Relative Strength of Strategy Elimination Procedures

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Executive Summary

We compare the relative strength of 4 procedures on finite strategic games:

iterated elimination of strategies that are

weakly/strictly

dominated by a

pure/mixed strategy.

Dominance by a Pure Strategy

	X	Y
A	$2, -$	$1, -$
B	$1, -$	$0, -$
C	$2, -$	$0, -$

- A strictly dominates B.
- A weakly dominates C.

Dominance by a Mixed Strategy

	X	Y
A	$2, -$	$0, -$
B	$0, -$	$2, -$
C	$0, -$	$0, -$
D	$1, -$	$0, -$

- $1/2A + 1/2B$ strictly dominates C.
- $1/2A + 1/2B$ weakly dominates D.

Iterated Elimination: Example

Consider

	L	M	R
T	$3, 2$	$2, 1$	$1, 0$
C	$2, 1$	$1, 1$	$4, 0$
B	$0, 4$	$0, 1$	$0, 0$

Which strategies are strictly dominated?

Iterated Elimination: Example, ctd

By eliminating B and R we get:

	L	M
T	$3,2$	$2,1$
C	$2,1$	$1,1$

Now C is strictly dominated by T, so we get:

	L	M
T	$3,2$	$2,1$

Now M is strictly dominated by L, so we get:

	L
T	$3,2$

4 Operators

Given: initial finite strategic game H.
G: a restriction of H ($G_i \subseteq H_i$).

- *LS*(G): outcome of eliminating from G all strategies strictly dominated by a pure strategy,
- *LW*(G): $\ldots$ weakly dominated by a pure strategy,
- *MLS*(G): $\ldots$ strictly dominated by a mixed strategy,
- *MLW*(G): $\ldots$ weakly dominated by a mixed strategy.

- Note For all G
  - *MLW*$(G) \subseteq$ *LW*$(G) \subseteq$ *LS*(G),
  - *MLW*$(G) \subseteq$ *MLS*$(G) \subseteq$ *LS*(G).

Iterated Elimination

- Do these inclusions extend to the outcomes of iterated elimination?
- None of these operators is monotonic.

Example

	X
A	$1, 0$
B	$0, 0$

Then

- $\mathit{LS}(H) = (\{A\}, \{X\})$,
- $\mathit{LS}(\{B\}, \{X\}) = (\{B\}, \{X\})$.
- So $(\{B\}, \{X\}) \subseteq H$, but not $\mathit{LS}(\{B\}, \{X\}) \subseteq \mathit{LS}(H)$.

Operators

T: operator on a finite lattice $(D, \subseteq)$.

- - $T^0 = D$,
  - T^k: k-fold iteration of T,
  - $T^\omega := \cap_{k \geq 0} T^k$.
- T is monotonic if

$$G \subseteq G' \text{ implies } T(G) \subseteq T(G').$$

Lemma T and U operators on a finite lattice $(D, \subseteq)$.

- For all G, $T(G) \subseteq U(G)$,
- at least one of T and U is monotonic.

Then $T^\omega \subseteq U^\omega$.

Approach

Given two strategy elimination operators Φ_l and Ψ_l such that for G

$$\Phi_l(G) \subseteq \Psi_l(G).$$

To prove

$$\Phi_l^\omega \subseteq \Psi_l^\omega$$

- we define their 'global' versions Φ_g and Ψ_g,
- prove $\Phi_g^\omega = \Phi_l^\omega$ and $\Psi_g^\omega = \Psi_l^\omega$,
- show that for all G

$$\Phi_g(G) \subseteq \Psi_g(G),$$

- show that at least one of Φ_g and Ψ_g is monotonic.

Global Operators

G: a restriction of H.
$s_i, s'_i \in H_i$.

- $s'_i \succ_G s_i$:

  $$\forall s_{-i} \in S_{-i}\ p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i})$$

- $s'_i \succ_G^w s_i$:
  $$\forall s_{-i} \in S_{-i}\ p_i(s'_i, s_{-i}) \geq p_i(s_i, s_{-i}),$$

  $$\exists s_{-i} \in S_{-i}\ p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}).$$

- *GS*$(G) := G'$, where

  $$G'_i := \{s_i \in G_i \mid \neg \exists s'_i \in H_i\ s'_i \succ_G s_i\}.$$

- Similar definitions for *GW*, *MGS*, *MGW*.

Strict Dominance

Lemma

- For all G
$$\textit{MLS}(G) \subseteq \textit{LS}(G).$$
- $\textit{GS}^{\omega} = \textit{LS}^{\omega}$.
- $\textit{MGS}^{\omega} = \textit{MLS}^{\omega}$.
(Brandenburger, Friedenberg and Keisler '06)
- For all G
$$\textit{MGS}(G) \subseteq \textit{GS}(G).$$
- *GS* and *MGS* are monotonic.

Conclusion: $\textit{MLS}^{\omega} \subseteq \textit{LS}^{\omega}$.

Weak Dominance

Lemma

- For all G
$$\mathit{MLW}(G) \subseteq \mathit{LW}(G).$$
- $\mathit{GW}^{\omega} = \mathit{LW}^{\omega}$.
- $\mathit{MGW}^{\omega} = \mathit{MLW}^{\omega}$.
(Brandenburger, Friedenberg and Keisler '06)
- For all G
$$\mathit{MGW}(G) \subseteq \mathit{GW}(G).$$
- *GS* and *MGS* are monotonic.

Conclusions: $\mathit{LW}^{\omega} \subseteq \mathit{LS}^{\omega}$ and $\mathit{MLW}^{\omega} \subseteq \mathit{MLS}^{\omega}$.

Weak Dominance, ctd

What about $MLW^{\omega} \subseteq LW^{\omega}$?

Consider

	X	Y	Z
A	$2,1$	$0,1$	$1,0$
B	$0,1$	$2,1$	$1,0$
C	$1,1$	$1,0$	$0,0$
D	$1,0$	$0,1$	$0,0$

Applying to *MLW* we get

	X	Y
A	$2,1$	$0,1$
B	$0,1$	$2,1$

Another application of *MLW* yields no change.

Weak Dominance, ctd

	X	Y	Z
A	$2, 1$	$0, 1$	$1, 0$
B	$0, 1$	$2, 1$	$1, 0$
C	$1, 1$	$1, 0$	$0, 0$
D	$1, 0$	$0, 1$	$0, 0$

Applying *LW* we first get

	X	Y
A	$2, 1$	$0, 1$
B	$0, 1$	$2, 1$
C	$1, 1$	$1, 0$

Weak Dominance, ctd

	X	Y
A	$2,1$	$0,1$
B	$0,1$	$2,1$
C	$1,1$	$1,0$

Applying *LW* again we get

	X
A	$2,1$
B	$0,1$
C	$1,1$

and then

	X
A	$2,1$

Rationalizability

- Rationalizability is defined as iterated elimination of globally never best responses to beliefs.
- Possible beliefs: pure strategies, uncorrelated mixed strategies or correlated mixed strategies.
- $GR(G) := G'$, where

$$G'_i := \{s_i \in G_i \mid \exists \mu_i \in G(\mathcal{B}_i) \forall s'_i \in H_i \, p_i(s_i, \mu_i) \geq p_i(s'_i, \mu_i)\}.$$

- This yields a monotonic operator.
- Consequently $GP^\omega \subseteq GU^\omega \subseteq GC^\omega$.
- Also $GC^\omega = MLS^\omega$.
  (Pearce '84)
- In particular $GP^\omega \subseteq LS^\omega$.

Epistemic Analysis

Theorem Take an arbitrary strategic game.
RAT$(\overline{\phi})$: each player i uses property ϕ_i to select his strategy ('each player i is ϕ_i-rational').
Suppose each ϕ_i is monotonic. Then the following sets of strategy profiles coincide:

- those that the players choose in the states in which **RAT**$(\overline{\phi})$ is common knowledge,
- those that the players choose in the states in which **RAT**$(\overline{\phi})$ is true and is common belief,
- those that remain after the iterated elimination of the strategies that are not ϕ_i-optimal.

The latter requires transfinite iterations.