Admissibility and unifiability in contact logics

Philippe Balbiani[1] and Çiğdem Gencer[2]

[1] Institut de recherche en informatique de Toulouse
CNRS — Université de Toulouse
[2] Department of Mathematics and Computer Science
Istanbul Kültür University

Overview

1. **Introduction**
2. **Syntax and semantics of contact logics**
3. **Axiomatization and decidability of contact logics**
4. **Admissibility: definitions**
5. **Admissibility: useful lemmas**
6. **Admissibility: decidability**
7. **Unifiability**
8. **Conclusion and open problems**

Introduction

Unifiability in L: Given a formula $\phi(x_1, \ldots, x_n)$

- Determine whether there exists formulas $\psi_1, \ldots, \psi_n$ such that $\phi(\psi_1, \ldots, \psi_n) \in L$

References:

- Ghilardi, S.: *Unification in intuitionistic logic.* Journal of Symbolic Logic **64** (1999) 859–880.
- Ghilardi, S.: *Best solving modal equations.* Annals of Pure and Applied Logic **102** (2000) 183–198.
- Rybakov, V.: *A criterion for admissibility of rules in the model system S4 and the intuitionistic logic.* Algebra and Logic **23** (1984) 369–384.
- Rybakov, V.: *Admissibility of Logical Inference Rules.* Elsevier (1997).

Introduction

Admissibility in L: Given an inference rule $\frac{\phi(x_1,\ldots,x_n)}{\psi(x_1,\ldots,x_n)}$

- Determine whether for all formulas $\chi_1, \ldots, \chi_n$, if $\phi(\chi_1, \ldots, \chi_n) \in L$, $\psi(\chi_1, \ldots, \chi_n) \in L$

References:

- Ghilardi, S.: *Unification in intuitionistic logic.* Journal of Symbolic Logic **64** (1999) 859–880.
- Ghilardi, S.: *Best solving modal equations.* Annals of Pure and Applied Logic **102** (2000) 183–198.
- Rybakov, V.: *A criterion for admissibility of rules in the model system S4 and the intuitionistic logic.* Algebra and Logic **23** (1984) 369–384.
- Rybakov, V.: *Admissibility of Logical Inference Rules.* Elsevier (1997).

Introduction

Contact logics: Logics for reasoning about the contact relations between regular subsets in a topological space

Syntax:

- Regular regions (x, y, etc)
- Boolean operations: empty region (0), complement of a region ($-a$), union of two regions ($a \sqcup b$)
- Binary predicates: contact ($C(a, b)$), equality ($a \equiv b$)

References:

- Dimov, G., Vakarelov, D.: *Contact algebras and region-based theory of space: a proximity approach — I.* Fundamenta Informaticæ**74** (2006) 209–249.
- Vakarelov, D.: *Region-based theory of space: algebras of regions, representation theory, and logics.* In: *Mathematical Problems from Applied Logic. Logics for the XXIst Century. II.* Springer (2007) 267–348.

Introduction

Contact logics: Logics for reasoning about the contact relations between regular subsets in a topological space

Semantics:

- Contact algebras of regions
- Contact algebras of some classes of topological spaces
- Kripke structures regarded as adjancency spaces

References:

- Dimov, G., Vakarelov, D.: *Contact algebras and region-based theory of space: a proximity approach — I.* Fundamenta Informaticæ**74** (2006) 209–249.
- Vakarelov, D.: *Region-based theory of space: algebras of regions, representation theory, and logics.* In: *Mathematical Problems from Applied Logic. Logics for the XXIst Century. II.* Springer (2007) 267–348.

Syntax and semantics of contact logics

Terms:

- $a, b ::= x \in AT \mid \mathbf{0} \mid -\mathbf{a} \mid (\mathbf{a} \sqcup \mathbf{b})$

Formulas:

- $\phi, \psi ::= \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \mathbf{C}(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} \equiv \mathbf{b}$

Intuitive readings of terms and formulas:

- $\mathbf{0}$: empty region
- $-\mathbf{a}$: complement of region *a*
- $\mathbf{a} \sqcup \mathbf{b}$: union of regions *a* and *b*
- $\mathbf{C}(\mathbf{a}, \mathbf{b})$: regions *a* and *b* are in contact
- $\mathbf{a} \equiv \mathbf{b}$: regions *a* and *b* are equal

Syntax and semantics of contact logics

Terms:

- $a, b ::= x \in AT \mid \mathbf{0} \mid -\mathbf{a} \mid (\mathbf{a} \sqcup \mathbf{b})$

Formulas:

- $\phi, \psi ::= \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \mathbf{C}(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} \equiv \mathbf{b}$

Examples:

- $x \not\equiv 0 \rightarrow C(x, x)$
- $C(x, y) \rightarrow C(x, z) \vee C(-z, y)$
- $x \not\equiv 0 \wedge -x \not\equiv 0 \rightarrow C(x, -x)$
- $C(x, x) \vee C(-x, -x)$
- $x \sqcap y \not\equiv 0 \rightarrow C(x, y) \vee C(-y, y)$

Syntax and semantics of contact logics

Frames: $\mathcal{F} = (W, R)$

- W is a nonempty set of points
- R is a binary relation on W

Models: $\mathcal{M} = (W, R, V)$

- (W, R) is a frame
- $V : \ x \in AT \mapsto V(x) \subseteq W$ interprets all atomic terms

Interpretation of terms in model $\mathcal{M} = (W, R, V)$:

- $(x)^{\mathcal{M}} = V(x)$
- $(0)^{\mathcal{M}} = \emptyset$
- $(-a)^{\mathcal{M}} = W \setminus (a)^{\mathcal{M}}$
- $(a \sqcup b)^{\mathcal{M}} = (a)^{\mathcal{M}} \cup (b)^{\mathcal{M}}$

Syntax and semantics of contact logics

Frames: $\mathcal{F} = (W, R)$

- W is a nonempty set of points
- R is a binary relation on W

Models: $\mathcal{M} = (W, R, V)$

- (W, R) is a frame
- $V : x \in AT \mapsto V(x) \subseteq W$ interprets all atomic terms

Satisfiability of formulas in model $\mathcal{M} = (W, R, V)$:

- $\mathcal{M} \not\models \bot$
- $\mathcal{M} \models \neg\phi$ iff $\mathcal{M} \not\models \phi$
- $\mathcal{M} \models \phi \vee \psi$ iff either $\mathcal{M} \models \phi$, or $\mathcal{M} \models \psi$
- $\mathcal{M} \models C(a, b)$ iff $((a)^{\mathcal{M}} \times (b)^{\mathcal{M}}) \cap R \neq \emptyset$
- $\mathcal{M} \models a \equiv b$ iff $(a)^{\mathcal{M}} = (b)^{\mathcal{M}}$

Syntax and semantics of contact logics

Terms:

- $a, b ::= x \in AT \mid \mathbf{0} \mid -\mathbf{a} \mid (\mathbf{a} \sqcup \mathbf{b})$

Formulas:

- $\phi, \psi ::= \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \mathbf{C}(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} \equiv \mathbf{b}$

Translation: $\tau : \ \phi \mapsto \tau(\phi) \in \mathcal{L}(\Box, [U])$

- $\tau(\bot) = \bot$
- $\tau(\neg\phi) = \neg\tau(\phi)$
- $\tau(\phi \vee \psi) = \tau(\phi) \vee \tau(\psi)$
- $\tau(C(a, b)) = \langle U \rangle (a \wedge \Diamond b)$
- $\tau(a \equiv b) = [U](a \leftrightarrow b)$

Proposition (soundness of τ):

- $\mathcal{M} \models \phi$ iff $\mathcal{M} \models \tau(\phi)$

Syntax and semantics of contact logics

Terms:

- $a, b ::= x \in AT \mid \mathbf{0} \mid -\mathbf{a} \mid (\mathbf{a} \sqcup \mathbf{b})$

Formulas:

- $\phi, \psi ::= \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \mathbf{C}(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} \equiv \mathbf{b}$

Proposition (correspondence):

- $\mathcal{F} \models \mathbf{x} \not\equiv \mathbf{0} \to \mathbf{C}(\mathbf{x}, \mathbf{x})$ iff $\mathcal{F}$ is **reflexive**
- $\mathcal{F} \models \mathbf{C}(\mathbf{x}, \mathbf{y}) \to \mathbf{C}(\mathbf{x}, \mathbf{z}) \vee \mathbf{C}(-\mathbf{z}, \mathbf{y})$ iff $\mathcal{F}$ is **dense**
- $\mathcal{F} \models \mathbf{x} \not\equiv \mathbf{0} \wedge -\mathbf{x} \not\equiv \mathbf{0} \to \mathbf{C}(\mathbf{x}, -\mathbf{x})$ iff $\mathcal{F}$ is **connected**
- $\mathcal{F} \models \mathbf{C}(\mathbf{x}, \mathbf{x}) \vee \mathbf{C}(-\mathbf{x}, -\mathbf{x})$ iff $\mathcal{F}$ is **non-2-colourable**
- $\mathcal{F} \models \mathbf{x} \sqcap \mathbf{y} \not\equiv \mathbf{0} \to \mathbf{C}(\mathbf{x}, \mathbf{y}) \vee \mathbf{C}(-\mathbf{y}, \mathbf{y})$ iff $\mathcal{F}$ is **looping**

Axiomatization and decidability of contact logics

Axiomatization: Let λ_0 be the axiomatic system consisting of

- $\phi \to (\psi \to \phi)$
- $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$
- $\ldots$
- $a \sqcup (b \sqcup c) \equiv (a \sqcup b) \sqcup c$
- $a \sqcup b \equiv b \sqcup a$
- $\ldots$
- $C(a, b) \to a \not\equiv 0$
- $C(a, b) \to b \not\equiv 0$
- $C(a, b) \wedge a \leq c \to C(c, b)$
- $C(a, b) \wedge b \leq c \to C(a, c)$
- $C(a \sqcup b, c) \to C(a, c) \vee C(b, c)$
- $C(a, b \sqcup c) \to C(a, b) \vee C(a, c)$
- Modus ponens

Axiomatization and decidability of contact logics

Proposition:

- λ_0 is complete with respect to the class of all frames

Proposition:

- $\lambda_0 + \mathbf{a} \not\equiv \mathbf{0} \to \mathbf{C}(\mathbf{a}, \mathbf{a})$ is complete with respect to the class of all **reflexive** frames
- $\lambda_0 + \mathbf{C}(\mathbf{a}, \mathbf{b}) \to \mathbf{C}(\mathbf{a}, \mathbf{c}) \vee \mathbf{C}(-\mathbf{c}, \mathbf{b})$ is complete with respect to the class of all **dense** frames
- $\lambda_0 + \mathbf{a} \not\equiv \mathbf{0} \wedge -\mathbf{a} \not\equiv \mathbf{0} \to \mathbf{C}(\mathbf{a}, -\mathbf{a})$ is complete with respect to the class of all **connected** frames
- $\lambda_0 + \mathbf{C}(\mathbf{a}, \mathbf{a}) \vee \mathbf{C}(-\mathbf{a}, -\mathbf{a})$ is complete with respect to the class of all **non-2-colourable** frames
- $\lambda_0 + \mathbf{a} \sqcap \mathbf{b} \not\equiv \mathbf{0} \to \mathbf{C}(\mathbf{a}, \mathbf{b}) \vee \mathbf{C}(-\mathbf{b}, \mathbf{b})$ is complete with respect to the class of all **looping** frames

Axiomatization and decidability of contact logics

Remark: Contact logic has a Kripke-type semantics

- Standard translation into a first-order language
- Bounded morphism
- Bisimulation
- Canonical model construction
- Canonicity
- Sahlqvist theorem
- Filtration method

Proposition:

- If $\mathcal{C}$ is a class of frames definable by a first-order sentence with at most 2 variables, $\mathcal{C}$-satisfiability is decidable in nondeterministic exponential time
- If there exists a finite set Γ of axiom schemas such that $\lambda = \lambda_0 + \Gamma$, λ is decidable

Admissibility: definitions

Let λ be an extension of λ_0

Inference rules: $\frac{\phi(x_1,\ldots,x_n)}{\psi(x_1,\ldots,x_n)}$

Admissibility: $\frac{\phi(x_1,\ldots,x_n)}{\psi(x_1,\ldots,x_n)}$ is λ-admissible iff

- for all terms $a_1, \ldots, a_n$, if $\phi(a_1, \ldots, a_n) \in \lambda$, $\psi(a_1, \ldots, a_n) \in \lambda$

Proposition: If $\frac{\phi(x_1,\ldots,x_n)}{\psi(x_1,\ldots,x_n)}$ is λ-admissible

- $\lambda + \frac{\phi(x_1,\ldots,x_n)}{\psi(x_1,\ldots,x_n)}$ and λ have the same theorems

Examples:

- $\frac{C(x,y)}{C(y,x)}$ is admissible in λ_0
- $\frac{x \not\equiv 0 \wedge y \not\equiv 0 \rightarrow C(x,y)}{x \not\equiv 0 \wedge y \not\equiv 0 \rightarrow x \sqcap y \not\equiv 0}$ is admissible in $\lambda_0 + a \not\equiv 0 \rightarrow C(a, a)$

Admissibility: useful lemmas

Let λ be an extension of λ_0

Remark: $\frac{\phi(x_1,\ldots,x_n)}{\psi(x_1,\ldots,x_n)}$ is non-λ-admissible iff

- there exists $(a_1,\ldots,a_n)$ in the set A_n of all n-tuples of terms such that $\phi(a_1,\ldots,a_n) \in \lambda$ and $\psi(a_1,\ldots,a_n) \notin \lambda$

Equivalence relation $\cong^n_\lambda$ on A_n: $(a_1,\ldots,a_n) \cong^n_\lambda (b_1,\ldots,b_n)$ iff

- for all formulas $\phi(x_1,\ldots,x_n)$, $\phi(a_1,\ldots,a_n) \in \lambda$ iff $\phi(b_1,\ldots,b_n) \in \lambda$

Lemma: $\cong^n_\lambda$ has finitely many equivalence classes on A_n
Remark: λ-admissibility is decidable if λ is decidable and

- a complete set of representatives for each class on A_n modulo $\cong^n_\lambda$ can be effectively computed

Admissibility: useful lemmas

Let λ be an extension of λ_0

Remark: λ-admissibility is decidable if λ is decidable and

- a complete set of representatives for each class on A_n modulo $\cong^n_\lambda$ can be effectively computed

Equivalence relation $\simeq^n_\lambda$ on A_n: $(a_1, \ldots, a_n) \simeq^n_\lambda (b_1, \ldots, b_n)$ iff

- for all C-free formulas $\phi(x_1, \ldots, x_n)$, $\phi(a_1, \ldots, a_n) \in \lambda$ iff $\phi(b_1, \ldots, b_n) \in \lambda$

Lemma: $\cong^n_\lambda \subseteq \simeq^n_\lambda$
Lemma: $\simeq^n_\lambda$ has finitely many equivalence classes on A_n
Lemma: If λ is balanced, $\cong^n_\lambda \supseteq \simeq^n_\lambda$

Admissibility: decidability

Let λ be an extension of λ_0

Lemma: If λ is decidable

- a complete set of representatives for each class on A_n modulo $\simeq_\lambda^n$ can be effectively computed

Proposition: If λ is decidable and λ is balanced

- λ-admissibility is decidable

Proof: Given $\frac{\phi(x_1,\ldots,x_n)}{\psi(x_1,\ldots,x_n)}$

- compute a complete set $(a_1^1,\ldots,a_n^1),\ldots,(a_1^N,\ldots,a_n^N)$ of representatives for each class on A_n modulo $\simeq_\lambda^n$
- if there exists a positive integer k such that $k \leq N$, $\phi(a_1^k,\ldots,a_n^k) \in \lambda$ and $\psi(a_1^k,\ldots,a_n^k) \notin \lambda$, return *false*, else return *true*

Unifiability

Let λ be an extension of λ_0

Unifiability: $\phi(x_1, \ldots, x_n)$ is λ-unifiable iff

- there exists terms $a_1, \ldots, a_n$ such that $\phi(a_1, \ldots, a_n) \in \lambda$

Proposition: The following conditions are equivalent when λ is consistent

- $\phi(x_1, \ldots, x_n)$ is λ-unifiable
- $\frac{\phi(x_1,\ldots,x_n)}{\bot}$ is non-λ-admissible

Examples:

- $C(a, b) \rightarrow c \not\equiv 0$ is unifiable in λ_0 when either a and c are BA-unifiable, or b and c are BA-unifiable
- $a_1 \not\equiv 0 \wedge a_2 \not\equiv 0 \rightarrow C(a_3, a_4)$ is unifiable in $\lambda_0 + a \not\equiv 0 \rightarrow C(a, a)$ when a_1, a_2, a_3 and a_4 are BA-unifiable

Unifiability

Let λ be an extension of λ_0

Unifiability: $\phi(x_1, \ldots, x_n)$ is λ-unifiable iff

- there exists terms $a_1, \ldots, a_n$ such that $\phi(a_1, \ldots, a_n) \in \lambda$

Lemma: The following conditions are equivalent

- $\phi(x_1, \ldots, x_n)$ is λ-unifiable
- there exists $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}^n$ such that $\phi(\epsilon_1, \ldots, \epsilon_n) \in \lambda$

Proposition: λ-unifiability is *NP*-complete

Proposition: The following conditions are equivalent when $C(1, 1) \in \lambda$

- $\phi(x_1, \ldots, x_n)$ is non-λ-unifiable
- $\phi(x_1, \ldots, x_n) \to \bigvee\{x_i \not\equiv 0 \wedge x_i \not\equiv 1 \colon 1 \leq i \leq n\} \in \lambda$

Conclusion and open problems

Conclusion:

1. **Proposition:** If λ is decidable and λ is balanced
   - λ-admissibility is decidable
2. **Proposition:** λ-unifiability is *NP*-complete

Conclusion and open problems

Open problems:

1. Exact complexity of λ-admissibility?
2. Construction of bases of λ-admissible inference rules?
3. Unification type of λ?
4. Decidability/complexity of λ-admissibility with parameters?
5. Decidability/complexity of λ-unifiability with parameters?

Conclusion and open problems

References:

1. Dimov, G., Vakarelov, D.: *Contact algebras and region-based theory of space: a proximity approach — I.* Fundamenta Informaticæ**74** (2006) 209–249.
2. Ghilardi, S.: *Unification in intuitionistic logic.* Journal of Symbolic Logic **64** (1999) 859–880.
3. Ghilardi, S.: *Best solving modal equations.* Annals of Pure and Applied Logic **102** (2000) 183–198.
4. Rybakov, V.: *A criterion for admissibility of rules in the model system S*4 *and the intuitionistic logic.* Algebra and Logic **23** (1984) 369–384.
5. Rybakov, V.: *Admissibility of Logical Inference Rules.* Elsevier (1997).
6. Vakarelov, D.: *Region-based theory of space: algebras of regions, representation theory, and logics.* In: *Mathematical Problems from Applied Logic. Logics for the XXIst Century. II.* Springer (2007) 267–348.