} \land_{c \neq c_{1} \ldots \neq c_{t} \in C} M_{a} (\land_{i=1}^{t} \neg c_{i_{a}})$

players can imagine others to hold other cards

$\text{don't know not} := \land_{a \neq b \in A} \land_{c \neq c_{1} \ldots \neq c_{t} \in C} (\land_{i=1}^{t} c'_{a_{i}} \rightarrow M_{a} (\land_{i=1}^{t} c'_{b_{i}}))$

Table 3: The theory $\text{kgames}_d^+$ for dealing $d \in A^C$

3.2 $I_d$ is a model of $\text{kgames}_d^+$

Fact 2

$I_d$ is a model of $\text{kgames}_d$.

The proof is along the same lines as that of fact 1 on page 5.

3.3 Dependencies among axioms

As in the previous section, we establish axiomatic dependencies informally, with natural deduction style proofs. Further, to illustrate nonequivalence of axioms,
we sometimes ‘go semantical’ and give countermodels.

### 3.3.1 See and dontsee

The following six axioms can all be seen as generalizations of see33 and dontsee33 respectively:

\[
\text{see} := \bigwedge_{a \in A} \bigwedge_{c \in C} (c \rightarrow K_a c_a)
\]

\[
\text{dontsee} := \bigwedge_{a \in A} \bigwedge_{c \in C} (\neg c \rightarrow K_a \neg c_a)
\]

\[
\text{seeall} := \bigwedge_{a \in A} \bigwedge_{i \neq j \in N \subseteq C} (\bigwedge_{i=1}^{l_a} c_a^i \rightarrow K_a \bigwedge_{i=1}^{l_a} c_a^j)
\]

\[
\text{dontseeall} := \bigwedge_{a \in A} \bigwedge_{i \neq j \in N \subseteq C} (\bigwedge_{i=1}^{l_a} \neg c_a^i \rightarrow K_a \bigwedge_{i=1}^{l_a} \neg c_a^j)
\]

\[
\text{seedeal} := \bigwedge_{a \in A} \bigwedge_{d \in D_a} (\delta_a^d \rightarrow K_a \delta_a^d)
\]

\[
\text{dontseedeal} := \bigwedge_{a \in A} \bigwedge_{d \in D_a} (\neg \delta_a^d \rightarrow K_a \neg \delta_a^d)
\]

Axioms see and dontsee seem to be the most straightforward generalizations: for every agent and for every single card, if a player holds it he knows that, and if he doesn’t hold it, he knows that too. They are therefore listed in kgames\(^+\). Instead, seeall and dontseeall express that, if a player holds a given number of cards, he knows them all, and that if he holds all others not, he knows that too. Axiom seedeal (for parameter dealing d) expresses that, if a player holds a given number of cards and all others not, he knows that, or in other words: he knows his local state. Axiom dontseedeal expresses that for all local states he doesn’t have, he knows that too.

Somewhat surprisingly, all six axioms are equivalent in kgames\(^+\). The proofs are simple and use the axioms dealings. Although see appears to be the most straightforward of all six, for other reasons we will retain seedeal instead.

### 3.3.2 Atmost and atleast

Similar to atleast, the following expresses that each player holds a fixed number of cards:

\[
\text{exactly} := \bigwedge_{a \in A} \neg c^i \neq \neg c^j \in C \bigwedge_{i=1}^{l_a} c_a^i
\]

It holds that atmost, atleast \(\vdash\) exactly. Informal proof:

Suppose not exactly. Then for some player a both \(\bigwedge_{i=1}^{l_a} c_a^i\) and \(\bigwedge_{j=1}^{l_a} c^j\) hold, where \(\exists i \leq l_a : \forall j \leq l_a : c^i \neq c^j\) (we may assume that \(l_a > 0\)). Then a holds more than \(l_a\) cards. That implies that some other player b holds less than \(l_b\) cards, using the axiom atmost that all cards are different. Therefore, for no selection of \(l_b\) cards does \(\bigwedge_{i=1}^{l_b} c_b^i\) hold. Therefore atmost doesn’t hold. Contradiction. Therefore exactly.

Just as for three players and three cards, we can derive atleast, atmost \(\vdash\) dealings and vice versa, similarly to the proof in the previous section. We will prefer dealings over atmost and atleast.

15
3.3.3 Ignorance: don’tknowthat, don’tknownot, and don’tknow

The form of ignorance in table 3 is the outcome of a process of gradual generalization of don’tknowthat and don’tknownot. We will repeat this process, in order to convince the reader of the inevitability of this outcome. Apart from don’tknowthat and don’tknownot, there is a third way to express ignorance as well, more related to the parameter dealing d: this is the axiom don’tknow. Fortunately, all three versions of ignorance are equivalent.

Don’tknowthat

In $3^+$ we only had to take into account one card per player. How to generalize to more than one card? Starting from our original ‘observations’ that for all different agents a and b and for all cards c we demand $\neg K_\alpha c_b$, we might, instead, now claim that a doesn’t know any combination of b’s cards: for any $c_1, \ldots, c_b$,

$$\neg K_\alpha c_b \vdash \neg K_\alpha \bigwedge_{i=1}^b c_i.$$

Unfortunately, unlike in the game for three players and three cards, the axiom that $\neg K_\alpha c_b$ for all cards c and different players a and b, isn’t strong enough. A counterexample is the game for three players 1, 2, 3 each holding two cards, with dealing $k|mn|op$, where 2 has told the others that he holds one of m and n. For any single card, it still holds that a player doesn’t know another player to have that card. But players 1 and 3 now clearly are less ignorant than they were initially. So we have to demand the players to be less informed than $\neg K_\alpha c_b$, i.e. $\neg K_\alpha (c_1 \lor \ldots \lor c_b^r)$, for some $r > 1$. This is a stronger claim, as, e.g.:

$$\neg K_\alpha (\bigvee_{i=1}^r c_i^1) \vdash \neg K_\alpha c_b^1$$

What is r, or, in other words: how large is our ignorance? We argue that, for a given dealing d, $r = |C| - |d^{-1}(a)| - |d^{-1}(b)|$. We start to illustrate that with an example:

Again, we look at the game for three players and six cards. Player 1 doesn’t know of any two cards that player 2 has one of those, but of some combinations of three cards he does: e.g. $K_1(m_1 \lor n_2 \lor o_2)$. Why? Suppose $\neg K_1(m_1 \lor n_2 \lor o_2)$, then $M_1(\neg m_2 \land (n_2 \lor o_2))$, in other words: it would be conceivable for player 1 that player 2 didn’t have all those cards. If that were true, and given that player 1 holds k and l himself, there would be only one card left for 2 to hold: p. But then player 2 would hold only 1 card. This contradicts at least (or dealings).

The extent of player a’s ignorance therefore is, that he doesn’t know that another player, b, has one from any of $|C| - |d^{-1}(a)| - |d^{-1}(b)|$ cards (as before, we write $\neg a \vdash b$ for $|C| - |d^{-1}(a)| - |d^{-1}(b)|$):
\[
\bigwedge_{a \neq b \in \mathbb{A}} \bigwedge_{c^1 \neq \ldots \neq c^{1-s} \in C} \neg K_a \bigvee_{i=1}^{t-a_b} c^i_b
\]

This, of course, is equivalent to:

\[
\text{donthknowthat} = \bigwedge_{a \neq b \in \mathbb{A}} \bigwedge_{c^1 \neq \ldots \neq c^{1-s} \in C} \bigwedge_{i=1}^{t-a_b} M_a \bigwedge_{i=1}^{t-a_b} \neg c^i_a
\]

Just as for don'tknowthat_{33}, we might have considered weakening the axiom don'tknowthat with the precondition ‘if player 1 doesn’t hold these cards’. This is don'tknowther:

\[
\text{donthknowther} = \bigwedge_{a \neq b \in \mathbb{A}} \bigwedge_{c^1 \neq \ldots \neq c^{1-s} \in C} \bigwedge_{i=1}^{t-a_b} (\bigwedge_{i=1}^{t-a_b} \neg c^i_a \rightarrow M_a \bigwedge_{i=1}^{t-a_b} \neg c^i_b)
\]

As previously, in the theory \textit{kgames}\textsuperscript{+}\textsubscript{a} the version without precondition is provably equivalent to the version with precondition:

\[
\text{kgames}\textsuperscript{+}\textsubscript{a} \vdash \text{donthknowther}. \text{ Trivial.}
\]

\textbf{kgames}\textsuperscript{+}\textsubscript{a} – don'tknowthat + don'tknowther \vdash don'tknowthat: Let \(a \neq b \in \mathbb{A}\), and \(c^1 \neq \ldots \neq c^{1-a_b} \in C\). If \(\bigwedge_{i=1}^{t-a_b} \neg c^i_a\), then from that and don'tknowther follows \(M_a \bigwedge_{i=1}^{t-a_b} \neg c^i_a\). Otherwise, \(\neg \bigwedge_{i=1}^{t-a_b} \neg c^i_a\). That is equivalent to \(\bigvee_{i=1}^{t-a_b} c^i_a\), i.e. a holds some \(r\) at most \(\geq a\), of the cards \(c^i\). For all of those, there are (different) cards in \(|C| \setminus \\{c^1, \ldots, c^{1-a_b}\}\) that \(a\) doesn’t hold. Note that this is possible, because \(|C| - \frac{a}{t-a_b} = \frac{a}{t-b}\), remove \(t-b\) cards from that, and there still remain \(\geq a\) to choose from. Once again, we now have a conjunction of \(\neg a\) cards \(c^1, \ldots, c_{t-a_b}\) that \(a\) doesn’t hold, and from that and don'tknowther follows \(M_a \bigwedge_{i=1}^{t-a_b} \neg c^i_a\). Now for the \(r\) cards that \(a\) holds from \(c^1, \ldots, c^{1-a_b}\), we have that \(a\) knows that \(b\) doesn’t hold them (ii). Combining (i) with (ii), we get that \(M_a \bigwedge_{i=1}^{t-a_b} \neg c^i_a\). This proves don'tknowthat. \(\blacksquare\)

\textbf{Dontknownot}

A similar process of gradual generalization leads from don'tknownot_{33} to don'tknownot.

By itself, \(\neg a \rightarrow M_a c_b\) for all cards \(c\) and different players \(a\) and \(b\), is not strong enough. We cannot derive that \(a\) can imagine \(b\) to hold a combination of two cards: just as in general we cannot derive \(M_a (\varphi \land \psi\) from \(M_a \varphi\) and \(M_b \psi\). E.g. in the example above for three players each holding two cards, 1 can imagine an atom \(m_0\) to be both false – \(M_1 \neg m_0\) – and true – \(M_1 m_0\), but not at the same time – \(\neg M_1 (m_2 \land \neg m_2)\).
On the other hand, in the same example we want, e.g., that \( M_1(m_2 \land n_2) \) holds. So, similarly to the generalization leading to \( \text{don't know what} \), we have to find the largest conjunction still conceivable. Obviously, we cannot imagine another player to hold more cards than the number we know him to hold. This number indeed is the required maximum:

\[
don't \text{knownot} = \bigwedge_{a \in A} \bigwedge_{c^i \in C} (\bigwedge_{i=1}^{i_b} \neg c^i_a \rightarrow M_a \bigwedge_{i=1}^{i_b} c^i_b)
\]

We are about to show that \( \text{don't knownot} \) is equivalent to \( \text{don't know that} \), but before that we introduce a third way to describe ignorance: \( \text{don't know} \).

**Dont know**

The axioms \( \text{don't knownot} \) and \( \text{don't know that} \) are unsatisfactory, because they are too much in terms relations *between* players. In the special case of three players and three cards, \( \text{don't knownot} \) and \( \text{don't know that} \) were more satisfactory. Because the players only had one card each, it appeared that the formulation of ignorance was for arbitrary (single) cards, and not strictly related to the number of cards of player.

Instead of referring to the amounts of cards of two different players, we might as well refer more directly to the entire dealing of cards. This is the case in \( \text{don't know} \). The following explanation might help to make it appear plausible: Prior to the state of the game where the cards have been dealt and where players have looked into their cards, is the state the cards have been dealt but the players haven’t seen their own cards yet. In that stage, all dealings in \( D_3 \) are possible for all players. (See also section 4.) Looking up cards then corresponds to revising that maximum ignorance \( \bigwedge_{a \in A} \bigwedge_{d \in D_3} M_a \delta_d \). This can be done by conditioning on it. The condition is that, after they have looked up their cards, players only consider dealings that correspond with their own cards. This is the axiom \( \text{don't know} \):

\[
\text{don't know} := \bigwedge_{a \in A} \bigwedge_{d \in D_3} (\delta^a_d \leftrightarrow M_a \delta_d)
\]

Fortunately, all three forms of ignorance are equivalent. This is surprising, because \( \text{don't knownot} \) and \( \text{don't know that} \) appear to describe complementary kinds of ignorance.

\( \text{don't knownot} \vdash \text{don't know that} \):

**Proof:** Suppose not. Then there are players \( a, b \) and cards \( c^1, \ldots, c^{i_b-a_b} \), where \( i - a_b = |C| - i_a - i_b \), such that \( K_a \bigvee_{i=1}^{i_b} c^i_b \).

Regardless of whether \( a \) holds some of these cards \( c^1, \ldots, c^{i_b-a_b} \) himself, because \( r = |C| - i_a - i_b \) there must be at least \( i_b \) cards other than those, that \( a \) doesn’t hold, suppose: \( ca^1, \ldots, ca^{i_b} \). In other words, we have that: \( \neg ca^1_a \land \ldots \land \neg ca^{i_b}_a \). Applying \( \text{don't knownot} \) we get \( M_a \bigwedge_{i=1}^{i_b} ca^i_b \).
Formula $M_a \bigwedge_{i=1}^{i_b} a_i^b$ means that $a$ considers $b$ to hold the $i_b$ cards $a_1^b, \ldots, a_i^b$.
Formula $K_a \bigvee_{i=1}^{m} c_i^a$ means that $a$ knows that $b$ holds at least one more card, namely from the (other!) cards $c_1, \ldots, c_m$. Therefore, $a$ considers $b$ to hold more than $i_b$ cards.
On the other hand, axioms atmost and atleast (or, equivalently, dealings) express that $b$ holds exactly $i_b$ cards. Contradiction.

\textbf{Proof:} Suppose player $a$ doesn’t have any of the cards $c_1, \ldots, c_i$: $\bigwedge_{i=1}^{i_b} \neg c_i^a$.
Instead, $a$ has cards $c_i^{i_b+1}, \ldots, c_i^{i_b+1}$. Let $d^*$ be a dealing of cards where $a$ has those cards and such that $b$ has all the cards $c_1, \ldots, c_i$, thus $\bigwedge_{i=1}^{i_b} c_i^b$. Formulas $c_i^{i_b+1} \wedge \ldots \wedge c_i^{i_b+1}$ is the subformula of $\delta_d^a$, consisting of all positive literals (i.e., atoms). Therefore, from $c_i^{i_b+1} \wedge \ldots \wedge c_i^{i_b+1}$ and dealings follows $\delta_d^a$. From that and $\text{dontknow}$ follows $M_a \delta_d$. Because $\bigwedge_{i=1}^{i_b} c_i^b$ is a subformula of the conjunction $\delta_d$, and because in general $\Diamond \varphi, \varphi \rightarrow \psi \vdash \Diamond \psi$, it follows that $M_a \bigwedge_{i=1}^{i_b} c_i^b$. Therefore $\bigwedge_{i=1}^{i_b} \neg c_i^a \rightarrow M_a \bigwedge_{i=1}^{i_b} c_i^b$. As the cards $c_1, \ldots, c_i$ were arbitrary, we have shown that $\text{dontknownot}$. 

\textbf{dontknowthat} $\vdash \text{dontknow}$:

\textbf{Proof:} Let $a \in A, d \in D_a$. Assume $\delta_d^a$. From that and see follows $K_a \delta_d^a$.
Suppose $\neg M_a \delta_d$. We have that $\neg M_a \delta_d \leftrightarrow K_a \neg \delta_d \leftrightarrow K_a (\neg \delta_1^d \wedge \ldots \wedge \neg \delta_n^d)$. We show that, for an arbitrary agent $b$, $\neg \delta_d^b$ leads to a contradiction, so that also $K_a (\neg \delta_1^d \wedge \ldots \wedge \neg \delta_n^d)$ leads to a contradiction.

Suppose $\neg \delta_d^b$, i.e., for some ordering of cards: $\neg c_1^b \wedge \neg c_2^b \wedge \ldots \wedge \neg c_i^b \wedge c_i^{i_b} \wedge c_i^{i_b+1} \wedge c_i^{i_b+1}$. We may assume that $a \neq b$, because otherwise we have a direct contradiction. Agent $a$ now knows a disjunction of $\left| C \right| - j b$ cards: $a$ has $j a$ cards himself. These cards $b$ doesn’t have. Also these cards are all different from the $i_b$ cards $c_1, c_2, \ldots, c_i$; that $b$ also doesn’t have. Therefore $b$ must have one of the $\left| C \right| - j a - i_b$ remaining cards, and $a$ knows that. This contradicts $\text{dontknowthat}$. Therefore $\delta_d^a \rightarrow M_a \delta_d$.

We prove $M_a \delta_d \rightarrow \delta_d^a$ by contraposition. Assume $\neg \delta_d^a$, i.e., $\neg \bigvee_{d \neq a} \delta_d^a$. From that, with dealings, follows $\bigvee_{d \neq a} \neg \delta_d^a$.
Suppose that $\delta_d^a$, and keep in mind that $\neg \delta_d^a \leftrightarrow \delta_d^a$, because $d' \neq a \leftrightarrow d$). From that and see follows, again, $K_a \delta_d^a$, and similarly to the previous argument, and because $d' \neq a \leftrightarrow d$, we derive $K_a \neg \delta_d^a$. This is equivalent to $\neg M_a \delta_d$. The last implies $\neg M_a \delta_d$ (because of contraposition of the general scheme $\Diamond (\varphi \wedge \psi) \rightarrow \Diamond \varphi)$.

We have now shown that $\text{dontknow} \vdash \text{dontknownot}, \text{dontknownot} \vdash \text{dontknowthat},$ and $\text{dontknowthat} \vdash \text{dontknow}$. Therefore, all three versions of ignorance are equivalent in the theory kgames$^+$.

\footnote{For better understanding we give an informal argument that from $\text{dontknowthat}$ follows $\text{dontknownot}$, even though we don’t need it, as we have proven it indirectly: Assume $\text{dontknowthat}$. Suppose $\text{dontknownot}$ doesn’t hold. I.e., suppose that $\neg c_1^a \wedge \ldots \neg c_i^b$ for some players.}
3.3.4 Boundary case in $\text{games}_d^+$

If there are only two players (1 and 2), both players have full knowledge of the dealing of cards. The axiom $\text{don't know that}$ 'disappears', as $|C| - |a - b| = 0$. In some of the proofs (namely, those about ignorance) we have essentially used that there are more than two players. $\text{don't know not}$ still holds, but now 1 cannot only imagine 2 to have some cards he does not have, but $\text{knows}$ it, because those are all the other cards.

If there are more than two players, it still can be the case that the cards are dealt over only two of those players, i.e. $\exists a, b \in A : |d^{-1}(a)| + |d^{-1}(b)| = |C|$. Suppose that is the case for players 1 and 2. Now, $\text{don't know that}$ disappears for just the combination of 1 and 2. Although 1 and 2 still have full knowledge of the dealing of cards, the other players haven't, and 1 and 2 know that.

3.4 The theory $\text{games}_d$

Given all the dependencies between axioms, that we have proven in the previous subsection, we are now left with the choice how to simplify the theory $\text{games}_d^+$. The axioms $\text{see deal}$, $\text{dealings}$ and $\text{don't know}$ suffice. Given that we prove in $\text{S}_{5n}$, so that for all $a$ and $d$: $K_d \delta_d^a \rightarrow \delta_d^a$, we further propose to combine $\text{see deal} = \bigwedge_{a \in A} \bigwedge_{d \in D_d} (\delta_d^a \rightarrow K_d \delta_d^a)$ and $\text{don't know} = \bigwedge_{a \in A} \bigwedge_{d \in D_d} (\delta_d^a \leftrightarrow M_a \delta_d)$ into one axiom $\text{see don't know}$. This appears to be the most elegant formulation of the theory. We therefore present the following as the theory $\text{games}_d$, a shorter but equivalent version of $\text{games}_d^+$:

\[
\begin{align*}
\text{dealings} & := \bigvee_{d \in D_d} \delta_d \\
\text{see don't know} & := \bigwedge_{a \in A} \bigwedge_{d \in D_d} (K_a \delta_d^a \leftrightarrow M_a \delta_d)
\end{align*}
\]

Table 4: The theory $\text{games}_d$, for dealing $d \in A^C$

\[a \neq b\] and cards $c^1, ..., c^b$, and that $\neg M_a(e^a_1 \land ... \land e^a_b)$. The last is equivalent to: $a$ knows that $b$ doesn’t hold at least one of these cards. As neither $a$ himself holds one of those cards, there must be a third player holding one of them. Now simplify the game by throwing all other players on one heap, so to speak: give all other cards in the hands of an imaginary player $b'$. Then $a$ knows that $b'$ holds one of $c^1, ..., c^b$: $K_a(e^a_1 \lor ... \lor e^a_b)$. As $|C| - |a - b'| = |b|$, this contradicts $\text{don't know that}$. Therefore $\text{don't know not}$. 

20
Just as for the case of three persons and three cards, we have shown that we cannot substantially weaken the theory, by deleting axioms or weakening axioms. We now show that we also do not need to strengthen the theory, because $I_d$ is its only model. Together, this shows that we have chosen the right model, and the right axioms, for describing the game state where a finite amount of cards are dealt over a finite amount of players.

3.5 All models of kgames$_d$ are bisimilar to $I_d$

Proposition 2
Let $M$ be an $S5_n$ model of theory kgames$_d$. Then $M$ is bisimilar to $I_d$.

Proof Write $M = \langle W^M, (\sim^M_a)_{a \in A}, V^M \rangle$. We have that $M \models$ kgames$_d$. Write $I_d = \langle D_d, (\sim_a)_{a \in A}, V \rangle$, for the intended initial model $I_d$. Observe that, because $M \models$ dealings, each world $w \in M$ has a valuation $V_w = V_d$ for some $d \in D_d$. Define relation $R \subseteq (M \times I_d)$ as follows:

$$\forall w \in M : \forall d \in D_d : R(w, d) \iff V_w = V_d$$

We prove that $R$ is a bisimulation between $M$ and $I_d$.

Forth:
Let $w, w' \in M$, let $d \in D_d$. Suppose that $R(w, d)$ and that, for an arbitrary $a \in A$: $w \sim_a w'$. We find an $R$-image of $w'$, in $D_d$, as follows:

Observe that $I_d, d \models \delta_d$. As $V_w = V_d$, also $M, w \models \delta_d$. Therefore $M, w \models M_a \delta_d$. From that and $M, w \models$ seedon't know follows $M, w \models K_a \delta_d$. From that and $w \sim_a w'$ follows $M, w' \models \delta_d$, i.e.: $M, w' \models V_d \sim_a \delta_d$. Therefore there is a $d \sim_a d$ such that $M, w' \models \delta_d$. That $d'$ is the required $R$-image of $w'$: note that $d \sim_a d'$, and that $V_d, w' = V_d$, because also, obviously, $I_d, d' \models \delta_d$.

Back:
Let $d, d' \in I_d$, let $w \in M$. Suppose that $R(w, d')$ and that, for an arbitrary $a \in A$, $d \sim_a d'$. We find an $R$-original of $d'$, in $M$, as follows:

$M, w \models \delta_d$
$\implies$ reflexivity
$M, w \models M_a \delta_d$
$\iff$ from seedon't know
$M, w \models K_a \delta_d$
$\iff$
$\forall w'' \sim_a w : M, w'' \models \delta_d$
$\iff$
$\forall w'' \sim_a w : M, w'' \models V_d \sim_a \delta_d$
$\iff$
$\forall w'' \sim_a w : \exists d' \sim_a d : M, w'' \models \delta_d$.
\(\Leftrightarrow\) \(\forall u'' \sim_a w : \exists d' \sim_a d' : M, u'' \models \delta_{d'}\)

\(\Leftrightarrow\) \(\forall u'' \sim_a w : M, u'' \models \sqrt{\delta_{d'}}\)

\(\Leftrightarrow\) \(\forall u'' \sim_a w : M, u'' \models \delta_{\delta_{d'}}\)

\(\Leftrightarrow\) \(M, w \models K_a \delta_{d'}\)

\(\Rightarrow\) \(M, w \models \delta_{\delta_{d'}}\)

\(\Rightarrow\) \(\exists u' \sim_a w : M, u' \models \delta_d\)

Any \(u'\) satisfying the last statement is an \(\mathcal{A}\)-original of \(d'\), as \(M, u' \models \delta_d\) says that \(V_{u'} = V_{d'}\).

Note that in the forth part of the proof, we have only essentially used that, for any agent \(a\) and dealing \(d\), \(M_a \delta_d \rightarrow K_a \delta_{d'}\), whereas in the back part of the proof, we have also essentially used the reverse: \(K_a \delta_{d'} \rightarrow M_a \delta_d\). Further, note that, in the proof, we only use reflexivity of models; the proposition therefore holds for all \(T_a\) models.

Instead of this direct proof, there is also an indirect proof. The indirect proof uses that the model \(I_a\) can be constructed by executing an action in a simpler model for card games. In the next section, 4, we now present this simpler model.

4 Axioms for players not seeing their cards

We have described and axiomatized the state of the game where the cards have been dealt and where players have looked into their cards. We called it the initial state of a knowledge game. This game state is preceded by a state where the cards have been dealt but where the players haven’t looked into their cards yet. Assume that everybody sees how many cards lie upside down in front of each player. Therefore players know the type of the actual dealing. They know how many cards they have, and how many cards everybody else, but they do not what these cards are. Still, it is enough for all players to know (compute) the set of relevant dealings given actual dealing \(d\), as \(D_a = D_{\text{type}(d)}\).

We call the model underlying this state the pre-initial model \(preI_a\). In the state \(preI_a, d\), all dealings in \(D_a\) are possible for all players, i.e. each player’s access on the set of relevant dealings is the universal relation:

\[\text{preI}_a = \{D_a, (\sim_a)_{a \in A}, V\}\]

where

\[\forall a \in A : \forall d, d' \in D_a : d \sim_a d'\]
and, as before:

$$\forall d \in D_d : \forall c_a \in P : V_d(c_a) = 1 \text{ iff } d(c) = a.$$ 

We continue with axiomatizing this model \( \text{pre}I_d \).

### 4.1 The theory \( \text{prekgames}_d \)
Let \( d \) be a dealing of cards. The theory \( \text{prekgames}_d \) is the conjunction of the axioms in table 5.

\[
\begin{align*}
\text{dealings} & := \bigvee_{d \in D_d} \delta_d \\
\text{don't know why} & := \bigwedge_{a \in A} \bigwedge_{d \in D_d} M_a \delta_d
\end{align*}
\]

Table 5: The theory \( \text{prekgames}_d \), for parameter dealing \( d \)

It will be obvious that \( \text{pre}I_d \) is a model of theory \( \text{prekgames} \) for parameter dealing \( d \). We now proceed as follows: first we prove that the theory describes this model, or, in other words, that all other models of \( \text{prekgames} \) are bisimilar to \( \text{pre}I_d \). Then we show that the model \( I_d \), describing the initial state of a knowledge game, can be constructed from \( \text{pre}I_d \) by executing a knowledge action type (as defined in [vD00a]). We can then prove the uniqueness of \( I_d \) without having to worry about its axiomatization! This is also convenient, because the bisimilarity proof to establish the uniqueness of \( I_d \) is more complex than the one below, to establish the uniqueness of the pre-initial state \( \text{pre}I_d \).

### 4.2 All models of \( \text{prekgames}_d \) are bisimilar to \( \text{pre}I_d \)

**Proposition 3**

Let \( d \in A^C \) be a dealing of cards. Let \( M \) be a model of \( \text{prekgames} \) for parameter \( d \). Then \( M \) is bisimilar to \( \text{pre}I_d \).

**Proof** Let \( M \) be a model of \( \text{prekgames} \) for parameter \( d \). Write \( M = \langle W^M, (\sim_a^M)_{a \in A}, V^M \rangle \). We remind the reader that \( \text{pre}I_d = \langle D_d, (\sim_a)_{a \in A}, V \rangle \).

First observe that, because \( M \models \text{dealings} \), each world \( w \in M \) has a valuation \( V_w = V_d \) for some \( d \in D_d \).

Define relation \( \mathfrak{R} \subseteq (M \times \text{pre}I_d) \) as follows:

$$\forall w \in M : \forall d \in D_d : \mathfrak{R}(w,d) \iff V_w = V_d$$

We prove that \( \mathfrak{R} \) is a bisimulation between \( M \) and \( \text{pre}I_d \).

**Forth:** Let \( w, w' \in M \). Let \( d \in D_d \). Suppose that \( \mathfrak{R}(w,d) \) and that, for an arbitrary agent \( a \in A \), \( w \sim_a^M w' \). From our observation on valuations in \( M \), it
follows that there is a dealing $d' \in D_d$ such that $V_{u'} = V_{d'}$. This dealing $d'$ is our required $\mathfrak{R}$-image in $D_d$: because $\sim_\alpha$ is universal on $D_d$ it trivially holds that $d \sim_\alpha d'$, and because $V_{u'} = V_{d'}$ we have that $\mathfrak{R}(w', d')$.

**Back:** Let $d, d' \in D_d$. Let $w \in M$. Suppose that $\mathfrak{R}(w, d)$ and that, for an arbitrary agent $a \in A$, $d \sim_\alpha d'$.

Suppose there is no $w' \in M$ such that $V_{u'} = V_{d'}$. In other words: there is no $w' \in M$ such that $M, w' \models \delta_{d'}$. Then, in particular there is no $w' \in M$ such that $w \sim_\alpha^M w'$ and $M, w' \models \delta_{d'}$, thus $M, w \not\models M, w \not\models$ don't know why. Contradiction.

Therefore there is such a $w' \in M$, and, as we have shown, there is even a $w'$ that is $\sim_\alpha^M$-related to $w$. This world $w'$ is our required $\mathfrak{R}$-original in $M$: we have that $w \sim_\alpha^M w'$, and because $V_{u'} = V_{d'}$ we have that $\mathfrak{R}(w', d')$. ■

### 4.3 Looking up cards

In [vD00a], see also [vD99], we presented the language DKL of dynamic knowledge logic, containing dynamic modal operators $[\alpha]$ for actions $\alpha \in KA$ and action types $\alpha \in KT$, and its interpretation. The action of all players looking up (turning) their cards, in a state where those cards have been dealt, is a KT action type. We call this action type $\text{lookup}_A$. KT actions and KA action types have a precise formal interpretation $[\ ]$, that is a relation between $S5$ models. If the relation is functional, we can use $[\ ]$ as a postfix unary operator. We now have that:

**Fact 3**

$\text{preI}_A[\text{lookup}_A] = I_A$

In other words: the model $I_A$ is the unique model resulting from executing an action of type $\text{lookup}_A$ in the model $\text{preI}_A$. Using fact 3, we have an ‘indirect’ proof of proposition 2 that $I_A$ is the unique model of $\text{kgames}$:

**Indirect proof of proposition 2:** Let $M$ be a model of $\text{kgames}$. For every agent $a$, add access for $a$ between all $A$-related worlds in $M$ that are not $\sim_\alpha$-related. The resulting model $M'$ is a model of $\text{prekgames}$. It holds that $M'[\text{lookup}_A] = M$. Because $M'$ is bisimilar to $\text{preI}_A$, $M'[\text{lookup}]$ is bisimilar to $\text{preI}_A[\text{lookup}]$, i.e., using fact 3, $M$ is bisimilar to $I_A$. ■

In the proof, we used that bisimilarity is invariant under execution of action types with a functional interpretation. For details, see [vD00a]. For the definition of $\text{lookup}_A$ and the proof that its interpretation is functional, see [vD00b].
5 Further observations

5.1 Modal fixed points

Even though we considered both models and axiomatizations for card game states, we ended up in axiomatizing two different S5n models, and their states. For modal models and states in general, their description (or, as it is called, ‘characteristic formula’) in an infinitary (propositional) modal logic can be computed by a fixed point construction. See [vB98], relating to [BM96]. In this subsection we show that our specific results are in accordance with this general construction.

We apply the construction in [vB98], Chapter 5 (Modal Fixed Points and Bisimulation), to the initial model $I_d$ of a knowledge game for dealing $d$. A given model $M$ can be described with the following fixed-point construction:

$$E(M) = \bigwedge_{w \in M} E(M, w) = \bigwedge_{w \in M} (p_w \rightarrow (\delta_w \wedge \bigwedge_{Rw} \hat{\delta}_w \wedge \Boxw p_w))$$

Here $\delta_w$ is the atomic description of world $w$, as usual, and all $p_w$ are fresh atoms.

Beyond that, if $M$ is finite, we can replace the atomic variables $p_w$ by a unique modal definition $\Delta_w$ of $w$ in $M$. Indeed, this is the case for knowledge game states. Initial (and pre-initial) states $(I_d, d)$ of knowledge games, such as (hexa, rwb), are finite, and all worlds even differ in their atomic description $\delta_d$. So $\delta_d$ already serves as a unique modal definition $\Delta_d$ of worlds $d \in I_d$. We can then describe a solution for the equation above by replacing all $p_w$ by $\delta_w$ (i.e. $p_d$ by $\delta_d$). Also switching to a multiagent epistemic language we get the equation:

$$E(I_d)[p_d := \delta_d] = E^\delta(I_d) = \bigwedge_{d \in I_d} \bigwedge_{a \in A} (\delta_d \rightarrow (\delta_d \wedge (\bigwedge_{d \sim a \delta'} \delta_a \wedge K_a \bigvee \delta_{\delta'})))$$

As $\delta_d \leftrightarrow \bigvee_{d \sim a \delta'} \delta_a$, and as $\delta_d$ in the consequent is superfluous, we get:

$$E^\delta(I_d) = \bigwedge_{d \in I_d} \bigwedge_{a \in A} (\delta_d \rightarrow ((\bigwedge_{d \sim a \delta'} \delta_a \wedge K_a \delta_a)))$$

For $I_d = \text{hexa}$ and $w = \text{rwb}$ we get the following (in the consequent, we also delete formulas expressing reflexivity):

$$E^\delta(\text{hexa}, \text{rwb}) = \delta_{\text{rwb}} \rightarrow (M_1 \delta_{\text{rwb}} \wedge M_2 \delta_{\text{rwb}} \wedge M_3 \delta_{\text{rwb}} \wedge K_1 r_1 \wedge K_2 w_2 \wedge K_3 b_3)$$

What is the relation between $\text{kgames}_d$ and $E^\delta(I_d)$? Because in $E^\delta(I_d)$ we replaced atomic (fresh) variables $p_d$ by atomic descriptions $\delta_d$, we have to assume explicitly that we ‘live’ in one of these worlds/dealings, i.e. that one of these descriptions holds. Differently said: $\text{dealings} = \bigvee_{d \in I_d} \delta_d$, given parameter dealing $d$, is an axiom. It holds that $\text{kgames}_d = \text{dealings} + \text{seedonknow}$ is logically equivalent to $\text{dealings} + E^\delta(I_d)$. 

25
\textbf{Proposition 4}  
\texttt{kgames}_d \leftrightarrow \texttt{dealings} + E^S(I_d)  

It suffices to prove that in \texttt{S5}_n plus \texttt{dealings}, \texttt{seedontknow} is equivalent to \texttt{E}^S(I_d):

\[
\bigwedge_{d \in I_d} \bigwedge_{a \in A} (\delta_d \rightarrow (\bigwedge_{d \sim a} M_a \delta_d) \land K_a \delta_d^a) \quad \Leftrightarrow \quad \bigwedge_{d \in I_d} \bigwedge_{a \in A} (K_a \delta_d^a \leftrightarrow M_a \delta_d)
\]

For the proof, see the appendix.

\textbf{Models versus states}

We close this subsection with applying another observation from \cite{vB98}, on the relation between model and state descriptions. Given the finite model description \(E^\Delta(M) = E(M)[\mu_w := \Delta_w]\), where \(\Delta_w\) is a unique modal description of world \(w\), a state from \(M\) is described by the formula

\[
\Delta_w \land \Box^* E^\Delta(M).
\]

Given that \texttt{kgames}_d is the description of the model \(I_d\), we can therefore describe any of its states \((I_d,d)\) by

\[
\delta_d \land \mathcal{C}_A(\texttt{kgames}_d).
\]

We even can replace the common knowledge operator by the iteration \(E^\text{\texttt{max}}\) (with \(E\) the general knowledge operator, and not the ‘description operator’ of a model), where max is the maximum length of a path between two dealings. (Except for some examples, we do not know the actual value of max; we conjecture that \(\text{max} \leq |A|\).)

An example: the state \((\texttt{hexa}, \texttt{rwb})\) can be described by \(\delta_{\texttt{rwb}} \land \mathcal{C}_{12333}\). As any world of \texttt{hexa} can be reached by a \(\{1,2,3\}\)-path \((\bigcup_{i \in \{1,2,3\}} \sim_i\text{-path})\) of at most length 2, we can replace this by \(\delta_{\texttt{rwb}} \land E_{123} E_{123} 33\).

Similarly, we can compute that \(\texttt{prekgames}_d = \texttt{dealings} + \texttt{dontknowany}\) is equivalent to \(\texttt{dealings} + E^S(\texttt{pre} I_d)\). Other than \texttt{kgames}, where we had to prove an equivalence, computing \(E^S(\texttt{pre} I_d)\) directly results in \texttt{dontknowany}.

\subsection{5.2 Frame characteristics}

We have axiomatized models for knowledge games. One might wonder whether the frames underlying these models cannot be characterized directly. Frames for knowledge games for \(n\) players can be seen as distributed systems for \(n\) processors, with holes. In \cite{Lom99, LvdMR00} a distributed system for \(n\) processors that only know their own states is called an \(n\)-dimensional hypercube. Hypercubes have the property of being \textit{weakly-directed}: for any world \(w\), and for any \(n\) (possibly different) worlds that the \(n\) agents can access from \(w\), there
is a world $w'$ accessible from those $n$ worlds for those $n$ agents (possibly in a different order). They are characterized by a multiagent modal axiom $\text{WD}$. See also [vD00c].

Figure 4, below, illustrates how $\text{hexa}$ relates to a three-dimensional hypercube, i.e., a cube. On the left, the model (‘hypercube’) of the distributed system for three processors 1, 2 and 3 each having three possible local state values $r$, $w$ and $b$. On the right, $\text{hexa}$, the model of an initial state of a game for three players 1, 2, and 3 and three cards $r$, $w$ and $b$. In the hypercube on the left, we have highlighted the worlds and the access that we find in $\text{hexa}$, on the right.

![Hypercube and hexa](image)

Figure 4: On the left, a three-dimensional hypercube. On the right, $\text{hexa}$.

### 5.3 Belief revision

In [vD00a] we define a multiagent modal language with dynamic operators for actions (‘knowledge changing operations’). These actions are interpreted as a relation between $S5$ models and $S5$ states. Instead, here we have axiomatized two different models, $I_d$ and $\text{pre}I_d$. We have seen that $\text{pre}I_d$ can be transformed into $I_d$ by a lookup action type. (Similarly game states ($\text{pre}I_d, d'$) can be transformed into game states ($I_d, d'$) by a lookup action.) We now have a relation between the axiomatizations of these models:

| $\models \text{pre} \text{games} \rightarrow [\text{lookup}] \text{games}$  
| $\Leftrightarrow$ by definition  
| $\forall M : M \models \text{pre} \text{games} \rightarrow [\text{lookup}] \text{games}$  
| $\Leftrightarrow$ as only $\text{pre}I_d$ is a model of $\text{pre} \text{games}$  
| $\text{pre}I_d \models [\text{lookup}] \text{games}$  
| $\Leftrightarrow$ by definition of interpretation in $\text{DKL}$  
| $\text{pre}I_d[\text{lookup}] \models \text{games}$  
| $\Leftrightarrow$ by fact 3  
| $I_d \models \text{games}$

Complementary to our semantic approach, that relates theories by the unique
models that they describe, one might consider to relate these theories in a more
direct way by a process of belief revision. Intuitively, we have already done that.
An example: In preI₄, axiom

\[ \text{don't know any} = \bigwedge_{a \in A} \bigwedge_{d' \in D_d} M_a \delta_{d'} \]

describes that all dealings are relevant. In the transition from preI₄ to I₄,
where all players look up their cards, we have to weaken this axiom: no longer
all dealings are conceivable to all players. An obvious way to weaken it, is to
conditionize the \( M_a \delta_{d'} \): after 'lookup', a dealing is only conceivable if your
known private state corresponds to it, and vice versa:

\[ \text{seedon't know} = \bigwedge_{a \in A} \bigwedge_{d' \in D_d} (K_a \delta'^{d'}_{d'} \leftrightarrow M_a \delta_{d'}). \]

Can these 'intuitions' be made more precise? We imagine axiom seedon't know
being computed from something like don't know any[lookup₄], where \( \varphi[\alpha] \) stands for: 'the revision of axiom \( \varphi \) as a consequence of the execution of action \( \alpha \)'. The
process of revising pregames would then be:

\[
\begin{align*}
\text{pregames₄[lookup₄]} &= (\text{dealings} \land \text{don't know any}[\text{lookup₄}]) \\
   &= \text{dealings}[\text{lookup₄}] \land \text{don't know any}[\text{lookup₄}] \\
   &= \text{dealings} \land \text{seedon't know}
\end{align*}
\]

This topic of 'direct' theory revision is also discussed in [vB00]. A procedure
is given for the special case of actions that are public announcements. In [vB00],
the process is not called 'theory revision', as we do, but syntactic relativization
of a formula.

5.3.1 Axiomatizing other game states

In order to axiomatize other game states, and in general other states resulting
from action execution, we obviously need a way of systematic belief revision.

An example: in a knowledge game state we can execute a show action (see
[vD99]). We have to revise once more our ignorance, and only consider dealings
that are consistent with both our private state and with that what we now
know from the private state of others (the card that we have seen). An extra
complication is that this involves subgroup common knowledge.

Just as for preI₄ and I₄, any game state resulting from executing a game
action in I₄ will still satisfy dealings and see. Only ignorance has to be revised.
Further, any game state will be finite, as it is constructed by a KT action
type from the finite model I₄, and it is also unique, because I₄ is unique and
bisimilarity is invariant under action execution.

We have not pursued this course any further. The area of knowledge games
seems to be a fruitful playing field for interactions between belief revision and
model updating. Because our action semantics is clear and simple, we have an
easily check proposed beliefs revisions.

28
6 Conclusion

We have axiomatized two different game states for card games. We started with the state where some cards are dealt over players and where players hold the cards in their hands, i.e. where they can see their own cards. For the state of three players and three cards, we showed that the preferred model hexa is described by the theory 33. For the state of any finite number of players and cards, in other words for any dealing d of cards over players, we showed that the preferred model l_4 is described by the theory l_games. In particular we have described various equivalent versions of the axiom see that expresses that a player knows the cards that he holds, and we have described three different axioms that express ignorance, that are all equivalent to each other. Prior to the state where players have picked up their cards from the table, is the state where cards have been dealt over players but where they haven’t picked them up yet. We have proven that its preferred model prel_4 is described by the axiomatization prel_games. We have shown that our results correspond to those of fixed point computations of the description of modal models.

Appendix

Proof from section 2.2

Proof of fact 1: hexa ⊨ 33+. From all axioms we prove a typical case. In the proofs, read hexa, w ⊨ φ for w ⊨ φ.

hexa ⊨ see33:

rwb ⊨ r_1 and rbw ⊨ r_1
⇔ rwb ⊨ K_1 r_1
⇒ rwb ⊨ r_1 → K_1 r_1

hexa ⊨ donsee33:

rwb ⊨ ¬ w_1 and rbw ⊨ ¬ w_1
⇔ rwb ⊨ ¬ w_1 and rbw ⊨ ¬ w_1
⇔ rwb ⊨ K_1 ¬ w_1
⇒ rwb ⊨ ¬ w_1 → K_1 ¬ w_1

hexa ⊨ atmost33:
\( \text{rwb} \models r_1 \text{ and } \text{rwb} \not\models r_2 \)
\[ \Rightarrow \]
\( \text{rwb} \not\models r_1 \land r_2 \)
\[ \Leftrightarrow \]
\( \text{rwb} \models \neg(r_1 \land r_2) \)

\text{hexa} \models \text{atleast33}:

\( \text{rwb} \models r_1 \)
\[ \Rightarrow \]
\( \text{rwb} \models r_1 \lor w_1 \lor b_1 \)

\text{hexa} \models \text{dontknowthat33}:

\( \text{bur} \not\models r_1 \)
\[ \Rightarrow \]
\( \text{bur} \not\models r_1 \text{ or } \text{rwb} \not\models r_1 \)
\[ \Leftrightarrow \]
\( \text{rwb} \not\models K_2 r_1 \)
\[ \Leftrightarrow \]
\( \text{rwb} \models \neg K_2 r_1 \)

\text{hexa} \models \text{dontknownot33}:

\( \text{rwb} \models r_1 \)
\[ \Rightarrow \]
\( \text{rwb} \models M_2 r_1 \)
\[ \Rightarrow \]
\( \text{rwb} \models \neg r_2 \rightarrow M_2 r_1 \)

\text{Proofs from section 2.3}

33\textsuperscript{+} \rightarrow \text{see33}\textsuperscript{+} \rightarrow \text{see33}:

We prove the case \( r_1 \rightarrow K_1 r_1 \).

Suppose \( r_1 \).

From \text{dealings33} follows \( r_1 \rightarrow \neg w_1 \). From \( r_1 \) and \( r_1 \rightarrow \neg w_1 \) follows \( \neg w_1 \).

From \( \neg w_1 \) and \text{dontsee33} follows \( K_1 \neg w_1 \).

From \text{dealings33} follows \( r_1 \rightarrow \neg b_1 \). From \( r_1 \) and \( r_1 \rightarrow \neg b_1 \) follows \( \neg b_1 \). From \( \neg b_1 \) and \text{dontsee33} follows \( K_1 \neg b_1 \).

From \( K_1 \neg w_1 \) and \( K_1 \neg b_1 \) follows \( K_1 (\neg w_1 \land \neg b_1) \). From \text{atleast33} follows \( r_1 \lor w_1 \lor b_1 \). From \( \neg w_1 \land \neg b_1 \) and \( r_1 \lor w_1 \lor b_1 \) follows \( r_1 \). Therefore from \( K_1 (\neg w_1 \land \neg b_1) \) and \( K_1 (r_1 \lor w_1 \lor b_1) \) (as \text{atleast33} is commonly known) follows \( K_1 r_1 \).

Therefore \( r_1 \rightarrow K_1 r_1 \).
atmost33, at least33 ⊬ function33:

We show by informal proof that at most33, at least33 ⊬ function33, for the case \( r_1 \lor r_2 \lor r_3 \). First we show that there is at most one player holding the red card, then we show that there is at least one player holding the red card.

Suppose more than one player holds the red card, e.g. \( r_1 \land r_2 \). This contradicts \( \neg (r_1 \land r_2) \), which is a conjunct from at most33. Therefore at most one player holds the red card.

Suppose nobody holds the red card, i.e. \( \neg r_1 \land \neg r_2 \land \neg r_3 \). From \( r_1 \lor w_1 \lor b_1 \) and \( \neg r_1 \) follows \( w_1 \lor b_1 \). From \( r_2 \lor w_2 \lor b_2 \) and \( \neg r_2 \) follows \( w_2 \lor b_2 \). From \( r_3 \lor w_3 \lor b_3 \) and \( \neg r_3 \) follows \( w_3 \lor b_3 \).

Either \( w_1 \) or \( \neg w_1 \).

Suppose \( \neg w_1 \). From \( \neg w_1 \) and \( w_1 \lor b_1 \) follows \( b_1 \). From \( b_1 \) and \( \neg (b_1 \land b_2) \) follows \( \neg b_2 \). From \( \neg b_2 \) and \( w_2 \lor b_2 \) follows \( w_2 \). From \( b_1 \) and \( \neg (b_1 \land b_3) \) follows \( \neg b_3 \). From \( \neg b_3 \) and \( w_3 \lor b_3 \) follows \( w_3 \). From \( w_2, w_3 \) and \( \neg (w_2 \land w_3) \) follows a contradiction.

Suppose \( w_1 \). From \( w_1 \) and \( \neg (w_1 \land w_2) \) follows \( \neg w_2 \). From \( \neg w_2 \) and \( w_2 \lor b_2 \) follows \( b_2 \). From \( b_2 \) and \( \neg (b_2 \land b_3) \) follows \( \neg b_3 \). From \( \neg b_3 \) and \( w_3 \lor b_3 \) follows \( w_3 \). From \( w_1, w_3 \) and \( \neg (w_1 \land w_3) \) follows a contradiction.

Therefore the assumption that nobody holds the red card leads to a contradiction. Therefore at least one player holds the red card.

Therefore exactly one player holds the red card: \( r_1 \lor r_2 \lor r_3 \).

at most33, at least33 ⊬ exactly33:

Suppose \( \neg (r_1 \lor w_1 \lor b_1) \). Then either player 1 doesn’t hold any cards, or player 1 holds more than 1 card. Player 1 not holding any cards is a contradiction with at least33. Therefore suppose that he holds more than 1: e.g. that \( r_1 \land w_1 \). From \( r_1 \) and \( \neg (r_1 \land r_2) \) follows \( \neg r_2 \). From \( \neg r_2 \) and \( \neg w_2 \) and \( r_2 \lor w_2 \lor b_2 \) follows \( b_2 \). From \( r_1 \) and \( \neg (r_1 \land r_3) \) follows \( \neg r_3 \). From \( \neg r_3 \) and \( r_3 \lor w_3 \lor b_3 \) follows \( w_3 \). From \( b_2 \) and \( b_3 \) and \( \neg (b_2 \land b_3) \) follows a contradiction. Therefore \( r_1 \lor w_1 \lor b_1 \).

dealings33 ⊬ at most33 ∧ at least33.

That dealings33 ⊬ at least33 is obvious, e.g. \( \delta_{\text{ruck}} \models r_1 \lor r_1 \lor w_1 \lor b_1 \), similarly for all other cases \( \varphi_d \), etc. That dealings33 ⊬ at most33 is also obvious, e.g. \( \varphi_{\text{ruck}} \models r_1 \land \neg r_2 \land \neg r_3 \land \neg (r_1 \land r_2) \). Similarly for all other cases \( \delta_d \), etc.

at most33, at least33 ⊬ dealings33:

Also at least33, at most33 ⊬ dealings33. This can be proven by reasoning from
all different cases from atleast33 that are consistent with atmost33:

It holds that atleast33 \models r_1 \lor w_1 \lor b_1. Suppose r_1. It holds that r_1, atmost33 \models \neg r_2 and r_1, atmost33 \models \neg r_3. Thus r_1 \land \neg r_2 \land \neg r_3.

Also atleast33 \models r_2 \lor w_2 \lor b_2. Assumption r_2 is contradictory. Therefore \neg w_2 \lor b_2. Suppose \neg w_2. Similarly to above, it follows that \neg w_1 \land w_2 \land \neg w_3.

Also atleast33 \models r_3 \lor w_3 \lor b_3. Only b_3 is consistent with the previous. Similarly to above, it follows that \neg b_1 \land \neg b_2 \land b_3.

The conjunction of r_1 \land \neg r_2 \land \neg r_3, \neg w_1 \land w_2 \land \neg w_3, and \neg b_1 \land \neg b_2 \land b_3 is the formula \delta_{s_{uw}}. Similarly for other cases. Thus dealings33.

atmost33, dontknowthen33 \models dontknownot33:

We prove the case \neg K_2 r_1. Either r_2 or \neg r_2. If \neg r_2 then from that and from \neg r_2 \rightarrow \neg K_2 r_1 follows \neg K_2 r_1. If r_2 then from that and from \neg (r_1 \land r_2) follows \neg r_1. If K_2 r_1 held then, because of reflexivity, r_1 would hold. Contradiction with \neg r_1. Therefore \neg K_2 r_1.

33^+ \models dontknownotthat33:

We prove the case \neg r_2 \rightarrow \neg K_2 \neg r_1. Suppose it doesn’t hold. Then both \neg r_2 and \neg K_2 \neg r_1. From dontsee33 and \neg r_2 follows K_2 \neg r_2. From function33 and \neg r_1 and \neg r_2 follows r_3. Therefore, from K_2 \neg r_1 and K_2 \neg r_2 follows K_2 r_3. From dontknowthat33 follows \neg K_2 r_3. Contradiction. Therefore \neg r_2 \rightarrow \neg K_2 \neg r_1.

33^+ \models dontknowthatthat33:

We prove the case \neg K_2 r_1. Assume K_2 r_1. We derive a contradiction. From K_2 r_1 and reflexivity follows r_1. A conjunct from atmost33 is \neg (r_1 \land r_2). From r_1 and \neg (r_1 \land r_2) follows \neg r_2. From \neg r_2 and dontknownot33 follows \neg K_2 \neg r_3. Also, from r_1 and \neg (r_1 \land r_3) follows \neg r_3, and therefore: from K_2 r_1 and atmost33 follows K_2 \neg r_3. Contradiction. Therefore \neg K_2 r_1.

Proofs from section 2.5.1

\sigma^{33} \models 33:

First observe that for any dealing xyz: \sigma_{xyz} \models \delta_{xyz}.

\sigma^{33} \models see33: Obvious.

\sigma^{33} \models dealings33: Obvious.

\sigma^{33} \models dontknowthat33: Suppose not. Then there are agents a and b and a card c such that K_a c_b. E.g. K_1 r_2. Then r_2. Therefore either \delta_{arb} or \delta_{brw}. Therefore either \sigma_{arb} or \sigma_{brw}.

32
Suppose \( \sigma_{\text{verb}} \). Then \( M_1 \delta_{\text{verb}} \). Therefore \( M_1 b_2 \), thus (as \( \sigma^{33} \vdash \text{dealings33} \)) \( M_1 \neg r_2 \), i.e. \( \neg K_1 r_2 \). Contradiction.

Similarly for \( \sigma_{\text{brw}} \). Then \( M_1 \delta_{\text{brw}} \). Therefore \( M_1 w_2 \), thus (as \( \sigma^{33} \vdash \text{dealings33} \)) \( M_1 \neg r_2 \), i.e. \( \neg K_1 r_2 \). Therefore, again a contradiction. 

\[ 33 \vdash \sigma^{33} : \]

Proof by cases from dealings33: Suppose \( \delta_{\text{verb}} \). From \( \delta_{\text{verb}} \) and see33 follows \( K_1 r_1 \); and from that and from dealings33 follows \( K_1 \delta_{\text{verb}}^0 \). Similarly for \( K_2 \delta_{\text{verb}}^0 \) and \( K_3 \delta_{\text{verb}}^0 \).

Now suppose \( \neg M_1 \delta_{\text{brw}} \). Because of \( r_1 \), it is obvious that \( \neg M_1 \delta_{\text{brw}} \), \( \neg M_1 \delta_{\text{brw}} \), \( \neg M_1 \delta_{\text{brw}} \). Therefore \( K_1 (\neg \delta_{\text{brw}} \land \neg \delta_{\text{brw}} \land \neg \delta_{\text{brw}} \land \neg \delta_{\text{brw}}) \). From that and dealings33 follows \( K_1 \delta_{\text{brw}} \), therefore \( K_1 w_2 \). From dontknowthat33 follows \( \neg K_1 w_2 \). Contradiction. Therefore \( M_1 \delta_{\text{brw}} \). Similarly for \( M_2 \delta_{\text{brw}} \) and \( M_3 \delta_{\text{brw}} \). Therefore \( \sigma^{33} \).

Similarly for other cases from dealings33. 

Proof from section 2.6

Proof of proposition 1: Let \( M = (W, \{ \sim_1, \sim_2, \sim_3 \}, V) \) be an \( S \delta_3 \) model of 33, i.e. \( M \models 33 \). Then \( M \) is bisimilar to hexa:

**Proof**  In our proof we use the notation: \( \text{hexa} = \langle W^h, \sim^h, V^h \rangle \), where \( W^h = \{ r\text{ub}, r\text{bw}, b\text{bw}, b\text{ub}, w\text{rb}, w\text{br} \} \), \( \sim_1^h = \{ \{ r\text{ub}, r\text{bw} \}, \{ b\text{bw}, b\text{ub} \}, \{ w\text{rb}, w\text{br} \} \} \), \( \sim_2^h = \{ \{ r\text{ub}, w\text{rb} \}, \{ w\text{br}, b\text{ub} \} \} \), \( \sim_3^h = \{ \{ r\text{ub}, w\text{br} \}, \{ w\text{rb}, b\text{bw} \} \} \), \( V_{ij}^h = V_{ijk} \) such that: \( V_{ijk}(i_1) = V_{ijk}(j_2) = V_{ijk}(k_3) = 1 \) and \( V_{ijk}(p) = 0 \) for all other (six) atomic propositions \( p \).

First an observation on valuations of worlds in \( M \). Because \( M, w \models \text{dealings33} \), and because each one of the six exclusive alternatives in dealings33 correspond to a valuation, any world \( w \in M \) has one of six different valuations \( V_{\text{verb}}, V_{\text{brw}}, V_{\text{brw}}, V_{\text{brw}}, V_{\text{verb}}, V_{\text{verb}} \).

Now define relation \( \mathcal{R} \subseteq (M \times \text{hexa}) \) as follows:

\[ \forall w \in M : \forall w^h \in \text{hexa} : \mathcal{R}(w, w^h) \Leftrightarrow V_w = V_{w^h}^h. \]

We prove that \( \mathcal{R} \) is a bisimulation between \( M \) and hexa.

**Forth:**

Let \( w, w' \in M \), let \( w^h \in \text{hexa} \). Suppose \( w \sim_1 w' \) and \( \mathcal{R}(w, w^h) \). We find an \( \mathcal{R} \)-image of \( w' \) for every valuation \( V_w \) on \( w \). First suppose \( V_w = V_{\text{verb}} \). From \( \mathcal{R}(w, w^h) \) follows \( V_{w^h}^h = V_w = V_{w^h} \). Therefore \( w^h = \text{verb} \).

As \( M \) is a model of 33, \( M \models \text{see33} \). From \( M, w \models \text{see33} \), follows \( M, w \models r_1 \to K_1 r_1 \). From \( V_w(r_1) = V_{\text{verb}}(r_1) = 1 \) and \( M, w \models r_1 \to K_1 r_1 \), follows \( M, w \models K_1 r_1 \). From \( M, w \models K_1 r_1 \), and \( w \sim_1 w' \) follows \( M, w' \models K_1 r_1 \). Therefore \( V_{w'} = V_{\text{verb}} \) or \( V_{w'} = V_{\text{verb}} \). If \( V_{w'} = V_{\text{verb}} \), choose \( \text{verb} \) as the \( \mathcal{R} \)-image of \( w' \) in
hexa: obviously \( rw \sim_r \) \( rwb \) and also \( \mathcal{R}(w', rwb) \). If \( V_{w'} = V_{rwb} \), choose \( rwb \) as the \( \mathcal{R} \)-image of \( w' \) in hexa: we now have \( rw \sim_r \) \( rwb \) and \( \mathcal{R}(w', rwb) \).

Similarly for the five other valuations \( V_w \) on \( w \). Similarly for \( i = 2 \) and \( i = 3 \).

Back:

Let \( w^h, w^h_s \in \text{hexa} \), let \( w \in M \). Suppose \( w^h \sim_1^h \) \( w^h_s \) and \( \mathcal{R}(w, w^h) \). We find an \( \mathcal{R} \)-original of \( w^h_s \) for every valuation \( V_{w^h} \) on \( w^h \). First suppose \( V_{w^h} = V_{rwb} \).

Obviously \( w^h = rwb \).

From \( rwb \sim_r \) \( w^h \) follows \( w^h = rwb \) or \( w^h = rwb \). If \( w^h = rwb \) choose \( w \) itself as the required \( \mathcal{R} \)-original of \( w^h_s \). As \( M \) is an \( S5 \) model, \( w \sim_1 \) \( w \), and we already assumed \( \mathcal{R}(w, rwb) \).

Otherwise \( w^h = rwb \). We derive a contradiction from the assumption that there is no \( w' \in M \) such that \( w \sim_1 \) \( w' \) and \( V_{w'} = V_{rwb} \).

Suppose so. In other words: for all \( w' \in M : w \sim_1 \) \( w' \) \( \Rightarrow \) \( V_{w'} = V_{rwb} \).

Suppose \( w \sim_1 \) \( w' \). As before, from see33 follows \( M, w' \models x1 \) r1 and from that and \( w \sim_1 \) \( w' \) follows \( M, w' \models x1 \) and therefore \( V_{w'} = V_{rwb} \) or \( V_{w'} = V_{rwb} \). From that and the assumption follows \( V_{w'} = V_{rwb} \), thus \( M, w' \models x2 \), and thus, as \( w' \) is an arbitrary \( 1 \)-accessible world from \( w' \in M \), \( M, w' \models x1 \) \( w2 \). However, also \( M \models \text{downtonwthat33} \), thus \( M, w \models \neg \text{K}_1 \) \( w2 \). Contradiction.

Therefore there is a \( w' \in M \) such that \( w \sim_1 \) \( w' \) and \( V_{w'} = V_{rwb} \). By definition we have \( \mathcal{R}(w', rwb) \). So we have found the required \( \mathcal{R} \)-original of \( rwb \).

Similarly for the five other valuations \( V_{w^h} \) on \( w^h \). Similarly for \( i = 2 \) and \( i = 3 \).

\[ \square \]

**Proof from section 5.1**

Proof of proposition 4: \( \text{lgames}_n \leftrightarrow \text{dealings} + E^g(I_d) \). It suffices to prove that in \( S5_n \) plus dealings, seedontknow is equivalent to \( E^g(I_d) \):

\[ \begin{align*}
\Lambda_d \in I_d & \land \alpha \in A (\delta_d \rightarrow (\Lambda_{d', \sim_\alpha} M_\alpha \delta_{d'})) \land K_\alpha \delta_d \\
\iff & \land_{d \in I_d} \land_{\alpha \in A} (K_\alpha \delta_d \leftrightarrow M_\alpha \delta_d)
\end{align*} \]

**Proof:** As usual we assume a somewhat informal, natural-deduction like, proof style.

(i) \( \Rightarrow \) (ii)

Let \( a \in A, d \in D_d \). First, we prove that \( K_\alpha \delta_d \rightarrow M_\alpha \delta_d \):

\[ \begin{align*}
K_\alpha \delta_d & \Rightarrow \\
\delta_d & \Rightarrow \\
\bigvee_{d' \sim_\alpha} \delta_{d'}
\end{align*} \]

Let \( d' \sim_\alpha d \) be an arbitrary dealing such that \( \delta_{d'} \). Then: 34
\[ \delta_d' \]
\[ \Rightarrow \]
\[ \land_{d'' \sim_a d'} M_a \delta_{d''} \]
\[ \Rightarrow \]
\[ M_a \delta_d \]

Therefore \( K_a \delta^a_d \Rightarrow M_a \delta_d \).

Next, we prove that \( M_a \delta_d \Rightarrow K_a \delta^a_d \), by contraposition. Start by observing that:

\[ \neg K_a \delta^a_d \]
\[ \Leftrightarrow \]
\[ \neg K_a (\lor_{d' \sim_a d} \delta_{d'}) \]
\[ \Leftrightarrow \]
\[ M_a (\lor_{d' \sim_a d} \neg \delta_{d'}) \]

Suppose \( M_a \delta_d \). Either \( \delta_d \) or \( \neg \delta_d \). If \( \delta_d \), then apply (i) and \( K_a \delta^a_d \) follows. If \( \neg \delta_d \), then from dealings it follows that \( \delta_{d''} \) for some \( d'' \neq d \). Again with (i), follows \( K_a \delta^a_{d''} \). We can have either \( d'' \sim_a d \) or \( d'' \not\sim_a d \). If \( d'' \sim_a d \) then \( \delta^a_{d''} \Rightarrow \delta^a_d \) and from that and \( K_a \delta^a_{d''} \) follows \( K_a \delta^a_d \). If \( d'' \not\sim_a d \) we derive a contradiction:

\[ K_a \delta^a_{d''} \]
\[ \Leftrightarrow \]
\[ K_a \lor_{d' \sim_a d''} \delta_{d''} \]
\[ \Leftrightarrow \]
\[ K_a \lor_{d' \not\sim_a d''} \neg \delta_{d''} \]
\[ \Leftrightarrow \]
\[ K_a \neg \delta_d \]
\[ \Leftrightarrow \]
\[ \neg M_a \delta_d \]

(ii) \( \Rightarrow \) (i)

Suppose \( \delta_d \). Then \( M_a \delta_d \). From that and (ii) follows \( K_a \delta^a_d \). From that, and because for all \( d' \sim_a d \) \( \delta^a_d \Leftrightarrow \delta^a_{d'} \), follows that for all \( d' \sim_a d \) \( K_a \delta^a_{d'} \). Using (ii) for all \( d' \sim_a d \), we get: for all \( d' \sim_a d \) \( M_a \delta_{d'} \). Thus \( \lor_{d' \sim_a d} M_a \delta_{d'} \).

References


