Positive Formulas, Minimal Logic and Uniform Interpolation

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Subject of the talk

We recall the syntax and semantics of intuitionistic propositional logic IPC and minimal propositional logic MPC.

We introduce the top-model property and show that it characterizes the positive formulas of IPC.

We define a revised version of the uniform interpolation theorem and prove it for the positive fragment and for MPC.

We give some applications to intuitionistic predicate logic IQC.

We say something about the corresponding morphisms.

Positive formulas

Positive formulas are the ones with only $\wedge, \vee, \rightarrow$ (no $\neg, \bot$) and in predicate logic with the quantifiers.

They are interesting because it has been proposed in the past that intuitionistic logic should do without negation (Griss and others), Brouwer did not agree with that but still it is interesting to see what logically can be done without.

Another reason to see what can be done without negation is that constructive proofs of the completeness of intuitionistic first order logic meet great difficulties (Gdel, Kreisel), but these difficulties can be overcome it one restricts to the positive fragment (Friedman, Veldman).

There is a close relationship between the positive fragment and minimal logic, results about the one entail results about the other.

Kripke Models

A Kripke frame $\mathfrak{F} = \langle W, R \rangle$ consists of a set of worlds W and a reflexive partial ordering R.

A Kripke model $\mathfrak{M} = \langle W, R, V \rangle$ consists of a Kripke frame with a valuation V mapping propositional variables to an upward closed subset of W.

- $\mathfrak{M}, w \Vdash p \iff w \in V(p)$,
- $\mathfrak{M}, w \Vdash \varphi \wedge \psi \iff \mathfrak{M}, w \Vdash \varphi$ and $\mathfrak{M}, w \Vdash \psi$,
- $\mathfrak{M}, w \Vdash \varphi \vee \psi \iff \mathfrak{M}, w \Vdash \varphi$ or $\mathfrak{M}, w \Vdash \psi$,
- $\mathfrak{M}, w \Vdash \varphi \to \psi \iff$ for all w' with wRw', if $\mathfrak{M}, w' \Vdash \varphi$ then $\mathfrak{M}, w' \Vdash \psi$,
- $\mathfrak{M}, w \nVdash \bot$.

We write $w \Vdash \varphi$. Easily shown: If wRw' and $w \Vdash \varphi$, then $w' \Vdash \varphi$ (persistence).

In a Kripke model first order logic IQC domains D_w are assigned to each node so that $w\,R\,w' \Rightarrow D_w \subseteq D_{w'}$. Interpretations of relations are added so that each world w becomes a classical model $\mathfrak{M}_w$ in such a way that if $w\,R\,w'$ then $\mathfrak{M}_w$ is a submodel of $\mathfrak{M}_{w'}$. This guarantees persistence.

Minimal (propositional) logic (MPC) is obtained from the positive fragment of intuitionistic propositional logic IPC by adding a weaker negation:

$$\neg\varphi \text{ is defined as } \varphi \to f$$

where f is a special propositional variable: f has no specific properties, in particular $f \to \varphi$ does not hold. The Hilbert system for MPC is the same as IPC but without $f \to \varphi$.

For the semantics, f is interpreted as an ordinary propositional variable. Therefore, we will get the semantics as the $[\vee, \wedge, \to]$-fragment in IPC, with a special propositional variable f.

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DEFINITION

A top model is defined as a Kripke model for intuitionistic logic that has a largest node t such that all atomic sentences are true in t.

Any model $\mathfrak{M}$ can be turned into its top model $\mathfrak{M}^+$ by adding a node t at the top of the model, and making all atomic sentences true in t. In case of first order logic, $D_x = \bigcup_{w \in W} D_w$.

DEFINITION

φ has the top-model property when the following holds:

$$\mathfrak{M}, w \models \varphi \iff \mathfrak{M}^+, w \models \varphi.$$

In words, satisfaction in the old and new model is the same for φ.

Lemma

Let t be the top of any top model, and let φ be a positive sentence. Then $t \models \varphi$.

Theorem

Let φ be a positive formula, and $\mathfrak{M}$ be any Kripke-model. Then, for all $w \in W$ of $\mathfrak{M}$,

$$\mathfrak{M}, w \models \varphi \iff \mathfrak{M}^+, w \models \varphi$$

(φ has the top-model property).

Jankov's logic KC is defined as $\mathrm{IPC} + \neg\varphi \vee \neg\neg\varphi$. Its first order variant QKC as $\mathrm{IQC} + \neg\varphi \vee \neg\neg\varphi$. QKC is valid on top models.

Corollary

If φ is positive, then $\vdash_{\mathrm{IPC}} \varphi \iff \vdash_{\mathrm{KC}} \varphi$ and $\vdash_{\mathrm{IQC}} \varphi \iff \vdash_{\mathrm{QKC}} \varphi$.

The logic DNS (Double Negation Shift) is defined as $\mathrm{IQC} + \forall x \neg\neg\varphi(x) \to \neg\neg\forall x \varphi(x)$. DNS is valid on top models.

Corollary

If φ is positive, then $\vdash_{\mathrm{IQC}} \varphi \iff \vdash_{\mathrm{QKC+DNS}} \varphi$.
DNS *cannot be expressed by a positive formula (even on frames).*

The logic $\mathrm{CD} = \mathrm{IQC} + \forall x(\varphi(x) \vee \psi) \to \forall x \varphi(x) \vee \psi$ Intuitionistic Predicate Logic with Constant Domains is complete for models with constant domains.

Corollary

Assume φ is positive. Then $\vdash_{\mathrm{IQC+CD}} \varphi \iff \vdash_{\mathrm{QKC+CD+DNS}} \varphi$.

Construction of φ^+

Theorem

For each φ of IQC there exists φ^+ such that either φ^+ is positive, or $\varphi^+ = \bot$, and such that for all $\mathfrak{M}, w$, φ and φ^+ behave in the same way on top models:

$$\mathfrak{M}^+, w \models \varphi \iff \mathfrak{M}^+, w \models \varphi^+$$

Idea of the proof:

φ^+ is constructed in stages. In stage 0 we remove all occurrences of $\bot$ (and $\top$) by using equivalences like
$(\bot \wedge \psi) \sim \bot$, $(\bot \vee \psi) \sim \psi$, $(\bot \to \psi) \sim \top$, $\quad (\psi \to \bot) \sim \neg\psi$ (!).
We may stumble on $\varphi = \bot$, but then we are done immediately.

In stage 1 we first concentrate on an innermost occurrence of $\neg$. We replace this subformula $\neg\psi$ by $\bot$, noting that on top models the positive ψ is true in the top, so $\neg\psi$ is false everywhere.

Continuing, we act in the even stages as in stage 0 and in the odd stages as in stage 1 until no occurrences of $\bot$ or $\neg$ are left, φ^+ has been reached.

Theorem

If φ is has the top-model property, then there exists ψ such that ψ is positive or $\bot$, and $\vdash_{\mathsf{IQC}} \varphi \leftrightarrow \psi$.

Idea of the proof:

If φ has the top model property then it does not only behave the same as φ^+ on top models, but everywhere and is therefore equivalent to it.

Sidecomment:

One can follow a similar approach to disjunctionless formulas, but so far only for the propositional calculus.

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THEOREM (UNIFORM INTERPOLATION THEOREM FOR THE POSITIVE FRAGMENT OF IPC)

The post-interpolant

For any positive formula $\varphi(\vec{p}, \vec{q})$ where $\vec{p}$, $\vec{q}$ are disjoint, there is a positive formula $\theta(\vec{p})$ (the uniform post-interpolant for $\varphi(\vec{p}, \vec{q})$) such that

- $\vdash_{\mathsf{IPC}} \varphi(\vec{p}, \vec{q}) \rightarrow \theta(\vec{p})$,
- *For any positive $\psi(\vec{p}, \vec{r})$ where $\vec{r}$ and $\vec{p}$, $\vec{q}$ are disjoint, if $\vdash_{\mathsf{IPC}} \varphi(\vec{p}, \vec{q}) \rightarrow \psi(\vec{p}, \vec{r})$, then $\vdash_{\mathsf{IPC}} \theta(\vec{p}) \rightarrow \psi(\vec{p}, \vec{r})$.*
  Moreover, $\theta(\vec{p})$ is $(\exists \vec{q}\, \varphi)^+$, where $\exists \vec{q}\, \varphi$ is the uniform post-interpolant for φ in full IPC.

Given a post-interpolant for the full calculus the proof follows the same line of thought as in the characterization above.

The result is not trivial. The post-interpolant of $(p \rightarrow q) \rightarrow p$ in the full logic is $\neg\neg p$. In the positive fragment it is $(\neg\neg p)^+ = \top$.

Theorem (Uniform Interpolation Theorem for the positive fragment of IPC)

The pre-interpolant

For any positive formula $\psi(\vec{p}, \vec{r})$ where $\vec{p}$, $\vec{r}$ are disjoint, one of the following two cases holds:

1. *There is a formula $\theta(\vec{p})$, (the uniform pre-interpolant for $\psi(\vec{p}, \vec{r})$) such that $\vdash_{\mathsf{IPC}} \theta(\vec{p}) \to \psi(\vec{p}, \vec{r})$, and for any $\varphi(\vec{p}, \vec{q})$ where $\vec{q}$ and $\vec{p}$, $\vec{r}$ are disjoint, if $\vdash_{\mathsf{IPC}} \varphi(\vec{p}, \vec{q}) \to \psi(\vec{p}, \vec{r})$, then $\vdash_{\mathsf{IPC}} \varphi(\vec{p}, \vec{q}) \to \theta(\vec{p})$. Moreover, $\theta(\vec{p})$ is $(\forall \vec{q}\, \varphi)^+$, where $\forall \vec{q}\, \varphi$ is the uniform pre-interpolant for φ in full* IPC.
2. *For any positive $\varphi(\vec{p}, \vec{q})$ where $\vec{q}$ and $\vec{p}$, $\vec{r}$ are disjoint, $\nvdash_{\mathsf{IPC}} \varphi(\vec{p}, \vec{q}) \to \psi(\vec{p}, \vec{r})$.*

Same proof again. No surprise that some formulas only have $\bot$ as a uniform pre-interpolant (case 2), same in classical logic: take $p \to r$ for $\psi(\vec{p}, \vec{r})$).

The result is not trivial. The pre-interpolant of $((p \to q) \to p) \to p$ in the full logic is $\neg\neg p \to p$. In the positive fragment it is $(\neg\neg p \to p)^+ = p$.

Of course the analogous results hold for minimal logic. (There is a difference with IPC in that, if we write negations instead of f the interpolants for negation-free formulas need no negation in full MPC.)

Outline

The right morphisms for models for the positive fragment are partial:

DEFINITION (APOSTOLOS TZIMOULIS, ZHIGUANG ZHAO)

A partial function $f : \langle W, R, V\rangle \to \langle W', R', V'\rangle$ is a positive morphism if

1. If $w, v \in dom(f)$ and wRv, then $f(w)R'f(v)$,
2. If $w \in dom(f)$ and $f(w)R'v$, then there exists some $u \in dom(f)$ such that $f(u) = v$ and $w\,R\,u$ (back condition),
3. If $v \in dom(f)$ and $w\,R\,v$, then $w \in dom(f)$,
4. If $w \in dom(f)$, then $w \in V(p) \iff f(w) \in V'(p)$,
5. $\{w \in W \mid \exists p\,(w \notin V(p))\} \subseteq dom(f)$.

Properties of the morphisms

Positive morphisms correspond to Chagrov and Zakharyaschev's dense subreductions and are related to the strong partial Esakia morphisms of the Bezhanishvili's.

Positive morphisms preserve positive formulas.

An *n*-universal model can be defined for which the positive morphisms have the same properties as p-morphisms have with respect to the ordinary *n*-universal model. deJ-type Jankov-formulas can be defined by taking the $^{+}$-form of these characteristic formulas in the ordinary *n*-universal model. They behave appropriately.

Again this applies directly to minimal logic.

Future work

Coordinate better with the companion Coumans-van Gool paper.

Especially by developing the morphisms further.
Can we get a characterization of the uniform interpolants analogous to the characterization of the uniform interpolants in the full fragment as bisimulation quanifiers?

Get more applications to first order logic. At the moment no real application of the characterization of the positive formulas exists in that case.

THANKS