

Epistemic Updates on Algebras

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Not amenable to a standard algebraic treatment.

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- added **modularity** (many choices of underlying logic);
- **finer-grained understanding** of epistemic updates.

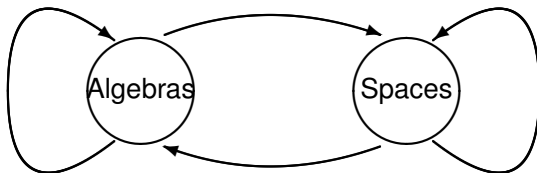
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Dualities are back-and-forth mappings between classes of mathematical objects which induce **systematic translations** of their constructs, relations and properties:

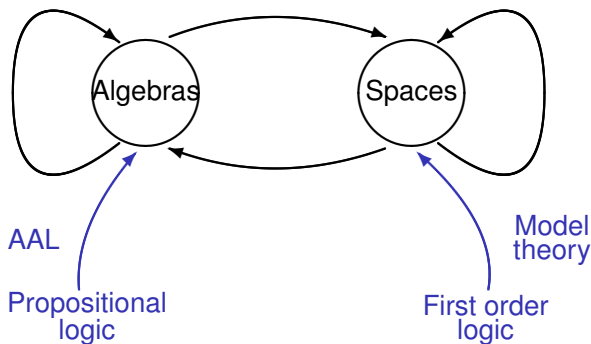
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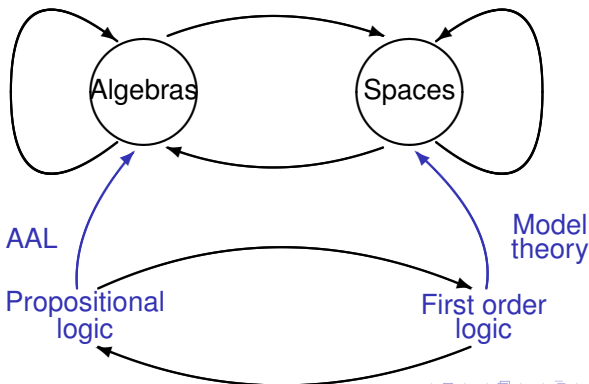
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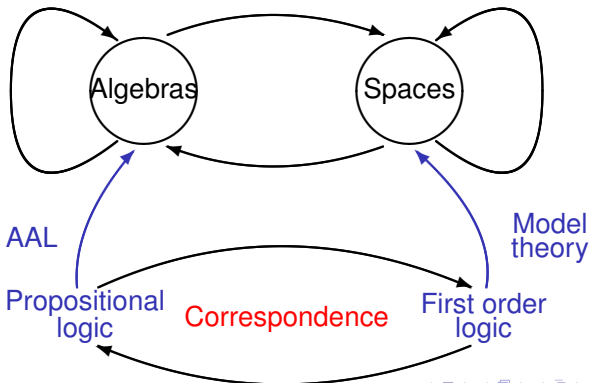
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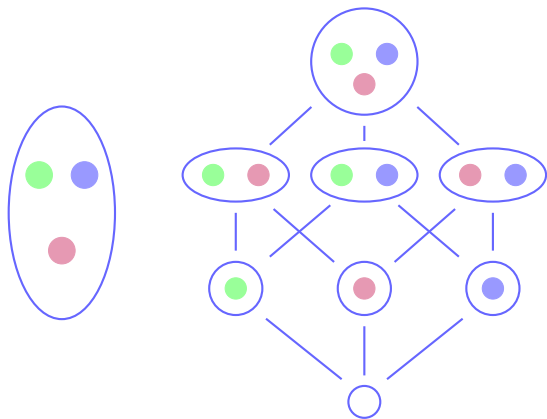


Key case: Sets and Boolean algebras

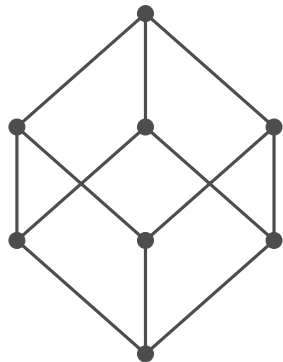
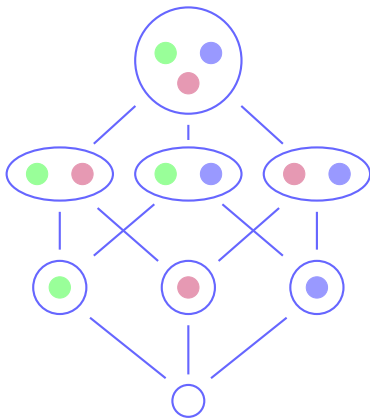
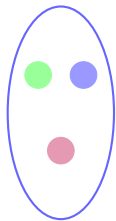
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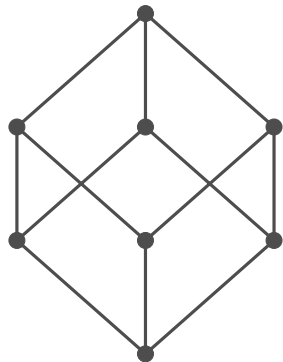
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- **Third step** Dualize back to relational models for non classically based logics.

Algebraic models

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$$\mathbb{A}^a := \mathbb{A} / \equiv_a$$

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Let $\pi_a : \mathbb{A} \rightarrow \mathbb{A}^a$ be the canonical projection.

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Crucial remark

Each \equiv_a -equivalence class has a canonical representant. Hence, the map $i' : \mathbb{A}^a \rightarrow \mathbb{A}$ given by $[b] \mapsto b \wedge a$ is injective.

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- If $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame \mathcal{F} , then $\mathbb{A}^a \cong_{BAO} \mathcal{F}^{a+}$.

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from which we get

$$\llbracket \langle \alpha \rangle \varphi \rrbracket_M = \llbracket \alpha \rrbracket_M \cap i[\llbracket \varphi \rrbracket_{M^\alpha}] = \llbracket \alpha \rrbracket_M \cap i'(\llbracket \varphi \rrbracket_{M^\alpha}). \quad (1)$$

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$M^\alpha := (\mathbb{A}^\alpha, V^\alpha)$ s.t. $\mathbb{A}^\alpha = \mathbb{A}^{\llbracket \alpha \rrbracket_M}$ and $V^\alpha : \text{AtProp} \rightarrow \mathbb{A}^\alpha$ is $\pi \circ V$, i.e. $\llbracket p \rrbracket_{M^\alpha} = V^\alpha(p) = \pi(V(p)) = \pi(\llbracket p \rrbracket_M)$ for every p .

Intuitionistic PAL

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Interaction with logical constants

$$\langle\alpha\rangle\perp = \perp$$

$$[\alpha]\top = \top$$

Preservation of facts

$$\langle\alpha\rangle p = \alpha \wedge p$$

$$[\alpha] p = \alpha \rightarrow p$$

Interaction with disjunction

$$\langle\alpha\rangle(\varphi \vee \psi) = \langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi$$

$$[\alpha](\varphi \vee \psi) = \alpha \rightarrow (\langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi)$$

Interaction with conjunction

$$\langle\alpha\rangle(\varphi \wedge \psi) = \langle\alpha\rangle\varphi \wedge \langle\alpha\rangle\psi$$

$$[\alpha](\varphi \wedge \psi) = [\alpha]\varphi \wedge [\alpha]\psi$$

Interaction with implication

$$\langle\alpha\rangle(\varphi \rightarrow \psi) = \alpha \wedge (\langle\alpha\rangle\varphi \rightarrow \langle\alpha\rangle\psi)$$

$$[\alpha](\varphi \rightarrow \psi) = \langle\alpha\rangle\varphi \rightarrow \langle\alpha\rangle\psi$$

Interaction with \diamond

$$\langle\alpha\rangle\diamond\varphi = \alpha \wedge \diamond\langle\alpha\rangle\varphi$$

$$[\alpha]\diamond\varphi = \alpha \rightarrow \diamond\langle\alpha\rangle\varphi$$

Interaction with \square

$$\langle\alpha\rangle\square\varphi = \alpha \wedge \square[\alpha]\varphi$$

$$[\alpha]\square\varphi = \alpha \rightarrow \square[\alpha]\varphi$$

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Work in progress:

Intuitionistic account of Muddy Children Puzzle.