Brief note: completeness and decidability of the logic of relativized end-extension set algebras

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Abstract

The most popular concrete algebras for predicate logic are (generated subalgebras) of cylindric set algebras. This research note is part of a larger project, that looks at an alternative class of concrete algebras, which use end-extensions instead of cylindrifications. An earlier manuscript gave reason to think that the generated subalgebras of the usual, full case would be essentially isomorphic. This research note shows some similarities between the two classes of algebras in the case of relativized algebras: there is a class of relativized end-extension set algebras, which has some of the key properties of relativized cylindric set algebras: models of the logic contain essentially arbitrary sets of assignments, the key difference between the axiomatization of the full and relativized case is the restriction of vacuous generalization to atomic formulas, and membership in the logic is decidable.

Introduction

The most popular concrete algebras for predicate logic are (generated subalgebras) of cylindric set algebras. Nevertheless, there have been repeated attempts in the literature to explore alternative classes of algebras for predicte logic such as the algebras of sets of finite sequences of Craig (1974), or the many-sorted cylindric algebras of Bernays (1959). The present note is part of this tradition, exploring the case of a class of algebras closely related to but different from cylindric set algebras, which are called end-extension set algebras. End-extension algebras are different from cylindric set algebras in that sets of ω sequences are related to one another by the predecessor operation instead of cylindrification. The corresponding propositional modal logic was axiomatized in Ben Shalom (1995) and a one-to-one translation was suggested between its formulas and predicate logic formulas (up to alphabetical variants).

A full end-extension set algebra is defined by:

$$\langle \mathcal{B}({}^{\omega}U), \mathrm{c}, \mathrm{Id}_i \rangle_{i-1<\omega}$$

with

$$\mathrm{c} =_{def} \mathrm{pred}$$

i.e., for every $x \in {}^{\omega}U$, $\mathrm{pred}x =_{def} \{\mathrm{pred}\alpha \mid \alpha \in x\}$, where $(\mathrm{pred}\alpha)_i =_{def} \alpha_{i+1}$, for all $\alpha \in {}^{\omega}U, i < \omega$.

$$\mathrm{Id}_i =_{def} \{\alpha \in {}^{\omega}U \mid \alpha_0 = \alpha_i\}$$

Following a convention in the cylindric algebra literature, we will use the following abbreviation:

$$\mathrm{s}_i x =_{\mathrm{def}} \mathrm{c}(\mathrm{Id}_i \wedge x)$$

Generated subalgebras:

Generators of algebras for a first-order language with a set $\mathcal{R}$ of relation symbols and $\rho{:}\mathcal{R} \to \omega$ a rank-function on $\mathcal{R}$ are of the form:

$$\{\mathrm{R} \times {}^{\omega}U \mid R \in \mathcal{R}, \mathrm{R} \in {}^{\rho(R)}U\}$$

Axiomatization

Axioms beyond the minimal modal logic K

$$[+1]: \quad t = \mathrm{c}[+1]t$$

$$U: \quad \mathrm{c}(\mathrm{Id}_i \wedge x) = -\mathrm{c}(\mathrm{Id}_i \wedge -x)$$

$$\mathrm{Id}: \quad \mathrm{Id}_i = \mathrm{c}(\mathrm{Id}_1 \wedge \mathrm{Id}_{i+1})$$

$$\rho: \quad x_j = \underbrace{\mathrm{c}(\mathrm{Id}_j \wedge \ldots \mathrm{c}(\mathrm{Id}_j \wedge}_{j \text{ times}} x_j \underbrace{)\ldots)}_{j \text{ times}}$$

Relativized set algebras:

A relativized end-extension set algebra is defined by:

$$\langle \mathcal{B}(V), \mathrm{c}, \mathrm{Id}_i \rangle_{i-1<\omega}$$

$$V \subseteq \text{ unraveling of } \langle {}^{\omega}U, \mathrm{c} \rangle$$

$$V \text{ closed under } \mathrm{s}_i, i-1<\omega$$

$$\mathrm{c}, \{\mathrm{Id}_i\}_{i-1<\omega} \text{ restricted to } V$$

with unraveling defined by the following:

Definition 1 *Let s be a point in the domain of a model $\mathcal{M} = \langle S, R \rangle$. $\mathcal{M}_s = \langle S_s, R_s \rangle$, the* unraveling of $\mathcal{M}$ around s *is defined inductively by:*

$$\begin{array}{lcl} S_0 & = & \{(s,0)\} \\ S_{n+1} & = & S_n \cup \{(as', n+1) \mid (a,n) \in S_n,\ a_0 R s'\} \\ S_s & = & \bigcup_{i\in\omega} S_i \\ (a,l) R_s (a',l') & \text{iff} & a' = as' \text{ for some } s' \in S \end{array}$$

Generated subalgebras:

Generators of a relativized algebra for a first-order language with a set $\mathcal{R}$ of relation symbols and $\rho{:}\mathcal{R} \to \omega$ a rank-function on $\mathcal{R}$ are of the form:

$$\{((\mathrm{R} \times {}^{\omega}U) \times \omega) \ \cap\ V \mid R \in \mathcal{R}, \mathrm{R} \in {}^{\rho(R)}U\}$$

Axiomatization

$[+1]'$: $t = \mathrm{c}[+1]t$ $\quad$ t atomic or negated atomic

U : $\mathrm{c}(\mathrm{Id}_i \wedge x) = -\mathrm{c}(\mathrm{Id}_i \wedge -x)$

Id : $\mathrm{Id}_i = \mathrm{c}(\mathrm{Id}_1 \wedge \mathrm{Id}_{i+1})$

ρ : $x_j = \underbrace{\mathrm{c}(\mathrm{Id}_j \wedge \ldots \mathrm{c}(\mathrm{Id}_j \wedge}_{j\text{ times}} x_j \underbrace{) \ldots)}_{j\text{ times}}$

Theorem 1 $\Pi' =_{def} K[+1]'U\mathrm{Id}\rho$ *is an axiomatization of the logic of relativized end-extension set algebras.*

proof:

Appendix A.

Theorem 2 *The logic of relativized end-extension set algebras is decidable.*

proof:

Appendix B.

Thus, relativized end-extension set algebras have some of the key properties of relativized cylindric set algebras, as summarized in van Benthem (1996): models of the logic contain essentially arbitrary sets of assignments, the key difference between the axiomatization of the full and relativized case is the restriction of vacuous generalization to atomic formulas, and membership in the logic is decidable.

A Completeness

Let γ^0 be a set of $\mathcal{L}$ formulas that is Π'-consistent.

We construct a tree model $\mathcal{M}^t = (S^t, R^t, V^t)$ and a point $s^t \in S$ such that

$$\mathcal{M}^t \models_{s^t} \gamma \text{ for all } \phi \in \gamma^0$$

Step 1: Since Q is countable, γ^0 can be extended to a maximally Π'-consistent set γ' in the usual way. Enumerate $\mathcal{L}$. Define

$$\begin{aligned}
\gamma_0 &= \gamma^0 \\
\gamma_{n+1} &= \begin{cases} \gamma_n \cup \{\phi_n\} & \text{if } \gamma_n \vdash_{\Pi'} \phi_n \\ \gamma_n \cup \{\neg\phi_n\} & \text{otherwise} \end{cases} \\
\gamma' &= \bigcup_{n \geq 0} \gamma_n
\end{aligned}$$

Step 2: Let $\mathcal{M}^{\Pi'}_{\Gamma'} = \langle S^{\Pi'}_{\gamma'}, R^{\Pi'}_{\gamma'}, V^{\Pi'}_{\gamma'} \rangle$ be the unraveling of the canonical model of Π' around γ', and for every (Σ, i) in $\mathcal{M}^{\Pi'}_{\gamma'}$ let $a_{\Sigma,i}$ be a new object.

Lemma 1 *For every $\delta, \delta' \in Id$,*

$$\vdash_{\Pi'} \Box(\delta \rightarrow \delta') \leftrightarrow \Box(\delta \leftrightarrow \delta')$$ [1]

$$\vdash_{\Pi'} \Diamond(\delta \wedge \delta') \leftrightarrow \Box(\delta \leftrightarrow \delta')$$

[1] All proofs in the appendices are in the corrseponding propositional modal logic, and $\Box$ is the universal modal operator.

Proof:

$$\begin{array}{lll}
\Box(\delta \rightarrow \delta') & & \vdash_{\Pi'} \\
\Diamond(\delta \wedge \delta') & U & \vdash_{\Pi'} \\
\Diamond(\delta' \wedge \delta) & & \vdash_{\Pi'} \\
\Box(\delta' \rightarrow \delta) & U &
\end{array}$$

□

Define the following binary relation on Id

$$\delta \sim \delta' \text{ iff } \Box(\delta \leftrightarrow \delta') \in \gamma'$$

$\sim$ is an equivalence relation on Id.

Let

$$A_{\gamma'} =_{def} Id/{\sim} \cup \{a_{\Sigma,i} \mid (\Sigma, i) \in \mathcal{M}_{\gamma'}^{\Pi'}\}$$

Step 3: Define a function f from $\mathcal{M}_{\gamma'}^{\Pi'}$ into pairs of the form (s, l), where s is a sequences of length ω over $A_{\gamma'}$, $l \in \omega$, by induction on i:

We write $f(\Sigma, l)$ for $f((\Sigma, l))$.

$$\begin{array}{rcl}
f_0(\gamma', 0) & = & (\ldots[\delta_k]\ldots[\delta_2][\delta_1], 0) \\
f_{n+1}(\Sigma\gamma, n+1) & = & \begin{cases} (ss_m, n+1) & \text{if } f_n(\Sigma, n) = (s, n),\ \delta_{m+1} \text{ is the minimal } \delta \in \gamma \\ (sa_{\Sigma\gamma, n+1}, n+1) & \text{if } f_n(\Sigma, n) = (s, n),\ \text{no such } \delta \text{ exists} \end{cases} \\
f & = & \bigcup_{i\in\omega} f_i
\end{array}$$

Definition 2 *$\mathcal{M}^t = \langle S^t, R^t, V^t \rangle$, is defined by:*

$$\begin{array}{lcl} S^t & = & \{f(\Sigma, l) \mid (\Sigma, l) \in \mathcal{M}_{\gamma'}^{\Pi'}\} \\ (s,l)R^t(s',l') & \text{iff} & s' = sa, a \in A_{\gamma'}, l' = l+1 \\ f(\Sigma, l) \in V^t(q) & \text{iff} & (\Sigma, l) \in V_{\gamma'}^{\Pi'}(q) \end{array}$$

Step 4: f is a bisimulation between the domains of $\mathcal{M}_{\Gamma'}^{\Pi'}$ and $\mathcal{M}^t$.

Lemma 2 *Let $(s,l) \in \mathcal{M}^t, (\Sigma, l) \in \mathcal{M}_{\gamma'}^{\Pi'}$ such that $(s,l) = f(\Sigma, l)$. For all $i, k \in \omega$,*

$$s_i = s_k \text{ iff } (\Sigma, l) \vdash_{\Pi'} \Box(\delta_{i+1} \leftrightarrow \delta_{k+1})$$

Proof:

By induction on l.

$l = 0$: by the construction.

$l = m+1$: by the construction, there is an $(s', m) \in \mathcal{M}^t, (\Sigma', m) \in \mathcal{M}_{\gamma'}^{\Pi'}$ such that $(s', m)R^t(s, m+1)$, $(\Sigma', m)R_{\gamma'}^{\Pi'}(\Sigma, m+1)$, and $(s', m) = f(\Sigma', m)$.

Subcase 1: $i, k \geq 1$

$$\begin{array}{lcc} s_i = s_k & & \Leftrightarrow \\ s'_{i-1} = s'_{k-1} & & \Leftrightarrow \\ (\Sigma', m) \vdash_{\Pi'} \Box(\delta_i \leftrightarrow \delta_k) & \text{IH} & \Leftrightarrow \\ (\Sigma', m) \vdash_{\Pi'} \Box\Box(\delta_{i+1} \leftrightarrow \delta_{k+1}) & [+1]' & \Leftrightarrow \\ (\Sigma, m+1) \vdash_{\Pi'} \Box(\delta_{i+1} \leftrightarrow \delta_{k+1}) & & \end{array}$$

Subcase 2: If $i = k = 0$, we are done. Otherwise, wlg $k > i = 0$

By the construction of $\mathcal{M}_{\gamma'}^{\Pi'}$, let δ_n be the minimal δ in Σ_0'.

$$\begin{array}{lll}
s_0 = s_k & & \Leftrightarrow \\
s_n = s_k & & \Leftrightarrow \\
s'_{n-1} = s'_{k-1} & & \Leftrightarrow \\
(\Sigma', m) \vdash_{\Pi'} \Box(\delta_n \leftrightarrow \delta_k) & \text{IH} & \Leftrightarrow \\
(\Sigma, m+1) \vdash_{\Pi'} \delta_k & & \Leftrightarrow \\
(\Sigma, m+1) \vdash_{\Pi'} \Box(\delta_{i+1} \leftrightarrow \delta_{k+1}) & \text{Id} &
\end{array}$$

□

Lemma 3 *For every* $(\Sigma, l) \in S_{\gamma'}^{\Pi'}$, $\delta_i \in Id$ *there is exactly one* $(\Sigma\gamma, l+1) \in S_{\gamma'}^{\Pi'}$ *such that* $(\Sigma\gamma, l+1) \vdash_{\Pi'} \delta_i$.

Proof:

Such a $(\Sigma\gamma, l+1)$ exists because $(\Sigma, l) \vdash_{\Pi'} \Box(\delta_i \rightarrow \top) \leftrightarrow \Diamond(\delta_i \wedge \top)$. Suppose there are two: $(\Sigma\gamma, l+1)$ and $(\Sigma\gamma'', l+1)$, then for every ϕ, $(\Sigma, l) \vdash_{\Pi'} \Box(\delta_i \rightarrow \phi) \leftrightarrow \Diamond(\delta_i \wedge \phi)$, so $(\Sigma\gamma, l+1)$ and $(\Sigma\gamma'', l+1)$ verify the same modal formulas, so γ and γ'' are the same Π'-maximal set. □

Lemma 4 f *is* $1-1$.

Proof:

For every $i \in \omega$, f_i maps objects of the form (Σ, i) to objects of the form (s, i), so it is enough to prove that every f_i is $1-1$.

By induction on i:

$i = 0$: obvious.

$i = m+1$: By the IH and the construction, the only relevant case is of the form $(sa, m+1) = f(\Sigma\gamma, m+1) = f(\Sigma\gamma'', m+1)$, with δ_{k+1} being the minimal $\delta \in \gamma$ and $\delta_{k'+1}$ being the minimal $\delta \in \gamma''$. So by the construction, $s_k = s_{k'}$, so by lemma 2 $(\Sigma, m) \vdash_{\Pi'} \Box(\delta_{k+1} \leftrightarrow \delta_{k'+1})$, so by lemma 3, $(\Sigma\gamma, m+1) = (\Sigma\gamma'', m+1)$. □

By definition 1 and lemma 4, f is a bisimulation between the domains of $\mathcal{M}_{\Gamma'}^{\Pi'}$ and $\mathcal{M}^t$.

Step 5: $\mathcal{M}^t$ is a relativized end-extension model.

Lemma 5 *For every $(s, l) \in S^t$ and δ_i*

$$(s, l) \models \delta_i \text{ iff } s_0 = s_i$$

Proof:

By lemma 2 and Id. □

Let $a_1, a_2, \ldots, a_h \in A_{\gamma'}$ be a positive number of element such that $a_1, a_2, \ldots, a_h$ appear together in the string part s of some $(s, l) \in S^t$. Let $t_{a_1, a_2, \ldots, a_h}$ be the (s, l) with the minimum l, such that every $a_g, 1 \leq g \leq h$ appears in s. By the construction such a $t_{a_1, a_2, \ldots, a_h}$ exists and is unique.

For every g, $1 \leq g \leq h$, let k_g be the difference between the level of $t_{a_1, a_2, \ldots, a_h}$ and t_{a_g}.

Lemma 6 *For every* $(sa_1a2\ldots a_j, l) \in S^t$ *and* p

$$(sa_1a2\ldots a_j, l) \models p \quad \text{iff}$$

$$f^{-1}(t_{a_1,a_2,\ldots,a_j}) \models \Box(\delta_{i_1+1} \to \Box(\delta_{i_2+2} \to \ldots \Box(\delta_{i_j+j} \to p \underbrace{)\ldots)}_{j \text{ times}}$$

Proof:

Let k the difference between the levels of $t_{a_1,a_2,\ldots,a_j}$ and $(sa_1a2\ldots a_j, l)$.

Subcase 1: $sa_1a2\ldots a_j = t_{a_1,a_2,\ldots,a_j} b_1 b2 \ldots b_m, m \geq j$.

$$\left.\begin{array}{l} t_{a_g} \\ \vdots \\ t_{a_1,a_2,\ldots,a_j} \end{array}\right\} k_g$$

$$\left.\begin{array}{l} b_1 \\ b_2 \\ \vdots \\ b_m \end{array}\right\} k$$

For all g, $1 \leq g \leq j$

$$(sa_1a_2\ldots a_j)_{j-g} = (sa_1a_2\ldots a_j)_{k+k_g} \qquad \Rightarrow$$

$$sa_1a_2\ldots a_j \models \Box(\delta_{j-g+1} \leftrightarrow \delta_{k+k_g+1}) \qquad \text{lemma 2} \quad \Rightarrow$$

$$sa_1a_2\ldots a_j \models \Box^g(\delta_j \leftrightarrow \delta_{k+k_g+g}) \qquad [+1']$$

So by lemma 1, we can replace every δ_{k+k_g+g} in the second line below by δ_j.

$$f^{-1}(t_{a_1,a_2,\ldots,a_j}) \models \Box(\delta_{k_1+1} \to \Box(\delta_{k_2+2} \to \ldots \Box(\delta_{k_j+j} \to p \underbrace{)\ldots)}_{j \text{ times}} \qquad \Leftrightarrow$$

$$f^{-1}(sa_1a_2\ldots a_j) \models \Box(\delta_{k_1+1+k} \to \Box(\delta_{k_2+2+k} \to \ldots \Box(\delta_{k_j+j+k} \to p \underbrace{)\ldots)}_{j \text{ times}} \quad [+1]' \quad \Leftrightarrow$$

$$f^{-1}(sa_1a_2\ldots a_j) \models \Box(\delta_j \to \Box(\delta_j \to \ldots \Box(\delta_j \to p \underbrace{)\ldots)}_{j \text{ times}} \qquad \Leftrightarrow$$

$$f^{-1}(sa_1a_2\ldots a_j) \models p$$

Subcase 2: $sa_1a2\ldots a_j = t_{a_1,a_2,\ldots,a_j}b_1b2\ldots b_m, m < j$.

For every $g < j - m$, $j - g = k + k_g$ by assumption. The rest is as in case 1. $\square$

By step 4 f is a bisimulation between the domains of $\mathcal{M}^{\Pi'}_{\Gamma'}$ and $\mathcal{M}^t$, and by step 5 $\mathcal{M}^t$ is a relativized end-extension model. So $\mathcal{M}^t$ is the desired relativized end-extension model and Π' is complete.

B Decidability

Definition 3 *A clipped relativized end-extension model for $\mathcal{L}$ is a triple $\mathcal{M} = (T, R, V)$, if is are a relativized end-extension model $\mathcal{M}' = (T', R', V')$ with root (s, l), $n, k \in \omega$ such that*

$$\begin{aligned} T &= \{(s', l') \in T' \mid l' \leq n + l\} \\ R &= R' \cap (T \times T) \\ V(p) &= V'(p) \cap T \end{aligned}$$

and for every s_i in the root (s,0) of T, $s_i = s_k$ for all $i \geq k$.

We call n the *depth* of the model $\mathcal{M}$.

Theorem 3 *For $\mathcal{L}$ with a finite P, every clipped relativized end-extension model can be bisimulated onto a finite clipped relativized end-extension model with the same root.*

Proof:

Let $\mathcal{M} = (T, R, V)$ be a clipped relativized end-extension model with root (s, l) and depth n.

The theorem is proved by induction on n:

$n = 0$: obvious

$n = k + 1$:

Step 1: Let U be the set of sons of (s, l). Every $(s', l + 1) \in U$ is the root of a clipped relativized end-extension model with depth n, so by the IH, it can be bisimulated onto a finite clipped relativized end-extension model with the same root.

Step 2: Let U' be the set of finite clipped relativized end-extension models obtained from relativized end-extension models in U in Step 1.

For every two relativized end-extension models $\mathcal{M}$ and $\mathcal{M}'$ in U' with roots t and t', define:

$$\mathcal{M} \sim \mathcal{M}' \textit{ iff } t, t' \textit{ verify the same modal formulas up to depth } k$$

$\sim$ is an equivalence relation. Since P is finite, and there is an h beyond which $\delta_i \leftrightarrow \delta_h$ for every $i \geq h$, it has only a finite number of classes. For each such class, let $(s'', l+1)$ be the root of a representative of the class. By lemma 5, for every member of a class there is a bisimulation that connects its root and the root of the representative.

Step 3: By composing the bisimuations in step 1 and 2, we get a bisimulation that connects every $(s', l+1)$ in the original relativized end-extension model to the root of one of a finite number of clipped relativized end-extension models with roots in U. Moreover, this bisimulation is onto, since for very representative in step 2, the bisimulation between it and itself can be assumed to be identity. The desired finite clipped relativized end-extension model is the one determined by the original root and the finite set of finite clipped relativized end-extension models with roots in U as its sons. □

By theorem 1, every formula ϕ is false in a relativized end-extension model $\mathcal{M}$ iff it is false in some finite clipped relativized end-extension model M'. Since Π' is complete for the class of relativized end-extension models models, non-membership in Π' is semi-decidable. As Π' is axiomatized,membership in Π' is semi-decidable too. So membership in Π' is decidable. □

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