Separating intermediate predicate logics of some linear orders

Norbert Preining
joint work with Arnold Beckmann

Japan Advanced Institute of Science and Technology

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Question

Take standard first order language.

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Same question with one (1) monadic predicate symbol?

The results

Theorem

If $0 \prec \alpha \prec \beta \prec \omega^\omega$ *with* $\beta \succeq \omega$*, then* $A_{\alpha,\beta} \in \mathsf{L}(\alpha)$*, but* $A_{\alpha,\beta} \notin \mathsf{L}(\beta)$.

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If $0 \prec \alpha \prec \beta \prec \omega^\omega$ *with* $\beta \succeq \omega$*, then* $A_{\alpha,\beta} \in L(\alpha)$*, but* $A_{\alpha,\beta} \notin L(\beta)$.

Theorem
If $0 \prec \alpha \prec \beta \prec \omega^\omega$*, then* $A^*_\alpha \in L(\alpha^*)$*, but* $A^*_\alpha \notin L(\beta^*)$.

Preliminaries

- $\mathcal{L}$ be a countable first-order language which includes the propositional constant $\bot$
- fix a universe of objects U
- Kripke frame (K, R) (usual conditions on domains and accessibility relation R), in addition assume R to be linear
- upward closed subsets of K: $Up(K)$, totally ordered by $\subseteq$
- smallest element $0_K = \emptyset$, largest element $1_K = K$
- intervals $[a, b]$ for $a, b \in Up(K)$
- LIN axiom: $(A \to B) \vee (B \to A)$
- CD axiom: $\forall x(A \vee B(x)) \to (A \vee \forall x B(x))$

Valuation

Let φ be a mapping from atomic formulas with constants for U into $Up(K)$.
Extension of φ to all well-formed formulas is defined as follows

- $\varphi(A \wedge B) = \varphi(A) \cap \varphi(B)$
- $\varphi(A \vee B) = \varphi(A) \cup \varphi(B)$
- $\varphi(A \to B) = \begin{cases} K & \varphi(A) \subseteq \varphi(B) \\ \varphi(B) & \text{otherwise} \end{cases}$
- $\varphi(\forall x A) = \bigcap\{\varphi(A(u)) \colon u \in U\}$
- $\varphi(\exists x A) = \bigcup\{\varphi(A(u)) \colon u \in U\}$

Definition of the logic

Definition

The *logic defined by a linear Kripke frame* $K = (W, R)$, denoted by $L(K)$, is the set of all $\mathcal{L}$-formulas A such that for all Kripke models (K, U) and all valuations φ of (K, U), $\varphi(A') = 1_K$, where A' is a closure of A.

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But: reasoning in Kripke frames is difficult, as we actually reason in the (linear) order of the upsets of the frame.

Fortunately in the linear case, we can switch sometimes to Gödel logics...

First Order Gödel Logics

Fix a truth value set $\{0, 1\} \subseteq V \subseteq [0, 1]$, V closed
Interpretation φ consists of

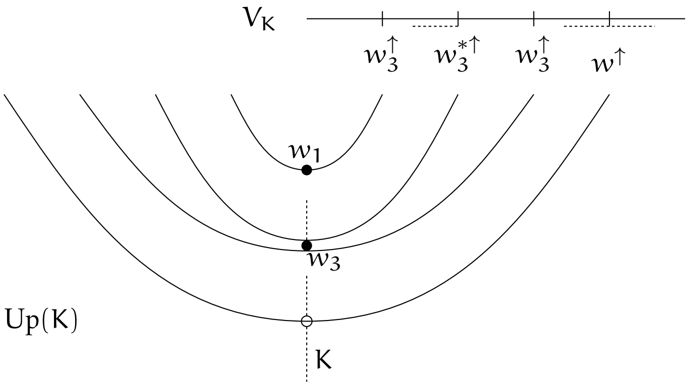
- a nonempty set U, the universe of φ
- for each k-ary predicate symbol P a function $P^\varphi : U^k \to V$
- for each variable x an object $x^\varphi \in U$

Extend the valuation to all formulas

- $\varphi(A \wedge B) = \min\{\varphi(A), \varphi(B)\}$ and
  $\varphi(A \vee B) = \max\{\varphi(A), \varphi(B)\}$
- $\varphi(A \to B) = \begin{cases} \varphi(B) & \text{if } \varphi(A) > \varphi(B) \\ 1 & \text{if } \varphi(A) \leqslant \varphi(B) \end{cases}$
- $\varphi(\forall x A(x)) = \inf\{\varphi(A(u)) \colon u \in U\}$
- $\varphi(\exists x A(x)) = \sup\{\varphi(A(u)) \colon u \in U\}$

Mapping Kripke worlds into the reals

Embed Up(K) into the truth value set such that the order and existing infima and suprema are preserved.



Equivalence result with linear Kripke frames

Gödel logic to Kripke frame

For each Gödel logic there is a countable linear Kripke frame such that the respective logics coincide.

Kripke frames to Gödel logic

For each countable linear Kripke frame there is a Gödel truth value set such that the respective logics coincide.

History

Timeline

finitely valued logics

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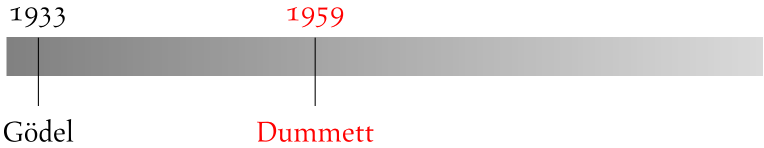

infinitely valued propositional Gödel logics

History

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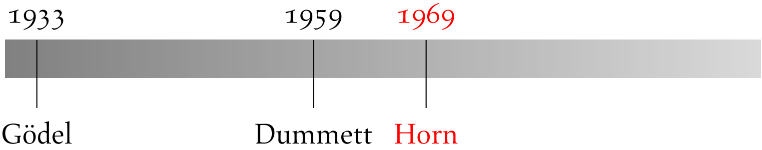

linearly ordered Heyting algebras

History

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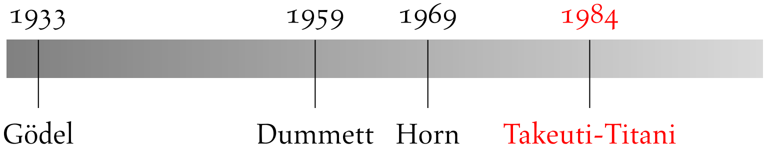
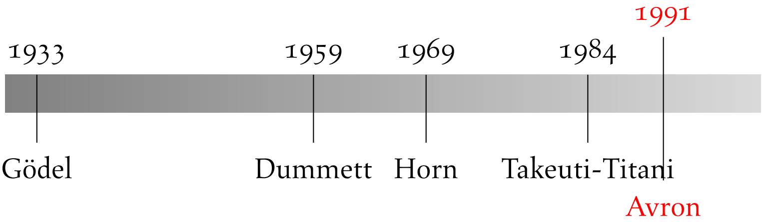


intuitionistic fuzzy logic

History

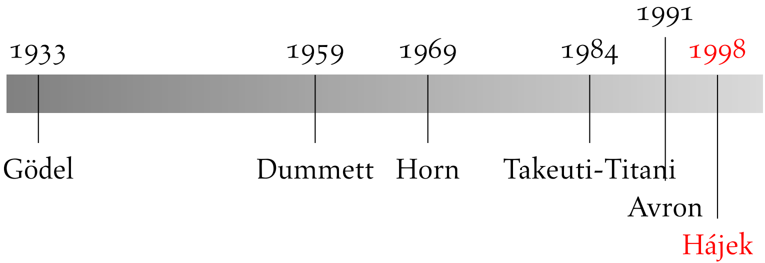
Timeline



hypersequent calculus

History

Timeline



t-norm based logics

History

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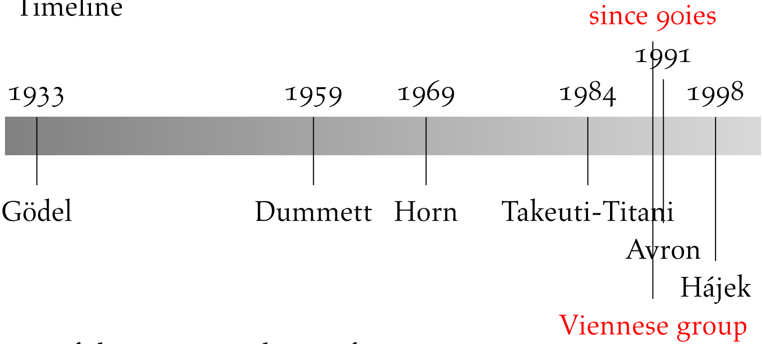

proof theory, #, Kripke, qp, fragments, . . .

Related work

- P. Minari, M. Takano, H. Ono. Intermediate predicate logics determined by ordinals. In *Journal of Symbolic Logic*, 55:3, pages 1099–1124, 1990.
- M. Baaz. Infinite-valued Gödel logics with 0-1-projections and relativizations. In *Gödel 96. Kurt Gödel's Legacy*, volume 6 of *LNL*, pages 23–33, 1996.
  (Separation of logics with different number of accumulation points.)
- N.P. Gödel logics and Cantor-Bendixon Analysis. In *Proceedings of LPAR'2002*, LNAI 2514, pages 327–336, 2002. (Separation of logics with different CB rank at 0 – the simple case here)

Descriptive Set Theory

Cantor-Bendixon Derivatives and Ranks

Polish spaces, i.e. separable, completely metrizable topological spaces. $\mathbb{R}$ is a Polish space: $X' = \{x \in X \colon x \text{ is limit point of } X\}$

Theorem (Cantor-Bendixon)

Let X be a polish space. For some countable ordinal α_0, $X^\alpha = X^{\alpha_0}$ for all $\alpha \geqslant \alpha_0$ (X^{α_0} is the perfect kernel).

CB Ranks for countable closed sets

- If X is countable, then $X^\infty = \emptyset$.
  (every perfect set has at least cardinality of the continuum)
- rank of an element: $\mathrm{rk}_{CB}(x) = \sup\{\alpha \colon x \in X^\alpha\}$
- rank of X: $\mathrm{rk}_{CB}(X) = \sup\{\mathrm{rk}_{CB}(x) \colon x \in X\}$

Logics under discussion

Kripke frame

For any ordinal $\kappa < \omega^\omega$ define two linear Kripke frames over constant domain $\mathsf{K}(\kappa)$ and $\mathsf{K}(\kappa^*)$ as

$$\mathsf{K}(\kappa) = (\kappa, \subseteq)$$
$$\mathsf{K}(\kappa^*) = (\kappa, \supseteq).$$

We consider the logics $\mathsf{L}(\kappa) = \mathsf{L}(\mathsf{K}(\kappa))$ and $\mathsf{L}(\kappa^*) = \mathsf{L}(\mathsf{K}(\kappa^*))$.

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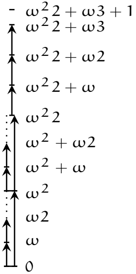
Theorem

The logics $\mathsf{L}(\alpha)$, $\mathsf{L}(\beta)$, $\mathsf{L}(\alpha^*)$, $\mathsf{L}(\beta^*)$ *for* $\omega \leqslant \alpha \neq \beta < \omega^\omega$ *can already be separated within the fragment of one monadic predicate symbol. (Finite cases are trivial)*

Kripke frames, upset order



$$\alpha = \omega^2 2 + \omega 3 + 1$$

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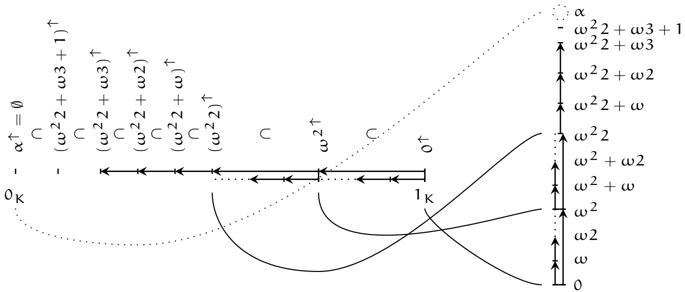

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Expressing orders

Relativized CB rank

Let $\mathrm{rk}_{\varphi\mathrm{CB}}(c) = \mathrm{rk}_{\mathrm{CB}}(c)$ in the closure of $\{\varphi(P(u)) \colon u \in U\}$

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Let $\mathrm{rk}_{\varphi\mathrm{CB}}(c) = \mathrm{rk}_{\mathrm{CB}}(c)$ in the closure of $\{\varphi(P(u)) : u \in U\}$

$A \prec B := (B \to A) \to B$

Evaluation: $\varphi(A \prec B) = \begin{cases} 1_K & \varphi(A) < \varphi(B) \\ \varphi(B) & \text{otherwise} \end{cases}$

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Evaluation: $\varphi(A \prec B) = \begin{cases} 1_K & \varphi(A) < \varphi(B) \\ \varphi(B) & \text{otherwise} \end{cases}$

$Q(c) := \forall x((Pc \prec Px) \to Px)$

Lemma:

$$\varphi(Q(c)) = \begin{cases} \varphi(P(c)) & \text{if } \varphi(P(c)) = 1_K \text{ or } \mathrm{rk}_{\varphi\mathrm{CB}}(c) \geqslant 1 \\ \mathrm{succ}(\varphi(P(c))) & \text{otherwise} \end{cases}$$

Expressing infima

Let

$$\begin{aligned} \mathrm{Inf}^0(x) &= \bot \to \bot \\ \mathrm{Inf}^{n+1}(x) &= \forall y((Px \prec Py) \to \exists z(\mathrm{Inf}^n(z) \wedge Px \prec Pz \prec Py)) \end{aligned}$$

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Core lemma

For $n > 0$ we have

$$\varphi(\mathrm{Inf}^n(c)) = \begin{cases} 1_K & \text{if } \varphi(P(c)) = 1_K \text{ or } \mathrm{rk}_{\varphi CB}(c) = n \\ \varphi(P(c)) & 0 < \mathrm{rk}_{\varphi CB}(c) < n \\ \mathrm{succ}(\varphi(P(c))) & \mathrm{rk}_{\varphi CB}(c) = 0 \end{cases}$$

Simple case – separation formula

In the following we consider only $\kappa = \omega^n$.

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Let

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Theorem

With the definitions from above, we have

$$A^n \notin L(K^n) \qquad (= G(V^n))$$

$$A^n \in L(K^m) \text{ for } m < n \qquad (= G(V^m))$$

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We have to give a counterexample, i.e., an evaluation that sends A^n to a value less then 1_K.

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Let $U = V^n$ and defined

$$\varphi(P(u)) = u$$

Then it is easy to see that for $x = 1$ and $y = 0$ we have

$$\begin{aligned}
\varphi(\mathrm{Inf}^n(1_K)) &= 1_K && \text{because 1 is always infima of all degrees} \\
\varphi(\mathrm{Inf}^n(0_K)) &= 1_K && \text{because } \mathrm{rk}_{\varphi CB}(0_K) = n \\
\varphi(Q(1_K)) &= 1_K && \text{see above} \\
\varphi(Q(0_K)) &= 0_K && \text{because } 0_K \text{ is not isolated}
\end{aligned}$$

and thus, $\varphi(A^n) = 0_K$.

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We have to show that for all possible valuations of x and y the inner formula is evaluated to 1_K.

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x	y	$\lvert x\rvert_{\varphi CB}$	$\lvert y\rvert_{\varphi CB}$	$Inf^n(x)$	$Inf^n(y)$	$Q(x)$	$Q(y)$	A^n
1	1	/	/	1	1	1	1	1
< 1	1	$0 < . < n$	/	x	1	x	1	1
		0	/	$succ(x)$	1	$succ(x)$	1	1
1	< 1	/	$0 < . < n$	1	y	1	y	1
		/	0	1	$succ(y)$	1	$succ(y)$	1
< 1	< 1	$0 < . < n$	$0 < . < n$	x	y	x	y	1
		$0 < . < n$	0	x	$succ(y)$	x	$succ(y)$	1
		0	$0 < . < n$	$succ(x)$	y	$succ(x)$	y	1
		0	0	$succ(x)$	$succ(y)$	$succ(x)$	$succ(y)$	1

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x	y	$\lvert x\rvert_{\varphi CB}$	$\lvert y\rvert_{\varphi CB}$	$Inf^n(x)$	$Inf^n(y)$	$Q(x)$	$Q(y)$	A^n
1	1	/	/	1	1	1	1	1
< 1	1	$0 < . < n$	/	x	1	x	1	1
		0	/	$succ(x)$	1	$succ(x)$	1	1
1	< 1	/	$0 < . < n$	1	y	1	y	1
		/	0	1	$succ(y)$	1	$succ(y)$	1
< 1	< 1	$0 < . < n$	$0 < . < n$	x	y	x	y	1
		$0 < . < n$	0	x	$succ(y)$	x	$succ(y)$	1
		0	$0 < . < n$	$succ(x)$	y	$succ(x)$	y	1
		0	0	$succ(x)$	$succ(y)$	$succ(x)$	$succ(y)$	1

This completes the proof for the simple case.

General case

Now assume we have to ordinals $\omega \preceq \alpha \prec \beta$

$$\alpha = \omega^n k_n + \cdots + \omega^0 k_0$$
$$\beta = \omega^n l_n + \cdots + \omega^0 l_0$$

for some finite $n, l_0, \ldots, l_n, k_0, \ldots, k_n$ with $n > 0$, with $n > 0$, $l_n > 0$, and since $\alpha < \beta$ there is maximal $d \leqslant n$ such that $k_d < l_d$. Let

$$\vec{x} = (x_1^{n+1}, x_1^n, \ldots, x_{l_n}^n, \ldots, x_1^d, \ldots, x_{l_d}^d),$$

General case cont.

For arbitrary variables, let

$$\text{chain}(x_1, \ldots, x_n) = (P(x_1) \to Q(x_2)) \vee \bigvee_{i=2}^{n-1} (P(x_i) \to P(x_{i+1})) \ .$$

General case cont.

For arbitrary variables, let

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Reminder:

$$\varphi(Q(c)) = \begin{cases} \varphi(P(c)) & \text{if } \varphi(P(c)) = 1_K \text{ or } \mathrm{rk}_{\varphi CB}(c) \geqslant 1 \\ \mathrm{succ}(\varphi(P(c)) & \text{otherwise} \end{cases}$$

General case cont.

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and define $A_{\alpha,\beta}(\vec{x})$ and $A_{\alpha,\beta}$ as follows:

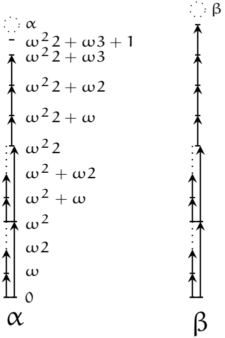
$$A_{\alpha,\beta}(\vec{x}) = \Big(\bigwedge_{u=d}^{n} \bigwedge_{i=1}^{l_u} \mathrm{Inf}^u(x_i^u) \Big) \to \mathrm{chain}(\vec{x})$$

and

$$A_{\alpha,\beta} = \forall \vec{x} A_{\alpha,\beta}(\vec{x}).$$

Example

$$\alpha = \omega^2 2 + \omega 3 + 1 \qquad \beta = \omega^2 2 + \omega 4 \qquad l_d = 1$$



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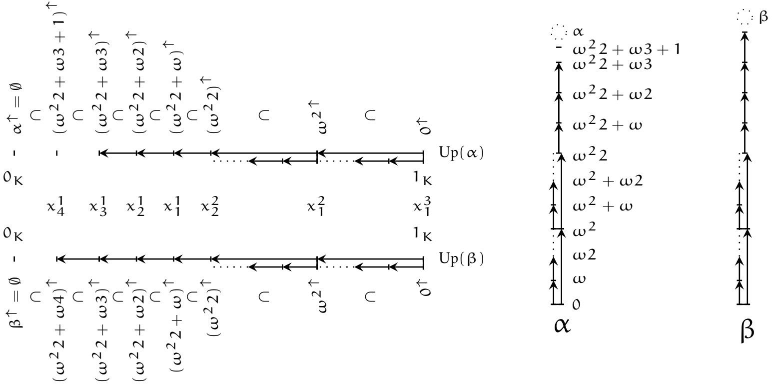

Separating the general case

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Future work - generalized CB-analysis

Combine the two methods (sup and inf ordering) to separate all logics in the class of *uniformly CB-structured* Kripke frames.

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At any CB derivation step, the set of isolated (i.e., to be dropped) points between any two remaining points (i.e., accumulation points at that level), is either an inf-set, a sup-set, or an inf-sup-set.

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Combine the two methods (sup and inf ordering) to separate all logics in the class of *uniformly CB-structured* Kripke frames.

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An infinite subset of isolated points of a linear order is an *inf-set* (*sup-set*; *inf-sup-set*) if it has a supremum (infimum; neither supremum nor infimum).