Minimal Revision and Classical Kripke Models: First Results

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Classical Multi-Modal Kripke Models

- Prop is a finite set of atomic propositions $p, q, r, \ldots$
- Ind is a set of indices $a, b, c, \ldots$

$R : \mathsf{Ind} \to \mathcal{P}(W \times W)$

$\uparrow$

- A multi-modal Kripke «frame» $\mathcal{F}$ is a pair $(\ W\ ,\ R\ )$

$w \in W$

$\uparrow$

- A «pointed model» is a triple $\langle\ \mathcal{F}\ ,\ V\ ,\ w\ \rangle$

$\downarrow$

$V : \mathsf{Prop} \to \mathcal{P}(W)$

Semantics

$\mathcal{L}^0$ is the set of all well-formed formulas (wffs) ϕ composed as follows:

$$\phi \ ::= \ p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_a \phi$$

where $p \in \mathsf{Prop}, a \in \mathsf{Ind}$.

$$\begin{aligned}
\langle \mathcal{F}, V, w \rangle \Vdash p &\iff p \in V(p) \\
\langle \mathcal{F}, V, w \rangle \Vdash \neg\phi &\iff \text{not } \langle \mathcal{F}, V, w \rangle \Vdash \phi \\
\langle \mathcal{F}, V, w \rangle \Vdash (\phi \wedge \psi) &\iff \langle \mathcal{F}, V, w \rangle \Vdash \phi \text{ and } \langle \mathcal{F}, V, w \rangle \Vdash \psi \\
\langle \mathcal{F}, V, w \rangle \Vdash \Box_a \phi &\iff \forall x : \text{if } R(a)wx \text{ then } \langle \mathcal{F}, V, x \rangle \Vdash \phi
\end{aligned}$$

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\end{aligned}$$

$$\langle \mathcal{F}, V, w \rangle \Vdash [\$\phi]\psi \iff \forall \langle \mathcal{F}', V', w' \rangle \in \mathcal{D} : \text{if } \langle \mathcal{F}, V, w \rangle \xrightarrow{\$\phi} \langle \mathcal{F}', V', w' \rangle \text{ then } \langle \mathcal{F}', V', w' \rangle \Vdash \psi$$

Also: $\Diamond \phi \overset{\text{def}}{=} \neg \Box \neg\phi$ and $\langle \$\phi \rangle \psi \overset{\text{def}}{=} \neg[\$\phi]\neg\psi$

Fixed-Frame Minimal Revision

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For all $\phi \in \mathcal{L}^{\dagger}$ and pointed models $\langle \mathcal{F}, V, w \rangle$ and $\langle \mathcal{F}, V^{\star}, w \rangle$, let $\langle \mathcal{F}, V, w \rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^{\star}, w \rangle$ if and only if

1. $\langle \mathcal{F}, V^{\star}, w \rangle \Vdash \phi$ and
2. There is no valuation V' for $\mathcal{F}$ such that
   (a) $\langle \mathcal{F}, V', w \rangle \Vdash \phi$ and
   (b) $\delta(\mathcal{F}, V, V') \subset \delta(\mathcal{F}, V, V^{\star})$.

$\mathcal{L}^{\dagger}$ $\qquad \phi \ ::= \ p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_a \phi \mid [\dagger\,\phi]\,\phi$

$$\langle \mathcal{F}, V, w \rangle \Vdash [\dagger\,\phi]\,\psi \iff \forall V' : \text{if } V' \text{ is a valuation function for } \mathcal{F} \text{ and } \langle \mathcal{F}, V, w \rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V', w \rangle \text{ then } \langle \mathcal{F}, V', w \rangle \Vdash \psi$$

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$\mathcal{L}^{\dagger}$

$$\phi \ ::= \ p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_a \phi \mid [\dagger \phi]\, \phi$$

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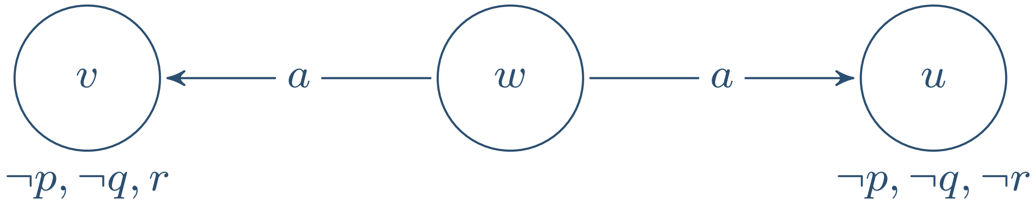
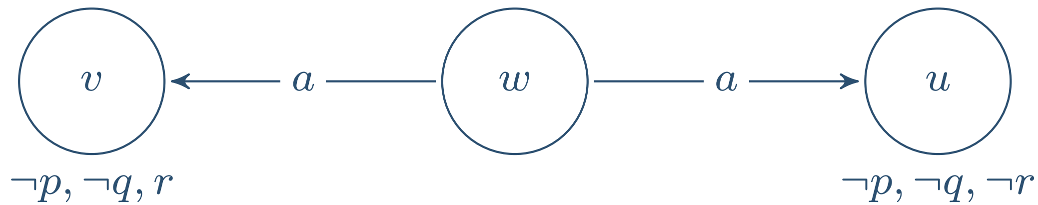
Example



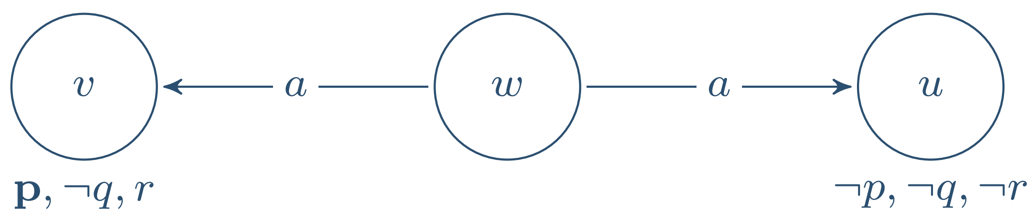
$$\Downarrow \dagger(\Diamond p \vee \Diamond(q \wedge r))$$

Example

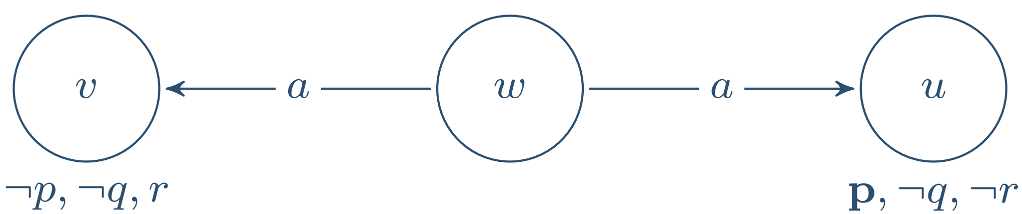


$$\Downarrow \dagger(\Diamond p \vee \Diamond(q \wedge r))$$

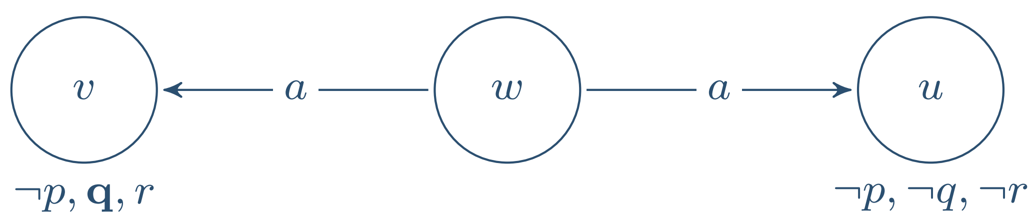
$\delta(\mathcal{F}, V, V') = \{(v, p)\}$:



$\delta(\mathcal{F}, V, V'') = \{(u, p)\}$:



$\delta(\mathcal{F}, V, V''') = \{(v, q)\}$:



Properties of $[\dagger\,\phi]$

- Not a Normal Modal Operator
  - Rule of Necessitation $\models \psi \implies \models [\dagger\,\phi]\,\psi$
  - K $\models([\dagger\,\phi](\psi \to \chi) \to ([\dagger\,\phi]\,\psi \to [\dagger\,\phi]\,\chi))$
  - Substitution fails $\models((p \wedge q) \to [\dagger\,\neg \mathrm{p}]\,q)$ but $\not\models((p \wedge p) \to [\dagger\,\neg \mathrm{p}]\,p)$
- AGM
  - Success $\models [\dagger\,\phi]\,\phi$
  - Finite Consistency $(\mathcal{F}, w) \models [\dagger\,\phi]\,\bot \implies (\mathcal{F}, w) \models \neg\phi$

    For all finite pointed models $\langle \mathcal{F}, V, w\rangle$, if a wff ϕ is satisfiable at $(\mathcal{F}, w)$ then there is a pointed model $\langle \mathcal{F}, V^\star, w\rangle$ such that $\langle \mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle \mathcal{F}, V^\star, w\rangle$.
  - Extensionality $\models(\phi \equiv \psi) \implies \models([\dagger\,\phi]\,\chi \equiv [\dagger\,\psi]\,\chi)$
  - Special Vacuity $\models((\phi \wedge \psi) \to [\dagger\,\phi]\,\psi)$
- Shift Unique and Shift Reflexive $\models [\dagger\,\phi]([\dagger\,\phi]\,\psi \equiv \psi)$

Generalized Minimal Revision

For all $\phi \in \mathcal{L}^{\ddagger}$ and all pointed models $\langle \mathcal{F}, V, w \rangle$ and $\langle \mathcal{F}^\star, V^\star, w^\star \rangle$, let $\langle \mathcal{F}, V, w \rangle \xrightarrow{\ddagger \phi} \langle \mathcal{F}^\star, V^\star, w^\star \rangle$ if and only if there is a valuation V' for $\mathcal{F}^\star$ such that

1. $\langle \mathcal{F}, V, w \rangle \underline{\leftrightarrow} \langle \mathcal{F}^\star, V', w^\star \rangle$ and
2. $\langle \mathcal{F}^\star, V', w^\star \rangle \xrightarrow{\dagger \phi} \langle \mathcal{F}^\star, V^\star, w^\star \rangle$.

$\mathcal{L}^{\ddagger}$ $\qquad \phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_a \phi \mid \langle \dagger \phi \rangle \phi \mid [\dagger \phi] \phi \mid \langle \ddagger \phi \rangle \phi \mid [\ddagger \phi] \phi.$

$$\langle \mathcal{F}, V, w \rangle \Vdash_\sigma [\ddagger \phi] \psi \iff \forall \langle \mathcal{F}^\star, V^\star, w^\star \rangle : \text{if } \langle \mathcal{F}^\star, V^\star, w^\star \rangle \text{ is a } \sigma\text{-model and } \langle \mathcal{F}, V, w \rangle \xrightarrow{\ddagger \phi} \langle \mathcal{F}^\star, V^\star, w^\star \rangle \text{ then } \langle \mathcal{F}^\star, V^\star, w^\star \rangle \Vdash_\sigma \psi$$

Generalized Minimal Revision

For all $\phi \in \mathcal{L}^{\ddagger}$ and all pointed models $\langle \mathcal{F}, V, w \rangle$ and $\langle \mathcal{F}^{\star}, V^{\star}, w^{\star} \rangle$, let $\langle \mathcal{F}, V, w \rangle \xrightarrow{\ddagger \phi} \langle \mathcal{F}^{\star}, V^{\star}, w^{\star} \rangle$ if and only if there is a valuation V' for $\mathcal{F}^{\star}$ such that

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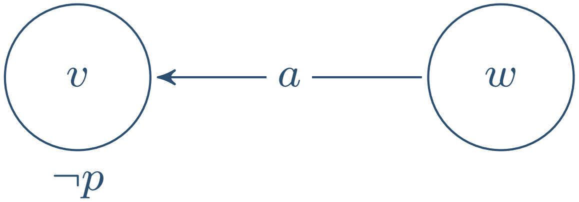
2. $\langle \mathcal{F}^{\star}, V', w^{\star} \rangle \xrightarrow{\dagger \phi} \langle \mathcal{F}^{\star}, V^{\star}, w^{\star} \rangle$.

$\mathcal{L}^{\ddagger}$ $$\phi \ ::= \ p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_a \phi \mid \langle \dagger \phi \rangle \phi \mid [\dagger \phi] \phi \mid \langle \ddagger \phi \rangle \phi \mid [\ddagger \phi] \phi.$$

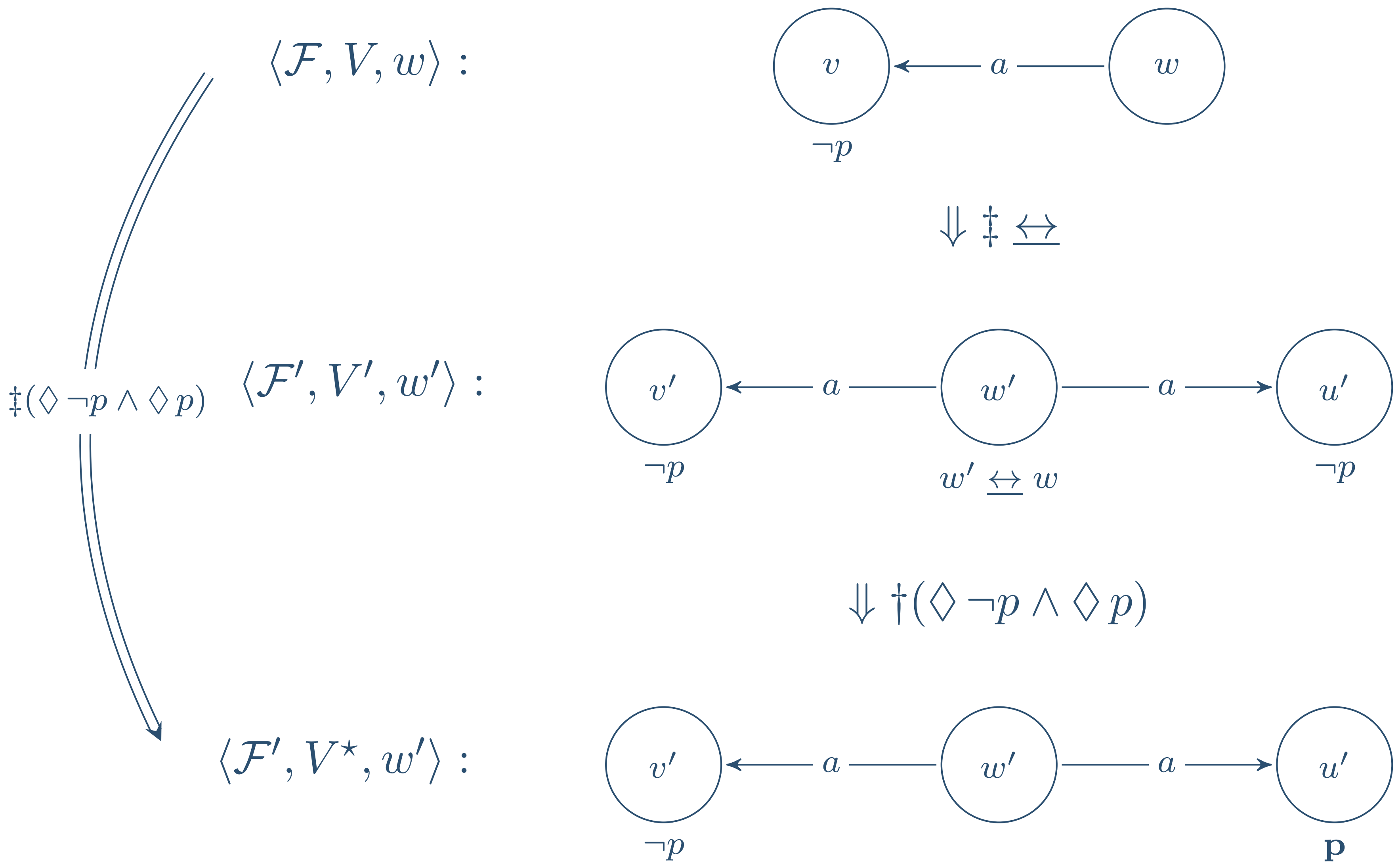
$$\langle \mathcal{F}, V, w \rangle \Vdash_{\sigma} [\ddagger \phi] \psi \iff \forall \langle \mathcal{F}^{\star}, V^{\star}, w^{\star} \rangle : \text{if } \langle \mathcal{F}^{\star}, V^{\star}, w^{\star} \rangle \text{ is a } \sigma\text{-model and } \langle \mathcal{F}, V, w \rangle \xrightarrow{\ddagger \phi} \langle \mathcal{F}^{\star}, V^{\star}, w^{\star} \rangle \text{ then } \langle \mathcal{F}^{\star}, V^{\star}, w^{\star} \rangle \Vdash_{\sigma} \psi$$

Example

$\langle \mathcal{F}, V, w \rangle$:



Example



How $\xrightarrow{\dagger\,\phi}$ and $\xrightarrow{\ddagger\,\phi}$ Relate

$\vDash_\sigma(\langle\dagger\,\phi\rangle\,\psi \rightarrow \langle\ddagger\,\phi\rangle\,\psi)$ and $\vDash_\sigma([\ddagger\,\phi]\,\psi \rightarrow [\dagger\,\phi]\,\psi)$

For all pointed models $\langle\mathcal{F}, V, w\rangle$ *and* $\langle\mathcal{F}, V^\star, w\rangle$ *it is the case that if* $\langle\mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle\mathcal{F}, V^*, w\rangle$ *then* $\langle\mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle\mathcal{F}, V^*, w\rangle$.

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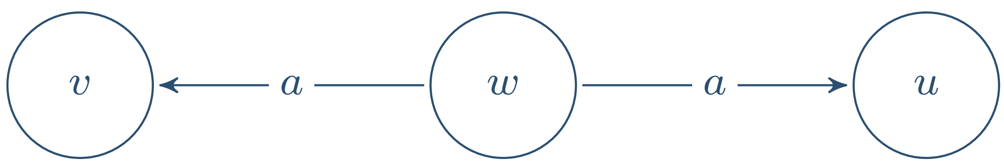
For all pointed models $\langle\mathcal{F}, V, w\rangle$ and $\langle\mathcal{F}, V^\star, w\rangle$ it is the case that if $\langle\mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle\mathcal{F}, V^*, w\rangle$ then $\langle\mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle\mathcal{F}, V^*, w\rangle$.

However: $\langle\mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle\mathcal{F}, V^*, w\rangle \not\Longrightarrow \langle\mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\phi} \langle\mathcal{F}, V^*, w\rangle$

$\mathcal{F}$:



Let $V(p) = \{v\}$ and $V'(p) = \{u\}$. It follows that

1. $\langle\mathcal{F}, V, w\rangle \underline{\leftrightarrow} \langle\mathcal{F}, V', w\rangle$
2. $\langle\mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\top} \langle\mathcal{F}, V, w\rangle$
3. From (2) and (3) it follows that $\langle\mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\top} \langle\mathcal{F}, V', w\rangle$

However, observe that not $\langle\mathcal{F}, V, w\rangle \xrightarrow{\dagger\,\top} \langle\mathcal{F}, V', w\rangle$

Properties of $[\ddagger\,\phi]$

- Almost a Normal Modal Operator
- AGM
  - Success $\quad \models_\sigma [\ddagger\,\phi]\,\phi$
  - Finite D-Consistency $\quad \models_\sigma [\ddagger\,\phi]\,\bot \implies \models_\sigma \neg\phi$

    If σ includes seriality and if $\phi \in \mathcal{L}^\ddagger$ is satisfiable in a finite σ-frame then for all finite pointed σ-models $\langle \mathcal{F}, V, w\rangle$ there is a pointed model $\langle \mathcal{F}^\star, V^\star, w^\star\rangle$ such that $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$.

  - Extensionality $\quad \models_\sigma (\phi \equiv \psi) \implies \models_\sigma ([\ddagger\,\phi]\,\chi \equiv [\ddagger\,\psi]\,\chi)$
  - Special Quasi-Vacuity $\quad \not\models_\sigma ((\phi \wedge \psi) \rightarrow [\ddagger\,\phi]\,\psi)$

    Nevertheless, for all pointed models $\langle \mathcal{F}, V, w\rangle$ and $\langle \mathcal{F}^\star, V^\star, w^\star\rangle$ such that $\langle \mathcal{F}, V, w\rangle \Vdash_\sigma \phi$ and $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$ it is the case that $\langle \mathcal{F}, V, w\rangle \underline{\leftrightarrow} \langle \mathcal{F}^\star, V^\star, w^\star\rangle$.

- Shift Quasi-Unique and Shift Reflexive $\quad \models_\sigma [\ddagger\,\phi]([\ddagger\,\phi]\,\psi \equiv \psi)$

$\not\models_\sigma ((\phi \wedge \psi) \rightarrow [\ddagger\, \phi]\, \psi)$

At least one instance of $((\phi \wedge \psi) \wedge \neg\, [\ddagger\, \phi]\, \psi)$ is satisfiable:
$((\top \wedge \langle\dagger(\Diamond_{\mathrm{a}}\, \mathrm{p} \wedge \Diamond_{\mathrm{a}}\, \neg\mathrm{p})\rangle\, \top) \wedge \neg\, [\ddagger\, \top]\, \langle\dagger(\Diamond_{\mathrm{a}}\, \mathrm{p} \wedge \Diamond_{\mathrm{a}}\, \neg\mathrm{p})\rangle\, \top)$

$\mathcal{F}$:

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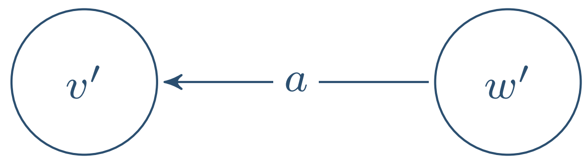

Let $V(p) = V'(p) = \emptyset$ for all $p \in \mathsf{Prop}$. It follows that

1. $\langle \mathcal{F}, V, w\rangle \underline{\leftrightarrow} \langle \mathcal{F}', V', w'\rangle$
2. $\langle \mathcal{F}', V', w'\rangle \xrightarrow{\dagger\, \top} \langle \mathcal{F}', V', w'\rangle$
3. From (1), (2): $\langle \mathcal{F}, V, w\rangle \xrightarrow{\ddagger\, \top} \langle \mathcal{F}', V', w'\rangle$

However, observe that
$\langle \mathcal{F}, V, w\rangle \Vdash_\sigma \langle\dagger(\Diamond_{\mathrm{a}}\, \mathrm{p} \wedge \Diamond_{\mathrm{a}}\, \neg\mathrm{p})\rangle\, \top$ but $\langle \mathcal{F}', V', w'\rangle \nVdash_\sigma \langle\dagger(\Diamond_{\mathrm{a}}\, \mathrm{p} \wedge \Diamond_{\mathrm{a}}\, \neg\mathrm{p})\rangle\, \top$

Counting Worlds

How to test whether three or more worlds are accessible from $\langle \mathcal{F}, V, w \rangle$ over a.

1. Take three incompatible non-modal formulas:
   $(p \wedge q), (p \wedge \neg q), (\neg p \wedge \neg q)$.

2. Prefix with a diamond:
   $\Diamond_a (p \wedge q), \Diamond_a (p \wedge \neg q), \Diamond_a (\neg p \wedge \neg q)$

3. Make it a conjunction:
   $((\Diamond_a (p \wedge q) \wedge \Diamond_a (p \wedge \neg q)) \wedge \Diamond_a (\neg p \wedge \neg q))$

4. Test if there's a valuation function that satisfies the conjunction:
   $\langle \mathcal{F}, V, w \rangle \Vdash_\sigma \langle \dagger ((\Diamond_{\mathrm{a}} (\mathrm{p} \wedge \mathrm{q}) \wedge \Diamond_{\mathrm{a}} (\mathrm{p} \wedge \neg \mathrm{q})) \wedge \Diamond_{\mathrm{a}} (\neg \mathrm{p} \wedge \neg \mathrm{q})) \rangle \top$

Characterizing Frame Conditions

Worlds accessible from $\langle \mathcal{F}, V, w \rangle$ over a in n steps are not accessible over a in m steps if and only if $\langle \mathcal{F}, V, w \rangle \Vdash_\sigma [\dagger\, \Box_{\mathrm{a}}^{\mathrm{n}}\, \mathrm{p}]\, [\dagger\, \Box_{\mathrm{a}}^{\mathrm{m}}\, \neg \mathrm{p}]\, \Box_a^n\, p$.[1]

$$\Downarrow n = 0, m = 1$$

All σ-frames are irreflexive if and only if $\vDash_\sigma [\dagger\, \mathrm{p}]\, [\dagger\, \Box_{\mathrm{a}}\, \neg \mathrm{p}]\, p$.

[1] Here $\Box_a^n\, \phi$ stands for $\underbrace{\Box_a \ldots \Box_a}_{n} \phi$.

Bisimilarity Operator

Once more

For all $\phi \in \mathcal{L}^{\ddagger}$ and all pointed models $\langle \mathcal{F}, V, w \rangle$ and $\langle \mathcal{F}^{\star}, V^{\star}, w^{\star} \rangle$, let $\langle \mathcal{F}, V, w \rangle \xrightarrow{\ddagger\,\phi} \langle \mathcal{F}^{\star}, V^{\star}, w^{\star} \rangle$ if and only if there is a valuation V' for $\mathcal{F}^{\star}$ such that

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Bisimulation operator

$$\begin{aligned}\langle \mathcal{F}, V, w \rangle \Vdash_\sigma [\underline{\leftrightarrow}]\phi \iff \forall \langle \mathcal{F}', V', w' \rangle : &\text{ if } \langle \mathcal{F}, V, w \rangle \underline{\leftrightarrow} \langle \mathcal{F}', V', w' \rangle \\ &\text{ and } \mathcal{F}' \text{ is a } \sigma\text{-frame} \\ &\text{ then } \langle \mathcal{F}', V', w' \rangle \Vdash_\sigma \phi\end{aligned}$$

$$\vDash_\sigma ([\ddagger \phi]\, \psi \leftrightarrow [\underline{\leftrightarrow}]\, [\dagger \phi]\, \psi)$$