A continuum of incomplete intermediate logics

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Abstract

Although in 1977 V.B. Shehtman constructed the first Kripke incomplete intermediate logic, no-one in known literature has completed his work by constructing a continuum of such logics. After a substantial reminder on how an incomplete logic can be obtained, I will construct a sequence of frames similar to those used by Jankov and Fine. None of these frames can be reduced by p-morphism to another; simultaneously, there are no p-morphisms from generated subframes of Fine-Shehtman frame onto any frame from the considered sequence. All of the frames satisfy all Shehtman's axioms. Therefore, by using characteristic formulae of the frames from the sequence it is possible to obtain the desired conclusion.

In 1970s, a number of important, deep and technically complicated results concerning relational semantics for modal logics was obtained by such authors as S. Thomason, K. Fine, M.S. Gerson, R.I. Goldblatt, J. F. A. K. van Benthem and W. Blok; it was the Golden Age of the subject, see [Bu82], [Bu83] and [ChZ97] for references and summaries of the most important works. The main goal of my paper is to attract attention to the fact that many important results lack superintuitionistic analogues, although the task of transferring them is highly nontrivial. This gap may be partially due to the fact that Kripke semantics never became so popular in the realm of intermediate logics as they are in the realm of modal logics, which are more suitable and flexible tools to deal with frames. There were fewer experts working on relational semantics for intuitionistic logics. One of the most distinguished persons in the field, V. B. Shehtman constructed in 1977 the first Kripke incomplete intermediate propositional logic. His construction was based mainly on the frame from [Fi74b], but he has very ingeniously used the formula introduced in [GdJ74]. Nevertheless, he did not follow

Fine's suggestion that it seems to be possible to construct a continuum of incomplete logics. Such a continuum of $S4$ logics was presented in [Ry77] in the same year as Shehtman's construction; it is known, however, that incompleteness of a modal logic does not imply incompleteness of its intuitionistic equivalent. In [On72] one may find information that there exists a continuum of incomplete predicate superintuitionistic logics. Unfortunately, the information is given without a proof; besides, it is far easier to construct an incomplete predicate superintuitionistic logic than to construct an incomplete propositional superintuitionistic logic. It is really surprising but up to this day no-one has published a proof that there exists a continuum of such logics. I am going to fill this gap.

In this paper I shall try to conform to the standard definitions and symbols which may be found, for example, in a monograph by Chagrov & Zakharyaschev [ChZ97]. Nevertheless, for the sake of convenience, let me remind the most standard ones. Unless otherwise stated, by a logic I shall mean a superintuitionistic (intermediate) logic.

Definition 1 A (Kripke) structure/frame *consists of a set and a relation of partial order* $\mathcal{F} = \langle W, \leqslant \rangle$.

Definition 2 A substructure/subframe *of a structure* $\mathcal{F} = \langle W, \leqslant \rangle$ *is a frame* $\mathcal{G} = \langle V, \leqslant_1 \rangle$ *where* $V \subseteq W$ *and* $\leqslant_1 = V^2 \cap \leqslant$.

Definition 3 A (Kripke) model *is an ordered pair* $\mathcal{M} = \langle \mathcal{F}, \mathcal{B} \rangle$ *consisting of a frame* $\mathcal{F} = \langle W, \leqslant \rangle$ *and a function* $\mathcal{B}$ *from a set of propositional variables to the set of upward closed subsets of* W. *Valuation is extended to all formulae in the usual way.*

Now I want to introduce two technical notions, weaker than *finite approximability* (*finite model property*) and stronger than *completeness*

Definition 4 *A logic is* fa-approximable *iff the set of its theorems coincides with the set of all formulae true in some class of rooted frames with no infinite antichains.*

Definition 5 *A logic is* ac-approximable *iff the set of its theorems coincides with the set of all formulae true in some class of frames with no infinite ascending chains — Chagrov & Zakharayaschev call such orders* Noetherian.

Professor A. Wroński suggested to me that fa-approximability implies ac-approximability. It would give rise to a following picture:

finite approximability $\Rightarrow$ fa-approximability $\Rightarrow$ ac-approximability $\Rightarrow$ completeness.

In my paper, I shall prove that there exists a continuum of propositional logics even outside the broadest class, i.e. class of all complete logics. Nevertheless, first let me describe how an incomplete logic can be obtained — it is an easy generalization of Shehtman's method [Sh77].

Theorem 1 *A logic L lacks ac-approximability iff its modal companion above* **Grz** *τL is incomplete.*

Proof It is enough to recall that **Grz** is complete with respect to all partial orders without infinite ascending chains. ⊣

Theorem 2 *If there exists any rule of the form*

$$\frac{(\psi \lor (\psi \to e(\chi))) \to \chi}{\chi}$$

(e is any uniform substitution) which is not admissible in some intermediate logic, then this logic lacks ac-approximability and thus lacks finite model property.

Proof (sketch) In any family of frames adequate for the logic (if there exists such) there must be a frame validating

$$(\psi \lor (\psi \to e(\chi))) \to \chi$$

with all substitutions (because the formula belongs to the logic) and refuting χ under some valuation. It can be easily seen that such a frame must contain an infinite ascending chain — see figure 1. ⊣

Corollary 3 *If an intermediate logic satisfies the assumptions of theorem 2, then its companion above* **Grz** *is incomplete.*

Proof A consequence of theorems 1 and 2. ⊣

In fact far more can be proven about such a logic — see my forthcoming paper [Li02].

Theorem 4 *If there exists any rule of the form*

$$\frac{\begin{array}{c}(\psi \lor (\psi \to e(\chi))) \to \chi \\ \psi \leftrightarrow \varsigma \to \tau \\ \tau \to e(\tau)\end{array}}{\chi}$$

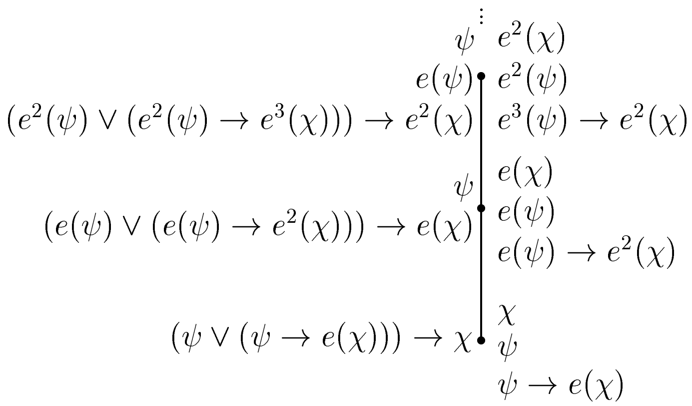

Figure 1: A model refuting χ but verifying $\psi \vee (\psi \to e(\chi)) \to \chi$

which is not admissible in a logic L, then in any class of frames adequate for L (if there exists any) there must be a structure containing an infinite comb *or* a willow *(see fig. 2) as a substructure; thus L must lack both ac-approximability and fa-approximability.*

Proof Similar to the proof of theorem 2 — see fig. 4. $\dashv$

Let me recall the celebrated Gabbay-de Jongh axioms [GdJ74]

$$\boldsymbol{bb_n} = \bigwedge_{i=0}^{n} ((p_i \to \bigvee_{j \neq i} p_j) \to \bigvee_{j \neq i} p_j) \to \bigvee_{i=0}^{n} p_i \qquad (n \geq 1)$$

which are complete with respect to the class of all finite frames of branching n. It is well known that they can be refuted in the infinite comb. Nevertheless, not every frame containing the infinite comb as a substructure refutes these axioms — see figure 3. Therefore the following theorem is nontrivial:

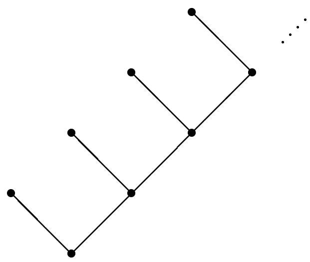

Figure 2: Infinite comb

Theorem 5 *If there exists any rule of the form*

$$\frac{\begin{array}{c}(\psi \lor (\psi \to e(\chi))) \to \chi \\ \psi \leftrightarrow \varsigma \to \tau \\ \varsigma \lor \tau \to e(\varsigma) \land e(\tau) \\ \chi \leftrightarrow \psi \lor e(\tau)\end{array}}{\chi} \tag{1}$$

which is not admissible in some intermediate logic L, then in any class of frames adequate for L (if there exists any) there must exist a structure refuting $\boldsymbol{bb_n}(n \geq 2)$. *Thus if L contains any of Gabbay-de Jongh axioms, it must be incomplete.*

Proof may be carried out in a manner similar to the one of Shehtman [Sh77], but it is needlessly complicated, e.g. with a superfluous use of transfinite induction. Therefore I would like to sketch more elegant and intuitive proof. Assume then that there is a frame $\mathcal{F}$ for L, a valuation $\mathcal{V}$ and a point x in $\mathcal{F}$ such that $x \not\models_{\mathcal{V}} \chi$. It is easy to check that x must be the root of submodel of $\langle \mathcal{F}, \mathcal{V} \rangle$ depicted on a picture 4. Now let me define a new valuation $\mathcal{B}$ based on $\mathcal{V}$ and inspired by the figure 4:

$$\mathcal{B}(p_0) := \bigcup_{n=3m} \mathcal{V}(e^n(\varsigma) \land \bigwedge_{k=0}^{n-1} e^k(\chi)) \cup \bigcap_{n \in \omega} \mathcal{V}(e^n(\psi)),$$

$$\mathcal{B}(p_1) := \bigcup_{n=3m+1} \mathcal{V}(e^n(\varsigma) \land \bigwedge_{k=0}^{n-1} e^k(\chi)) \cup \bigcap_{n \in \omega} \mathcal{V}(e^n(\psi)),$$

$$\mathcal{B}(p_2) := \bigcup_{n=3m+2} \mathcal{V}(e^n(\varsigma) \land \bigwedge_{k=0}^{n-1} e^k(\chi)) \cup \bigcap_{n \in \omega} \mathcal{V}(e^n(\psi)).$$

Axioms of L and the figure 4 assure us that sets $\mathcal{B}(p_0)$, $\mathcal{B}(p_1)$ and $\mathcal{B}(p_2)$ are distinct and nonempty. It is easily seen that the consequent of $\boldsymbol{bb_2}$ is

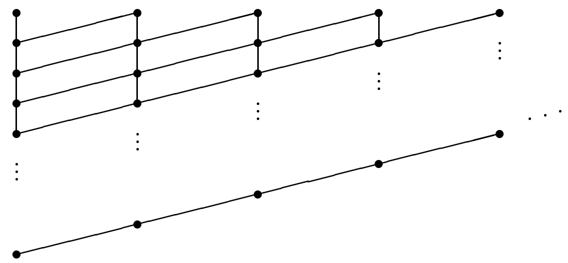

Figure 3: A structure containing the infinite comb as a substructure where Gabbay-de Jongh axiom $\boldsymbol{bb_2}$ is true

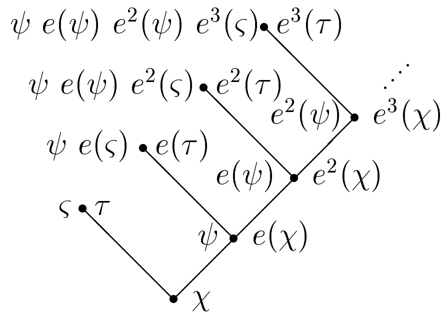

Figure 4: A submodel of $\langle \mathcal{F}, \mathcal{V} \rangle$ whose root refutes χ.

refuted at x under the valuation $\mathcal{B}$. Now suppose that there is some $y \geqslant x$ such that some conjunct of the premise of $\boldsymbol{bb_2}$ is *classically* refuted at y, e.g. $y \vDash_{\mathcal{B}} p_0 \to p_1 \vee p_2$ and $y \nvDash_{\mathcal{B}} p_1 \vee p_2$. If there is some $n \in \omega$ such that $y \vDash_{\mathcal{V}} e^n(\varsigma)$ then by the axioms of L $y \vDash_{\mathcal{V}} e^m(\chi)$ for any $m \leq n$ and $y \vDash_{\mathcal{B}} p_0 \vee p_1 \vee p_2$, which leads to an immediate contradiction. Hence for no $n \in \omega$, $y \vDash_{\mathcal{V}} e^n(\varsigma \vee \tau)$. But now there must be some $n \in \omega$ such that $y \nvDash_{\mathcal{V}} e^n(\psi)$ (otherwise $y \in \bigcap\limits_{n \in \omega} \mathcal{V}(e^n(\psi))$). Hence $y \nvDash_{\mathcal{V}} e^n(\chi)$ and it is possible to construct an infinite comb similar to the one from the figure 4 whose root is y. But it is easy to find some point from this comb which belongs to $\mathcal{B}(p_0)$ and does not belong to $\mathcal{B}(p_1 \vee p_2)$. ⊣

It may be worth mentioning that the rule 1 is as a matter of fact inspired by the form of axioms in later Shehtman's paper [Sh80]. In his paper from 1977 [Sh77] the axioms were more complicated and to make Shehtman's 1977 theorem a consequence of theorem 5 — as I am going to do — the rule 1 should be replaced by the following one:

$$\frac{\begin{array}{c} (\psi \vee (\psi \to e(\chi))) \to \chi \\ \psi \leftrightarrow \varsigma \to \tau \\ \tau \to e(\tau) \\ \chi \leftrightarrow \psi \vee e(\psi) \\ e(\psi) \to \psi \vee e(\tau) \end{array}}{\chi} \tag{2}$$

Now let me consider a family of formulae introduced by Shehtman:

$$\beta_{-1} = p, \qquad \gamma_{-1} = q, \qquad \beta_0 = q \to p, \qquad \gamma_0 = p \to q,$$

$$\beta_{n+1} = \gamma_n \to \beta_n \vee \gamma_{n-1}, \qquad \gamma_{n+1} = \beta_n \to \gamma_n \vee \beta_{n-1},$$

$$\alpha_n = \beta_{n+2} \wedge \gamma_{n+2} \to \beta_{n+1} \vee \gamma_{n+1} \qquad (n \in \omega),$$

$$\eta = \alpha_0 \to \alpha_1 \vee \alpha_2, \qquad \epsilon = \alpha_0 \vee \alpha_1,$$
$$\delta = \eta \to \epsilon, \qquad \kappa = \alpha_1 \to \alpha_0 \vee \beta_2$$

If ς stands for $\beta_2 \wedge \gamma_2$, τ stands for $\beta_1 \vee \gamma_1$ and e is defined as follows:

$$e(p) := q \vee (q \to p), \qquad e(q) := p \vee (p \to q),$$

then the following observation allows me to use the variant of theorem 5 concerning the rule 2

α_0	is of the form ψ, i.e.	$\varsigma \to \tau$,
ϵ	is of the form χ, i.e.	$\psi \vee e(\psi)$,
δ	is equivalent to	$(\psi \vee (\psi \to e(\chi))) \to \chi$,
κ	intuitionistically implies	$e(\psi) \to \psi \vee e(\tau)$,
$\tau \to e(\tau)$	is an **Int**-tautology.	

Of course, it would be also possible to use the theorem 5 without any modification. In this case one should define ϵ as $\alpha_0 \vee \beta_2 \vee \gamma_2$ or even $\alpha_0 \vee \beta_2$, δ as $(\alpha_0 \to \alpha_1 \vee \beta_3) \to \alpha_0 \vee \beta_2$ and no κ is needed at all. Nevertheless, I am going to stick to the first paper of Shehtman to make references easier; the paper from 1980 [Sh80] is less known.

Lemma 6 *Axioms δ and κ are true in a structure known as the* Fine frame *(see figure 5). The axiom $\boldsymbol{bb_n}$ is true in a general frame based on the Fine frame and generated by the two upward closed singletons. The same general frame refutes the axiom ϵ.*

Proof is quite easy and may be found, for example, in [Sh77]. $\dashv$

Corollary 7 (Shehtman) *An intermediate logic L determined by the axioms δ, κ, and $\boldsymbol{bb_2}$ is incomplete.*

Proof A consequence or theorem 5 and lemma 6. $\dashv$

Now I may construct a continuum of incomplete logics inspired by the ideas from classical Kit Fine's papers [Fi74a], [Fi74b]. I will construct a sequence of frames $\mathcal{F}_n$ (see fig. 6) very similar to the sequence from [Fi74a].

Lemma 8 *For any $n \in \omega$, $\mathcal{F}_n \models \delta \wedge \kappa \wedge \boldsymbol{bb_2}$. Besides, $\mathcal{F}_n \models \epsilon$.*

Proof The fact that Gabbay-de Jongh axioms are true in all of those frames is obvious. It is impossible to refute simultaneously α_0 and α_1 in any of the frames which implies that $\mathcal{F}_n \models \delta \wedge \epsilon$. Validity of κ is checked similarly as in case of the Fine frame. $\dashv$

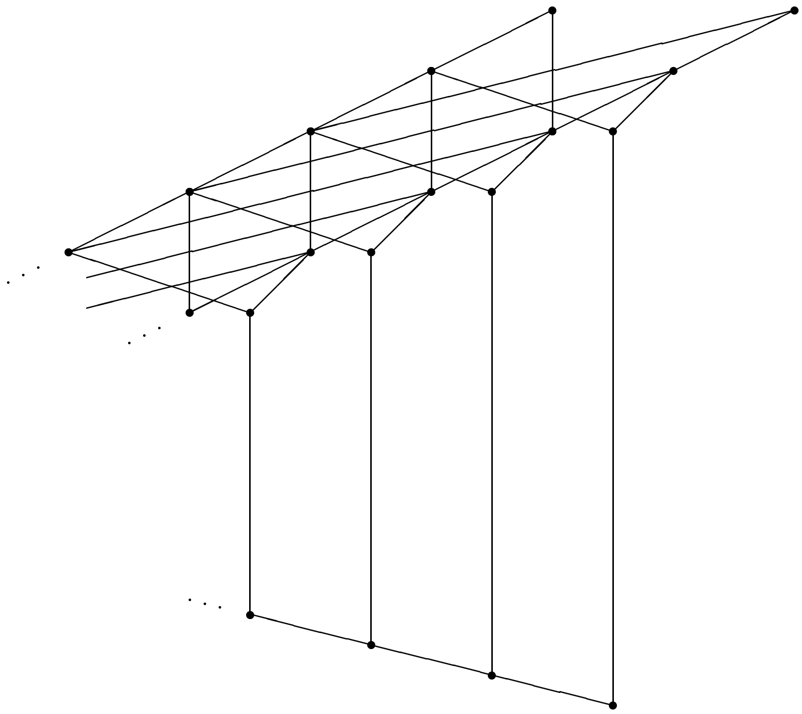

Figure 5: The Fine frame

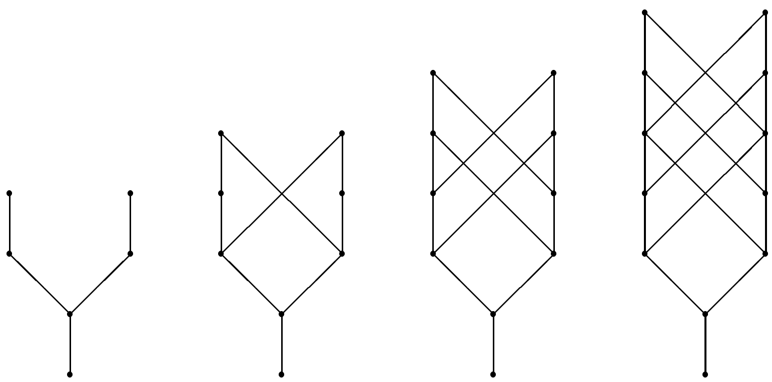

Figure 6: Frames $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$

Lemma 9 *For any $n \in \omega$, there exists no p-morphism from any generated subframe of $\mathcal{F}_n$ onto $\mathcal{F}_m$ ($m \neq n$). In other words,*

$$\mathcal{F}_n \models \beta^{\#}(\mathcal{F}_m, \bot)(m \neq n),$$

where $\beta^{\#}(\mathcal{F}_m, \bot)$ is a Jankov formula for $\mathcal{F}_m$.

Proof is similar to the one in [Fi74a] (by induction). ⊣

Lemma 10 *For any $n \in \omega$, there exists no p-morphism from any generated subframe of the Fine frame onto $\mathcal{F}_n$. In other words, Jankov formulae for all the sequence are satisfied in the Fine frame.*

Proof as above. ⊣

Theorem 11 *Distinct subsets of natural numbers generate distinct intermediate logics whose axioms are δ, κ, $\boldsymbol{bb_2}$ and Jankov formulae of those frames from the sequence whose indices belong to a given subset of ω. All of those logics are incomplete.*

Proof The fact that those logic are all distinct is a consequence of lemmata 8 and 9. The fact that these logics are incomplete follows from the theorems 5 and lemmata 6, 10 — a suitable inference rule is not admissible in any of the logics. ⊣

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References

[Bu82] R.A. Bull. Review. *Journal of Symbolic Logic*, 47:440-445, 1982

[Bu83] R.A. Bull. Review. *Journal of Symbolic Logic*, 48:488-495, 1983

[ChZ97] A.V. Chagrov and M.V. Zakharyaschev. *Modal Logic.* Clarendon Press, Oxford 1997.

[Fi74a] K. Fine. An Ascending Chain of $S4$ Logics.*Theoria*, 40:110-116, 1974.

[Fi74b] K. Fine. An Incomplete Logic Containing $S4$.*Theoria*, 40:23-29, 1974.

[GdJ74] D.M. Gabbay and D.H.J. de Jongh. A Sequence of Decidable Finitely Axiomatizable Intermediate Logics with the Disjunction Property. *Journal of Symbolic Logic*, 39:67-78, 1974.

[Ja68] V.A. Jankov. Constructing a Sequence of Strongly Independent Superintuitionistic Propositional Calculi. *Soviet Mathematics Doklady*, 9:806-807, 1968.

[Li02] T. Litak. Incompleteness revisited. Forthcoming.

[On72] H. Ono. A Study of Intermediate Predicate Logics. *Publ. RIMS, Kyoto University*, 8:619-649, 1972/73.

[Ry77] V.V. Rybakov. Noncompact Extensions of the Logic $S4$. *Algebra and logic*, 16:321-334, 1977.

[Sh77] V.B. Shehtman. On Incomplete Propositional Logics. *Soviet Mathematics Doklady*, 18:985-989, 1977.

[Sh80] V.B. Shehtman. Topological models of propositional logics. *Semiotics and Information Science*, 15:74–98, 1980. (Russian)